ISSN: 1401-5617

An example of improved hypercontractivity

Mats Erik Andersson

Research Reports in Mathematics
Number 17, 2000

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.matematik.su.se/reports/2000/17

Date of publication: December 19, 2000
2000 Mathematics Subject Classification: Primary 46E35, Secondary 47D99.
Keywords: Logarithmic Sobolev inequality, semigroup of operators.

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.matematik.su.se
info@matematik.su.se

AN EXAMPLE OF IMPROVED HYPERCONTRACTIVITY.

Mats Erik Andersson

The purpose of this presentation is to provide an example where improved information on hypercontractivity can be achieved. Improved in the sense that the result emanating from a logarithmic Sobolev inequality is bootstrapped to get a better degree of hypercontractivity.

The first section is devoted to the construction of a class of probability spaces together with operator semigroups that display a spectral gap between their only non-trivial eigenvalue and the corresponding logarithmic Sobolev exponent. The construction generalizes a three-point space used by the author in [A]. With a different interpretation similar probability spaces were built by Diaconis and Saloff-Coste in [DS]. All necessary background on hypercontractivity and logarithmic Sobolev inequalities are conveniently found in the exposition [G2].

The second section introduces two variants of Gross' theorem on how hypercontractivity may be deduced from logarithmic Sobolev inequalities. These variants will be be combined in section three in order to derive stronger hypercontractivity claims in any situation where a spectral gap is present, in particular to the model spaces from the first section. It should be stressed that the first and second sections are independent and that they are brought together in this document for the sole purpose of demonstrating that the mechanisms in section two can be applied to some non-trivial settings.

A class of operator semigroups based on simplices.

The underlying simplex. Denote the normalized surface measure on the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ by σ_n. It is clear that in the space of linear functions on $\mathbb{R}^n$ the function $k_n(x,y) = (1 + n\, x \cdot y)/(n+1)$ is the reproducing kernel on S^{n-1}. Clearly, the unit sphere is too large, as a point set, to avoid interdependencies of function values for linear functions. Let us reduce the underlying set to a simplex in order to get a situation admitting unique determination of linear functions.

Lemma 1. *There is a discrete, finite set $Q_n \subseteq S^{n-1}$, $|Q_n| = n+1$, $\sum_{x \in Q_n} x = 0$, such that all distances between different $x, y \in Q_n$ is constantly $\rho_n = \sqrt{2(n+1)/n}$.*

Excepting the first half of section two, the material in this presentation was developed during three useful months of the late summer and early Autumn of 1998 in Wrocław, Poland, where I visited Prof. Marek Bożejko. The visit was organized within an exchange program between the Polish and Royal Swedish Academies of Natural Sciences. I gratefully commend these two institutes as well as the Department of Mathematics at the University of Wrocław for a very influential and stimulating experience. To a still larger degree my gratitude goes to Marek Bożejko for having shared his knowledge and inspired me also in other directions not presented here.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Proof. Let $c_n = n^{-1}\sqrt{n^2-1}$ and consider

$$h_n : \mathbb{R}^{n-1} \to \mathbb{R}^n,\ x = (x_1, \dots, x_{n-1}) \mapsto (c_n x_1, \dots, c_n x_{n-1}, -\tfrac{1}{n}).$$

Thus $|h_n(x)|^2 = \frac{n^2-1}{n^2}|x|^2 + \frac{1}{n^2}$, so $h_n\big|_{S^{n-2}}$ is an isometric injection.

Consider $Q_1 = \{1, -1\}$ and $Q_n = h_n(Q_{n-1}) \cup \{(0, \dots, 0, 1)\}$. Inductively one has $|Q_n| = n+1$ and $Q_n \subseteq S^{n-1}$, due to the above isometry. Writing $X_n = \sum_{x \in Q_n} x$, the relations $X_1 = 0$ and $X_n = (c_n X_{n-1}, -1) + (0, \dots, 0, 1)$ follow. By induction all X_n are zero. Here $(c_n X_{n-1}, -1)$ has the coordinates of $c_n X_{n-1}$ in the first $n-1$ places.

Now suppose a particular Q_{n-1} possesses the property – as Q_1 trivially does – that the distance between its elements is constant; say ρ_{n-1}. Successive projection on the first $n-2, \dots, 2, 1$ coordinates, demonstrates that each Q_k, $k \leqslant n-1$, has constant distance ρ_k. Clearly, $\rho_k = c_k \rho_{k-1}$, so an explicit value obtains:

$$\rho_{n-1} = c_{n-1} \dots c_2 \rho_1 = \sqrt{\tfrac{2n}{n-1}}.$$

Between two elements in Q_n, both generated from Q_{n-1}, the distance is $c_n \rho_{n-1} = \sqrt{2(n+1)/n}$. From $(0, \dots, 0, 1)$ to any other element in Q_n, the distance is

$$\left[c_n^2 + (1 + \tfrac{1}{n})^2\right]^{1/2} = \sqrt{\tfrac{2(n+1)}{n}}.$$

This completes the proof of the claim.

Proposition 2. *Every linear function $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto a_0 + a_1 x_1 + \dots + a_n x_n$ is given by the representation $f(x) = \sum_{y \in Q_n} f(y)\, k_n(x, y)$.*

Proof. Any two $x, y \in Q_n$ together with the origin span a triangle. After rotations all these triangles are congruent by the lemma above. Hence $x \cdot y$ is constant for $y \in Q_n \setminus \{x\}$ and $x \in Q_n$. Due to $\sum_{y \in Q_n \setminus \{x\}} y = -x$ the common value is $-1/n$ and so

$$k_n(x, y) = \begin{cases} 1, & x = y \in Q_n, \\ 0, & x \neq y, \text{ both in } Q_n. \end{cases}$$

Based on the reproducing property on S^{n-1}, the above values of k_n on $Q_n \times Q_n$ validate the claimed representation of linear functions.

Corollary 3. *For any $y \in Q_n$ the positivity $k_n(\cdot, y) \geqslant 0$ obtains throughout the closed convex hull $\overline{\operatorname{conv}}\, Q_n$.*

This immediate result will be instrumental later on.

Construction of an operator semigroup. We next aim at the construction of a semigroup acting on all linear function defined on $\mathbb{R}^n$; the underlying space will be Q_n. Take for μ_n the discrete probability measure $\sum_{\omega \in Q_n} (n+1)^{-1}\, \delta_\omega$. Every function norm $\|f\|_2$ is taken with respect to this measure; likewise for inner products. Writing $K_n(x, \omega) = 1 + n x \cdot \omega$, Proposition 2 expresses for all linear functions the reproducing formula

$$f(x) = \int_{Q_n} f(\omega) K_n(x, \omega)\, d\mu_n(\omega).$$

This means that any function defined on Q_n has a unique extension to $\mathbb{R}^n$ as a linear function, and vice versa. By Corollary 3, the linear extension is positive on $\overline{\operatorname{conv}}\, Q_n$ if and only it is positive on Q_n. (This statement should not be understood as expressing only strict positivity!)

The (linear) functions, positive on Q_n, and of mean value 1 there can be written

$$(*)\qquad f(x) = 1 + \sum_{\omega\in Q_n} \alpha_\omega\, nx\cdot\omega, \quad \alpha_\omega \geqslant 0,\ \sum \alpha_\omega = 1,$$

since they are convex combinations of $K_n(\cdot,\omega)$, $\omega \in Q_n$. The projection on the subspace of constant functions is $Ef = \int f\, d\mu_n$; denote the complementary projection $A = E^\perp = I - E$. The definition

$$P_r = E + rA, \quad P_r f(x) = f(0) + \sum_{\omega\in Q_n} r\alpha_\omega\, nx\cdot\omega$$

yields a multiplicatively parametrized semigroup for $0 \leqslant r \leqslant 1$. The corresponding infinitesimal generator is of course A.

For $f \geqslant 0$ the logarithmic Sobolev inequality leads to the quantity

$$Q(f) = \frac{\int f^2 \log f\, d\mu_n - \|f\|_2^2 \log \|f\|_2}{\langle Af, f\rangle} = \frac{\int f^2 \log f\, d\mu_n - \frac{1}{2}\|f\|_2^2 \log \|f\|_2^2}{\|f\|_2^2 - 1},$$

where the last equality follows upon the normalization $f(0) = Ef = 1$. Since every non-trivial eigenvalue of A is 1, the identity $\langle Af, f\rangle = \|f\|^2 - 1 = \|Af\|_2^2$ obtains. Observe that Jensen's inequality ensures that $Q(f)$ is positive for non-constant functions.

Problem. Determine $\sup\{Q(f)\,;\, f \text{ linear}, f \geqslant 0 \text{ on } Q_n, f(0) = 1\}$.

The supremum is attained since the the underlying set is a compact subset of an $(n+1)$-dimensional linear variety. For convenience a special positive cone is introduced as follows:

$$N = \{\, f : \mathbb{R}^n \to \mathbb{R} \text{ linear}, f \geqslant 0 \text{ on } Q_n, f(0) = 1\}.$$

Any $f \in N$ has a representation as in $(*)$. For functions $f, g \in N$ we consider a variational functional defined as

$$\operatorname{Var}_f(g) = \big(\langle f, g\rangle - 1\big)\, Q(f) - \int gf \log f\, d\mu_n + \langle f, g\rangle \log \|f\|_2.$$

Proposition 4. *$f \in N$ is a critical point for Q exactly when $\operatorname{Var}_f(g) = 0$, all $g \in N$.*

Proof. If $f, g \in N$, $0 \leqslant \rho \leqslant 1$, then also $\rho f + (1-\rho)g \in N$. Let $\Omega h = \frac{\partial}{\partial\rho} h(\rho)\big|_{\rho=1^-}$. Obviously we have for any critical point f of Q that $\Omega\big[Q\big(\rho f + (1-\rho)g\big)\big] = 0$ for all $g \in N$. In order to apply this observation, we fix an element $g \in N$ and write $R_\rho f = \rho f + (1-\rho)g$, whence $R_\rho f\big|_{\rho=1} = f$. We need an explicit expression for $\Omega\big[Q(R_\rho f)\big]$. The identity

$$Q(R_\rho f) = \frac{\int (R_\rho f)^2 \log R_\rho f\, d\mu_n - \frac{1}{2}\|R_\rho f\|_2^2 \log \|R_\rho f\|_2^2}{\|R_\rho f\|_2^2 - 1}$$

yields as a first stepping stone the following expansion.

$$\begin{aligned}\Omega\big[Q(R_\rho f)\big] =& \frac{\int \Omega[R_\rho f]\{2f\log f + f\}\,d\mu_2 - \frac{1}{2}\Omega[\|R_\rho f\|_2^2]\{\log\|f\|_2^2+1\}}{\|f\|_2^2-1} \\ &\qquad - \frac{\Omega[\|R_\rho f\|_2^2]}{\|f\|_2^2-1}\cdot\frac{\int f^2\log f\,d\mu_n - \frac{1}{2}\|f\|_2^2\log\|f\|_2^2}{\|f\|_2^2-1}.\end{aligned}$$

Since $\Omega[R_\rho f] = f - g$, it is straightforward to verify $\Omega[\|R_\rho f\|_2^2] = 2\|f\| - 2\langle f,g\rangle$. Thus the above simplifies to

$$\begin{aligned}\Omega[Q(R_\rho f)] =& \frac{\int (f-g)\{2f\log f + f\}\,d\mu_n - \{\|f\|_2^2 - \langle f,g\rangle\}\{1+\log\|f\|_2^2\}}{\|f\|_2^2-1} \\ &- 2Q(f)\,\frac{\|f\|_2^2-\langle f,g\rangle}{\|f\|_2^2-1}.\end{aligned}$$

The first numerator can be simplified to

$$2\,Q(f)\{\|f\|_2^2-1\} + \langle f,g\rangle\log\|f\|_2^2 - 2\int fg\log f\,d\mu_n.$$

Hence it follows that

$$\begin{aligned}\Omega\big[Q(R_\rho f)\big]\,\{\|f\|_2^2-1\} &= 2Q(f)\,\big\{\|f\|_2^2 - 1 - \|f\|_2^2 + \langle f,g\rangle\big\} + \langle f,g\rangle\log\|f\|_2^2 \\ &\qquad - 2\int fg\log g\,d\mu_n \\ &= 2\,\mathrm{Var}_f(g).\end{aligned}$$

The extremality of $Q(f)$ forces $\Omega[Q(R_\rho f)] = 0$, whence $\mathrm{Var}_f(g) = 0$. Since $g \in N$ was arbitrary, one half of the claim follows. The remaining part is simpler still, just retrace the above argument backwards and disregard the intermediate derivations.

Corollary 5. *For any critical point of $Q(f)$, the relation $\int f\log f\,d\mu_n = \log\|f\|_2$ holds.*

Proof. Summing $\mathrm{Var}_f(K_n(\cdot,\omega)) = 0$ over all $\omega \in Q_n$ yields

$$\sum\nolimits_{\omega\in Q_n}\big(f(\omega)-1\big)\,Q(f) - \sum\nolimits_{\omega\in Q_n} f(\omega)\log f(\omega) + \sum\nolimits_{\omega\in Q_n} f(\omega)\log\|f\|_2 = 0,$$

which reduces to

$$0\cdot Q(f) - (n+1)\int f\log f\,d\mu_n + (n+1)\log\|f\|_2 = 0$$

since $f \in N$. This was the claim.

Remark. For $f \in N$, the measure $d\nu = f\,d\mu_n$ is a probability measure. Since $\log t$ is concave, Jensen's inequality gives

$$\int f\log f\,d\mu_n = \int \log f\,d\nu \leqslant \log\int f\,d\nu = 2\log\|f\|_2.$$

The corollary thus claims that, for any extremal function f, $\int f\log f\,d\mu_n$ takes half its maximal possible value as *de facto* value.

Proposition 6. *The following two statements are equivalent.*

(1) f *is a critical point for* Q,

(2) $\displaystyle\int g\, f \log f \, d\mu_n = \langle f, g\rangle \big\{ \log \|f\|_2 + Q(f) \big\} - Q(f)$, *all* $g \in N$.

Proof. According to Proposition 4, $f \in N$ is a critical point if and only if $\mathrm{Var}_f \equiv 0$. When explicitly writing $\mathrm{Var}_f(g) = 0$ the relation

$$\int f g \log f \, d\mu_n = \big(\langle f, g\rangle - 1\big)\, Q(f) + \langle f, g\rangle \, \log \|f\|_2$$

appears, which readily is seen to coincide with the second property.

Lemma 7. *For* $f \in N$, *the value* $f(\omega) = 1$ *is attained if and only if* $\alpha_\omega = 1/(n{+}1)$.

Proof. The representation $f(x) = 1 + \sum_{\omega \in Q_n} \alpha_\omega \, n x \cdot \omega$ with $\sum \alpha_\omega = 1$ clearly gives the equivalences (use $x \cdot \omega = -1/n$ on Q_n, $x \neq \omega$)

$$\begin{aligned} f(\omega) = 1 \quad &\Leftrightarrow \quad \sum\nolimits_{x \in Q_n} \alpha_x \, n x \cdot \omega = 0 \quad \Leftrightarrow \quad n\alpha_\omega - \sum\nolimits_{x \neq \omega} \alpha_x = 0 \\ &\Leftrightarrow \quad n\,\alpha_\omega - (1 - \alpha_\omega) = 0 \quad \Leftrightarrow \quad (n+1)\alpha_\omega = 1, \end{aligned}$$

which resolves the claim.

Proposition 8. *If* $f \in N$ *is a critical point of* Q, *then either* $f \equiv 1$ *or* $f \neq 1$ *on* Q_n. *Equivalently, either all* α_ω *take the value* $1/(n+1)$, *or none at all.*

Proof. Assume $f \in N$ to be a critical point with $f(\omega) = 1$ for some $\omega \in Q_n$. Applying Proposition 6 with $g(x) = K_n(x, \omega)$ yields $0 = 1 \cdot \big\{ \log \|f\|_2 + Q(f) \big\} - Q(f)$, which says $\|f\|_2 = 1$. The known case of equality in the Cauchy–Schwarz inequality

$$1 = \int f \, d\mu_n \leqslant \Big(\int d\mu_n \Big)^{1/2} \Big(\int f^2 \, d\mu_n \Big)^{1/2} = 1$$

demonstrates that necessarily $f \equiv 1$ on Q_n. The claimed alternatives are therefore clear. The preceding lemma provides the stated reinterpretations.

Proposition 9. *Either of the following two conditions for non-constant* $f \in N$ *are equivalent to the two properties in Proposition 6.*

(3) $\dfrac{f(\omega) \log f(\omega) - f(\omega) \log \|f\|_2}{f(\omega) - 1} = C$ *is independent of* ω,

(4) $\dfrac{f(\omega) \log f(\omega) - f(\omega) \int f \log f \, d\mu_n}{f(\omega) - 1} = Q(f)$ *for all* $\omega \in Q_n$.

In case any of them is true, the value $C = Q(f)$ *in* (3) *necessarily follows.*

Proof. Observe that for a non-constant critical point $f \in N$, one has $f \neq 1$, so both quotients in the statement are defined for all $\omega \in Q_n$; this was the purpose of Proposition 8.

Consider first the equivalence between Proposition 6, property (2), and the above (3). Thanks to linearity and convexity in the argument $g \in N$, (2) is equivalent to $\mathrm{Var}_f(\, K_n(\cdot, \omega)\,) = 0$ for all $\omega \in Q_n$, that is to say

$$f(\omega) \log f(\omega) = \big(f(\omega) - 1\big)\, Q(f) + f(\omega) \log \|f\|_2, \quad \text{for all } \omega \in Q_n.$$

Hence (3) follows from (2).

In case (3) applies, an identity appears:

$$f(\omega)^2 \log f(\omega) - f(\omega)^2 \log \|f\|_2 = C\, f(\omega)^2 - C\, f(\omega).$$

Using $\int f\, d\mu_n = 1$, an integration against μ_n provides

$$\int f^2 \log f\, d\mu_n - \|f\|_2^2 \log \|f\|_2 = C\big(\|f\|_2^2 - 1\big).$$

Hence $C = Q(f)$, so (2) holds for every $g = K_n(\cdot, \omega)$, $\omega \in Q_n$. Taking convex combinations of them, the full validity of (2) has been derived from (3).

Next, taking a non-constant critical point f, the equivalence (1)$\Leftrightarrow$(3) shows, in view of $\int f \log f\, d\mu_n = \log \|f\|_2$ from Corollary 5 as well as $C = Q(f)$, that the two quotients in the statement coincide. Hence (3) implies (4).

Conversely, the independence of ω in (4) implies

$$f(\omega)^2 \log f(\omega) - f(\omega)^2 \int f \log f\, d\mu_n = Q(f)\, \big\{ f(\omega)^2 - f(\omega) \big\}, \quad \text{all } \omega \in Q_n.$$

Integration and use of $\int f\, d\mu_n = 1$ yield

$$\begin{aligned} \int f^2 \log f\, d\mu_n - \|f\|_2^2 \int f \log f\, d\mu_n &= Q(f)\, \big(\|f\|_2^2 - 1\big) \\ &= \int f^2 \log f\, d\mu_n - \|f\|_2^2 \log \|f\|_2. \end{aligned}$$

It follows that $\int f \log f\, d\mu_n = \log \|f\|_2$, so property (4) expresses the same as (3) does. The proof is complete.

Proposition 10. *If $f \in N$ is a critical point of Q, then either $f \equiv 1$ or f takes exactly two values on Q_n.*

Proof. According to Proposition 8 we may assume that $f \neq 1$ and $Q(f) > 0$. Consider the function

$$R(q) = \big(1 + \tfrac{1}{q}\big) \log(1+q) - \big(1 + \tfrac{1}{q}\big) \log \|f\|_2 = \big(1 + \tfrac{1}{q}\big) \log \tfrac{1+q}{\|f\|_2}.$$

According to Proposition 9 we have $R(\, f(\omega) - 1\,) = Q(f)$ for each $\omega \in Q_n$. Furthermore, the proof of Proposition 8 establishes $\|f\|_2 > 1$ in the present situation. Differentiation yields

$$R'(q) = q^{-2} \big\{ q - \log(1+q) + \log \|f\|_2 \big\} > 0 \quad \text{on} \quad]-1, 0[\, \cup\,]0, \infty[\,,$$

since $\log \|f\|_2 > 0$ and $q - \log(1+q) > 0$ in that set. Hence $R(q)$ is strictly increasing in each of the two indicated, connected intervals.

From $(1+q)\log(1+q) = q + \mathcal{O}(q^2)$ follows $\lim_{q \to 0\pm} R(q) = \mp\infty$; in addition, $\lim_{q \to -1^+} (1 + q^{-1}) \log(1+q) = 0$ readily yields $\lim_{q \to -1^+} R(q) = 0$, whereas $\lim_{q \to +\infty} R(q) = +\infty$ is obvious. We conclude that R is strictly increasing from 0 to $+\infty$ inside $]-1, 0[$ and is likewise strictly increasing from $-\infty$ to $+\infty$ in the interval $]0, \infty[$. This observation says, as desired, that $R(f(\omega) - 1) = Q(f)$ entails exactly two values for $f(\omega)$ as ω ranges through Q_n.

Lemma 11. *Any $f \in N$, which takes only two values on Q_n, can be written $f = P_r f_k$ for some $r \in]0,1]$. Here $f_k \in N$ is determined by the property that for some $1 \leqslant k \leqslant n$ exactly k different α_ω take the value $1/k$, whereas the remaining ones are zero.*

Proof. From $f(x) = \sum_{\omega\in Q_n} \alpha_\omega K_n(x,\omega) = \alpha_x(n+1)$ and the assumption that only two value are attained, it follows that there is a disjoint dissection $Q_n = K \cup L$ with $\omega \mapsto \alpha_\omega$ constant on each K and L; let the common values be α_K and α_L on K and L respectively. We may without loss assume that $|K| \geqslant 1$ and $\alpha_K > \alpha_L \geqslant 0$.

Define next f_k by its parameters $\alpha_\omega(f_k) = 1/k$ for all $\omega \in K$ and 0 for all $\omega \in L$; here $k = |K|$. Clearly $f_k \in N$ and also

$$P_r f_k(x) = 1 + rnk^{-1} \sum_{\omega\in K} x\cdot\omega.$$

It follows that

$$P_r f_k(x) = \begin{cases} 1 + \frac{n+1-k}{k}\, r, & x \in K, \\ 1 - r, & x \in L. \end{cases}$$

For the given $f \in N$ we can determine $0 < r \leqslant 1$ such that $f \geqslant 1 - r$ and such that equality attains exactly $n + 1 - k = |L|$ times, namely on L. The remaining value $1 + a$ is thus determined by

$$n + 1 = \sum_{\omega\in Q_n} f(\omega) = k(1 + a) + (n + 1 - k)(1 - r),$$

clearly containing $a = r(n + 1 - k)/k$. One readily sees that this entails $f = P_r f_k$ identically, which was the claim.

Remark. It is an easy matter to calculate

$$\|P_r f_k\|_2^2 = 1 + \tfrac{n+1-k}{k}\, r^2,$$

which will be needed presently. The preceding Lemma says also that any critical point on the boundary of N must be one of the special functions f_k.

Proposition 12. *The only critical points for $Q(f)$ in the interior of N, are the functions $P_r f_k$ with $r = (s-1)/2s$ and $s = (n+1-k)/k$.*

Proof. According to Lemma 11 and suitable, positive values of r and s, any critical point (i.e., function on Q_n) attains only the values $1-r$ and $1+sr$. In fact, precisely $n + 1 - k$ and k times, respectively.

According to Proposition 9, statement (3), and the preceding remark, f gets to be a critical point only if

$$(1 + sr)\log(1 + sr) + s(1 - r)\log(1 - r) - \tfrac{1}{2}(1 + s)\log(1 + sr^2) = 0$$

is satisfied for $r \in]-1/s, 1[$. The relation arose through the equating of two outcomes of (3), first with $f(\omega) = 1 - r$ and then $1 + sr$. Denote the left-hand side of the equation by $U(r,s)$. It is readily calculated that $U(0,s) = U(\frac{s-1}{2s}, s) = 0$, which says that the claimed values are indeed relevant. We need to exclude further zeros in order to have given a proof of the proposition. An elementary but tedious argument will reveal that $r \mapsto U(r,s)$ has a third order zero in $r = 0$ and exactly one additional zero.

Fix s and differentiate to get

$$\frac{1}{s}\frac{\partial U}{\partial r} = \log\frac{1+sr}{1-r} - \frac{(1+s)r}{1+sr^2}.$$

The change of variables $r \mapsto t = (1+sr)/(1-r)$ is monotonely increasing and takes $]-1/s, 0[$ to $]0,\infty[$; in particular $r = 0 \leftrightarrow t = 1$. We will assume this interpretation to be in effect. Consider now the functions

$$U_1(t,s) = \frac{\partial}{\partial r}U(r,s) = \log t - \frac{(t-1)(t+s)}{t^2+s},$$

$$M(t,s) = (t^2+s)\,U_1(t,s) = (t^2+s)\log t - (t-1)(t+s).$$

Two differentiations yield

$$\frac{\partial M}{\partial t} = 2t\log t - t + 1 - s(1-\tfrac{1}{t}) \quad\text{and}\quad \frac{\partial^2 M}{\partial t^2} = 2\log t + 1 - \tfrac{s}{t^2}.$$

Thus $\frac{\partial^2 M}{\partial t^2}$ is strictly increasing from $-\infty$ to ∞ for $0 < t < \infty$. Due to $\frac{\partial^2 M}{\partial t^2}(1,s) = 1-s$, it is clear that $\frac{\partial^2 M}{\partial t^2}$ has sign tableau $-0+$ with the zero within $]1,\infty[$ in case $s > 1$, whereas $s < 1$ makes the zero fall inside $]0,1[$.

Observing $\frac{\partial M}{\partial t}\big|_{t=1} = 0$ and $\frac{\partial M}{\partial t}\big|_{t=0} = \frac{\partial M}{\partial t}\big|_{t=+\infty} = +\infty$, we record a sign tableau $+0-0+$ for $\frac{\partial M}{\partial t}$. The point $t = 1$ is always a zero, whereas the second zero falls in $]1,\infty[$ or $]0,1[$ according as to which of the cases $s > 1$ or $0 < s < 1$, respectively, applies.

Based on $M(1,s) = 0$, $M(0,s) = -\infty$, and $M(+\infty,s) = +\infty$, we deduce that $t \mapsto M(t,s)$ has a double zero in $t = 1$ and a simple zero $t_1(s) > 1$ in case $s > 1$ or a likewise simple zero $t_2(s) < 1$ in case $0 < s < 1$. Further zeros are missing. This means that $t \mapsto M(t,s)$ enjoys a sign tableau $-0-0+$ for $s > 1$ and $-0+0+$ for $s < 1$.

These last properties of $M(t,s)$ can immediately be transplanted back to say the same for $r \mapsto \frac{\partial}{\partial r}U(r,s)$. In particular, incorporating the evaluations $U(0,s) = 0$, $U(1,s) = \frac{1+s}{2}\log(1+s)$, and $U(-1/s,s) = \frac{1+s}{2}\log\frac{1+s}{s}$, one deduces that $r \mapsto U(r,s)$ has a third order zero at the origin and at most one additional zero, which must be $(s-1)/2s$.

Membership in N for the critical point is encoded in the relation

$$(n+1-k)(1-r) + k(1+sr) = n+1.$$

Refering to $r = (s-1)/2s$ as just derived, it is an easy matter to conclude that

$$s = (n+1-k)/k \quad\text{and}\quad s = 1$$

are the only solutions for given k. The latter option must be excluded since it forces $r = 0$ and thus a constant function. This means that the proof of the proposition has been completed.

Theorem 13. *At any critical point the value of $Q(f)$ does not exceed $\frac{n+1}{2(n-1)}\log n$. Moreover, this particular value is attained.*

Proof. The remark above describes the critical points at the boundary of N as some f_k, attaining the value $(n+1)/k$ on Q_n exactly k times and being zero elsewhere; here $1 \leq k \leq n$. Clearly

$$\|f_k\|_2^2 = \tfrac{n+1}{k}, \qquad Q(f_k) = \tfrac{n+1}{k}\log\tfrac{n+1}{k}\Big/2\big[\tfrac{n+1}{k}-1\big].$$

Since $m(x) = x\log x/(x-1)$ is strictly increasing in $]1,\infty[$, it is clear that

$$Q(f_k) = \tfrac{1}{2}\,m\big(\tfrac{n+1}{k}\big) \leq \tfrac{1}{2}\,m(n+1) = \frac{n+1}{2n}\log(n+1), \quad \text{all } 1 \leq k \leq n.$$

Now the calculations can proceed with the critical points of Q interior to N. According to Propositions 9 and 12 candidates for $\sup Q(f)$ are the numbers

$$\begin{aligned}
&(sr)^{-1}\big\{(1{+}sr)\log(1+sr) - \tfrac{1}{2}(1+sr)\log(1+sr^2)\big\}\Big|_{r=\frac{s-1}{2s}} \\
&\qquad = \frac{2}{s-1}\Big\{\frac{s+1}{2}\log\frac{s+1}{2} - \frac{s+1}{4}\log\frac{(s+1)^2}{4s}\Big\} \\
&\qquad = \frac{s+1}{2(s-1)}\log s,
\end{aligned}$$

evaluated with $s = (n+1-k)/k$ for $k = 1,\dots,n$.

Observe first that $s \mapsto s^{-1}$ leaves $\frac{s+1}{2(s-1)}\log s$ invariant, so we may restrict work to $s \geqslant 1$, that is to $1 \leqslant k \leqslant (n+1)/2$. Next we record the expressions

$$\frac{d}{ds}\,\frac{s+1}{s-1}\log s = \frac{1}{s(s-1)^2}\big(s^2-1-2s\log s\big) \quad \text{and}$$

$$\frac{d}{ds}\big(s^2-1-2s\log s\big) = 2\big(s-1-\log s\big) > 0 \quad \text{for } s > 1.$$

Hence $s \mapsto \frac{s+1}{2(s-1)}\log s$ is increasing on $[1,\infty[$. The viable alternatives for $s = (n+1-k)/k$ arise as conditioned by $s \leqslant n$, so the critical points interior to N give for $Q(f)$ at most the value

$$\frac{s+1}{2(s-1)}\log s\Big|_{s=(n+1-k)/k} \leq \frac{n+1}{2(n-1)}\log n.$$

Now the contributions from the critical points can be compared. Clearly

$$\frac{n+1}{2(n-1)}\log n > \frac{n+1}{2n}\log(n+1), \quad \text{for all } n \geq 2.$$

Thus the maximal value of $Q(f)$ at interior critical points of N exceeds the values at the boundary. The claim has consequently been fully verified.

Remark. It is straightforward to verify the lower part of this inequality:

$$\tfrac{1}{2}\log(n+2) < \sup_{f\in N} Q(f) \leq \tfrac{3}{2}\log n.$$

These are about the simplest estimates available. Further and somewhat more detailed analysis of (Q_n, P_r) will appear in the last section.

Recall also that the above semigroups P_r have spectral gap 1, whereas their log-Sobolev exponents are $\frac{n+1}{2(n-1)}\log n$. Clearly this latter quantity can be made arbitrarily large, so a family of semigroups are at hand with large distance to the spectral gap. They should be able to act as test cases in Rothaus' theory on hypercontractivity.

Refined logarithmic Sobolev inequalities

The setting is now that of a finite measure space (Ω, μ), a positivity-preserving semigroup e^{-tH}, and the corresponding infinitesimal generator H, supposedly non-negative and self-adjoint on $L^2(\mu)$. The companion quadratic form is $Q(u,v) = \langle H^{1/2}u, H^{1/2}v\rangle_{L^2(\mu)}$. A p-th logarithmic Sobolev inequality is a statement that

$$\int f^p \log f \, d\mu \leqslant c(p)\, Q(f, f^{p-1}) + \|f\|_p^p \log \|f\|_p,$$

for all positive f in $\mathrm{Dom}(H^{1/2})$. The non-negativity of H extends the validity also to complex valued functions f. It is convenient to say that the above inequality has index p. Here we take interest in the least possible quantity $c(p)$, called the principal coefficient function. In case $c(p)$ is not minimal it is simply called coefficient function.

More general forms of the inequality are common and this particular case has “local norm zero”. For more general cases, involving local norm contributions, one may refer to the original paper [G] by Gross as well as to his excellent exposition [G2].

The precise connection to hypercontractivity was established by Gross:

Theorem (Gross [G]**)** *Suppose $c(p)$ is a coefficient function for a logarithmic Sobolev inequality. For each p in $]a,b[$, let $q = q(t,p)$ be the solution of the initial value problem*

$$dq/dt = q/c(q), \quad q(0,p) = p, \quad t \geqslant 0.$$

Then $\|e^{-tH}\|_{L^p \to L^q} \leqslant 1$. $\square$

A common application of this is to determine $c(2)$ and then to apply the following result of Stroock in order to extend index 2 to some result for index p.

Theorem (Stroock, from [G2]) *With assumptions as above, each positive f belonging to the domain* $\mathrm{Dom}(H^{1/2})$ *yields for $p \geqslant 2$*

$$Q(f^{p/2}, f^{p/2}) \leqslant \tfrac{1}{4} p^2 (p-1)^{-1} Q(f, f^{p-1}).$$

Specifically, $c(p) \leqslant (p/2)(p-1)^{-1} c(2)$ in case $p \geqslant 2$.

The hypercontractivity $\|e^{-tH}\|_{p\to q} \leqslant 1$ for $e^{-t} \leqslant \{(p-1)/(q-1)\}^{c(2)/2}$ follows from Gross' theorem. The next two paragraphs will investigate to what extent the exponent $c(2)/2$ may be replaced by something smaller.

Estimating logarithmic Sobolev exponents. We need to clarify the behaviour of $c(p)$ as dependent on p. A generalization of Stroock's theorem is appropriate as a first source of information.

Theorem 14. *Let (Ω, μ) be a finite measure space and H a non-negative self-adjoint operator on $L^2(\mu)$, such that the semigroup e^{-tH} is positivity preserving and simultaneously a contraction on $L^\infty(\mu)$. Write $\mathcal{D} = \mathrm{Dom}(H^{1/2})$, $Q(f,g) = \langle H^{1/2} f, H^{1/2} g\rangle_{L^2(\mu)}$. If now $2 \leqslant p \leqslant q < \infty$ or $1 < q \leqslant p \leqslant 2$, and $f \geqslant 0$, $f \in L^\infty \cap \mathcal{D}$, then*

$$Q\left(f^{q/p}, f^{(1-1/p)q}\right) \leqslant \frac{q^2(p-1)}{p^2(q-1)}\, Q(f, f^{q-1}),$$

provided all the used powers of f belong to $\mathcal{D}$.

Remark. It is well known that $g \geqslant 0$, $g \in L^\infty \cap \mathcal{D}$ implies $g^r \in \mathcal{D}$ for all $r \geqslant 1$. Hence, for the case $2 \leqslant p \leqslant q$ the assumption $f \in \mathcal{D} \cap L^\infty$ suffices for applicability, whereas $f^{q-1} \in \mathcal{D} \cap L^\infty$ for $1 < q \leqslant p \leqslant 2$ will do perfectly.

To get the desired result a real analysis inequality is needed, whose proof is fairly straightforward to complete and is thus excluded.

Lemma 15. *Consider α and β in $]0,1[$ such that α lies between β and $1-\beta$. Then an inequality obtains.*

$$(s^\alpha - 1)(s^{1-\alpha} - 1) \leqslant c_{\alpha\beta}\, (s^\beta - 1)(s^{1-\beta} - 1), \text{ all } s > 0.$$

Here $c_{\alpha\beta} = \alpha(1-\alpha)/\beta(1-\beta)$ and equality is attained only for $s = 1$.

Proof of Theorem 14. Write $P_t = e^{-tH}$ and $\sigma_t = P_t 1$, whence $1 - \sigma_t \geqslant 0$ due to contractivity. It is well known that $Q(u,v) = \lim_{t\to 0^+} t^{-1}\langle (I - P_t)u, v\rangle$ and that

$$\begin{aligned}\langle (I-P_t)u, v\rangle = \tfrac{1}{2} \int P_t[\{u - u(x)\}\{v - v(x)\}](x)\, d\mu(x) \\ + \int \{1 - \sigma_t(x)\} u(x) v(x)\, d\mu(x).\end{aligned} \tag{1}$$

From the inequality in Lemma 15 it follows that for $s > 0$ the relation

$$(s^{1/p} - 1)(s^{1-1/p} - 1) \leqslant \tfrac{q^2(p-1)}{p^2(q-1)} (s^{1/q} - 1)(s^{1-1/q} - 1)$$

holds. In consequence, for each $\eta, \xi \geqslant 0$

$$(\eta^{q/p} - \xi^{q/p})(\eta^{(1-1/p)q} - \xi^{(1-1/p)q}) \leqslant \tfrac{q^2(p-1)}{p^2(q-1)} (\eta - \xi)(\eta^{q-1} - \xi^{q-1}). \tag{2}$$

Since both relations between p and q guarantee $q^2(p-1)/p^2(q-1) \geqslant 1$, the action of P_t on (2) yields, with two references to (1),

$$\begin{aligned}&\langle (I - P_t) f^{q/p}, f^{(1-1/p)q}\rangle \\ &\quad = \tfrac{1}{2} \int P_t[\{f^{q/p} - f(x)^{q/p}\}\{f^{(1-1/p)q} - f(x)^{(1-1/p)q}\}](x)\, d\mu(x) \\ &\qquad + \int \{1 - \sigma_t(x)\} f(x)^{q/p} f(x)^{(1-1/p)q}\, d\mu(x) \\ &\quad \leqslant \tfrac{1}{2} \tfrac{q^2(p-1)}{p^2(q-1)} \int P_t[\{f - f(x)\}\{f^{q-1} - f(x)^{q-1}\}](x)\, d\mu(x) \\ &\qquad + \int \{1 - \sigma_t(x)\} f(x) f(x)^{q-1}\, d\mu(x) \\ &\quad \leqslant \tfrac{q^2(p-1)}{p^2(q-1)} \langle (I - P_t) f, f^{q-1}\rangle.\end{aligned}$$

Division by t and identification of limits as $t \to 0^+$ finally prove $Q(f^{q/p}, f^{(1-1/p)q}) \leqslant \frac{q^2(p-1)}{p^2(q-1)}\, Q(f, f^{q-1})$, the intended inequality.

Now we are ready to improve our understanding of the principal coefficient function $c(p)$ in the logarithmic Sobolev inequality.

Theorem 16. *Let $c(p)$ be the principal coefficient function. The modified function $(1-1/p)\,c(p)$ is never increasing for $p \geqslant 2$ and never decreasing for $1 < p \leqslant 2$, i.e.,*

$$2 \leqslant p \leqslant q \ \text{ or } \ 1 < q \leqslant p \leqslant 2 \quad \text{give} \quad (1-1/q)c(q) \leqslant (1-1/p)c(p).$$

Proof. Consider $g \geqslant 0$, $g, g^{q-1} \in \mathcal{D} \cap L^\infty$. Applying the p-th logarithmic Sobolev inequality to $f = g^{q/p}$, and an ensuing use of Theorem 14, provides the calculation

$$\begin{aligned} \int g^q \log g \, d\mu &\leqslant \frac{p}{q}\, c(p)\, Q(g^{q/p}, g^{(1-1/p)q}) + \|g\|_q^q \log \|g\|_q \\ &\leqslant c(p) \frac{q(p-1)}{p(q-1)}\, Q(g, g^{q-1}) + \|g\|_q^q \log \|g\|_q. \end{aligned}$$

This means that $c(q) \leqslant \frac{q(p-1)}{p(q-1)}\, c(p)$, which is the claimed monotonicity. $\square$

Time dependent logarithmic Sobolev inequalities. As a next step we will modify Gross' theorem to better take into account the development of orbits with time, i.e., the behaviour of $t \mapsto e^{-tH} f$.

The range of the semigroup at a fixed time is essential in what follows:

$$\mathcal{D}_t = \big\{ e^{-tH} f \,;\, f \in L^2 \big\}.$$

Clearly $e^{-sH}\mathcal{D}_t = \mathcal{D}_{s+t}$ for all $s, t \geqq 0$. In particular,

$$s \geqslant t \geqslant 0 \quad \text{implies} \quad \mathcal{D}_s \subseteq \mathcal{D}_t \subseteq L^2.$$

The time dependent principal coefficient function $c(t,p)$ is the infimum of all c such that a logarithmic Sobolev inequality holds:

$$\text{(lSi)} \qquad \int f^p \log f \, d\mu - \|f\|_p^p \log \|f\|_p \leqslant c \, \langle Hf, f^{p-1} \rangle, \quad \text{all } f \in \mathcal{D}_t,\ f \geq 0.$$

Clearly $t \mapsto c(t,p)$ is decreasing, possibly constant on some intervals, and also $\lim_{t\to+\infty} c(t,2) = 1$. Assume for the moment that $c(t,q)$ is such that the differential equation

$$\text{(de)} \qquad \begin{cases} c(t,q)\, \dfrac{dq}{dt} = q, \\ \quad q(0,p) = p \end{cases}$$

has a solution $q = q(t,p)$, where p is a fixed parameter. Thus we have that

$$\frac{d}{dt} \log q(t,p) = \frac{1}{c(t, q(t,p))}.$$

We can read word by word Gross' proof of [G], Theorem 1, in order to verify the following line of argument. Let $\mathcal{D} = \mathrm{Dom}(H^{1/2})$. Then $f \in \mathcal{D}$ makes $f(t) = e^{-tH} f$

a member of $\mathcal{D}_t$ for all $t \geqslant 0$. Setting $\alpha(t) = \|f(t)\|_{q(t,p)}$ Gross' calculations demonstrate

$$\begin{aligned}\frac{d\alpha(t)}{dt} &= \|f(t)\|_q^{1-q}\Big[\frac{1}{c(t,q)}\Big\{\int f(t)^q \log f(t)\,d\mu - \|f(t)\|_q^q \log \|f(t)\|_q\Big\} \\ &\qquad\qquad - \langle Hf(t), f(t)^{q-1}\rangle\Big] \\ &\leqslant 0 \quad \text{for } t \geqslant 0,\ q = q(t,p).\end{aligned}$$

Thus $\alpha(t) \leqslant \alpha(0)$, which says for each $f \in \mathcal{D}$ that $\|f(t)\|_{q(t,p)} \leqslant \|f(0)\|_{q(0,p)} = \|f\|_p$. In other words,

$$\|e^{-tH} f\|_{q(t,p)} \leqslant \|f\|_p, \quad \text{all } f \in L^p$$

when taking the density of $\mathcal{D}$ in L^p into consideration.

Consider now any function $c = c(t,p)$ satisfying the logarithmic Sobolev inequality (lSi) and simultaneously making (de) solvable with solution $q = q(t,p)$. Retracing our earlier steps mannered on Gross' argumentation, it is clear that everything in the argument remains fully valid for this particular – not necessarily optimal – function $c(t,p)$. Hence we have established a variant of Gross' original theorem.

Theorem 17. *Let the positive function $c(t,p)$, $t \geqslant 0$, $2 \leqslant p < p_0$, be such that the logarithmic Sobolev inequality* (lSi) *holds with $c = c(t,p)$, and simultaneously such that* (de) *is solvable with solution $q = q(t,p)$ for $2 \leq p < p_0$. Then e^{-tH} is hypercontractive to the extent that*

$$\|e^{-tH} f\|_{L^{q(t,p)}} \leqslant \|f\|_p, \ \textit{all } f \in L^p \textit{ and } 2 \leq p < p_0.$$

The bootstrapped log-Sobolev inequality. Let once more $c(t,p)$ denote the least possible value of c in the log-Sobolev inequality (lSi). Looking back into the proof of Theorem 14, it is clear that the same proof may be performed verbatim for the smaller class of $f \in \mathcal{D} \cap \mathcal{D}_s$, $f \geqslant 0$, for each fixed $s \geqslant 0$. Hence the proof of Theorem 16 can also be reinterpreted so as to only use $g, g^{q-1} \in \mathcal{D} \cap \mathcal{D}_s \cap L^\infty$. This being done we have established a variant of Theorem 16.

Proposition 18. *Let $c = c(t,p)$ be the least positive function satisfying* (lSi)*. Then each $t \geqslant 0$ has the property that*

$$2 \leqslant p \leqslant q \quad \textit{implies} \quad (1 - 1/q)\, c(t,q) \leqslant (1 - 1/p)\, c(t,p).$$

Since the principal coefficient function is difficult to compute as soon as $p \neq 2$, it is the following corollary that gives an applicable statement.

Corollary 19. *Suppose that $c(t,2)$ satisfies for $t \geqslant 0$*

$$\int f^2 \log f\, d\mu - \|f\|_2^2 \log \|f\|_2 \leqslant c(t,2)\,\langle Hf, f\rangle, \quad \textit{all } f \in \mathcal{D}_t,\ f \geq 0.$$

Then the extension $\tilde{c}(t,p) = \frac{p}{2(p-1)}\, c(t,2)$ of $c(t,2)$ to all $p \geqslant 2$, satisfies the general log-Sobolev inequality (lSi) *for $p \geqslant 2$ and $t \geqslant 0$.*

Proof. According to Proposition 18 applied to $p \geq 2$, we find

$$c(t,p) \leqslant \frac{p}{2(p-1)}\, c(t,p) = \tilde{c}(t,p).$$

Thus (lSi) is satisfied with $c = \tilde{c}(t,p)$ and the claim follows.

Remark. It must be pointed out at this stage, that the corollary is interesting only in case $c(t,2)>1$. For it is well known that $c(2)=1$ implies $c(p)=\frac{p}{2(p-1)}$ identically in the original setting and similarly one quickly sees by the same perturbation argument that $c(t,2)=1$ forces $c(t,p)=p/2(p-1)$.

Fix now one continuous and positive function $c(t,2)$, defined for all $t\geqslant 0$, that satisfies the inequality as stated in Corollary 19. We may suppose $t\mapsto c(t,2)$ to be non-increasing since the sets $\mathcal{D}_t$ decrease with t. The intention is to apply Theorem 17 with $c=\tilde{c}(t,p)$ as defined above, since the extension $\tilde{c}(t,p)$ satisfies (lSi) according to Corollary 19.

Consider for this purpose the initial condition $\tilde{q}(0,p)=p$ and the differential equation

$$\frac{d\tilde{q}}{dt}=\frac{\tilde{q}}{\tilde{c}(t,\tilde{q})}=\frac{2(\tilde{q}-1)}{c(t,2)},\quad \text{i.e.,}\quad \frac{d}{dt}\log(\tilde{q}-1)=\frac{2}{c(t,2)}.$$

The assumptions on $c(t,2)$ make $\log(\tilde{q}-1)$ strictly increasing and convex. Hence there exists a continuous inverse function $\tilde{t}(q,p)$ such that $\tilde{q}(\,\tilde{t}(q,p),p)=q$ for $q\geqslant p$. Clearly $q\mapsto\tilde{t}(q,p)$ is increasing. According to Theorem 17, $\tilde{t}=\tilde{t}(q,p)$ gives

$$\|e^{-\tilde{t}H}f\|_{L^q}=\|e^{-\tilde{t}H}f\|_{L^{\tilde{q}(\tilde{t},p)}}\leqslant\|f\|_{L^p}.$$

By the semigroup property, $t\geqslant\tilde{t}=\tilde{t}(q,p)$ implies

$$\|e^{-tH}f\|_{L^q}=\left\|e^{-\tilde{t}H}\left[e^{-(t-\tilde{t})H}f\right]\right\|_{L^q}\leqslant\left\|e^{-(t-\tilde{t})H}f\right\|_{L^p}\leqslant\|f\|_{L^p}.$$

This means that we have proved a hypercontractivity result.

Proposition 20. *Let $c(t,2)$ be a positive, continuous, and non-increasing function for $t\geqslant 0$. Assume that Corollary 19 applies and let $\tilde{q}=\tilde{q}(t,p)$ and $\tilde{t}=\tilde{t}(q,p)$ be as defined above. Then e^{-tH} is hypercontractive to the extent that for given $q\geqslant p\geqslant 2$*

$$t\geqslant\tilde{t}\quad\textit{implies}\quad\|e^{-tH}f\|_{L^q}\leqslant\|f\|_{L^p},\ \textit{all } f\in L^p.$$

It is this result that after an interpretation of the quantity $\tilde{t}(q,p)$ will deliver refined information on hypercontractivity.

Keep the above assumptions on $c(t,2)$ and write for visibility $c_{\max}=c(0,2)$, which is the maximum of all values attained by $c(t,2)$. Take $t_{\max}$ to be the largest number such that for $t\in[0,t_{\max}]$ we have $c(t,2)=c_{\max}$. Thus there is an exact value $\tilde{q}-1=(p-1)\exp(2t/c_{\max})$ for $0\leqslant t\leqslant t_{\max}$. This suggests an auxiliary function

$$R(t)=\big\{\tilde{q}(t,p)-1\big\}\,\exp\left\{-\frac{2t}{c_{\max}}\right\},$$

which has the properties

$$\begin{cases}\dfrac{d}{dt}\log R(t)=\dfrac{2}{c(t,2)}-\dfrac{2}{c_{\max}}\geqslant 0,\\ R(0)=p-1,\quad \log R(t)\text{ is convex and non-decreasing.}\end{cases}$$

In particular it follows that

$$\tilde{q}(t,p)-1\geqslant(p-1)\,\exp\frac{2t}{c_{\max}},\quad t\geqslant 0,$$

with equality exactly for $0 \leqslant t \leqslant t_{\max}$. Reformulated, this expresses

$$e^{-t} \geqslant \left(\frac{p-1}{\tilde{q}(t,p)-1}\right)^{c_{\max}/2}.$$

Take for notational simplicity – but semantically somewhat abusive – the quantity $\tilde{q}_{\max}$ to be

$$\tilde{q}_{\max}(p) = \tilde{q}(t_{\max}, p) = 1 + (p-1)\exp\frac{2t_{\max}}{c_{\max}}.$$

The previous display is preferably understood as

$$e^{-\tilde{t}(q,p)} \geqslant \left(\frac{p-1}{q-1}\right)^{c_{\max}/2},$$

simply by choosing $t = \tilde{t}(q,p)$. We know that strict inequality holds in this inequality as soon as $q > \tilde{q}_{\max}(p)$.

Corollary 21. *In the above setting and accompanying notation, for any $q > \tilde{q}_{\max}(p)$ the parameter interval for e^{-t} making e^{-tH} hypercontractive $L^p \to L^q$ is strictly larger than the interval expressed by $e^{-t} \leqslant [(p-1)/(q-1)]^{c_{\max}/2}$.*

Remark. It is precisely this latter interval that results from the standard application of Stroock's theorem, as noted at the beginning of this section.

Application to the simplicial semigroups

Naturally our last step in this development is to apply Corollary 21 to the particular class of semigroups constructed in the first section. Trivially we see that the infinitesimal generator A constructed there is idempotent and $e^{-tA} = E + e^{-t}A = P_{e^{-t}}$ for $t \geqslant 0$; in accordance with earlier notation it is convenient to let $r = e^{-t}$ parameterize $e^{-tA} \leftrightarrow P_r$. Studying a particular Q_n, thus fixing $n \geqslant 1$, an application of Corollary 21 demands calculation of $c_{\max}$, $t_{\max}$, and $\tilde{q}_{\max}$ as dependent on n for whichever choice of $c(t,2)$ we care to make. It is clear that

$$c(t,2) = \sup\left\{Q(f)\,;\, f \in e^{-tA}N\right\}$$

is the least possible, still valid choice. Of course $e^{-tA}N = \{e^{-tA}g\,;\, g \in N\}$. Looking back, the set N can be seen as a compact subset of $\mathbb{R}^n_+$, via the identification of $f \in N$ with its parameters $\{\alpha_\omega\}$, $\omega \in Q_n$, $\alpha_\omega \geqslant 0$, and $\sum \alpha_\omega = 1$.

Obviously $\mathcal{D}_t \cap \{Ef = 1\} = e^{-tA}N = P_rN$ in this context. A little lemma takes care of membership in $\mathcal{D}_t \cap \{Ef = 1\}$.

Lemma 22. *For non-constant $f \in N$ the membership $f \in P_rN$ holds if and only if $r \geqslant 1 - \min f$. Hence $f \in \mathcal{D}_t \cap \{Ef = 1\}$ if and only if $t \leqslant \log[1/(1-\min f)]$.*

Proof. Suppose $f = P_rg$ for $g \in N$. Then $f = 1 + r(g-1)$ and since $g \geqslant 0$

$$1 - \min f = r(1 - \min g) \leqslant r.$$

Let on the other hand, for given $f \geqslant 0$ non-constant, $r \in\,]0,1]$ be the unique number such that the thereby introduced $\tilde{g}$ enjoys the property

$$\tilde{g} = \frac{1}{r}(f - Ef) \geqslant -Ef$$

with equality being attained somewhere on Q_n. We see that $g = Ef + \tilde{g}$ gives $f = P_r g$. To wit, $E\tilde{g} = 0$ so

$$P_r g = Ef + r\tilde{g} = Ef + f - Ef = f.$$

Moreover, $f \in N$ produces $1 = Ef = Eg$ and $\min g = 1 + \min \tilde{g} = 1 - 1 = 0$, thus showing $g \in N$ to be at hand.

Finally, the construction of $\tilde{g}$ has $r = 1 - \min f$ as the effectuated choice. This completes the demonstration of the first part of the statement, whereas the second is a mere reformulation based on $r = e^{-t}$.

By the proof of Theorem 13 and the statement of Proposition 12 the maximum of $Q(f)$ for $f \in N$ is attained only by $\tilde{f} = P_\rho f_1$, where $\rho = (n-1)/2n$ and $f_1 \in N$ is one of the $n+1$ possible functions having value $n+1$ at one point of Q_n and being zero elsewhere. This means there is one $\xi \in Q_n$ with $a_\xi(f_1) = 1$ and $a_\omega(f_1) = 0$ otherwise. Thus

$$\tilde{f}(\omega) = P_\rho f_1(\omega) = \begin{cases} \frac{n+1}{2}, & \omega = \xi, \\ \frac{n+1}{2n}, & \omega \neq \xi. \end{cases}$$

Since $1 - \min \tilde{f} = \frac{n-1}{2n}$, Lemma 22 demonstrates $\tilde{f} \in P_r N$ if and only if $r \geq \frac{n-1}{2n}$. Hence the present setting on Q_n with semigroup P_r has

$$r_{\max} = e^{-t_{\max}} = \frac{n-1}{2n}, \qquad c_{\max} = \frac{n+1}{2(n-1)} \log n.$$

It follows that

$$\log\left(\frac{\tilde{q}_{\max}(p) - 1}{p-1}\right) = \frac{2t_{\max}}{c_{\max}} = \frac{4(n-1)\log\frac{2n}{n-1}}{(n+1)\log n}$$

Using an auxillary function $T(x) = (1-x)(1+x)^{-1}\log\left[2/(1-x)\right]$, this being suggested by $2t_{\max}/c_{\max} = 4T(1/n)/\log n$, it is not too difficult to establish that $T(x)$ is strictly decreasing on $]0,1[$ and is bounded by $\log 2$ there. It follows that in this situation

$$\log\left(\frac{\tilde{q}_{\max}(p) - 1}{p-1}\right) < \frac{4\log 2}{\log n}.$$

It is therefore time to express this our main application in full detail. A reference to Corollary 21 with the just calculated estimate proves the last result in this paper.

Theorem 23. *Consider the simplices Q_n and the multiplicative semigroups P_r on $L^2(Q_n)$ constructed earlier. For any $q > p \geq 2$ such that*

$$\frac{q-1}{p-1} \geq 16^{1/\log n}$$

the hypercontractivity $\|P_r f\|_q \leq \|f\|_p$ obtains for a parameter interval strictly larger than that indicated by $r \leq \left[(p-1)/(q-1)\right]^{\frac{(n+1)\log n}{4(n-1)}}$.

To complete the proof it suffices to observe that the calculations above established the estimate

$$\frac{\tilde{q}_{\max}(p) - 1}{p-1} < 16^{1/\log n}.$$

Thus the added condition on $(q-1)/(p-1)$ in the statement of the present theorem ensures that the bootstrapped hypercontractivity in Corollary 21 is applicable. Then the statement is merely a copy of that result.

Remark. It is hardly worth the effort to derive quantitative estimates on $Q(P_r f)$ for $r < r_{\max}$ in order to improve on the qualitative conclusion of Theorem 23. In principle such elaboration is possible though, and would quantify the size of an enhanced interval.

References

[A] M.E. Andersson, Remarks on hypercontractivity for the smallest groups, In: *Complex analysis and differential equations.*, Proc. of the Marcus Wallenberg symposium in honor of Matts Essen, Uppsala, June 15–18, 1997, Acta Universitatis Upsaliensis, 1999, pp. 51–60.

[DS] P. Diaconis and L. Saloff-Coste, Logarithmic Sobolev inequalities for finite Markov chains, *Ann. Appl. Probab.* **6** (1996), no. 3, 695–750.

[G] L. Gross, Logarithmic Sobolev inequalities, *Amer. J. Math.* **97** (1975), 1061–1083.

[G2] ———, Logarithmic Sobolev inequalities and contractive properties of semigroups, In: *Dirichlet forms*, Varenna 1992, LNM 1563, Springer Verlag, pp. 54–88.

Mats Erik Andersson, Matematiska institutionen, Stockholms universitet, SE 106 91 Stockholm

E-mail address: matsa@matematik.su.se