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# Construction of one-dimensional rings with fixed values of $t(R)\lambda_R(R/\mathfrak{C}) - \lambda_R(\overline{R}/R)$

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October 12, 2000

## 1 Introduction

Let $R$ be a local, Noetherian, one-dimensional domain; assume also that $R$ is analytically irreducible or, equivalently, that the integral closure $\overline{R}$ of $R$ in its quotient field is a DVR and is a finite $R$-module. It is natural to associate to $R$ a value semigroup $v(R)$ which is a subsemigroup of $\mathbb{N}$ and it is well known that there is a strict connection between $R$ and $v(R)$, when $R$ and $\overline{R}$ have the same residue field (cf. [K, Ms]).

More generally, when $R$ is not a domain, but just a reduced ring, if we assume that $\overline{R}$ is a finite $R$-module (or, equivalently, $R$ analytically unramified), it is again possible to associate to $R$ a value semigroup, which, in this case, is a subsemigroup of $\mathbb{N}^d$, where $d$ is the number of maximal primes of $\overline{R}$ (cf. [D, D'A]).

An important class of examples of such rings is given by the local rings of an algebric curve over an algebraically closed field in a singular point.

The key fact that allows to connect a ring to its value semigroup is that it is possible to compute the lenght $\lambda_R(I/J)$ (where $I \supseteq J$ are ideals of $R$) in terms of the semigroup. In this context one can consider the inequality $\lambda_R(\overline{R}/R) \leq t(R)\lambda_R(R/\mathfrak{C})$ (cf. [Ms, Proposition 3] and [De, Proposition 2.1]), where $\mathfrak{C} = (\overline{R} : R)$ is the conductor of $R$ and $t(R)$ is the Cohen-Macaulay type.

If $l^*(R) = t(R)\lambda_R(R/\mathfrak{C}) - \lambda_R(\overline{R}/R)$, it is proved in [D'A-De, Proposition 2.1] that $0 \leq l^*(R) \leq (t(R)-1)(\lambda_R(R/\mathfrak{C})-1)$. It is possible to give a characterization of rings satisfying the condition $l^*(R) \leq t(R) - 2$ and, assuming also that $t(R) = e(R) - 1$, of rings satisfying the condition $l^*(R) < t(R)$ (where $e(R)$ is the multiplicity of $R$; cf. [De, Theorems 2.3-2.10]). In the case $l^*(R) \geq t(R)$ there are results involving the type and the multiplicity of the ring $R$ and the lenght $\lambda(R/(\mathfrak{C} + xR))$ (where $xR$ is a minimal reduction of the maximal ideal; cf. [De-L-M, Theorem 2.2]), but a complete classification of such rings seems out of reach at present.

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It is natural to ask whether, fixed three natural numbers $n$, $t$, $l^*$ such that $n \geq 1$, $t \geq 1$ and $0 \leq l^* \leq (t-1)(n-1)$, there exists a ring $R$ such that $\lambda_R(R/\mathfrak{C}) = n$, $t(R) = t$ and $l^*(R) = l^*$.

The main goal of this paper is to give a positive answer to this question, giving a way to construct such rings. To make this construction we assume that the rings are complete and Arf. While the first assumption gives just a simplification of notation (cf. [D'A]), the second one allows to move the problem from the ring level to the semigroup level, giving the notion of Arf semigroup as in [B-D'A-F]. The main ingredient of the construction is the multiplicity tree of a ring (or of an Arf semigroup), introduced in [B-D'A-F].

In the next section we give all the preliminaries to our construction; in particular we explicitly give the way to read all the integers involved in our inequality, in terms of multiplicity tree (cf. Proposition 2.5). In section 3 we give the construction of the multiplicity trees satisfying the conditions requested (cf. Theorem 3.1). In section 4 we produce an example of the construction for particular values of $n$, $t$ and $l^*$ (cf. Example 4.1) and we study the case $l^*(R) = t(R)$, showing that if $\lambda(\overline{R}/\mathfrak{C})$ is large enough, then there is no analytically irreducible ring $R$ such that $l^*(R) = t(R)$ (cf. Proposition 4.7); this fact implies that, in order to get a positive answer to the main question, it is necessary to consider reduced rings and not only analytically irreducible domains.

# 2 Preliminaries

Throughout the rest of this paper we will assume that $(R, \mathfrak{m})$ is a local, complete, one-dimensional, reduced, Noetherian ring; we will denote by $\overline{R}$ the integral closure of $R$ in its total ring of fractions $Q$ and we assume that $R \neq \overline{R}$; notice that $\overline{R}$ is a finite $R$-module.

Under these hypotheses $\overline{R}$ is semilocal and it is a finite $R$-module (cf. [Ma, Theorem 10.2]); moreover the number $d$ of maximal ideals of $\overline{R}$ equals the number of minimal primes of $R$ (cf. [D'A, Proposition 1.1]). We will denote by $\mathfrak{m}_i$ the maximal ideals of $\overline{R}$ and by $\mathfrak{p}_i$ the minimal primes of $R$. We have the following commutative diagram

$$\begin{array}{ccc}
R & \longrightarrow & R/\mathfrak{p}_1 \times \cdots \times R/\mathfrak{p}_d \\
\downarrow & & \downarrow \\
\overline{R} & \xrightarrow{\simeq} & V_1 \times \cdots \times V_d \\
\downarrow & & \downarrow \\
Q & \xrightarrow{\simeq} & Q(R/\mathfrak{p}_1) \times \cdots \times Q(R/\mathfrak{p}_d)
\end{array}$$

where $V_i = \overline{(R/\mathfrak{p}_i)} = \overline{R}_{\mathfrak{m}_i}$, the integral closure of $R/\mathfrak{p}_i$ in its quotient field $Q(R/\mathfrak{p}_i)$, is a DVR. We will denote by $t_i$ a uniformizing parameter of $V_i$ and by

$v_i$ the valuation function associated to $V_i$. We also assume that $R/\mathfrak{m} \simeq V_i/(t_i)$ for every $i$ and that $|R/\mathfrak{m}| \geq d$.
For any $x = (x_1, \ldots, x_d) \in Q \setminus Z$ (where $Z$ is the set of zero divisors of $Q$) we define $v(x) = (v_1(x_1), \ldots, v_d(x_d))$. Hence we can define $v(R) = \{v(r) \mid r \in R \setminus Z\}$; more generally, for every regular fractional ideal $I$ of $R$ (where regular means that $I$ contains a nonzero divisor), we set $v(I) = \{v(i) \mid i \in I \setminus Z\}$.

With these hypotheses and notation, we recall first the following results that will be used in the sequel (cf. [B-D'A-F]):

- $v(R)$ is an additive subsemigroup of $\mathbb{N}^d$ and $v(I)$ is a semigroup ideal of $v(R)$ (i.e. $v(I) + s \subseteq v(I)$, for every $s \in v(R)$).
- Considering the usual product ordering in $\mathbb{N}^d$, that is $(\alpha_1, \ldots, \alpha_d) \leq (\beta_1, \ldots, \beta_d)$ if and only if $\alpha_i \leq \beta_i$ for $i = 1, \ldots, d$, the set of values $v(I)$ of a regular fractional ideal $I$ contains an element of smallest value, i.e. $\min v(I)$ exists. We will denote it by $\mathbf{m}_{v(I)}$.
- There exists a $\boldsymbol{\delta} \in \mathbb{N}^d$ such that $\boldsymbol{\delta} + \mathbb{N}^d \subseteq v(R)$.
- The ideals of $\overline{R}$ are of the form $\overline{R}(\boldsymbol{\delta}) = \{x \in \overline{R} \mid v(x) \geq \boldsymbol{\delta}\}$; the conductor $\mathfrak{C} = R : \overline{R}$ equals the largest ideal $\overline{R}(\boldsymbol{\delta})$ contained in $R$.
- If $I \subseteq J$ are two regular fractional ideals of $R$, then $\lambda_R(J/I)$ can be calculated by means of the sets of values $v(J)$ and $v(I)$. More precisely, if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in v(I), \boldsymbol{\alpha} < \boldsymbol{\beta}$, we let $d_{v(I)}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ denote the common length of a saturated chain of elements of $v(I)$ from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$. Let $\mathbf{m}_{v(I)}, \mathbf{m}_{v(J)}$ be the minimal elements in $v(I)$ and $v(J)$ respectively. Then for any sufficiently large $\boldsymbol{\alpha}$ we set $d(v(J) \setminus v(I)) = d_{v(J)}(\mathbf{m}_{v(J)}, \boldsymbol{\alpha}) - d_{v(I)}(\mathbf{m}_{v(I)}, \boldsymbol{\alpha})$. This definition is independent of the choice of $\boldsymbol{\alpha}$. Then we have $\lambda_R(J/I) = d(v(J) \setminus v(I))$.
- If $\mathbf{m}_{v(\mathfrak{m})} = (\alpha_1, \ldots, \alpha_d)$, the multiplicity of $R$ is $e(R) = \alpha_1 + \cdots + \alpha_d$.
- $t(R) = \lambda(\mathfrak{m}^{-1}/R) \leq e(R) - 1$.

Notice that these hypotheses are slightly different by the hypotheses of [De] and [D'A-De]; however the inequalities $\lambda_R(\overline{R}/R) \leq t(R)\lambda_R(R/\mathfrak{C})$ and $0 \leq l^*(R) \leq (t(R) - 1)(\lambda_R(R/\mathfrak{C}) - 1)$ are still true, with the same proof: the existence of the canonical ideal of $R$ follows from the fact that $R$ is complete and reduced (cf. [H-K, Satz 6,21]) and if $I$ is a regular ideal of $R$ and $x \in I$ is an element of minimal value, then $xR$ is a minimal reduction of $I$ (cf. [D'A, Remarks 2.1 (2)]).

In [B-D'A-F] has been introduced the notion of multiplicity tree of a ring. Recall that, if $I$ is an ideal of $R$, the blowing up $R^I$ of $I$ is $\cup_{n>0}(I^n : I^n)$. We have $(I^n : I^n) \subseteq (I^{n+1} : I^{n+1})$ for each $n$, and $R^I = (I^{n_0} : I^{n_0})$ for some $n_0$, since $R$ is Noetherian. Recall that we can associate to $R$, as in [L, p. 666], a sequence of semilocal rings $R = R_0 \subseteq R_1 \subseteq \cdots$ where $R_{i+1}$ is obtained from $R_i$ by blowing up $\mathrm{rad}(R_i)$, the Jacobson radical of $R_i$. We call this sequence the

*Lipman sequence.* Since, in our hypotheses, $\overline{R}$ is a finitely generated $R$-module, this sequence stabilizes for some $n$ and $R_h = \overline{R}$, for $h \geq n$. Recall also that, given a maximal ideal $\mathfrak{m}_j$ of $\overline{R}$ the *branch sequence* of $R$ along $\mathfrak{m}_j$ is the sequence of rings $(R_i)_{\mathfrak{m}_j \cap R_i}$ (cf. [L, p. 669])

**Example 2.1** Let $K$ be a field and let $R = K[[(t, u^2), (t, u^7), (t^2, u^7)]] \subset K[[t]] \times K[[u]]$; we get the following Lipman sequence: $R_1 = K[[t]] \times K[[u^2, u^5]]$, $R_2 = K[[t]] \times K[[u^2, u^3]]$, $R_3 = R_4 = \cdots = \overline{R} = K[[t]] \times K[[u]]$.

It is possible to associate to a local ring $R$ with $\overline{R} = V_1 \times \cdots \times V_d$ a rooted tree, called the *blowing up tree* of $R$, in the following way: the nodes are all local rings occuring in all branch sequences. The root (at level 0) is $R$, and on level 1 there are the localizations (at its maximal ideals) of $R_1 = R^{\mathfrak{m}} = R^{\mathrm{rad}(R)}$, and so on. If $U$ is a local ring in the tree and $\overline{U} = V_{i_1} \times \cdots \times V_{i_k}$, then $U$ has $k$ minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_k$. The vector $\mathbf{e}(U) = (e_1(U), \ldots, e_d(U))$ (where $e_j(U) = 0$ if $j \notin \{i_1, \ldots, i_k\}$ and $e_{i_j}(U) = e(U/\mathfrak{q}_j), j = 1, \ldots, k$) is said the *fine multiplicity* of $U$ (thus the usual multiplicity of $U$ is $\sum_{i=1}^{d} e_i(U)$). If we replace the local rings in the tree with their fine multiplicities, we get the *multiplicity tree* of $R$. We denote the nodes of the level $i$ of the multiplicity tree by $\mathbf{e}^1_{(i)}, \ldots, \mathbf{e}^{l^i}_{(i)}$. In the example above we get the blowing up tree and the multiplicity tree depicted in Fig. 1.

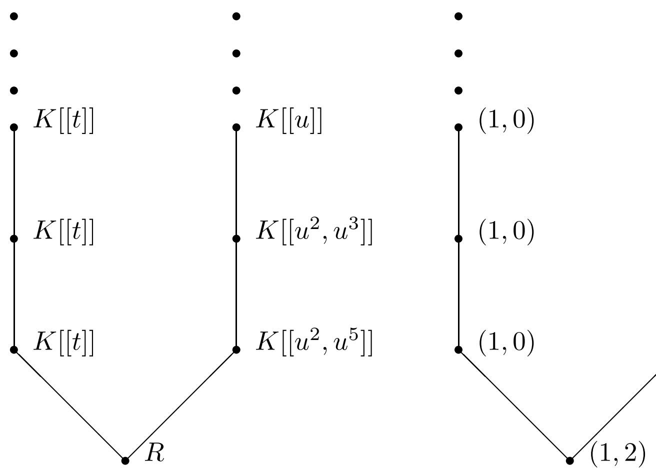


Fig. 1

**Remark 2.2** Notice that, since we assumed that $\overline{R}$ is a product of DVR's, each ring $R_i$ of the Lipman sequence associated to $R$ is the direct product of its localizations at maximal ideals [B-D'A-F, Corollary 3.2], i.e., the direct product of the local rings which appear at level $i$ in the blowing up tree.

In [B-D'A-F] is given a numerical characterization of those trees which are multiplicity trees of a ring:

**Theorem 2.3** *[B-D'A-F, Theorem 5.11] Let* $\mathbf{T}$ *be a tree of vectors* $\{\mathbf{e}^j_{(i)} = (e^j_{i,1}, \ldots, e^j_{i,d})\}$ *of* $\mathbb{N}^d$ *(where* $\mathbf{e}_{(0)}$ *is the root of the tree and the index* $(i)$ *denotes the level of the nodes in the tree). The following conditions are equivalent for* $\mathbf{T}$*:*
*1)* $\mathbf{T}$ *is the multiplicity tree of a ring.*
*2)* $\mathbf{T}$ *satisfies the three conditions a), b) and c) below:*

*a) There exists* $n \in \mathbb{N}$ *such that, for* $m \geq n$, $\mathbf{e}^j_{(m)} = (0, \ldots, 0, 1, 0, \ldots, 0)$ *(the nonzero coordinate in the* $j$*-th position) for any* $j = 1, \ldots, d$.

*b)* $e^j_{i,h} = 0$ *if and only if* $\mathbf{e}^j_{(i)}$ *is not in the* $h$*-th branch of* $\mathbf{T}$ *(the* $h$*-th branch of the tree is the unique maximal path containing the* $h$*-th unit vectors).*

*c)* $\mathbf{e}^j_{(i)} = \sum_{\mathbf{e} \in \mathbf{U} \setminus \mathbf{e}^j_{(i)}} \mathbf{e}$ *for some finite subtree* $\mathbf{U}$ *of* $\mathbf{T}$*, rooted in* $\mathbf{e}^j_{(i)}$.

The connection between rings and their value semigroups is particularly strict for Arf rings. A ring $R$ is said to be *Arf* if every regular integrally closed ideal is stable (cf. [L]; recall that a regular ideal $I$ is *stable* if $(I : I) = z^{-1}I$ for some nonzero divisor $z \in I$). Under our hypotheses the integrally closed ideals are of the form $R(\boldsymbol{\alpha}) = \{r \in R \mid v(r) \geq \boldsymbol{\alpha}\}$ (where $\boldsymbol{\alpha} \in v(R)$; cf. [D'A, Remarks 2.1.2]); in this case the element $z$ has to be an element of value $v(z) = \boldsymbol{\alpha}$. Similarly it is possible to define Arf semigroups: if $S = v(R)$ is the value semigroup of a ring and $\boldsymbol{\alpha} \in S$, define $S(\boldsymbol{\alpha}) = \{\boldsymbol{\beta} \in S \mid \boldsymbol{\beta} \geq \boldsymbol{\alpha}\}$; then $S$ is said to be *Arf* if, for every $\boldsymbol{\alpha} \in S$, $S(\boldsymbol{\alpha}) - S(\boldsymbol{\alpha}) := \{\boldsymbol{\beta} \in \mathbb{Z} \mid \boldsymbol{\beta} + S(\boldsymbol{\alpha}) \subseteq S(\boldsymbol{\alpha})\} = S(\boldsymbol{\alpha}) - \boldsymbol{\alpha}$ (cf. [B-D'A-F, Section 3]). For Arf semigroups it is possible to define the multiplicity tree (cf. [B-D'A-F, Section 5]) and the following result holds:

**Proposition 2.4** *[B-D'A-F, Proposition 5.10] The following statements are equivalent:*
*(1)* $R$ *is Arf.*
*(2)* $S = v(R)$ *is Arf and the multiplicity trees of* $R$ *and* $S$ *are the same.*

Moreover an Arf semigroup is completely described by its multiplicity tree (cf. [B-D'A-F, Proposition 5.9]); hence, in the case of Arf rings, it allows to compute all the numbers involved in the inequality $\lambda(\overline{R}/R) \leq t(R)\lambda(R/\mathfrak{C})$. More precisely we have the following

**Proposition 2.5** *Let* $\mathbf{T}$ *be the multiplicity tree of an Arf ring* $R$ *and let* $\mathbf{T}_{\mathfrak{C}}$ *be the subtree consisting of all the nodes of* $\mathbf{T}$ *which are non-unit vectors.*
*1) If* $n(\mathbf{T})$ *is the number of nodes of* $\mathbf{T}_{\mathfrak{C}}$*, then* $\lambda(R/\mathfrak{C}) = n(\mathbf{T})$.
*2) If* $\mathbf{e}_{(0)} = (e_{(0),1}, \ldots, e_{(0),d})$ *is the root of* $\mathbf{T}$*, then* $t(R) = e(R) - 1 = e_{(0),1} + \cdots + e_{(0),d} - 1$.
*3) If* $\mathbf{e}^j_{(i)} = (e^j_{(i),1}, \ldots, e^j_{(i),d})$ *are the nodes of* $\mathbf{T}$*, then*

$$\lambda\left(\frac{\overline{R}}{\mathfrak{C}}\right) = \sum_{\mathbf{e}^j_{(i)} \in \mathbf{T}_{\mathfrak{C}}} \left(\sum_{h=1}^{d} e^j_{(i),h}\right)$$

*and*

$$l^*(R) = \sum_{\mathbf{e}^j_{(i)} \in \mathbf{T}_{\mathfrak{C}} \setminus \{\mathbf{e}_{(0)}\}} \left( e(R) - \sum_{h=1}^{d} e^j_{(i),h} \right).$$

**Proof**. 1) By [B-D'A-F, Proposition 5.9], $v(R) = \{\mathbf{0}\} \bigcup_{\mathbf{T}'} \left\{ \sum_{\mathbf{e}^j_{(i)} \in \mathbf{T}'} \boldsymbol{e}^j_{(i)} \right\}$, where $\mathbf{0} \in \mathbb{N}^d$ and $\mathbf{T}'$ ranges over all finite subtrees of $\mathbf{T}$ rooted in $\mathbf{e}_{(0)}$. Hence a chain of points of $v(R)$ is obtained considering a chain of subtrees rooted in $\mathbf{e}_{(0)}$: $\mathbf{T}_1 \subseteq \mathbf{T}_2 \subseteq \cdots \subseteq \mathbf{T}_h$. To get a saturated chain, the subtree $\mathbf{T}_i$ has to be obtained by $\mathbf{T}_{i-1}$ adding exactly one node of $\mathbf{T}$. Since $\lambda_R(R/\mathfrak{C})$ equals the lenght of a saturated chain of points of $v(R)$ between $\mathbf{0}$ and $\boldsymbol{\delta}$, then we get the conclusion.

2) By Proposition 3.17 in [B-D'A-F], we have that $t(R) = e(R) - 1$; the second equality follows by the definition of multiplicity tree.

3) The first equality comes from [B-D'A-F, Corollary 5.13]. As for the second, by definition of $l^*(R)$ and from the points 1) and 2) of this proposition, it follows that

$$l^*(R) = t(R)\lambda_R(R/\mathfrak{C}) - \lambda_R(\overline{R}/R) = (t(R)+1)\lambda_R(R/\mathfrak{C}) - \lambda_R(\overline{R}/\mathfrak{C}) =$$

$$e(R)n(\mathbf{T}) - \sum_{\mathbf{e}^j_{(i)} \in \mathbf{T}_{\mathfrak{C}}} \left( \sum_{h=1}^{d} e^j_{(i),h} \right) = \sum_{\mathbf{e}^j_{(i)} \in \mathbf{T}_{\mathfrak{C}} \setminus \{\mathbf{e}_{(0)}\}} \left( e(R) - \sum_{h=1}^{d} e^j_{(i),h} \right).$$

If $R$ is a ring, it is possible to define its Arf closure, $R'$ (cf. [L, Proposition-Definition 3.1]), and the multiplicity trees of $R$ and $R'$ coincide (cf. [B-D'A-F, Proposition 5.3]). Hence we have:

**Corollary 2.6** *Let* $\mathbf{T}$ *be a tree of vectors of* $\mathbb{N}^d$. *The following conditions are equivalent for* $\mathbf{T}$*:*
*1)* $\mathbf{T}$ *is the multiplicity tree of a ring.*
*2)* $\mathbf{T}$ *is the multiplicity tree of an Arf ring.*

# 3 The main Theorem

Now we are ready to prove the main theorem of this paper.

**Theorem 3.1** *If* $n$, $t$, $l^*$ *are three fixed natural numbers such that* $n \geq 1$, $t \geq 1$ *and* $0 \leq l^* \leq (t-1)(n-1)$, *then there exists a ring* $R$, *satysfying the hypoteses of this paper, such that* $\lambda_R(R/\mathfrak{C}) = n$, $t(R) = t$ *and* $l^*(R) = l^*$.

**Proof**. We will prove that there exists an Arf ring $R$ satisfying the statement of this Theorem. Hence, by Proposition 2.5 and Corollary 2.6, it is enought to construct a multiplicity tree $\mathbf{T}$ of a ring $R$ such that $n(\mathbf{T}) = n$, $e_{(0),1} + \cdots + e_{(0),d} = t+1$ and $\sum_{\mathbf{e}^j_{(i)} \in \mathbf{T}_{\mathfrak{C}} \setminus \{\mathbf{e}_{(0)}\}} \left( t+1 - \sum_{h=1}^{d} e^j_{(i),h} \right) = l^*$. Let $l^*$, $n$ and $t$

be three integers with $n \geq 1$, $t \geq 1$ and $0 \leq l^* \leq (t-1)(n-1)$; if $n = 1$, then $l^* = 0$ and for any $t$, the multiplicity tree with one branch, whose nodes are $t+1, 1, 1, \dots$, satisfies the conditions of Theorem 2.3 and, if $R$ is an Arf ring with this multiplicity tree, then, by Proposition 2.5, $l^*(R) = 0$, $\lambda_R(R/\mathfrak{C}) = n(\mathbf{T})$=1 and $t(R) = t$. Hence we can assume $n > 1$. Let be $k$ an integer such that $1 \leq k \leq n-1$ and let $\mathbf{T}$ be the multiplicity tree depicted in Fig. 2.

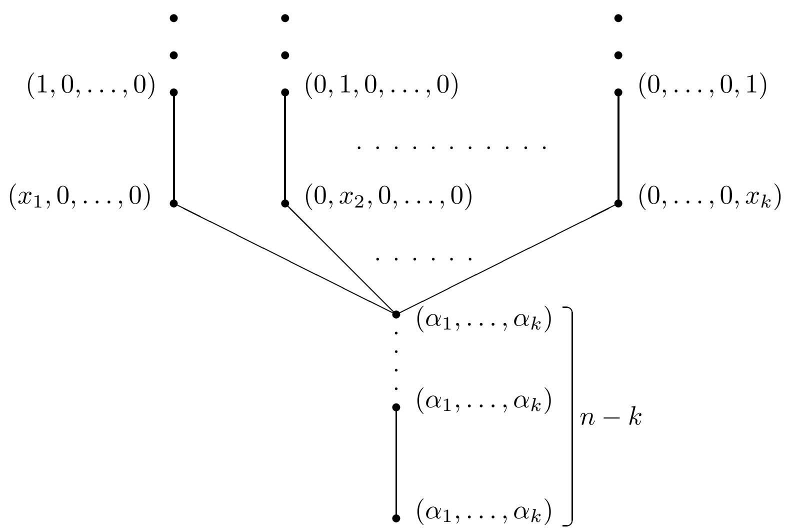


Fig. 2

In this tree $\mathbf{e}_{(0)} = \cdots = \mathbf{e}_{(n-k-1)} = (\alpha_1, \dots, \alpha_k)$, where $\alpha_1 + \cdots + \alpha_k = t+1$ and $\alpha_i \geq 2$, and $\mathbf{e}^j_{(n-k+1)} = (0, \dots, 0, x_j, 0, \dots, 0)$, where $x_j$ is in the j-th place and $1 < x_j \leq \alpha_j$. By Theorem 2.3 and Corollary 2.6, $\mathbf{T}$ is the multiplicity tree of an Arf ring $R$. Hence by Proposition 2.5, $\lambda_R(R/\mathfrak{C}) = n(\mathbf{T}) = n$, $t(R) = t$ and $l^*(R) = (k-1)e(R) + (\alpha_1 - x_1) + \cdots + (\alpha_k - x_k)$, so $l^*(R)$ can assume all the values between $(k-1)e(R) = (k-1)(t+1)$ and $(k-1)e(R) + e(R) - 2k = ke(R) - 2k = k(t-1)$. By Proposition 2.5, we have $e(R) \geq 2k$ and $t(R) \geq 2k-1$, but the inequality $(k-1)(t(R)+1) \leq k(t(R)-1)$ implies that $t(R) \geq 2k-1$. Hence our construction covers all the possible values of $t$, when $n > 1$ and $(k-1)(t+1) \leq l^* \leq k(t-1)$.

Since for $n = 2$, $l^* \leq t-1$, it remains to construct suitable multiplicity trees for the cases $n > 2$ and $k(t-1)+1 \leq l^*(R) \leq k(t+1)-1$ where $1 \leq k \leq n-2$. Assume that $l^* = k(t-1)+1+h$, where $h$ is an integer with $0 \leq h \leq 2k-2$. Now consider the tree $\mathbf{T}$ depicted in Fig. 3, where the number of coordinates of the vectors is $t+1$ (and $t \geq 2$). This tree satisfies the conditions of Theorem 2.3, hence if $R$ is an Arf ring with $\mathbf{T}$ as multiplicity tree, by Proposition 2.5, $\lambda_R(R/\mathfrak{C}) = n(\mathbf{T}) = n$, $t(R) = t$ and $l^*(R) = h+1+k(t-1) = l^*$.

Notice that in the tree depicted in Fig. 3 we have $n \geq 1+h+1+k = k+h+2$; therefore we still have to consider the cases $n-h-2 < k \leq n-2$, with $h > 0$.

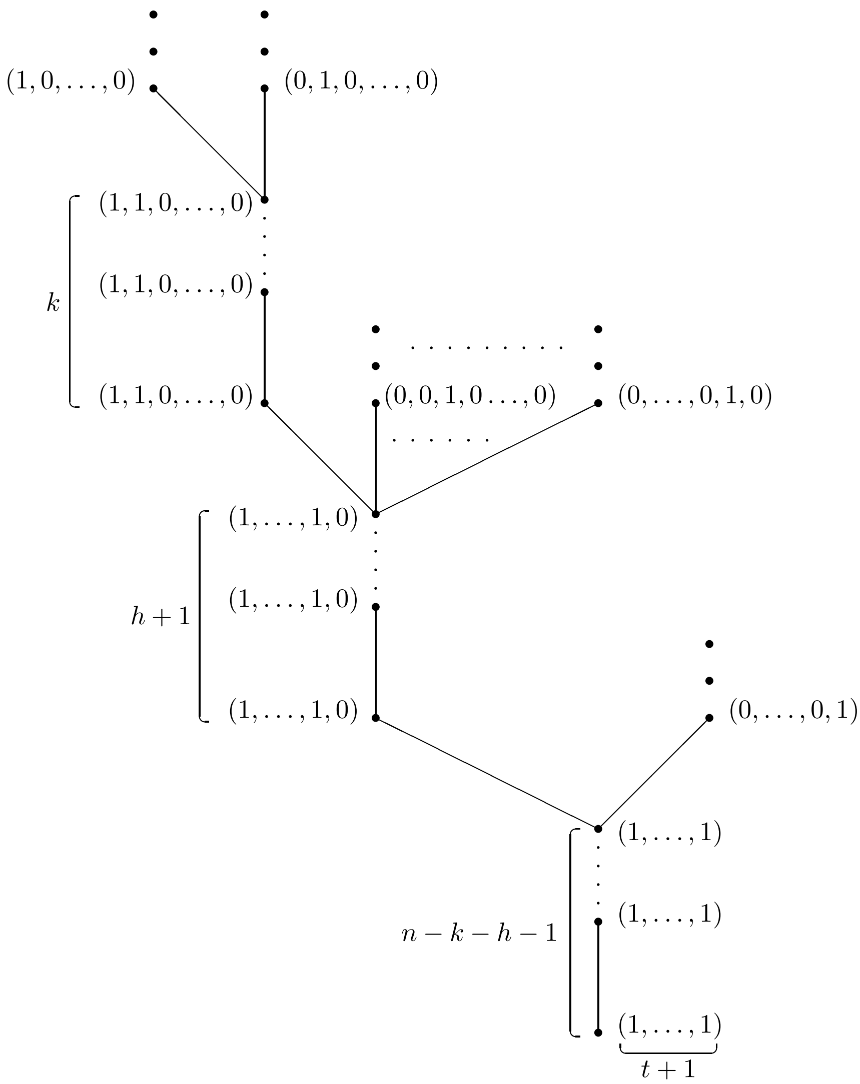


Fig. 3

Now, if $k = n-2-x$ with $0 \leq x \leq h-1$, let $q$ and $r$ be the integers such that $h+1 = q(x+1)+r$ with $1 \leq r \leq x+1$ and set $\psi(x) = q+1$; consider the tree $\mathbf{T}$ depicted in Fig. 4, where the number of coordinates of the vectors is $t+1$.

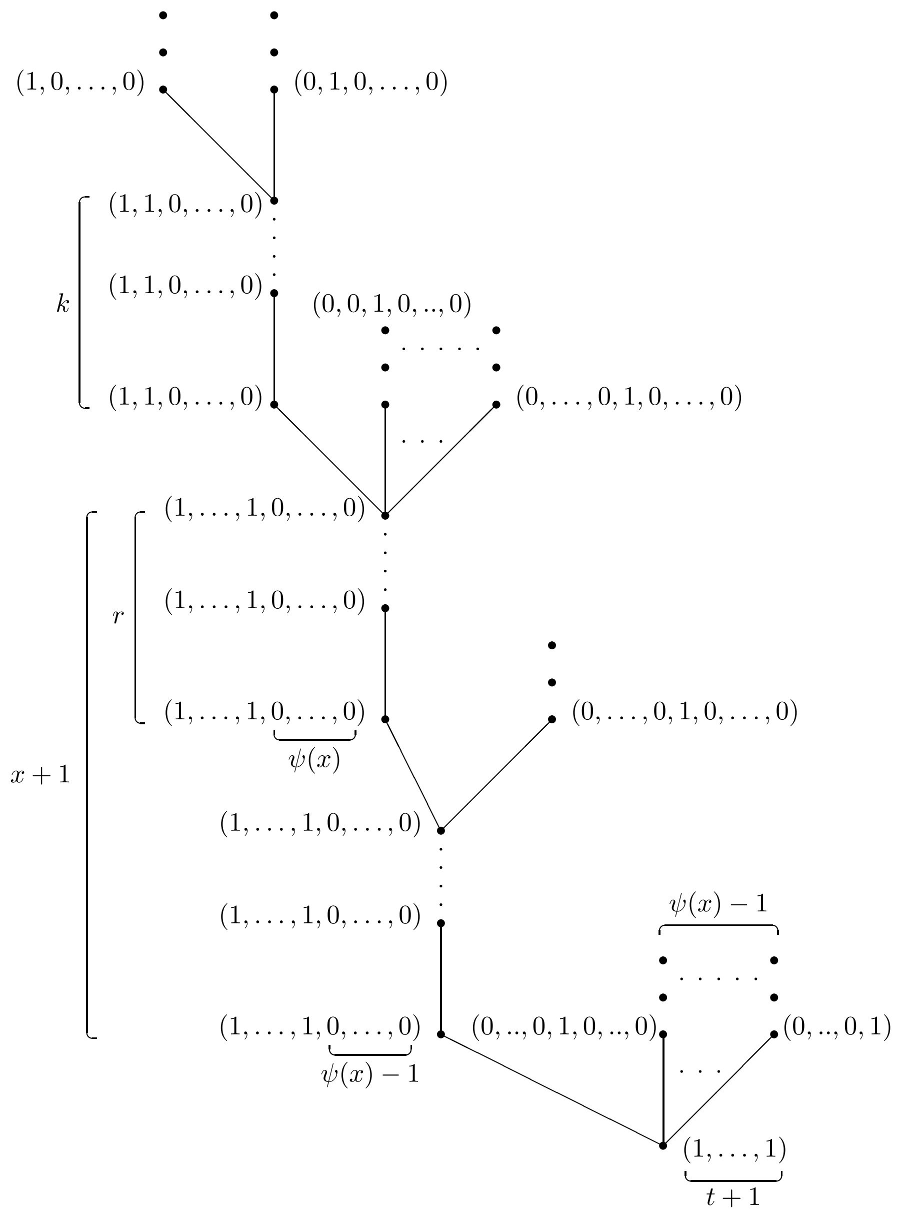


Fig. 4

By Theorem 2.3 and Corollary 2.6, $\mathbf{T}$ is the multiplicity tree of an Arf ring $R$. Hence by Proposition 2.5, $\lambda_R(R/\mathfrak{C}) = n(\mathbf{T}) = n$, $t(R) = t$ and $l^*(R) = (\psi(x)-1)(x+1-r)+\psi(x)r+(t-1)k = q(x+1-r)+(q+1)r+tk-k = q(x+1)+r+tk-k = h+1+tk-k = l^*$. By construction $t \geq \psi(x)+1$ and, by

$l^* = k(t-1)+h+1 \leq (n-1)(t-1)$, it follows $t \geq \frac{n-k+h}{n-k-1}$. Hence, in order to cover all the possible values of $t$, we have to show that the following inequalities hold: $\psi(x) < \frac{n-k+h}{n-k-1} \leq \psi(x)+1$. Since $n = k+2+x$, then $\frac{n-k+h}{n-k-1} = \frac{x+2+h}{x+1} = 1+\frac{h+1}{x+1}$ and by definition of $\psi$, we have $\psi(x) < 1+\frac{h+1}{x+1} \leq \psi(x)+1$.

# 4 Remarks and examples

In the next example, fixed two integers $n$ and $t$, we examine how the three constructions of the proof of Theorem 3.1 arise depending on the values of $l^*$.

**Example 4.1** Let $n = 4$ and $t = 5$. We have $0 \leq l^* \leq 12 = 3 \cdot 4$, hence the possible values of $k$ are 1, 2 and 3.

We use the first construction for $l^* = 0, 1, 2, 3, 4$ ($k = 1$), for $l^* = 6, 7, 8$ ($k = 2$) and $l^* = 12$ ($k = 3$).

We use the second construction for $l^* = 5$ ($k = 1$; here $h = 0$ and $n \geq k+h+2$) and for $l^* = 9$ ($k = 2$, here $h = 0$ and $n \geq k+h+2$).

We use the third construction for $l^* = 10, 11$ ($k = 2$, here $h = 1, 2$, respectively, and $n \leq k+h+1$).

We give an explicit construction of the multiplicity trees and of the rings for the values $l^* = 8, 9, 10$.

$\mathbf{l^* = 8}$. In this case we have more than one choice. A possible multiplicity tree is depicted in Fig. 5 and it is obtained by the tree depicted in Fig. 2, setting $n = 4$, $k = 2$, and $\alpha_1 = \alpha_2 = 3$ (so that $\alpha_1 + \alpha_2 - 1 = 5 = t$).

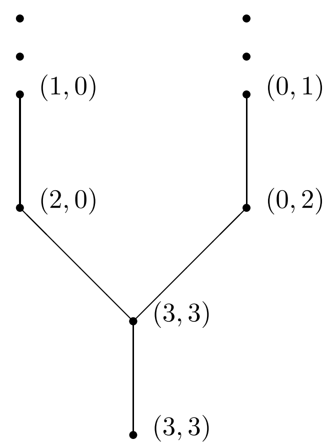


Fig. 5

From this multiplicity tree, using the proof of Corollary 5.8 in [B-D'A-F], we construct the ring $R$ in the following way: we start with $R^0 = K[[t]] \times K[[u]]$, where $t, u$ are indeterminates over a field $K$; then we set $R^1 = (K + t^2K[[t]]) \times (K + u^2K[[u]]) = U_1 \times U_2$, $R^2 = (1,1)K + (t^3U_1 \times u^3U_2)$ and $R = R^3 =$

$(1,1)K+(t^3,u^3)R^2 = (1,1)K+(t^3,u^3)K+((t^6K+t^8K[[t]])\times(u^6K+u^8K[[u]]))$. In Fig. 6 is depicted the blowing up tree of $R$.

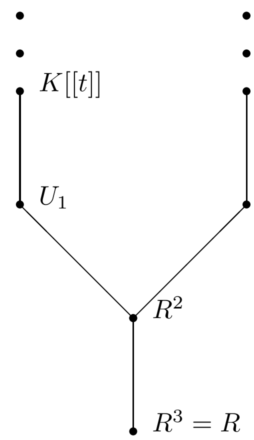


Fig. 6

$\mathbf{l^* = 9}$. The multiplicity tree is depicted in Fig. 7 and it is obtained by the tree in Fig. 3, setting $n=4$, $k=2$, $h=0$ and $t=5$.

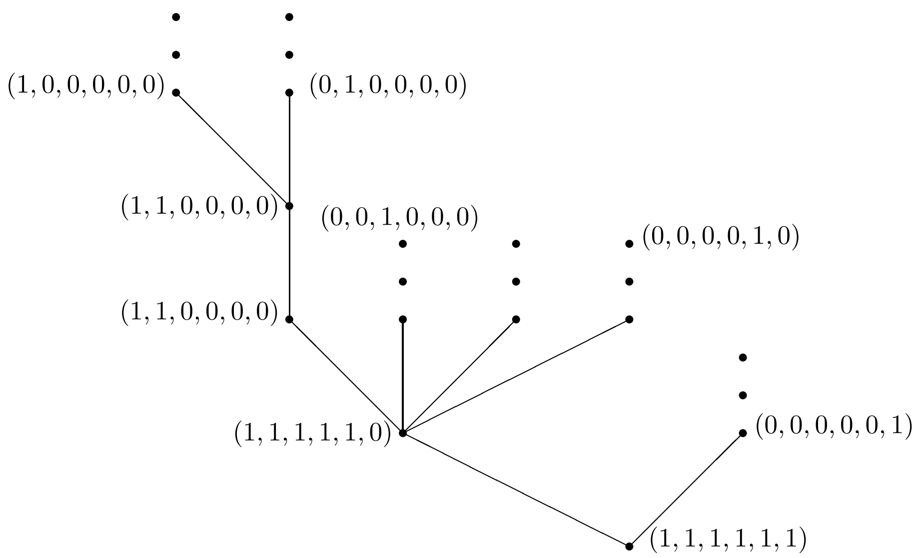


Fig. 7

From this multiplicity tree, using the proof of Corollary 5.8 in [B-D'A-F], we construct the ring $R$ in the following way: we start with $R^0 = K[[t_1]]\times K[[t_2]]\times$

$K[[t_3]] \times K[[t_4]] \times K[[t_5]] \times K[[t_6]]$, where $t_i$ are indeterminates over a field $K$ for every $i = 1, \dots, 6$ and $|K| \geq 6$; then we set $R^1 = ((1,1)K + (t_1 K[[t_1]] \times t_2 K[[t_2]])) \times K[[t_3]] \times K[[t_4]] \times K[[t_5]] \times K[[t_6]] = U_1 \times K[[t_3]] \times K[[t_4]] \times K[[t_5]] \times K[[t_6]]$, $R^2 = ((1,1)K + (t_1,t_2)U_1)) \times K[[t_3]] \times K[[t_4]] \times K[[t_5]] \times K[[t_6]] = U_2 \times K[[t_3]] \times K[[t_4]] \times K[[t_5]] \times K[[t_6]]$, $R^3 = ((1,1,1,1,1)K + ((t_1,t_2)U_2 \times t_3 K[[t_3]] \times t_4 K[[t_4]] \times t_5 K[[t_5]])) \times K[[t_6]] = U_3 \times K[[t_6]]$ and $R = R^4 = (1,1,1,1,1,1)K + ((t_1,t_2,t_3,t_4,t_5)U_3 \times t_6 K[[t_6]])$. In Fig. 8 is depicted the blowing up tree of $R$.

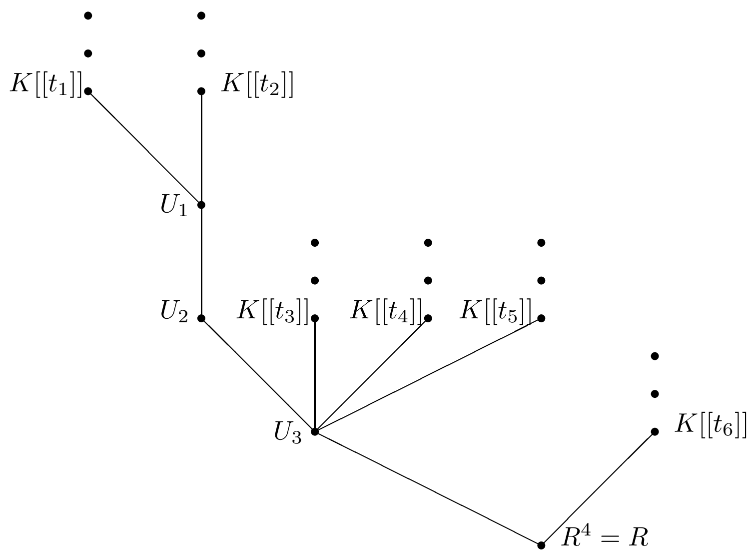


Fig. 8

$\mathbf{l^* = 10}$. The multiplicity tree is obtained (cf. Fig. 9) by the tree depicted in Fig. 4, setting $n = 4$, $k = 2$, $h = 1$ (hence $x = 0$ and $r = 0$) and $t = 5$.

From this multiplicity tree, using the proof of Corollary 5.8 in [B-D'A-F], we construct the ring $R$ in the following way: we start with $R^0 = K[[t_1]] \times K[[t_2]] \times K[[t_3]] \times K[[t_4]] \times K[[t_5]] \times K[[t_6]]$, where $t_i$ are indeterminates over a field $K$ for every $i = 1, \dots, 6$ and $|K| \geq 6$; then we set $R^1 = ((1,1)K + (t_1 K[[t_1]] \times t_2 K[[t_2]])) \times K[[t_3]] \times K[[t_4]] \times K[[t_5]] \times K[[t_6]] = U_1 \times K[[t_3]] \times K[[t_4]] \times K[[t_5]] \times K[[t_6]]$, $R^2 = ((1,1)K + (t_1,t_2)U_1)) \times K[[t_3]] \times K[[t_4]] \times K[[t_5]] \times K[[t_6]] = U_2 \times K[[t_3]] \times K[[t_4]] \times K[[t_5]] \times K[[t_6]]$, $R^3 = ((1,1,1,1)K + ((t_1,t_2)U_2 \times t_3 K[[t_3]] \times t_4 K[[t_4]])) \times K[[t_5]] \times K[[t_6]] = U_3 \times K[[t_5]] \times K[[t_6]]$ and $R = R^4 = (1,1,1,1,1,1)k + ((t_1,t_2,t_3,t_4)U_3 \times t_5 K[[t_5]] \times t_6 K[[t_6]])$. In Fig. 10 is depicted the blowing up tree of $R$.

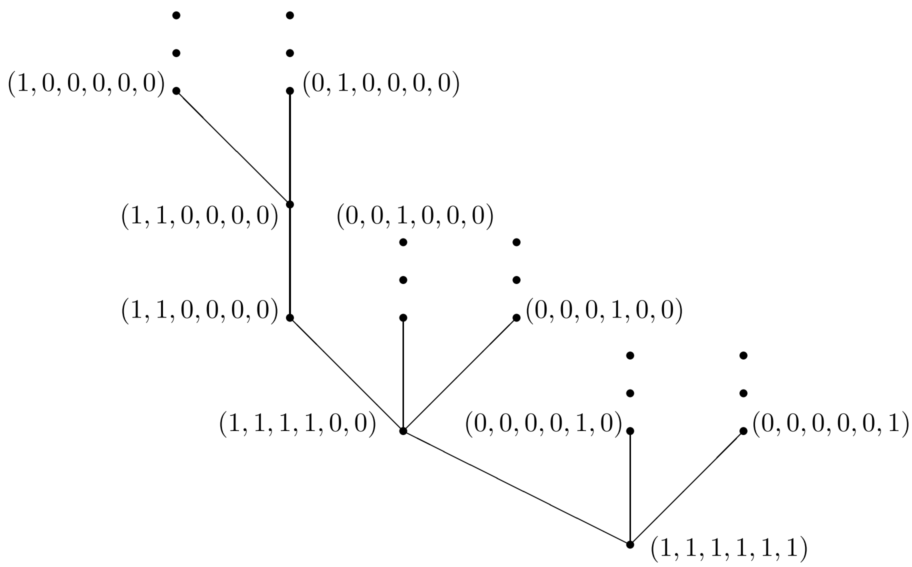


Fig. 9

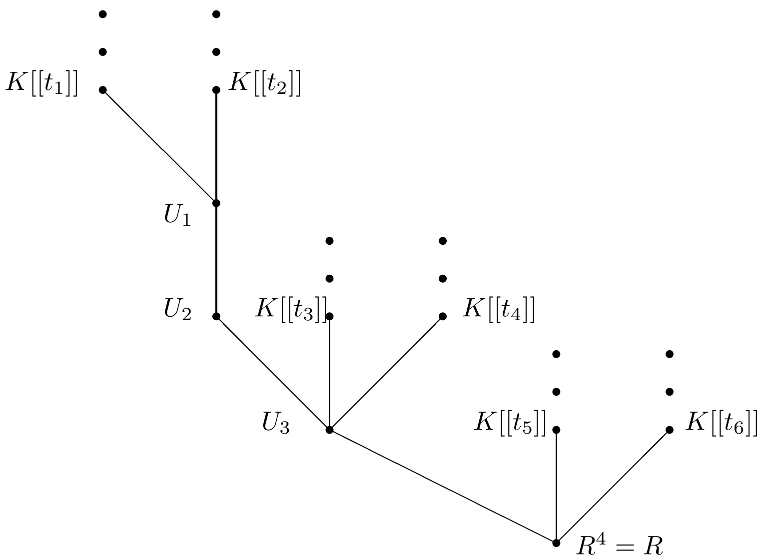


Fig. 10

We now show that the use of subsemigroups of $\mathbb{N}^d$, with $d > 1$, is necessary to construct rings $R$ with fixed values of $\lambda_R(R/\mathfrak{C})$, $t(R)$ and $l^*(R)$. More precisely we will focus our attention to the case $l^*(R) = t(R)$ and we will show that, if $n$ is an integer large enough, then, for every $t \geq 1$, there is no local, one-dimensional, Noetherian, complete domain $R$ with $\lambda_R(R/\mathfrak{C}) = n$ and $l^*(R) = t(R) = t$. Actually we will not need to assume that $R$ is complete, but only $R$ analytically irreducible (i.e. the completion $\widehat{R}$ of $R$ with respect ot the $\mathfrak{m}$-adic topology is a domain; cf. Proposition 4.7).

The following results are given in [De-L-M] under slightly different hypotheses. The assumption in [De] (and hence also in [De-L-M]) that $R$ is excellent is used to get the existence of the canonical module of $R$ isomorphic to an $\mathfrak{m}$-primary ideal and the infinite residue field implies the existence of principal minimal reductions of regular ideals. Under our hypotheses the existence of the canonical ideal of $R$ follows from the fact that $R$ is complete and reduced (cf. [H-K, Satz 6,21]). Moreover we do not need to assume (as in [De-L-M, Corollary 2.13]) that the field $K$ is infinite, since if $I$ is a regular ideal of $R$ and $x \in I$ is an element of minimal value, then $xR$ is a minimal reduction of $I$ (cf. [D'A, Remarks 2.1 (2)]).

**Proposition 4.2** *[De-L-M, Proposition 2.12] Assume $l^*(R) = t(R)$ and let $x \in \mathfrak{m}$ be an element of minimal value $v(x) = \mathbf{m}_{v(\mathfrak{m})}$. Only the following values for $\lambda(R/(\mathfrak{C} + xR))$ and $t(R)$ are possible:*

*(a) $\lambda(R/(\mathfrak{C} + xR)) = 3$, $t(R) = 2$ and $e(R) = 5$;*

*(b) $\lambda(R/(\mathfrak{C} + xR)) = 2$ and $t(R) = e(R) - 2$;*

*(c) $\lambda(R/(\mathfrak{C} + xR)) = 1$ and $t(R) = e(R) - 1$ .*

**Corollary 4.3** *[De-L-M, Corollary 2.13] Let $K$ a field and $R = K[[u^s \mid s \in S]]$, where $S \subseteq \mathbb{N}$ is a numerical semigroup. Assume $l^*(R) = t(R)$ and let $x \in \mathfrak{m}$ be an element of minimal value $v(x) = \mathbf{m}_{v(\mathfrak{m})}$. We have the following possibilities for $\lambda(R/(\mathfrak{C} + xR))$:*

*(a) $\lambda(R/(\mathfrak{C} + xR)) = 3$, $t(R) = 2$ and $e(R) = 5$;*

*(b) $\lambda(R/(\mathfrak{C} + xR)) = 2$ and $t(R) = e(R) - 2$ .*

In [De-L-M] examples of rings of the form $R = K[[t^s \mid s \in S]]$ ($S \subseteq \mathbb{N}$) are given for the case (a) (cf. [De-L-M, Example 2.14]) and for the case (b) (cf. [De-L-M, Example 2.11 (1)]). With the construction of Theorem 3.1 we get examples of rings for the case (c) of Proposition 4.2, with value semigroup contained in $\mathbb{N}^e$, where $e \geq 3$ is the multiplicity of $R$ (cf. Fig 3, with $k = 1$ and $h = 0$).

Let $(R, \mathfrak{m})$ be a local, Noetherian, one-dimesional, analytically irreducible domain. Assume also that, if $(\overline{R}, \mathfrak{n})$ is the integral closure of $R$ in its quotient field, then $R \neq \overline{R}$ and $R/\mathfrak{m} \simeq \overline{R}/\mathfrak{n}$. Under these hypotheses the value semigroup of $R$ is $S = v(R) = \{0, s_1, s_2, \ldots, s_{n(S)-1}, s_{n(S)}, \rightarrow\}$, where $s_1 = e(R)$ and the arrow means that every integer greater than or equal to $s_n$ belongs to $S$. The largest positive integer not in $S$ is called the *Frobenius number* of $S$ and is denoted by $g(S)$. Moreover if $I \supseteq J$ are two fractional ideals of $R$, then $\lambda(I/J) = |v(I) \setminus v(J)|$; in particular we have that $\lambda(R/\mathfrak{C}) = |\{0, 1, \ldots, g(S)\} \cap S| = n(S)$ and that $\lambda(\overline{R}/R) = |\mathbb{N} \setminus S| = g(S)+1-n(S)$. Moreover it is defined the *type* of $S$ as $t(S) = |T| = |\{q \in \mathbb{N} \setminus S \mid q + s \in S, \forall s \in S \setminus \{0\}\}|$ (for all the definitions and results about numerical semigroups we refer to [B-D-F]). Hence we can define $l^*(S) = n(S)t(S) - |\mathbb{N} \setminus S|$. Since $t(S) \geq t(R)$ (cf. [B-D-F, Theorem II.1.16]), we have that $l^*(S) \geq l^*(R)$ and in [B-D-F, Example II.1.19] is given an example for which the inequality $t(S) > t(R)$ holds. However in the particular case $R = K[[u^s \mid s \in S]]$, $t(R) = t(S)$ and hence $l^*(R) = l^*(S)$. Hence Corollary 4.3 can be translated to the semigroup level as the following statement:

**Corollary 4.4** *Let $S \subseteq \mathbb{N}$ be a numerical semigroup. Assume $l^*(S) = t(S)$, let $e = s_1$ be the smallest non zero element of $S$ and set $e + S = \{e + s \mid s \in S\}$. We have the following possibilities for $|S \setminus ((e + S) \cup \{g(S) + 1, \rightarrow\})|$:*

*(a)* $|S \setminus ((e + S) \cup \{g(S) + 1, \rightarrow\})| = 3$, $t(S) = 2$ *and* $e = 5$*;*

*(b)* $|S \setminus ((e + S) \cup \{g(S) + 1, \rightarrow\})| = 2$ *and* $t(S) = e - 2$ .

Now we are ready to study the numerical semigroups $S$ such that $l^*(S) = t(S)$. Notice that, since the inequality $l^*(R) \leq (\lambda(R/\mathfrak{C}) - 1)(t(R) - 1)$ holds for the rings of the form $R = K[[u^s \mid s \in S]]$, then the corresponding inequality $l^*(S) \leq (n(S) - 1)(t(S) - 1)$ holds for numerical semigroups. It follows that, if $l^*(S) = t(S)$, then $t(S) > 1$.

**Proposition 4.5** *If $n$ is an integer large enough ($n > 14$), then, for any numerical semigroup $S$ such that $n(S) = n$, $l^*(S) \neq t(S)$.*

**Proof**. We will prove that if $l^*(S) = t(S)$, then $n(S)$ is bounded by 14. Let $g = g(S)$ and $n = n(S)$. By Corollary 4.4 we have two possibilities for $|S \setminus ((e + S) \cup \{g + 1, \rightarrow\})|$.

If $S$ satisfies case (a) of Corollary 4.4, then $|S \setminus ((e + S) \cup \{g + 1, \rightarrow\})| = 3$, $t(S) = 2$ and $e = 5$; by the first condition, there exist exactly two non zero elements of $S$, $f$ and $h$, such that $f < h < g$, $f - 5 \notin S$ and $h - 5 \notin S$. It follows that, if $2f < g$, then $2f = h + 5\alpha$ (with $\alpha \geq 0$) and, if $f + h < g$, then $f + h = 5\beta$ (with $\beta > 0$). But if both these two equalities hold, then $h = 5\beta - f = 2f - 5\alpha$, that is $3f = 5(\alpha + \beta)$, which is a contradiction since 5 does not divide $f$. Therefore $f + h \geq g + 1$.

Moreover $\{g - 4, \ldots, g\} = \{5(p - 1), f + 5r_1, h + 5r_2, i_1, i_2\}$, where $i_1, i_2 \notin S$, $0 \leq r_2 \leq r_1$ and $p$ is the integer such that $5(p - 1) < g < 5p$. Since $i_1, i_2 \in T = \{q \in \mathbb{N} \setminus S \mid q + s \in S, \forall s \in S \setminus \{0\}\}$ and $t(S) = |T| = 2$, then $f - 5 \notin T$ and $h - 5 \notin T$; hence there exist two elements $q_1, q_2 \in S \setminus \{0\}$ such that $(f - 5) + q_1 \notin S$ and $(h - 5) + q_2 \notin S$. This implies $q_1, q_2 \in \{f, h\}$ (in fact any element of $S$ smaller than $g$ is of the form $5\alpha$, $f + 5\alpha$ or $h + 5\alpha$, with $\alpha \geq 0$); in particular $f + h - 5 \leq g$ (otherwise, if $f + h - 5 > g$, then also $2h - 5 > g$ and $q_2$ does not exit). It follows that $g < f + h \leq g + 5$.

Let $k$ and $j$ be the integers such that $5k < f < 5(k+1)$ and $5j < h < 5(j+1)$. With this notation $5(k+j) < f+h < 5(k+j+2)$. By $g+1 \leq f+h < 5(k+j+2)$ and by definition of $p$, we have $p \leq k+j+2$. On the other side, since $5(k+j) < f+h \leq g+5 < 5(p+1)$, we have $k+j \leq p$.

Since $l^*(S) = t(S)n(S) - |\mathbb{N}\setminus S| = t(S)$, then $|\mathbb{N}\setminus S| = t(S)(n-1) = 2(n-1)$. Moreover, by definition of $p$, $k$ and $j$, we have $n = p+p-k-1+p-j-1+\varepsilon = 3p-k-j-2+\varepsilon$ (where $\varepsilon = |\{5(p-1)+1,\dots,g\}\cap S| = 0,1,2$) and $|\mathbb{N}\setminus S| = 4k+3(j-k)+2(p-j-1)+\gamma = k+j+2p-2+\gamma$ (where $\gamma = |\{5(p-1)+1,\dots,g\}\cap \mathbb{N}\setminus S| = 1,2$). It follows that:

$$|\mathbb{N}\setminus S| = 2(n-1) \Longleftrightarrow k+j+2p-2+\gamma = 6p-2(k+j-2)+2\varepsilon-2 \Longleftrightarrow$$
$$4p = 3(k+j)+\gamma-2\varepsilon+4 \Longrightarrow p \leq 6$$

(the last inequality comes from the inequalities $k+j \leq p$, $\varepsilon \geq 0$ and $\gamma \leq 2$).

It follows that, since $k+j \geq p-2$ and $\varepsilon \leq 2$, then $n = 3p-k-j-2+\varepsilon \leq 2p+2 \leq 14$.

If $S$ satisfies case (b) of Corollary 4.4, then $t(S) = e-2$ (hence, since $t(S) > 1$, we can assume $e > 3$) and $|S\setminus((e+S)\cup\{g+1,\rightarrow\})| = 2$; hence there exists exactly one non zero element $f$ in $S$, such that $f < g$ and $f-e \notin S$. Hence, by the uniqueness of $f$, either $2f \geq g+1$ or, if $2f < g$, then $2f = \alpha e$ (with $\alpha > 2$).

Moreover $\{g+1-e,\dots,g\} = \{e(p-1), f+er, i_1, i_2, \dots, i_{e-2}\}$, where $p$ is the integer such that $(p-1)e < g < pe$ and $i_1, i_2, \dots, i_{e-2} \notin S$. Since $i_1, i_2, \dots, i_{e-2} \in T = \{q \in \mathbb{N}\setminus S \mid q+s \in S, \forall s \in S\setminus\{0\}\}$ and $t(S) = |T| = e-2$, then $f-e \notin T$; it follows that there exists an element $q \in S\setminus\{0\}$ such that $(f-e)+q \notin S$. This implies that $2f-e \notin S$; hence $2f-e \leq g$, that is $f \leq (g+e)/2$. On the other hand, $2f \geq g+1$, otherwise $2f-e = \alpha e - e = (\alpha-1)e \in S$; it follows that $f \geq (g+1)/2$.

We denote by $k$ the integer such that $ke < f < (k+1)e$. By definition of $k$, we have $(k+1)e > f \geq (g+1)/2$ and by definition of $p$ we have $2(k+1) \geq p$, that is $k \geq (p/2)-1$. Moreover, since $f \leq (g+e)/2$, by definition of $k$, we have $ke < f \leq (g+e)/2$, that is $(2k-1)e < g$. By definition of $p$, we have $2k-1 \leq p-1$, hence $k \leq p/2$.

Therefore $(p/2)-1 \leq k \leq p/2$ and we have to consider only three different cases, $k = (p/2)-1, p/2$ (when $p$ is even) and $k = (p-1)/2$ (when $p$ is odd).

Assume $p$ even and $k = p/2$. Since $l^*(S) = t(S)n(S) - |\mathbb{N}\setminus S| = t(S)$, then $|\mathbb{N}\setminus S| = t(S)(n-1) = (e-2)(n-1)$. Moreover, by definition of $p$ and $k$, we have $n = p+p-k-1+\varepsilon = (3/2)p-1+\varepsilon$ (where $\varepsilon = |\{(p-1)e+1,\dots,g\}\cap S| = 0,1)$ and $|\mathbb{N}\setminus S| = (e-1)p/2+(e-2)(p/2-1)+\gamma$ (where $\gamma = |\{(p-1)e+1,\dots,g\}\cap\mathbb{N}\setminus S| = 1,2,\dots,e-2$). It follows that:

$$|\mathbb{N}\setminus S| = (e-2)(n-1) \Longleftrightarrow (1/2)pe-(3/2)p = \gamma-\varepsilon(e-2)+e-2 \Longleftrightarrow$$
$$p(e-3) = 2(\gamma-\varepsilon(e-2)+e-2).$$

Since $\varepsilon \geq 0$ and $\gamma \leq e-2$, it follows that $p \leq 4(e-2)/(e-3)$. Hence, since $p$ is an even integer, if $e = 4$, then $p \leq 8$, if $e = 5$, then $p \leq 6$ and, if $e \geq 6$, then $p \leq 4$. In any case, since $\varepsilon \leq 1$, then $n = (3/2)p-1+\varepsilon \leq 12$.

We now consider the case $p$ even and $k = p/2 - 1$. Since $l^*(S) = t(S)n(S) - |\mathbb{N} \setminus S| = t(S)$, then $|\mathbb{N} \setminus S| = t(S)(n-1) = (e-2)(n-1)$. Moreover, by definition of $p$ and $k$, we have $n = p + p - k - 1 + \varepsilon = (3/2)p + \varepsilon$ (where $\varepsilon = |\{(p-1)e+1, \ldots, g\} \cap S| = 0, 1)$ and $|\mathbb{N} \setminus S| = (e-1)(p/2-1) + (e-2)p/2 + \gamma$ (where $\gamma = |\{(p-1)e+1, \ldots, g\} \cap \mathbb{N} \setminus S| \in \{1, 2 \ldots, e-2\}$). It follows that:

$$|\mathbb{N} \setminus S| = (e-2)(n-1) \iff (1/2)pe - (3/2)p = \gamma - \varepsilon(e-2) - 1 \iff$$
$$p(e-3) = 2(\gamma - \varepsilon(e-2) - 1).$$

Since $p$ is an even positive integer, $\varepsilon \geq 0$ and $\gamma \leq e-2$, it follows that $p = 2$. But this implies $k = p/2 - 1 = 0$ which is a contradiction to $f > e$.

Finally we consider the case $p$ odd and $k = (p-1)/2$. Since $l^*(S) = t(S)n(S) - |\mathbb{N} \setminus S| = t(S)$, then $|\mathbb{N} \setminus S| = t(S)(n-1) = (e-2)(n-1)$. Moreover, by definition of $p$ and $k$, we have $n = p+p-k-1+\varepsilon = p+(p-1)/2+\varepsilon$ (where $\varepsilon = |\{(p-1)e+1, \ldots, g\} \cap S| = 0, 1)$ and $|\mathbb{N} \setminus S| = (e-1)(p-1)/2 + (e-2)(p-1)/2 + \gamma$ (where $\gamma = |\{(p-1)e+1, \ldots, g\} \cap \mathbb{N} \setminus S| \in \{1, 2 \ldots, e-2\}$). It follows that:

$$|\mathbb{N} \setminus S| = (e-2)(n-1) \iff pe - 3p = 2\gamma - 2\varepsilon(e-2) + e - 3 \iff$$
$$p(e-3) = 2(\gamma - \varepsilon(e-2)) + e - 3.$$

Since $\varepsilon \geq 0$ and $\gamma \leq e-2$, it follows that $p \leq (3e-7)/(e-3)$ and, since $p$ is an odd integer larger than 1, we get that $p = 3, 5$ when $e = 4$ and $p = 3$ when $e \geq 5$.

It follows that, since $\varepsilon \leq 1$, then $n = p + (p-1)/2 + \varepsilon \leq 8$.

**Remark 4.6** We could make the statement of Proposition 4.5 more precise studying which semigroups can be constructed for every single value of $p$.

If $S$ satisfies case (a), it is proved in Proposition 4.5 that $p \leq 6$. Since $p \geq 2$ we have to consider five cases.

If $p = 2$, then, by definition of $p$, $2 \cdot 5 = 10 \geq g + 1$. The elements $f$ and $h$ introduced in this proof are smaller than $g + 1$. Hence $n = 4$ and $g + 1 = |\mathbb{N} \setminus S| + n = 2(n-1) + n = 10$. In order to get $t(S) = 2$ we find the following semigroups $\{0, 5, 6, 7, 10, \rightarrow\}$ and $\{0, 5, 6, 8, 10, \rightarrow\}$.

If $p = 3$, then, by definition of $p$, $3 \cdot 5 = 15 \geq g + 1$. Moreover for the integer $k$ and $j$ introduced in the proof of Proposition 4.5 we have $j, k \geq 1$ and $j + k \leq p$. Hence $n = 3p - k - j - 2 + \varepsilon = 4, 5, 6, 7$ (since $\varepsilon = 0, 1, 2$). If $n = 4$, then $g + 1 = |\mathbb{N} \setminus S| + n = 2(n-1) + n = 10$ which is a contradiction to $10 = (p-1)e < g + 1$. If $n = 5$, then $g + 1 = |\mathbb{N} \setminus S| + n = 2(n-1) + n = 13$; in order to get $t(S) = 2$ we find only the semigroup $\{0, 5, 8, 9, 10, 13, \rightarrow\}$. If $n = 6, 7$, then $g + 1 = |\mathbb{N} \setminus S| + n = 2(n-1) + n \geq 16$, a contradiction to $15 \geq g + 1$.

With similar arguments we get that there are no semigroups satisfying conditions of the case (a) with $4 \leq p \leq 6$.

Analyzing analogously the case (b), it is possible to find semigroups verifying $l^*(S) = t(S)$ only for particular subcases.

If $p$ is even and $k = p/2$ we get, for $e = 4$ and $p = 4$, the semigroup $S = \{0, 4, 8, 9, 12, 13, 16, \rightarrow\}$ (here $n(S) = 6$). While, for $e \geq 4$ and $p = 2$, we get infinite semigroups $S$ (with $n(S) = 4$): for example $S = \{0, e, e+1, 2e-1, \rightarrow\}$.

If $p$ is odd and $k = (p-1)/2$, we obtain, for $e = 4$ and $p = 5$, the semigroup $S = \{0, 4, 8, 11, 12, 15, 16, 19, \rightarrow\}$ (here $n(S) = 7$). While, for $e > 4$ and $p = 3$, we get the semigroups $S = \{0, e, 2e-2, 2e, 3e-2, \rightarrow\}$ (with $n(S) = 4$).

Using Proposition 4.5 we can prove that, if $n$ is an integer large enough, then, for every $t \geq 1$, there is no analytically irreducible domain $R$ with $\lambda_R(R/\mathfrak{C}) = n$ and $l^*(R) = t(R) = t$ (we remark that we do not need to assume $R$ complete, since also in this case canonical ideal and principal minimal reductions exist, as it is shown in [D'A-De]). This result is not an immediate consequence of Proposition 4.5, since it is possible that $l^*(R) = t(R)$ while $l^*(v(R)) \neq t(v(R))$; the example introduced in [B-D-F, Example II.1.19] works to show this fact: let $K$ be a field of characteristic different by 2 and let $R = K[[t^4, t^6 + t^7, t^{11}]]$. The value semigroup of $R$ is $v(R) = \{0, 4, 6, 8, 10, \rightarrow\}$ and $t(v(R)) = 3$ since $T = \{q \in \mathbb{N} \setminus S \mid q + s \in S, \forall s \in S \setminus \{0\}\} = \{2, 7, 9\}$. Moreover $|\mathbb{N} \setminus v(R)| = 6$ and $n(v(R)) = 4$; hence $l^*(v(R)) = 3 \cdot 4 - 6 = 6 \neq 3$. On the other hand, as it is shown in [B-D-F, Example II.1.19], $t(R) = 2$, thus $l^*(R) = 2 \cdot 4 - 6 = 2$.

**Proposition 4.7** *Let $(R, \mathfrak{m})$ be a local, Noetherian, one-dimesional, analytically irreducible domain. Assume also that, if $(\overline{R}, \mathfrak{n})$ is the integral closure of $R$ in its quotient field, then $R \neq \overline{R}$ and $R/\mathfrak{m} \simeq \overline{R}/\mathfrak{n}$. If $n$ is an integer large enough ($n > 14$) and $\lambda(R/\mathfrak{C}) = n$, then $l^*(R) \neq t(R)$.*

**Proof**. Let $S = v(R)$ be the value semigroup of $R$ and $e = e(R) = s_1$. We will prove that, if $l^*(R) = t(R)$, then $\lambda(R/\mathfrak{C}) \leq 14$.

If $l^*(R) = t(R)$, by Proposition 4.2, only the following values for $\lambda(R/(\mathfrak{C} + xR))$ and $t(R)$ are possible:

(a) $\lambda(R/(\mathfrak{C} + xR)) = 3$, $t(R) = 2$ and $e = 5$;

(b) $\lambda(R/(\mathfrak{C} + xR)) = 2$ and $t(R) = e - 2$;

(c) $\lambda(R/(\mathfrak{C} + xR)) = 1$ and $t(R) = e - 1$ .

If case (c) holds for $R$, then $e - 1 = t(R) \leq t(S) \leq e - 1$ implies that $t(R) = t(S) = e - 1$ (for the inequality $t(S) \leq e - 1$ cf. [B-D-F, Remark I.2.7 (a)]). Hence $l^*(S) = l^*(R)$ and therefore $l^*(S) = t(S) = e - 1$, but, by Corollary 4.4, this is not possible; it follows that $R$ cannot satisfy case (c).

If case (a) holds for $R$, then $2 = t(R) \leq t(S) \leq 5 - 1$. If $t(S) = 2$ we have that $l^*(S) = l^*(R)$, so $l^*(S) = t(S)$; hence we can apply Proposition 4.5 (case (a)) and we get $\lambda(R/\mathfrak{C}) = n(S) \leq 14$.

Assume that $t(S) = 3, 4$. We have $e = 5$ and $3 = \lambda(R/(\mathfrak{C} + xR)) = |S \setminus ((5 + S) \cup \{g + 1, \rightarrow\})|$; hence there exist exactly two non zero elements of $S$, $f$ and $h$, such that $f < h < g$, $f - 5 \notin S$ and $h - 5 \notin S$. It follows that, if $2f < g$, then $2f = h + 5\alpha$ (with $\alpha \geq 0$) and, if $f + h < g$, then $f + h = 5\beta$ (with $\beta > 0$). But if both these two equalities hold, then $h = 5\beta - f = 2f - 5\alpha$, that is $3f = 5(\alpha + \beta)$, which is a contradiction since 5 does not divide $f$. Therefore $f + h \geq g + 1$.

Moreover $\{g-4,\dots,g\}=\{5(p-1), f+5r_1, h+5r_2, i_1, i_2\}$, where $i_1,i_2\notin S$, $0\le r_2\le r_1$ and $p$ is the integer such that $5(p-1)<g<5p$. Since $i_1,i_2\in T=\{q\in\mathbb{N}\setminus S \mid q+s\in S, \forall s\in S\setminus\{0\}\}$ and $t(S)=|T|=3,4$, then at least one of the integers $f-5$ and $h-5$ belongs to $T$; hence $f+h-5\in S$. If $f+h-5\ge g+1$ then also $2h-5\ge g+1$ and this implies that $t(R)\ge 3$: in fact by $\lambda(R/xR+\mathfrak{C})=3$ it follows that $\lambda(\mathfrak{m}/xR+\mathfrak{C})=2$, hence there exist two elements $y,z\in R$ such that $\mathfrak{m}=xR+yR+zR+\mathfrak{C}$ and we can assume that $v(y)=f$ and $v(z)=h$, since, if $y_1,y_2\in R$ and $v(y_1)=v(y_2)$, then there exists a unity $u\in R$ such that $v(y_1-uy_2)>v(y_1)$ (cf. [K, Theorem]); moreover, since $\mathfrak{C}=\{r\in\overline{R}|\ v(r)\ge g+1\}$ (cf. [K, Theorem]), by $f+h-5\ge g+1$ and $2h-5\ge g+1$, it follows that $z/x\in\mathfrak{m}^{-1}$. But this is a contradiction to $t(R)=2$. Hence $f+h-5\le g$ and therefore $g<f+h\le g+5$.

Since $l^*(R)=t(R)$ or, equivalently, $t(R)(\lambda(R/\mathfrak{C})-1)=\lambda(\overline{R}/R)$, then $2(n(S)-1)=|\mathbb{N}\setminus S|$ and, since $g<f+h\le g+5$, we can use the same argument as in the proof of Proposition 4.5 (case (a)) to get $n\le 14$.

If case (b) holds for $R$, then $e-2=t(R)\le t(S)\le e-1$. If $t(S)=e-2$ we have that $l^*(S)=l^*(R)$, so $l^*(S)=t(S)$; hence we can apply Proposition 4.5 (case (b)) and we get $\lambda(R/\mathfrak{C})=n(S)\le 12$.

Assume that $t(S)=e-1$. We have $2=\lambda(R/(\mathfrak{C}+xR))=|S\setminus((e+S)\cup\{g+1,\rightarrow\})|$; hence there exists exactly one non zero element $f$ in $S$, such that $f<g$ and $f-e\notin S$. Hence, by the uniqueness of $f$, either $2f\ge g+1$ or, if $2f<g$, then $2f=\alpha e$ (with $\alpha>2$).

Moreover $\{g+1-e,\dots,g\}=\{e(p-1), f+er, i_1,i_2,\dots,i_{e-2}\}$, where $p$ is the integer such that $(p-1)e<g<pe$ and $i_1,i_2,\dots,i_{e-2}\notin S$. Since $i_1,i_2,\dots,i_{e-2}\in T=\{q\in\mathbb{N}\setminus S\mid q+s\in S,\forall s\in S\setminus\{0\}\}$ and $t(S)=|T|=e-1$, then $f-e\in T$; it follows that $2f-e\in S$. If $2f-e\ge g+1$, then $t(R)=e-1$: in fact by $\lambda(R/xR+\mathfrak{C})$ it follows that there exits $y\in R$ such that $\mathfrak{m}=xR+yR+\mathfrak{C}$ and we can assume that $v(y)=f$; but $2f-e\ge g+1$ implies that $y/x\in\mathfrak{m}^{-1}$. But this is a contradiction to $t(R)=e-2$. It follows that $2f-e\le g$, that is $f\le(g+e)/2$.

Since $l^*(R)=t(R)$ or, equivalently, $t(R)(\lambda(R/\mathfrak{C})-1)=\lambda(\overline{R}/R)$, then $(e-2)(n(S)-1)=|\mathbb{N}\setminus S|$. Therefore, if $2f\ge g+1$ (that is $f\ge(g+1)/2$), we can use the same argument as in the proof of Proposition 4.5 (case (b)) to get $n\le 12$.

Otherwise, if $2f<g$, then $2f=\alpha e$ (with $\alpha>2$). Denote by $k$ the integer such that $ke<f<(k+1)e$ and by $p$ the integer such that $(p-1)e<g\le pe$. By definition of $p$ and $k$, we have $n=p+p-k-1+\varepsilon$ (where $\varepsilon=|\{(p-1)e+1,\dots,g\}\cap S|=0,1$) and $|\mathbb{N}\setminus S|=(e-1)k+(e-2)(p-k-1)+\gamma$ (where $\gamma=|\{(p-1)e+1,\dots,g\}\cap\mathbb{N}\setminus S|=1,2,\dots,e-2$). It follows that:

$$|\mathbb{N}\setminus S|=(e-2)(n-1)\Longleftrightarrow p(e-2)-(e-2)+\varepsilon(e-2)=k(e-1)+\gamma$$
$$\Longrightarrow (p-1)(e-2)\le k(e-1)+(e-2)\Longrightarrow k\ge(p-2)(e-2)/(e-1)$$

(where the first implication follows by $\varepsilon\ge 0$ and $\gamma\le e-2$).

Since $ke<f$, then $2ke<2f<g$; hence $p>2k\ge 2(p-2)(e-2)/(e-1)$ or equivalently $p(e-1)>2(p-2)(e-2)$. It follows that $p(e-3)<4(e-2)$; hence, by $e \geq 4$ (since $e-2 = t(R) > 1$), it follows that $p < 8$ and $n = 2p-k-1+\varepsilon \geq 13$ (since $k \geq 1$ and $\varepsilon \leq 1$).

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