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On \mathcal{H}_{-4} -perturbations of self-adjoint operators.

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Abstract. Supersingular \mathcal{H}_{-4} -perturbations of positive self-adjoint operators are studied. It is proven that such singular perturbations can be described by non self-adjoint operators with real spectrum. The resolvent of the perturbed operator is calculated using generalization of Krein's formula for self-adjoint extensions.

1 Introduction.

Finite rank perturbations of self-adjoint operator are used to obtain operators with complicated structure of the spectrum which are exactly solvable. Such perturbations are called singular if the domains of the perturbed and unperturbed operators are different. Such perturbations appear for example during the investigation of point interactions [AGH-KH88,DO88]. The first rigorous mathematical treatment of such Hamiltonian was suggested by F.Berezin and L.Faddeev in 1961 [BF61]. It was realized that finite rank perturbations of self-adjoint operators defined by vectors not from the original Hilbert space can lead to interesting exactly solvable models. The well-known Krein's formula relating the resolvents of two different self-adjoint extensions of one symmetric operator play very important rôle in the studies of the spectral properties of these operators [Kr44,Nai43,GMT98,K99,AK00]. The perturbed operator can be defined by first restricting the original operator to a certain symmetric one and then extending it to another self-adjoint operator. Such models having numerous advantages cannot be used to model all interesting phenomena. For example the Laplace operator with point interaction in \mathbf{R}^3 has nontrivial scattering matrix in the s -channel only. In order to obtain models enabling to describe more complicated scattering phenomena, one has to consider finite rank perturbations determined by extremely singular vectors. The restriction-extension procedure described above can be used in the case where the perturbation is determined by the vectors being bounded linear functionals on the domain of the original operator. It has been realized that more singular perturbations can be defined only using certain extension of the original Hilbert space. In [Sh92,Sh88,TvD91,DLSHZ00] rank one supersingular perturbations were defined using self-adjoint operators acting in Pontryagin spaces. It was shown that the spectral properties of these models are described by generalized Nevanlinna functions with a finite number of negative squares [HdS97]. Similar ideas were used in [KP95,PP91,PP84,Pop92] where concrete problems of mathematical physics were attacked. Different physicists and mathematicians tried to define supersingular perturbations [DD81,DD86,Karp92,And99].

In our previous paper [KW00] supersingular rank one perturbation of positive self-adjoint operator has been defined without any use of spaces with indefinite metrics. But our approach was limited to the case of so-called \mathcal{H}_{-3} perturbations, i.e. perturbations determined by vectors from the Hilbert space \mathcal{H}_{-3} from the scale of Hilbert spaces associated with the original positive self-adjoint operator.¹ In the current paper we are able to make one step further and describe all supersingular perturbations from the class \mathcal{H}_{-4} . These perturbations can be constructed in a certain extended Hilbert space. But no self-adjoint operator can be associated with such perturbations. It is shown that such singular perturbations are described by non self-adjoint operators with real spectrum. We obtain formula for the resolvent of such operator, which is similar to celebrated Krein's formula. The spectral properties of the operators are described by generalized Nevanlinna functions.

The paper is organized as follows. In Section 2 all necessary preliminary facts concerning finite rank singular and supersingular perturbations are reviewed. Maximal operator corresponding to rank one \mathcal{H}_{-4} -perturbation is calculated and investigated in Section 3. The family of operators corresponding to such singular perturbation is obtained in Section 4. Some conclusions and possibilities for further developments are discussed in the last section.

2 Preliminaries.

Current paper is devoted to the construction of the operator describing rank one supersingular perturbation of a given positive self-adjoint operator A acting in a certain Hilbert space \mathcal{H} . The perturbed operator is given by the following formal expression

$$A_\alpha = A + \alpha \langle \varphi, \cdot \rangle \varphi, \quad (2.1)$$

where φ is a vector from the scale of Hilbert spaces associated with the operator A and α is a real coupling constant describing the strength of the perturbation.² Consider first the case $\varphi \in \mathcal{H}$. The perturbation $\alpha \langle \varphi, \cdot \rangle \varphi$ is a bounded symmetric operator and the perturbed operator A_α is self-adjoint on the domain of the original operator A . The resolvent of the perturbed operator is given by

$$\frac{1}{A_\alpha - \lambda} = \frac{1}{A - \lambda} - \frac{1}{\frac{1}{\alpha} + \langle \varphi, \frac{1}{A - \lambda} \varphi \rangle} \left\langle \frac{1}{A - \lambda} \varphi, \cdot \right\rangle \frac{1}{A - \lambda} \varphi. \quad (2.2)$$

All spectral properties of the perturbed operator A_α are described by the Nevanlinna function $Q(\lambda) = \langle \varphi, \frac{1}{A - \lambda} \varphi \rangle$ (see for example [AK00]).

¹ The scale of Hilbert spaces is defined rigorously in Section 2.

² Recent developments in this area are described in details in [AK97, AK97-2, GS95, KS95, KK00, KN98, Si95, AK00].

Consider now the scale of Hilbert spaces \mathcal{H}_s associated with the positive operator A . The norm in each space \mathcal{H}_s is defined by

$$\|U\|_{\mathcal{H}_s}^2 = \langle U, (A+1)^s U \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in the original Hilbert space \mathcal{H} . In order to avoid misunderstanding only the scale of Hilbert spaces associated with the original operator A and the original Hilbert space \mathcal{H} will be considered throughout the paper. All perturbations defined by vectors φ not from the original Hilbert space \mathcal{H} are called singular. These perturbations are characterized by the fact that the domain of the perturbed operator does not coincide with the domain of the original one. In the case $\varphi \in \mathcal{H}_{-1} \setminus \mathcal{H}$ the perturbation is relatively form bounded with respect to the sesquilinear form of the operator A and the perturbed operator can be determined using the form perturbation technique. The resolvent of the perturbed operator is again given by (2.2). The main difference is that the domain of the perturbed operator does not coincide with the domain of the original operator in general, but the perturbed operator is uniquely defined as a self-adjoint operator in the original Hilbert space \mathcal{H} [Si95,AK97]. Another way to define the perturbed operator is using the extension theory for symmetric operators. It is obvious that the perturbed and original operators coincide on the linear set of functions U satisfying the condition

$$\langle \varphi, U \rangle = 0. \tag{2.3}$$

Then the perturbed operator is an extension of the original operator restricted to this linear set. If $\varphi \in \mathcal{H}_{-1} \setminus \mathcal{H}$ the restricted operator is a symmetric operator with the deficiency indices $(1, 1)$. Its self-adjoint extension corresponding to the formal expression (2.1) is uniquely defined. The resolvent of the perturbed operator can be described using Krein's formula [Kr44,Nai43], which coincides with (2.2) in this case.

The case $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ has to be treated using the extension theory for symmetric operators, since the perturbation is not form bounded with respect to the original operator. The restricted symmetric operator can be defined in a way similar to \mathcal{H}_{-1} . But the perturbed operator is not uniquely defined anymore. One can only conclude that the perturbed operator is equal to one of the self-adjoint extensions of the restricted operator. All such operators can be parametrized by one real parameter $\gamma \in \mathbf{R} \cup \{\infty\}$ as follows

$$\frac{1}{A^\gamma - \lambda} = \frac{1}{A - \lambda} - \frac{1}{\gamma + \langle \varphi, \frac{1+\lambda}{A-\lambda} \frac{1}{A+1} \varphi \rangle} \left\langle \frac{1}{A-\lambda} \varphi, \cdot \right\rangle \frac{1}{A-\lambda} \varphi. \tag{2.4}$$

The relation between the real parameter γ describing the self-adjoint extensions of the restricted operator and the additive real parameter α appearing in formula (2.1) cannot be established without additional assumptions like homogeneity of the original operator and the perturbation vector. ³ The Nevanlinna function $Q(\lambda) =$

³ This approach has been developed in [AK97,AK97-2].

$\langle \varphi, \frac{1+\lambda}{A-\lambda} \frac{1}{A+1} \varphi \rangle$ can be considered as a regularization of the resolvent $\langle \varphi, \frac{1}{A-\lambda} \varphi \rangle$ which is not defined in the case $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$

$$Q(\lambda) = \langle \varphi, \frac{1+\lambda}{A-\lambda} \frac{1}{A+1} \varphi \rangle \stackrel{\text{formally}}{=} \langle \varphi, \frac{1}{A-\lambda} \varphi \rangle - \langle \varphi, \frac{1}{A+1} \varphi \rangle. \quad (2.5)$$

Observe that the two scalar products appearing in the right hand side of the last formula are not defined for $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, but their difference is in contrast well defined. The Nevanlinna function $\langle \varphi, \frac{1+A\lambda}{A-\lambda} \frac{1}{A^2+1} \varphi \rangle$ just coincides with Krein's Q -function appearing in the formula for the difference between the resolvents of two different self-adjoint extensions of one symmetric operator with the deficiency indices $(1, 1)$ [Kr44, Nai43].

The next step is to consider $\varphi \in \mathcal{H}_{-3}$. The restriction defined by (2.3) is defined only if one considers the original operator A as an operator acting in the Hilbert space \mathcal{H}_1 . Then the domain of the unperturbed operator A coincides with the space \mathcal{H}_3 and the restriction (2.3) determines a symmetric operator. From another hand formula (2.2) is valid only if one considers the extended Hilbert space containing vectors $\frac{1}{A-\lambda} \varphi \in \mathcal{H}_{-1}$. It appears that such extension is in fact one-dimensional, since

$$\frac{1}{A-\lambda} \varphi - \frac{1}{A-\mu} \varphi = (\varphi - \mu) \frac{1}{(A-\lambda)(A-\mu)} \varphi \in \mathcal{H}_1.$$

Hence it is enough to include the one dimensional subspace generated by the vector $\frac{1}{A+1} \varphi$ only. Hence the perturbed operator can be defined in the Hilbert space $\mathbf{H}_{-3} = \mathcal{H}_1 \oplus \mathbf{C}$ equipped with the natural embedding ρ_{-3}

$$\begin{aligned} \rho_{-3} : \mathbf{H}_{-3} &\rightarrow \mathcal{H}_{-1} \\ \mathbf{U} = (U, u_1) &\mapsto U + u_1 \frac{1}{A+1} \varphi. \end{aligned} \quad (2.6)$$

The perturbed operator corresponding to the formal expression (2.1) has been constructed in [KW00] by first defining certain maximal operator acting in \mathbf{H} and then restricting it to a self-adjoint operator. The maximal operator determined uniquely by the equality

$$\rho \mathbf{A} \mathbf{U} = A \rho \mathbf{U}|_{\text{mod } \varphi} \quad (2.7)$$

is similar to the adjoint operator appearing in the restriction-extension procedure used to construct \mathcal{H}_{-2} -perturbations. The set of self-adjoint restrictions of the maximal operator are described by one real parameter. Therefore formula (2.1) does not determine the perturbed operator uniquely, but a one parameter family of operators like in the case of \mathcal{H}_{-2} -perturbations. The resolvent of the perturbed operator restricted to the original Hilbert space is given by the formula

$$\rho \frac{1}{\mathbf{A}_\theta - \lambda} |_{\mathcal{H}_1} = \frac{1}{A-\lambda} - \frac{1}{(\lambda+1) \cot \theta + \langle \varphi, \frac{1}{A-\lambda} \frac{(\lambda+1)^2}{(A+1)^2} \varphi \rangle - 1} \left\langle \frac{1}{A-\lambda} \varphi, \cdot \right\rangle \frac{1}{A-\lambda} \varphi, \quad (2.8)$$

where $\theta \in [0, \pi)$ is the real number parametrizing the restrictions. The similarity between formulas (2.2) and (2.8) is obvious. The function

$$Q(\lambda) = \left\langle \varphi, \frac{1}{A - \lambda} \frac{(\lambda + 1)^2}{(A + 1)^2} \varphi \right\rangle$$

$$\stackrel{\text{formally}}{=} \left\langle \varphi, \frac{1}{A - \lambda} \varphi \right\rangle - \left\langle \varphi, \frac{1}{A + 1} \varphi \right\rangle - (\lambda + 1) \left\langle \varphi, \frac{1}{(A + 1)^2} \varphi \right\rangle \quad (2.9)$$

is a double regularization of the resolvent function. This function describes the spectral properties of the self-adjoint perturbed operator.

The aim of the current paper is to describe rank one singular perturbations of positive self-adjoint operators in the case $\varphi \in \mathcal{H}_{-4} \setminus \mathcal{H}_{-3}$. Our original aim was simply to generalize the ideas developed in [KW00] for the case of more singular perturbations. The main difference with the case $\varphi \in \mathcal{H}_{-3}$ is that the original operator A should be considered as an operator acting in the Hilbert space \mathcal{H}_2 from the scale of Hilbert spaces. Moreover this Hilbert space should be extended to include not only the vector

$$g_1 = \frac{1}{A + 1} \varphi \in \mathcal{H}_{-2}$$

but the vector

$$g_2 = \frac{1}{(A + 1)^2} \varphi \in \mathcal{H}$$

as well. Hence one has to consider the Hilbert space

$$\mathbf{H}_{-4} = \mathcal{H}_2 \oplus \mathcal{C}^2 \quad (2.10)$$

equipped with the standard imbedding

$$\begin{aligned} \rho_{-4} : \mathbf{H}_{-4} &\rightarrow \mathcal{H}_{-2} \\ \mathbf{U} = (U, u_2, u_1) &\mapsto U + u_2 \frac{1}{(A+1)^2} + u_1 \frac{1}{A+1} \varphi. \end{aligned} \quad (2.11)$$

The maximal operator can be defined using formula (2.7) in the way similar to \mathcal{H}_{-3} -perturbations. The main difference is that any symmetric restriction of the maximal operator is not self-adjoint. Hence no self-adjoint operator corresponds to formal expression (2.1). Instead one can consider the restrictions of the maximal operator that are regular operators. Regular operators are characterized by the fact that their domain coincides with the domain of their adjoint. The class of regular operators includes the self-adjoint operators. All regular restrictions of the maximal operator are parametrized by one real parameter in the way similar to \mathcal{H}_{-3} perturbations. The real and imaginary parts of these operators are calculated explicitly in the current paper. The resolvent of the perturbed operator is also calculated and it is shown that the spectrum of the perturbed operator is pure real. The resolvent restricted to the original Hilbert space is given by the formula similar to Krein's formula (2.4) and all spectral properties of the perturbed operator are described by Nevanlinna function Q given by (2.9).

3 The extended Hilbert space and the maximal operator.

Following the ideas expressed in Section 2, consider the Hilbert space⁴ $\mathbf{H} \equiv \mathbf{H}_{-4} = \mathcal{H}_2 \oplus \mathbf{C}^2$ equipped with the scalar product

$$\begin{aligned} \ll \mathbf{U}, \mathbf{V} \gg &= \langle U, V \rangle_{\mathcal{H}_2} + \bar{u}_2 v_2 + \bar{u}_1 v_1 \\ &= \langle U, (1 + A)^2 V \rangle + \bar{u}_2 v_2 + \bar{u}_1 v_1. \end{aligned} \quad (3.1)$$

One can consider different scalar product in \mathbf{H} but we decided to study the simplest case in order to avoid not necessary complications. The maximal operator \mathbf{A} being analog of the adjoint operator in the extension theory and acting in \mathbf{H} is defined using formula (2.7). The operator A appearing in this formula is the extension of the original operator A to the space \mathcal{H}_{-2} . This operator maps the space \mathcal{H}_{-2} onto the space \mathcal{H}_{-4} . To describe the domain of the operator \mathbf{A} one needs to consider the vector

$$g_3 = \frac{1}{A+1} g_2 = \frac{1}{(A+1)^3} \varphi. \quad (3.2)$$

Lemma 3.1. *Let $\varphi \in \mathcal{H}_{-4}$, then the maximal operator \mathbf{A} determined by (2.7) in \mathbf{H} is defined on the domain*

$$\text{Dom}(\mathbf{A}) = \{ \mathbf{U} = (U_r + u_3 g_3, u_2, u_1), U_r \in \mathcal{H}_4, u_3, u_2, u_1 \in \mathbf{C} \} \quad (3.3)$$

by the formula

$$\mathbf{A} \begin{pmatrix} U_r + u_3 g_3 \\ u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} AU_r - u_3 g_3 \\ u_3 - u_2 \\ u_2 - u_1 \end{pmatrix}. \quad (3.4)$$

Proof. Consider any vector \mathbf{U} from the domain of the operator \mathbf{A} and let us denote its image by $\mathbf{V} = (V, v_2, v_1)$. Then equality (2.7) can be written as follows

$$V + v_2 g_2 + v_1 g_1 \stackrel{\text{mod } \varphi}{=} AU - u_2 g_2 + (u_2 - u_1) g_1 + u_1 \varphi. \quad (3.5)$$

Taking into account that this equality holds modulus φ and that $AU \in \mathcal{H}_0$ we conclude that

$$v_1 = u_2 - u_1. \quad (3.6)$$

This implies

$$V + v_2 g_2 = AU - u_2 g_2,$$

⁴ We are going to drop the subindex -4 in the notations for the extended Hilbert space and standard imbedding introduced in (2.10) and (2.11).

where the usual equality sign is used, since the functions appearing on both sides of the equation belong to the original Hilbert space \mathcal{H} . This equality can be written as

$$V + U + v_2g_2 + u_2g_2 = (A + 1)U$$

and therefore

$$U = \frac{1}{A + 1}(V - U) + (v_2 - u_2)\frac{1}{A + 1}g_2.$$

It follows that the element U possesses the following representation

$$U = U_r + u_3g_3,$$

where $U_r \in \mathcal{H}_4$, $u_3 \in \mathbf{C}$ and g_3 is given by (3.2). Then equality (3.5) can be written as

$$V + v_2g_2 + v_1g_1 = AU_r - u_3g_3 + (u_3 - u_2)g_2 + (u_2 - u_1)g_1,$$

and one can deduce that

$$v_2 = u_3 - u_2, \quad V = AU_r - u_3g_3.$$

The Lemma is proven. \square

The spectrum of the operator \mathbf{A} covers the whole complex plane. Really consider any complex number λ . Then the element

$$\mathbf{U} = \begin{pmatrix} \frac{(1 + \lambda)^3}{A - \lambda}g_3 + (1 + \lambda)^2g_3 \\ (1 + \lambda) \\ 1 \end{pmatrix} = \begin{pmatrix} (1 + \lambda)^2\frac{A + 1}{A - \lambda}g_3 \\ (1 + \lambda) \\ 1 \end{pmatrix}$$

solves the equation $\mathbf{A}\mathbf{U} = \lambda\mathbf{U}$. Note that the last formula reads as follows in the special case $\lambda = -1$

$$\mathbf{A} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let us calculate the adjoint operator \mathbf{A}^* .

Lemma 3.2. *The operator \mathbf{A}^* , adjoint to \mathbf{A} , is defined on the domain*

$$\text{Dom}(\mathbf{A}^*) = \{ \mathbf{U} = (U_r, u_2, u_1); U_r \in \mathcal{H}_4, u_2, u_1 \in \mathbf{C}, \\ u_2 = \langle \varphi, U_r \rangle \} \quad (3.7)$$

by the formula

$$\mathbf{A}^* \begin{pmatrix} U_r \\ u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} AU_r \\ u_1 - u_2 \\ -u_1 \end{pmatrix}. \quad (3.8)$$

Proof. Consider arbitrary elements $\mathbf{U} \in \text{Dom}(\mathbf{A})$ and $\mathbf{V} = (V, v_2, v_1) \in \mathbf{H}$. The sesquilinear form of the operator \mathbf{A} can be written as follows

$$\begin{aligned} \ll (\mathbf{A} + 1)\mathbf{U}, \mathbf{V} \gg &= \langle (A + 1)U_r, (1 + A)^2V \rangle + \bar{u}_3v_2 + \bar{u}_2v_1 \\ &= \langle U_r + u_3g_3, (1 + A)^3V \rangle + \bar{u}_3 \{ -\langle g_3, (1 + A)^3V \rangle + v_2 \} \\ &\quad + \bar{u}_2v_1. \end{aligned} \tag{3.9}$$

Consider the subset of elements $\mathbf{U} \in \text{Dom}(\mathbf{A})$ with $u_1 = u_2 = u_3 = 0$. Then the first term in the last formula is a bounded functional with respect to $\mathbf{U} \in \mathbf{H}$ if and only if $V = V_r \in \mathcal{H}_4$. Consider next arbitrary $\mathbf{U} \in \text{Dom}(\mathbf{A})$. The first and the last two terms are bounded linear functionals with respect to $\mathbf{U} \in \mathbf{H}$, but $\mathbf{U} \mapsto u_3$ is not a bounded linear functional. Therefore we conclude that $\ll \mathbf{A}\mathbf{U}, \mathbf{V} \gg$ is a bounded linear functional with respect to \mathbf{U} only if the following (boundary) condition is satisfied

$$v_2 = \langle g_3, (1 + A)^3V_r \rangle \equiv \langle \varphi, V_r \rangle. \tag{3.10}$$

We conclude that the domain of the adjoint operator \mathbf{A}^* consists of elements $\mathbf{V} \in \mathbf{H}$ possessing the representation

$$\mathbf{V} = (V_r, v_2, v_1), \quad V_r \in \mathcal{H}_4, v_{1,2} \in \mathbf{C}$$

and satisfying (3.10). Under these conditions formula (3.9) reads as follows

$$\ll \mathbf{A}\mathbf{U}, \mathbf{V} \gg = \langle U_r + u_3g_3, (1 + A)^2AV_r \rangle + \bar{u}_2(v_1 - v_2) + \bar{u}_1(-v_1).$$

This formula implies that the action of the adjoint operator is given by (3.8) and the formulated conditions on the elements from its domain are not only necessary but also sufficient. The Lemma is proven. \square

The domain of \mathbf{A}^* is included in the domain of \mathbf{A} . But the operator \mathbf{A}^* is not a restriction of \mathbf{A} . Therefore no restriction of the operator \mathbf{A} is self-adjoint like in the case of \mathcal{H}_{-3} -perturbations. One can prove this fact directly using the boundary form of the operator \mathbf{A} .

Lemma 3.3. *The boundary form of the maximal operator \mathbf{A} is given by*

$$\ll \mathbf{A}\mathbf{U}, \mathbf{V} \gg - \ll \mathbf{U}, \mathbf{A}\mathbf{V} \gg = \left\langle \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \langle \varphi, U_r \rangle \\ u_3 \\ u_2 \\ u_1 \end{pmatrix}, \begin{pmatrix} \langle \varphi, V_r \rangle \\ v_3 \\ v_2 \\ v_1 \end{pmatrix} \right\rangle \tag{3.11}$$

Proof. The following straightforward calculations prove the Lemma

$$\begin{aligned}
 & \ll \mathbf{AU}, \mathbf{V} \gg - \ll \mathbf{U}, \mathbf{AV} \gg \\
 &= \ll \begin{pmatrix} AU_r - u_3 g_3 \\ u_3 - u_2 \\ u_2 - u_1 \end{pmatrix}, \begin{pmatrix} V_r + v_3 g_3 \\ v_2 \\ v_1 \end{pmatrix} \gg - \ll \begin{pmatrix} U_r + u_3 g_3 \\ u_2 \\ u_1 \end{pmatrix}, \begin{pmatrix} AV_r - v_3 g_3 \\ v_3 - v_2 \\ v_2 - v_1 \end{pmatrix} \gg \\
 &= \langle AU_r - u_3 g_3, (1+A)^2(V_r + v_3 g_3) \rangle + (\bar{u}_3 - \bar{u}_2)v_2 + (\bar{u}_2 - \bar{u}_1)v_1 \\
 &\quad - \langle U_r + u_3 g_3, (1+A)^2(AV_r - v_3 g_3) \rangle - \bar{u}(v_3 - v_2) - \bar{u}_1(v_2 - v_1) \\
 &= -\bar{u}_3 \langle \varphi, V_r \rangle + \langle U_r, \varphi \rangle v_3 + \bar{u}_3 v_2 + \bar{u}_2 v_1 - \bar{u}_2 v_3 - \bar{u}_1 v_2.
 \end{aligned}$$

We have used that $\varphi = (1+A)^3 g_3$ in these calculations. The Lemma is proven. \square

The sesquilinear boundary form of the maximal operator can also be presented in the form

$$\begin{aligned}
 & \ll \mathbf{AU}, \mathbf{V} \gg - \ll \mathbf{U}, \mathbf{AV} \gg \\
 &= \bar{u}_3 (v_2 - \langle \varphi, V_r \rangle) - \overline{(u_2 - \langle \varphi, U_r \rangle)} v_3 + \bar{u}_2 v_1 - \bar{u}_1 v_2. \tag{3.12}
 \end{aligned}$$

The matrix describing the boundary form

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is symplectic and has rank four.⁵ Any symmetric restriction of the operator \mathbf{A} is described by at least two boundary conditions. Such restriction cannot be self-adjoint, since the kernel of the operator $\mathbf{A} - \lambda$, $\Im \lambda \neq 0$ has dimension 1. Therefore no restriction of the operator \mathbf{A} is a self-adjoint operator in the Hilbert space \mathbf{H} . It follows that no self-adjoint operator corresponds to formal expression (2.1) in the case $\varphi \in \mathcal{H}_{-4} \setminus \mathcal{H}_{-3}$. In what follows we are going to show that the operator corresponding to (2.1) can be determined in the class of regular operators.

4 Supersingular perturbation as a regular operator.

In this section we are going to define an operator corresponding to (2.1) in the class of regular operators. We call an operator acting in the Hilbert space **regular** if it is densely defined and the domain of the operator coincides with the domain of the adjoint one. The operators corresponding to (2.1) will be defined by restricting the maximal operator \mathbf{A} to a certain regular operator.

⁵ Its characteristic determinant is equal to $\lambda^4 + 3\lambda^2 + 1$.

Theorem 4.1. *All regular restrictions of the operator \mathbf{A} are described by the following boundary condition*

$$a\langle\varphi, U_r\rangle + bu_3 + cu_2 = 0, \quad (4.1)$$

where (a, b, c) is a three dimensional nonzero vector $(a, b, c) \in \mathbf{R}^3$ orthogonal to the vector $(1, 0, 1)$.

Proof. The domain $\text{Dom}(\mathbf{A})$ consists of all elements $\mathbf{U} \in \mathbf{H}$ possessing the following representation

$$\mathbf{U} = (U_r + u_3g_3, u_2, u_1),$$

where $U_r \in \mathcal{H}_4$, $u_1, u_2, u_3 \in \mathbf{C}$. The domain $\text{Dom}(\mathbf{A}^*)$ of the adjoint operator is a subdomain of $\text{Dom}(\mathbf{A})$ described by the boundary conditions

$$\begin{cases} u_3 = 0, \\ u_2 = \langle\varphi, U_r\rangle. \end{cases}$$

Therefore the quotient space $\text{Dom}(\mathbf{A})/\text{Dom}(\mathbf{A}^*)$ has dimension 2 and any non-trivial restriction⁶ of the domain $\text{Dom}(\mathbf{A})$ to a linear subset containing $\text{Dom}(\mathbf{A}^*)$ is described by the boundary condition

$$a\langle\varphi, U_r\rangle + bu_3 + cu_2 = 0, \quad (4.2)$$

where a, b , and c are arbitrary complex numbers, not all equal to zero simultaneously $|a|^2 + |b|^2 + |c|^2 \neq 0$. Let us denote the corresponding subspace by \mathbf{D} and the restriction of the operator \mathbf{A} to this subspace by $\mathbf{A}|_{\mathbf{D}}$. The operator $\mathbf{A}|_{\mathbf{D}}$ is a restriction of the operator \mathbf{A} . Therefore the adjoint operator $\mathbf{A}|_{\mathbf{D}}^*$ is an extension of the operator \mathbf{A}^* . The sesquilinear form of the operator $\mathbf{A}|_{\mathbf{D}}$ is given by formula (3.9), where now $\mathbf{U} \in \mathbf{D}$. Consider vectors \mathbf{U} with $u_2 = u_3 = \langle\varphi, U_r\rangle = 0$. Then the scalar product

$$\langle U_r, (A + 1)^3 V \rangle$$

generates a bounded linear functional with respect to $(U_r, 0, 0) \in \mathbf{H}$ and the standard norm in \mathbf{H} if and only if the following representation holds⁷

$$(A + 1)^3 V = c\varphi + \tilde{f},$$

where $c \in \mathbf{C}$, $\tilde{f} \in \mathcal{H}_{-2}$. This implies that

$$V = cg_3 + \frac{1}{(A + 1)^3} \tilde{f},$$

⁶ Nontrivial restriction in this context is the one having domain different from both $\text{Dom}(\mathbf{A}^*)$ and $\text{Dom}(\mathbf{A})$.

⁷ Remember that U_r is orthogonal to φ .

and it follows that the vector V possesses the representation

$$V = V_r + v_3 g_3,$$

where $V_r \in \mathcal{H}_4$, $v_3 \in \mathbf{C}$. Then the sesquilinear form is given by

$$\begin{aligned} & \ll (\mathbf{A} + 1)\mathbf{U}, \mathbf{V} \gg \\ & = \langle U_r + u_3 g_3, (A + 1)^3 V_r \rangle - \bar{u}_3 \langle \varphi, V_r \rangle + v_3 \langle U_r, \varphi \rangle + \bar{u}_3 v_2 + \bar{u}_2 v_1. \end{aligned}$$

Let us consider separately three cases covering all possible values of the parameters a, b , and c .

1. Consider first the general case $a \neq 0$. Taking into account

$$\langle \varphi, U_r \rangle = -\frac{b}{a} u_3 - \frac{c}{a} u_2$$

one can get

$$\begin{aligned} & \ll (\mathbf{A} + 1)\mathbf{U}, \mathbf{V} \gg \\ & = \langle U_r + u_3 g_3, (A + 1)^3 V_r \rangle + \bar{u}_3 \left(-\langle \varphi, V_r \rangle - \frac{\bar{b}}{a} v_3 + v_2 \right) + \bar{u}_2 \left(v_1 - \frac{\bar{c}}{a} v_3 \right). \end{aligned}$$

The last expression determines a bounded linear functional if and only if the following relation holds

$$\bar{a} \langle \varphi, V_r \rangle + \bar{b} v_3 - \bar{a} v_2 = 0.$$

This condition coincides with (4.2) if and only if

$$a = -c$$

and the complex numbers a and b have the same phase. Hence without loss of generality the constants a, b, c can be chosen real and such that $c = -a$, i.e. $\langle (a, b, c), (1, 0, 1) \rangle = 0$.

2. Consider now the case $a = 0, b \neq 0$. The boundary condition has the following form

$$u_3 = -\frac{c}{b} u_2. \tag{4.3}$$

Hence the sesquilinear form of the operator is given by

$$\begin{aligned} & \ll (\mathbf{A} + 1)\mathbf{U}, \mathbf{V} \gg \\ & = \langle U_r + u_3 g_3, (A + 1)^3 V_r \rangle + \bar{u}_2 \left[\frac{\bar{c}}{b} (\langle \varphi, V_r \rangle - v_2) + v_1 \right] + \langle U_r, \varphi \rangle v_3. \end{aligned}$$

This form defines bounded linear functional with respect to $\mathbf{U} \in \mathbf{H}$ if and only if

$$v_3 = 0,$$

since $\langle U_r, \varphi \rangle$ is not a bounded linear functional. The last condition coincides with (4.3) only if $c = 0$. Then condition (4.3) reads as follows

$$bu_3 = 0.$$

Without loss of generality the constant b can be chosen real.

3. Consider the last possible case $a = 0 = b$, $c \neq 0$. The sesquilinear form given by

$$\ll (\mathbf{A} + 1)\mathbf{U}, \mathbf{V} \gg = \langle U_r + u_3 g_3, (A + 1)^3 V_r \rangle + \bar{u}_3 (v_2 - \langle \varphi, V_r \rangle) + v_3 \langle U_r, \varphi \rangle,$$

determines bounded linear functional if and only if the following conditions are satisfied

$$v_3 = 0 \text{ and } v_2 = \langle \varphi, V_r \rangle.$$

These conditions never coincide with the condition $u_2 = 0$. Hence this boundary condition does not define any regular restriction of the operator \mathbf{A} .

The boundary conditions described in 1 and 2 cover all boundary conditions of the form (4.1) with nonzero real vectors (a, b, c) orthogonal to $(1, 0, 1)$. The Theorem is proven. \square

The last theorem states that all regular restrictions of the operator \mathbf{A} are described by real three dimensional vectors (a, b, c) subject to the orthogonality condition $(a, b, c) \perp (1, 0, 1)$. The length of the vector (a, b, c) plays no rôle and therefore all boundary conditions can be parametrized by one real parameter - "angle" $\theta \in [0, \pi)$ as follows:

$$\sin \theta \langle \varphi, U_r \rangle + \cos \theta u_3 - \sin \theta u_2 = 0. \quad (4.4)$$

The following definition will be used.

Definition 4.1. The operator \mathbf{A}_θ is the restriction of the maximal operator \mathbf{A} to the set of functions satisfying boundary conditions (4.4).

Let us calculate the operator adjoint to \mathbf{A}_θ . The domain of this operator coincides with the domain $\text{Dom}(\mathbf{A}_\theta)$. The sesquilinear form of the operator \mathbf{A}_θ can be presented by the following expression using the fact, that the functions from the domains of the operators \mathbf{A}_θ and \mathbf{A}_θ^* satisfy (4.4)

$$\ll (\mathbf{A}_\theta + 1)\mathbf{U}, \mathbf{V} \gg = \langle U_r + u_3 g_3, (A + 1)^3 V_r \rangle + \bar{u}_2 v_3 + \bar{u}_2 v_1.$$

Hence the action of the operator \mathbf{A}_θ^* is given by

$$\mathbf{A}_\theta^* \begin{pmatrix} V_r + v_3 g_3 \\ v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} AV_r - v_3 g_3 \\ v_3 + v_1 - v_2 \\ -v_1 \end{pmatrix}. \quad (4.5)$$

The operator \mathbf{A}_θ and its adjoint are related by

$$\mathbf{A}_\theta - \mathbf{A}_\theta^* = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}. \quad (4.6)$$

Thus the regular operator corresponding to the formal expression (2.1) is not defined uniquely. Like in the case of \mathcal{H}_{-2} and \mathcal{H}_{-3} -perturbations one parameter family of operators has been constructed. The real and imaginary parts of the operator \mathbf{A}_θ are given by

$$\mathbf{A}_\theta = \Re \mathbf{A}_\theta + i \Im \mathbf{A}_\theta;$$

$$(\Re \mathbf{A}_\theta) \begin{pmatrix} U_r + u_3 g_3 \\ u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} AU_r - u_3 g_3 \\ u_3 - u_2 + \frac{1}{2}u_1 \\ \frac{1}{2}u_2 - u_1 \end{pmatrix};$$

$$\Im \mathbf{A}_\theta = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}. \quad (4.7)$$

The imaginary part of \mathbf{A}_θ is a bounded operator. Therefore the operator $\Re \mathbf{A}_\theta$ given by (4.7) is a self-adjoint operator on the domain $\text{Dom}(\mathbf{A}_\theta)$. The following theorem proves that the spectrum of the operator \mathbf{A}_θ is pure real even if the operator itself is not self-adjoint.

Theorem 4.2. *The resolvent of the operator \mathbf{A}_θ for all nonreal λ is given by the 3×3 bounded matrix operator*

$$\frac{1}{\mathbf{A}_\theta - \lambda} = \mathbf{R}(\lambda), \quad (4.8)$$

having the following components corresponding to the orthogonal decomposition of the Hilbert space $\mathbf{H} = \mathcal{H}_2 \oplus \mathbf{C} \oplus \mathbf{C} \ni \mathbf{U} = (U, u_2, u_1)$

$$R_{00}(\lambda) = \frac{1}{A-\lambda} - (1+\lambda)^2 \frac{\sin \theta}{\Delta} \frac{1}{A-\lambda} g_2 \left\langle \frac{1}{A-\lambda} \varphi, \cdot \right\rangle;$$

$$R_{20}(\lambda) = -(1+\lambda) \frac{\sin \theta}{\Delta} \left\langle \frac{1}{A-\lambda} \varphi, \cdot \right\rangle;$$

$$R_{10}(\lambda) = -\frac{\sin \theta}{\Delta} \left\langle \frac{1}{A-\lambda} \varphi, \cdot \right\rangle;$$

$$R_{02}(\lambda) = -(1+\lambda) \frac{\sin \theta}{\Delta} \frac{1}{A-\lambda} g_2;$$

$$R_{22}(\lambda) = -\frac{1}{\Delta} \left((1+\lambda) \left\langle \varphi, \frac{1+\lambda}{A-\lambda} g_3 \right\rangle \sin \theta + (1+\lambda) \cos \theta \right);$$

$$R_{12}(\lambda) = -\frac{1}{\Delta} \left(\left\langle \varphi, \frac{1+\lambda}{A-\lambda} g_3 \right\rangle \sin \theta + \cos \theta \right);$$

$$R_{01}(\lambda) = R_{21}(\lambda) = 0;$$

$$R_{11}(\lambda) = -\frac{1}{\Delta} \left((1+\lambda) \left\langle \varphi, \frac{1+\lambda}{A-\lambda} g_3 \right\rangle \sin \theta + (1+\lambda) \cos \theta - \sin \theta \right),$$

where the subindex 0 corresponds to the component $U \in \mathcal{H}_2$ and

$$\Delta(\lambda, \theta) = (1+\lambda)^2 \left\{ \left(\left\langle \varphi, \frac{1+\lambda}{A-\lambda} g_3 \right\rangle - \frac{1}{1+\lambda} \right) \sin \theta + \cos \theta \right\}. \quad (4.9)$$

Proof. Consider arbitrary $\mathbf{F} = (F, f_2, f_1) \in \mathbf{H}$. Then the resolvent equation

$$(\mathbf{A} - \lambda) \begin{pmatrix} U_r + u_3 g_3 \\ u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} F \\ f_2 \\ f_1 \end{pmatrix} \quad (4.10)$$

together with the boundary condition (4.4) imply that

$$\begin{pmatrix} 1 & -\left\langle \varphi, \frac{1+\lambda}{A-\lambda} g_3 \right\rangle & 0 & 0 \\ 0 & 1 & -(1+\lambda) & 0 \\ 0 & 0 & 1 & -(1+\lambda) \\ \sin \theta & \cos \theta & -\sin \theta & 0 \end{pmatrix} \begin{pmatrix} \langle \varphi, U_r \rangle \\ u_3 \\ u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} \langle \varphi, \frac{1}{A-\lambda} F \rangle \\ f_2 \\ f_1 \\ 0 \end{pmatrix}$$

The determinant of the matrix appearing in the last equation given by (4.9). The determinant is equal to zero for nonreal λ only if the expression in brackets is

equal to zero. The imaginary part of this expression can be calculated explicitly

$$I = y \left\{ \langle g_2, \frac{(A+1)^2}{(A-x)^2 + y^2} g_2 \rangle + \frac{1}{(1+x)^2 + y^2} \right\},$$

where we used the following notations for the imaginary and real parts of λ : $\lambda = x + iy$. Both items are positive and their sum cannot be equal to zero. Hence the determinant of the matrix is different from zero for all nonreal λ . Therefore the linear system has always unique solution for those λ :

$$\begin{aligned} \langle \varphi, U_r \rangle &= \frac{-1}{\Delta} \left\{ ((1+\lambda) \sin \theta - (1+\lambda)^2 \cos \theta) \langle \varphi, \frac{1}{A-\lambda} F \rangle \right. \\ &\quad \left. + \langle \varphi, \frac{1+\lambda}{A-\lambda} g_3 \rangle \sin \theta f_2 \right\}; \\ u_3 &= \frac{-1}{\Delta} \left\{ (1+\lambda)^2 \sin \theta \langle \varphi, \frac{1}{A-\lambda} F \rangle + (1+\lambda) \sin \theta f_2 \right\}; \\ u_2 &= \frac{-1}{\Delta} \left\{ (1+\lambda) \sin \theta \langle \varphi, \frac{1}{A-\lambda} F \rangle \right. \\ &\quad \left. + ((1+\lambda) \langle \varphi, \frac{1+\lambda}{A-\lambda} g_3 \rangle + (1+\lambda) \cos \theta) f_2 \right\}; \\ u_1 &= \frac{-1}{\Delta} \left\{ \sin \theta \langle \varphi, \frac{1}{A-\lambda} F \rangle + (\cos \theta + \sin \theta \langle \varphi, \frac{1+\lambda}{A-\lambda} g_3 \rangle) f_2 \right. \\ &\quad \left. \left((1+\lambda) \cos \theta - \sin \theta - (1+\lambda) \langle \varphi, \frac{1+\lambda}{A-\lambda} g_3 \rangle \sin \theta \right) f_1 \right\}. \end{aligned} \tag{4.11}$$

The component U can be calculated from the first equation (4.10)

$$\begin{aligned} U &= u_r + u_3 g_3 \\ &= \frac{1}{A-\lambda} F + (1+\lambda) u_3 \frac{1}{A-\lambda} g_3 + u_3 g_3 \\ &= \frac{1}{A-\lambda} F + u_3 \frac{1}{A-\lambda} g_2. \end{aligned}$$

Hence the resolvent of the operator \mathbf{A} is given by formula (4.8) for all nonreal λ . The Theorem is proven. \square

The theorem implies that the spectrum of the operator \mathbf{A}_θ is real. Consider the restriction of the resolvent to the subspace $\mathcal{H}_2 \subset \mathbf{H}$ combined with the embedding

$$\rho \frac{1}{\mathbf{A}_\theta - \lambda} |_{\mathcal{H}_2} = \frac{1}{A - \lambda} - \frac{1}{(\lambda + 1)^2 \left\{ \cot \theta + \langle \varphi, \frac{1+\lambda}{A-\lambda} \frac{1}{(A+1)^3} \varphi \rangle - \frac{1}{1+\lambda} \right\}} \left\langle \frac{1}{A-\lambda} \varphi, \cdot \right\rangle \frac{1}{A-\lambda} \varphi, \quad (4.12)$$

The last formula is analogous to Krein's formula connecting the resolvents of two self-adjoint extensions of one symmetric operator and is very similar to formula (2.8) describing the restricted resolvent of the self-adjoint operator corresponding to the singular \mathcal{H}_{-3} -perturbation. The spectral properties of the operator \mathbf{A}_θ are described by the function

$$\begin{aligned} Q(\lambda) &= \langle \varphi, \frac{1}{A-\lambda} \frac{(\lambda+1)^3}{(A+1)^3} \varphi \rangle \\ &\stackrel{\text{formally}}{=} \langle \varphi, \frac{1}{A-\lambda} \varphi \rangle - \langle \varphi, \frac{1}{A+1} \varphi \rangle - (1+\lambda) \langle \varphi, \frac{1}{(A+1)^2} \varphi \rangle \\ &\quad - (1+\lambda)^2 \langle \varphi, \frac{1}{(A+1)^3} \varphi \rangle, \end{aligned}$$

which is a triple regularized resolvent function.

5 Conclusions.

Rank one supersingular \mathcal{H}_{-4} -perturbation of a positive self-adjoint operator has been determined in the class of regular operator. It has been shown that such operator cannot be defined in the class of self-adjoint operators. The spectrum of the perturbed operator is pure real and it follows that the properties of this operator are similar to those of self-adjoint operators. One has to study the question whether this regular operator is similar to a certain self-adjoint operator. The method suggested in this paper can be generalized to determine even more singular perturbations of positive self-adjoint operator corresponding to vectors $\varphi \in \mathcal{H}_{-n}$, $n \geq 5$. This program will be carried out in one of the forthcoming publications.

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