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Integral Geometry Problem with Incomplete Data for Tensor Fields in a Complex Space.

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Abstract

The paper is devoted to the problem of reconstructing a tensor field in C^n, if its ray transform is known along all complex lines, intersecting a given complex curve. A procedure for recovering the solenoidal part of the tensor field is given.

1 Introduction and some theory of tensor fields in a complex space

For a major reference to integral geometry of tensor fields we refer the reader to the book [2]. In the paper [3] the author considered an integral geometry problem with incomplete data for symmetric tensor fields in a real space. (See [1] for the references to other papers on the integral geometry problems with incomplete data.) In the current article we are going to study a similar problem for tensors in a complex space. The problem for the complete collection of data was considered in the author's dissertation [4], as well as in [5]. We will need to recall some theory from these papers.

Let $p, q \geq 0$ be integers and T_p^q be the space of bidegree (p, q) tensors on C^n, i.e. the functions $f : \underbrace{C^n \times \ldots \times C^n}_{p} \times \underbrace{C^n \times \ldots \times C^n}_{q} \to C$, which are C-linear with respect to each of the first p variables and C-antilinear with respect to the last q. Let S_p^q be a space of tensors, symmetric with respect to the collections of the first p and the last q variables separately. There is a canonical projection $\sigma : T_p^q \to S_p^q$:

$$\sigma f(z_1, \ldots, z_p, w_1, \ldots, w_q) = \frac{1}{p!q!} \sum_{\pi \in \Pi_p} \sum_{\delta \in \Pi_q} f(z_{\pi 1}, \ldots, z_{\pi p}, w_{\delta 1}, \ldots, w_{\delta q}),$$

where Π_p, Π_q are permutation groups. We write each tensor $f \in T_p^q$ in the form

$$f = f_{i_1 \ldots i_p}^{j_1 \ldots j_q} dz^{i_1} \otimes \ldots \otimes dz^{i_p} \otimes d\bar{z}^{j_1} \otimes \ldots \otimes d\bar{z}^{j_q}.$$

Henceforth we will use the Einstein summation convention — summation with respect to the pairs of repeated indices, independently running from 1 to n. The numbers $f_{i_1 \ldots i_p}^{j_1 \ldots j_q}$ are called the coordinates (or the components) of the tensor f. A map $C^n \to T_p^q$ is called a *tensor field on* C^n. By $C^\infty(T_p^q)$ and $\mathcal{S}(T_p^q)$ we denote the spaces of tensor fields on C^n with smooth and rapidly decreasing components respectively. We will need the following operators, defined in coordinates:

$$(d_l f)_{i_1 \ldots i_{p+1}}^{j_1 \ldots j_q} = \sigma\left(\frac{\partial}{\partial z_{i_{p+1}}} f_{i_1 \ldots i_p}^{j_1 \ldots j_q}\right), \quad (\bar{d}_u f)_{i_1 \ldots i_p}^{j_1 \ldots j_{q+1}} = \sigma\left(\frac{\partial}{\partial \bar{z}_{j_{q+1}}} f_{i_1 \ldots i_p}^{j_1 \ldots j_q}\right),$$

$$(\bar{\delta}_l f)_{i_1 \ldots i_{p-1}}^{j_1 \ldots j_q} = \sum_{i=1}^{n} \frac{\partial}{\partial \bar{z}_i} f_{i_1 \ldots i_{p-1} i}^{j_1 \ldots j_q}, \quad (\delta_u f)_{i_1 \ldots i_p}^{j_1 \ldots j_{q-1}} = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} f_{i_1 \ldots i_p}^{j_1 \ldots j_{q-1} j}.$$

The operators d are the operators of inner differentiation of the different kinds ("l" - lower, "u" - upper), δ — the divergence operators.

Here, as usual,

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - \sqrt{-1}\frac{\partial}{\partial y_k}\right) \ , \quad \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + \sqrt{-1}\frac{\partial}{\partial y_k}\right).$$

Let $C_0^n = C^n \setminus \{0\}$.

Definition 1.1 *The ray transform of a tensor field $g \in C^\infty(S_p^q)$ is the function Ig, defined on $C^n \times C_0^n$ by the expression:*

$$Ig(z,\xi) = \int\limits_C g_{i_1\dots i_p}^{j_1\dots j_q}(z+t\xi)\xi^{i_1}\dots\xi^{i_p}\bar{\xi}^{j_1}\dots\bar{\xi}^{j_q}\,dS(t),$$

where $dS(t)$ is the area form on C, and we assume the absolute convergence of all integrals involved.

The problem we will be dealing with is to reconstruct g from Ig. It turns out that the operator I has a nontrivial kernel. In $\mathcal{S}(S_p^q)$ it consists exactly of the tensor fields of the form $d_l v + \bar{d}_u w$, where $v \in C^\infty(S_{p-1}^q), w \in C^\infty(S_p^{q-1})$ and v, w vanish sufficiently fast at infinity.

We need the following statement.

Theorem 1.2 *For a tensor field $g \in \mathcal{S}(S_p^q)$ there exists a unique tensor field $f \in C^\infty(S_p^q)$ such that for some tensor fields $v \in C^\infty(S_{p-1}^q), w \in C^\infty(S_p^{q-1})$ one has*

$$g = f + d_l v + \bar{d}_u w\ ,\ \bar{\delta}_l f = 0\ , \delta_u f = 0\ ;\ \ f, v, w \to 0\ \text{ as } |z| \to \infty\ .$$

The field f is called the *solenoidal part* of g, we denote $f = {}^s g$. It turns out that by knowing Ig we can reconstruct $f = {}^s g$, and there is in [4], [5] an explicit inversion formula.

We will be using the following version of the Fourier transform for tensor fields: in coordinates,

$$\hat{g}_{i_1\dots i_p}^{j_1\dots j_q}(\zeta) = (2\pi)^{-n}\int\limits_{C^n} e^{-\frac{\sqrt{-1}}{2}(\langle z,\zeta\rangle + \langle\zeta,z\rangle)} g_{i_1\dots i_p}^{j_1\dots j_q}(z)\,dV_{2n}(z),$$

where $\langle\cdot,\cdot\rangle$ is the standard Hermitian form on C^n and $dV_{2n}(z)$ is the volume form on C^n.

We have the following expression for $\hat{f}$ in terms of $\hat{g}$ ($f = {}^s g$), which will be useful later:

$$\hat{f}_{i_1\dots i_p}^{j_1\dots j_q}(\zeta) = \varepsilon_{i_1}^{k_1}(\zeta)\dots\varepsilon_{i_p}^{k_p}(\zeta)\,\varepsilon_{l_1}^{j_1}(\zeta)\dots\varepsilon_{l_q}^{j_q}(\zeta)\,\hat{g}_{k_1\dots k_p}^{l_1\dots l_q}(\zeta), \tag{1}$$

where

$$\varepsilon_i^j(\zeta) = \delta_i^j - \frac{\bar{\zeta}^i\zeta^j}{|\zeta|^2},$$

δ_i^j — the Kroneker symbol.

2 Integral geometry problem with incomplete data, reconstruction of the solenoidal part

For $n \geq 3$ let $\gamma \subset C^n$ be a C^1-smooth complex, but not necessarily holomorphic curve, parametrized as follows:

$$x = \phi(\lambda),\ \lambda \in \Lambda \subset C,\ \phi \in C^1(\Lambda).$$

Problem. *Let $g \in \mathcal{S}(S^q_p)$. Reconstruct its solenoidal part $f = {}^s g$ by the known values $Ig(z,\xi)$ for all $z \in \gamma$, $\xi \in C^n_0$.*

To formulate a condition on γ we need to consider the following algebraic setting. Let $P(z)$ be an arbitrary degree m polynomial on C^N, which is not necessarily holomorphic:

$$P(z) = \sum_{l+r\leq m} p^{(l,r)j_1\ldots j_r}_{\quad i_1\ldots i_l} z^{i_1}\ldots z^{i_l}\bar{z}^{j_1}\ldots\bar{z}^{j_r},\quad p^{(l,r)} \in S^r_l.$$

Altogether there are $\mathcal{L}_{N,m} := \binom{2N+m}{m}$ independent coefficients (taking into account symmetries).

Definition 2.1 *A collection of $\mathcal{L}_{N,m}$ points in C^N : $b_1,\ldots,b_{\mathcal{L}_{N,m}}$ is called defining of order m, if a polynomial $P(z)$ is determined uniquely by its values $P(b_j),\ j = 1,\ldots,\mathcal{L}_{N,m}$.*

Almost all collections are defining in the sense that they form in $(C^n)^{\mathcal{L}_{N,m}}$ the complement of an algebraic hypersurface.

Definition 2.2 *We say that a complex curve γ satisfies the complex Kirillov-Tuy condition of order $m \geq 1$, if for every $z \in C^n$, $\eta \in S^{2n-1}$ ($|\eta| = 1$) we can find a defining collection of order m: $a_1(z,\eta),\ldots,a_{\mathcal{L}_{n-1,m}}(z,\eta)$ in the intersection of the complex hyperplane $\langle a,\eta\rangle = \langle z,\eta\rangle$ with γ. (Defining, that is, for the polynomials on this hyperplane.)*

Theorem 2.3 *Let $\gamma \subset C^n$ $(n \geq 3)$ be a C^1-smooth complex curve, satisfying the complex Kirillov-Tuy condition of order $(p+q)$. If $g \in \mathcal{S}(S^q_p)$, then its solenoidal part $f = {}^s g$ can be uniquely reconstructed by the known values $Ig(z,\xi)$ for all $z \in \gamma$, $\xi \in C^n_0$.*

Proof.

We notice the following homogeneity property of Ig with respect to the second variable:

$$Ig(z,\tau\xi) = \frac{\tau^p \bar{\tau}^q}{|\tau|^2} Ig(z,\xi).$$

Thus for a fixed z we can treat $Ig(z,\cdot)$ as a tempered distribution from $\mathcal{S}'(C^n)$ and consider its Fourier transform.

Lemma 2.4 *We have the following formula in $\mathcal{S}'(C^n)$:*

$$(Ig)^{\wedge}(a,\eta) = \lim_{H\to\infty} \sum_{l=0}^{p}\sum_{r=0}^{q} \sum_{1\le\alpha_1<\ldots<\alpha_l\le p} \sum_{1\le\gamma_1<\ldots<\gamma_r\le q} (-1)^{l+r} a^{i_{\alpha_1}} \ldots a^{i_{\alpha_l}} \times$$

$$\times \bar{a}^{j_{\gamma_1}} \ldots \bar{a}^{j_{\gamma_r}} \int\limits_{|\rho|\le H} |\rho|^{2n-4} \rho^p \bar{\rho}^q e^{\frac{\sqrt{-1}}{2}(\langle\rho a,\eta\rangle + \langle\eta,\rho a\rangle)} \times$$

$$\times (z^{i_{\beta_1}} \ldots z^{i_{\beta_{p-l}}} \bar{z}^{j_{\delta_1}} \ldots \bar{z}^{j_{\delta_{q-r}}} g^{j_1\ldots j_q}_{i_1\ldots i_p}(z))^{\wedge} (\bar{\rho}\eta)\, dS(\rho). \tag{2}$$

Here we set $\{\beta_1 \ldots \beta_{p-l}\} = \{1 \ldots p\} \setminus \{\alpha_1 \ldots \alpha_l\}$, $1 \le \beta_1 < \ldots < \beta_{p-l} \le p$; $\{\delta_1 \ldots \delta_{q-r}\} = \{1 \ldots q\} \setminus \{\gamma_1 \ldots \gamma_r\}$, $1 \le \delta_1 < \ldots < \delta_{q-r} \le q$. The limit is taken in the weak sense in $\mathcal{S}'(C^n)$.

Proof of Lemma 2.4.

We need to apply both parts of (2) to a test function $\psi(\eta) \in \mathcal{S}(C^n)$. The left-hand side will then be

$$\langle (Ig)^{\wedge}(a,\eta), \psi(\eta)\rangle = \langle Ig(a,y), \widehat{\psi}(y)\rangle. \tag{3}$$

Consider the right-hand side before taking the limit:

$$\int\limits_{C^n} \sum_{l=0}^{p}\sum_{r=0}^{q} \sum_{1\le\alpha_1<\ldots<\alpha_l\le p} \sum_{1\le\gamma_1<\ldots<\gamma_r\le q} (-1)^{l+r} a^{i_{\alpha_1}} \ldots a^{i_{\alpha_l}} \bar{a}^{j_{\gamma_1}} \ldots \bar{a}^{j_{\gamma_r}} \times$$

$$\times \int\limits_{|\rho|\le H} |\rho|^{2n-4} \rho^p \bar{\rho}^q e^{\frac{\sqrt{-1}}{2}(\langle\rho a,\eta\rangle + \langle\eta,\rho a\rangle)} \times$$

$$\times (z^{i_{\beta_1}} \ldots z^{i_{\beta_{p-l}}} \bar{z}^{j_{\delta_1}} \ldots \bar{z}^{j_{\delta_{q-r}}} g^{j_1\ldots j_q}_{i_1\ldots i_p}(z))^{\wedge} (\bar{\rho}\eta)\, dS(\rho)\, \psi(\eta)\, dV_{2n}(\eta) =$$

$$= \int\limits_{|\rho|\le H} |\rho|^{2n-4} \rho^p \bar{\rho}^q \int\limits_{C^n} \sum_{l=0}^{p}\sum_{r=0}^{q} \sum_{1\le\alpha_1<\ldots<\alpha_l\le p} \sum_{1\le\gamma_1<\ldots<\gamma_r\le q} (-1)^{l+r} a^{i_{\alpha_1}} \ldots a^{i_{\alpha_l}} \times$$

$$\times\bar{a}^{j_{\gamma_1}}\dots\bar{a}^{j_{\gamma_r}}z^{i_{\beta_1}}\dots z^{i_{\beta_{p-l}}}\bar{z}^{j_{\delta_1}}\dots\bar{z}^{j_{\delta_{q-r}}}g^{j_1\dots j_q}_{i_1\dots i_p}(z)\times$$
$$\times(2\pi)^{-n}\int\limits_{C^n} e^{\frac{\sqrt{-1}}{2}(\langle\rho(z-a),\eta\rangle+\langle\eta,\rho(z-a)\rangle)}\psi(\eta)\,dV_{2n}(\eta)\,dV_{2n}(z)\,dS(\rho).$$

We can change the order of integration, because ψ and all the components of g are from the Schwartz space.

The last expression above equals

$$\int\limits_{|\rho|\le H}|\rho|^{2n-4}\rho^p\bar{\rho}^q\int\limits_{C^n}(z-a)^{i_1}\dots(z-a)^{i_p}(\overline{z-a})^{j_1}\dots(\overline{z-a})^{j_q}g^{j_1\dots j_q}_{i_1\dots i_p}(z)\times$$
$$\times\widehat{\psi}(\rho(z-a))\,dV_{2n}(z)\,dS(\rho).$$

Introducing variable change $y=\rho(z-a)$ and $t=1/\rho$, we obtain

$$\int\limits_{|\rho|\le H}\int\limits_{C^n}g^{j_1\dots j_q}_{i_1\dots i_p}(a+\frac{1}{\rho}y)y^{i_1}\dots y^{i_p}\bar{y}^{j_1}\dots\bar{y}^{j_q}\widehat{\psi}(y)\cdot dV_{2n}(y)\frac{1}{|\rho|^4}\,dS(\rho)=$$
$$=\int\limits_{|t|\ge H^{-1}}\int\limits_{C^n}g^{j_1\dots j_q}_{i_1\dots i_p}(a+ty)y^{i_1}\dots y^{i_p}\bar{y}^{j_1}\dots\bar{y}^{j_q}\widehat{\psi}(y)\,dV_{2n}(y)\,dS(t). \tag{4}$$

The integral above converges absolutely, i.e.

$$\int\limits_{C^n}\int\limits_{C}|g^{j_1\dots j_q}_{i_1\dots i_p}(a+ty)||y^{i_1}|\dots|y^{i_p}||\bar{y}^{j_1}|\dots|\bar{y}^{j_q}||\widehat{\psi}(y)|\,dS(t)\,dV_{2n}(y)<\infty,$$

because the function

$$y\to\int\limits_{C}|g^{j_1\dots j_q}_{i_1\dots i_p}(a+ty)||y^{i_1}|\dots|y^{i_p}||\bar{y}^{j_1}|\dots|\bar{y}^{j_q}|\,dS(t)$$

is positively homogeneous of the degree $(p+q-2)$ and $\widehat{\psi}\in\mathcal{S}(C^n)$.

Thus in (4) we can take the limit as $H\to\infty$ and obtain

$$\int\limits_{C^n}\int\limits_{C}g^{j_1\dots j_q}_{i_1\dots i_p}(a+ty)y^{i_1}\dots y^{i_p}\bar{y}^{j_1}\dots\bar{y}^{j_q}\,dS(t)\,\widehat{\psi}(y)\,dV_{2n}(y)=$$
$$=\int\limits_{C^n}Ig(a,y)\widehat{\psi}(y)\,dV_{2n}(y)=\langle Ig(a,y),\widehat{\psi}(y)\rangle,$$

which is the same as in (3).

This proves Lemma 2.4.

We notice that in the right-hand side of (2) we have a pointwise limit as $H \to \infty$ in the domain $\{\eta \in C^n \mid \eta \neq 0\}$, because the corresponding Fourier transform is rapidly decreasing (if $\eta = 0$, then the limit does not exist). We will need to show that the restriction of the distribution $(Ig\overset{\wedge}{)}\,(a,\eta)$ to this domain coincides with the regular distribution, defined by the pointwise limit.

Each term in (2) up to a coefficient has the form

$$\lim_{H\to\infty} \int\limits_{|\rho|\le H} |\rho|^{2n-4} \rho^p \bar\rho^q \, e^{\frac{\sqrt{-1}}{2}(\langle \rho a,\eta\rangle + \langle \eta, \rho a\rangle)} \, \widehat{G}(\bar\rho\eta)\, dS(\rho), \tag{5}$$

$$G(z) = z^{i_{\beta_1}} \dots z^{i_{\beta_{p-l}}} \bar z^{j_{\delta_1}} \dots \bar z^{j_{\delta_{q-r}}} \, g^{j_1\dots j_q}_{i_1\dots i_p}(z).$$

The components of g are from the Schwartz space, therefore $G(z)$ and $\widehat{G}(z)$ are rapidly decreasing and

$$|\widehat{G}(\zeta)| \le \frac{C_M}{1 + |\zeta|^M},$$

for every M.

So, in (5) we have for each $\eta \neq 0$ the following value

$$\int\limits_{C} |\rho|^{2n-4} \rho^p \bar\rho^q \, e^{\frac{\sqrt{-1}}{2}(\langle \rho a,\eta\rangle + \langle \eta, \rho a\rangle)} \, \widehat{G}(\bar\rho\eta)\, dS(\rho).$$

Take a test function $\psi \in \mathcal{S}(C^n)$ with $\operatorname{supp}\psi \subset C^n_0$. Then for some $r > 0$ we have $|\eta| \ge r$ on $\operatorname{supp}\psi$. Consider the following expression

$$\int\limits_{C^n} \int\limits_{|\rho|\le H} |\rho|^{2n-4} \rho^p \bar\rho^q \, e^{\frac{\sqrt{-1}}{2}(\langle \rho a,\eta\rangle + \langle \eta, \rho a\rangle)} \, \widehat{G}(\bar\rho\eta) dS(\rho) \cdot \psi(\eta)\, dV_{2n}(\eta). \tag{6}$$

We have the following estimate for each $\eta \in \operatorname{supp}\psi$:

$$\left| \int\limits_{|\rho|\le H} |\rho|^{2n-4} \rho^p \bar\rho^q \, e^{\frac{\sqrt{-1}}{2}(\langle \rho a,\eta\rangle + \langle \eta, \rho a\rangle)} \, \widehat{G}(\bar\rho\eta)\, dS(\rho) \right| \le$$

$$\le \int\limits_{|\rho|\le H} |\rho|^{2n+p+q-4} \frac{C_M}{1 + |\rho|^M |\eta|^M}\, dS(\rho) \le$$

$$\leq \int_C |\rho|^{2n+p+q-4} \frac{C_M}{1+|\rho|^M r^M}\, dS(\rho) = C(M) < \infty,$$

if M is sufficiently large for the last integral to converge.

Since ψ belongs to the Schwartz space and because of the Lebesgue dominated convegence theorem, we can take the pointwise limit under the integral sign over C^n in (6) and get

$$\int_{C^n} \int_C |\rho|^{2n-4} \rho^p \bar\rho^q e^{\frac{\sqrt{-1}}{2}(\langle \rho a, \eta\rangle + \langle \eta, \rho a\rangle)}\, \widehat{G}(\bar\rho\eta)\, dS(\rho) \cdot \psi(\eta)\, dV_{2n}(\eta).$$

By the hypothesis of the theorem, we therefore know the following expression for every $a \in \gamma$ and $\eta \in S^{2n-1}$:

$$\sum_{l=0}^{p} \sum_{r=0}^{q} \sum_{1\leq\alpha_1<\ldots<\alpha_l\leq p} \sum_{1\leq\gamma_1<\ldots<\gamma_r\leq q} (-1)^{l+r} a^{i_{\alpha_1}} \ldots a^{i_{\alpha_l}} \bar a^{j_{\gamma_1}} \ldots \bar a^{j_{\gamma_r}} \int_C |\rho|^{2n-4} \rho^p \bar\rho^q \times$$

$$\times e^{\frac{\sqrt{-1}}{2}(\langle \rho a, \eta\rangle + \langle \eta, \rho a\rangle)} \widehat{\left(z^{i_{\beta_1}} \ldots z^{i_{\beta_{p-l}}} \bar z^{j_{\delta_1}} \ldots \bar z^{j_{\delta_{q-r}}}\, g^{j_1\ldots j_q}_{i_1\ldots i_p}(z)\right)}(\bar\rho\eta)\, dS(\rho). \qquad (7)$$

We fix an arbitrary $z_0 \in C^n$ and $\eta \in S^{2n-1}$. By the hypothesis we can find a defining collection of points $a_1(z_0,\eta), \ldots, a_{\mathcal{L}_{n-1,m}}(z_0,\eta)$ in the intersection of the hyperplane $\langle a, \eta\rangle = \langle z_0, \eta\rangle$ with γ. Note that the restriction of the expression in (7) to this hyperplane is a polynomial $P(a)$ on it (because there we have $\langle \rho a, \eta\rangle = \rho\langle a, \eta\rangle = \rho\langle z_0, \eta\rangle$ and the dependence on a is purely polynomial).

The values $P(a_j(z_0,\eta))$ are known, because $a_j(z_0,\eta) \in \gamma$. Therefore $P(a)$ is known on the whole hyperplane.

We introduce the following polynomial $\widetilde{P}(a)$, defined everywhere on C^n:

$$\widetilde{P}(a) = P(z_0 + \pi_\eta(a - z_0)),$$

where $\pi_\eta(z) = z - \langle z, \eta\rangle\eta$ is the orthogonal projection to the complement $\eta^\perp$ of η with respect to the Hermitian form. It is clear that $\widetilde{P}$ is known on C^n. Its homogeneous part of the highest degree $(p+q)$ has the form

$$(-1)^{p+q} \int_C |\rho|^{2n-4} \rho^p \bar\rho^q\, e^{\frac{\sqrt{-1}}{2}(\langle \rho z_0, \eta\rangle + \langle \eta, \rho z_0\rangle)}\, \hat g^{j_1\ldots j_q}_{i_1\ldots i_p}(\bar\rho\eta)\, dS(\rho) \times$$

$$\times (a^{i_1} - \langle a, \eta\rangle \eta^{i_1}) \ldots (a^{i_p} - \langle a, \eta\rangle \eta^{i_p})\, (\bar a^{j_1} - \langle \eta, a\rangle \bar\eta^{j_1}) \ldots (\bar a^{j_q} - \langle \eta, a\rangle \bar\eta^{j_q}) =$$

$$= (-1)^{p+q} \int\limits_C |\rho|^{2n-4} \rho^p \bar{\rho}^q e^{\frac{\sqrt{-1}}{2}(\langle \rho z_0, \eta\rangle + \langle \eta, \rho z_0\rangle)} \hat{g}^{j_1 \ldots j_q}_{i_1 \ldots i_p}(\bar{\rho}\eta)\, dS(\rho) \times$$

$$\times \varepsilon^{i_1}_{k_1}(\eta) \ldots \varepsilon^{i_p}_{k_p}(\eta)\;\; \varepsilon^{l_1}_{j_1}(\eta) \ldots \varepsilon^{l_q}_{j_q}(\eta) a^{k_1} \ldots a^{k_p} \bar{a}^{l_1} \ldots \bar{a}^{l_q} =$$

$$= (-1)^{p+q} \int\limits_C |\rho|^{2n-4} \rho^p \bar{\rho}^q\; e^{\frac{\sqrt{-1}}{2}(\langle \rho z_0, \eta\rangle + \langle \eta, \rho z_0\rangle)}\; \hat{f}^{l_1 \ldots l_q}_{k_1 \ldots k_p}(\bar{\rho}\eta)\, dS(\rho) \times$$

$$\times a^{k_1} \ldots a^{k_p} \bar{a}^{l_1} \ldots \bar{a}^{l_q}, \tag{8}$$

where $f = {}^s g$ is the solenoidal part of g. (See the formula (1) and use $|\eta| = 1$ and $\varepsilon^j_i(\bar{\rho}\eta) = \varepsilon^j_i(\eta)$.)

Thus, we know all the coefficients in (8):

$$\int\limits_C |\rho|^{2n-4} \rho^p \bar{\rho}^q\; e^{\frac{\sqrt{-1}}{2}(\langle \rho z_0, \eta\rangle + \langle \eta\;, \rho z_0\rangle)}\; \hat{f}^{l_1 \ldots l_q}_{k_1 \ldots k_p}(\bar{\rho}\eta)\, dS(\rho).$$

Consider now a fixed $\eta_0 \in S^{2n-1}$ and introduce the variable $\mu = \bar{\rho}$. If we take $z_0 = \lambda \eta_0,\ \lambda \in C$, we therefore obtain

$$\int\limits_C |\mu|^{2n-4} \bar{\mu}^p \mu^q\; e^{\frac{\sqrt{-1}}{2}(\langle z_0, \mu\eta_0\rangle + \langle \mu\eta_0, z_0\rangle)}\; \hat{f}^{l_1 \ldots l_q}_{k_1 \ldots k_p}(\mu\eta_0)\, dS(\mu) =$$

$$= \int\limits_C e^{\frac{\sqrt{-1}}{2}(\lambda\bar{\mu} + \mu\bar{\lambda})}\; |\mu|^{2n-4} \bar{\mu}^p \mu^q\; \hat{f}^{l_1 \ldots l_q}_{k_1 \ldots k_p}(\mu\eta_0)\, dS(\mu).$$

Noting that $\frac{\sqrt{-1}}{2}(\lambda\bar{\mu} + \mu\bar{\lambda}) = \sqrt{-1}\,\mathrm{Re}(\lambda\bar{\mu})$, we recognize here the 2-dimensional Fourier transform (up to a coefficient) of the function

$$\mu \longrightarrow |\mu|^{2n-4} \bar{\mu}^p \mu^q \hat{f}^{l_1 \ldots l_q}_{k_1 \ldots k_p}(\mu\eta_0).$$

The value $\lambda \in C$ can be taken arbitrary, therefore this Fourier transform is known on C. Applying the inversion formula for it, we find $\hat{f}^{l_1 \ldots l_q}_{k_1 \ldots k_p}(\mu\eta_0)$ for all $\mu \in C$ (and all $\eta_0 \in S^{2n-1}$). Then, applying Fourier inversion in C^n, we obtain all the components $f^{l_1 \ldots l_q}_{k_1 \ldots k_p}$ of the solenoidal part f. This completes the proof of Theorem 2.3.

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