ISSN: 1401-5617

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Research Reports in Mathematics
Number 13, 2000

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Electronic versions of this document are available at
http://www.matematik.su.se/reports/2000/13

Date of publication: September 8, 2000
1991 Mathematics Subject Classification: Primary 42A55 Secondary 42A16.

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THE KNOWN MEASURES WITH L^1-BOUNDED PARTIAL SUMS BELONG TO $\operatorname{Rad} L^1$.

Mats Erik Andersson

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It has long been known that any formal Fourier series $\nu \sim \sum c_k e^{ik\theta}$ with the property that the partial sums $s_N\nu = \sum_{-N}^{N} c_k e_k$ are uniformly bounded in L^1-norm, must in fact be the Fourier–Stieltjes series of a measure $\nu \in M_0(\mathbb{T})$. For convenience the characters e_k are defined by $e_k(\theta) = e^{ik\theta}$. The algebra $M_0(\mathbb{T})$ consists of all measures in the measure algebra $M(\mathbb{T})$, whose Fourier coefficients vanish at infinity. Let henceforth M_b denote the vector space of all measures ν with $\sup_N \|s_N\nu\|_1$ finite. Clearly every $\nu \in M_b$ is a continuous measure.

The above result was achieved by Helson [H] and settled a conjecture by Steinhaus. The first to construct a singular measure in M_b was Weiss [W]. Next, Katznelson [K] developed a variant with the added touch that all partial sums be positive. Later on also Brown and Hewitt [B-H] have given a general construction, producing singular measures with positive partial sums and prescribed decay of Fourier coefficients. As known to the present author, no further publication adresses the construction of measures with L^1-bounded partial sums.

It is the intent of this paper to display the fact that the above three papers produce measures in the radical $\operatorname{Rad} L^1$. More precisely the result is as follows.

Synopsis. *The singular measures known in the literature to belong to M_b, do all all here the property $\nu * \nu \in L^2(\mathbb{T})$.*

The three published constructions demand separate handling, so are confined to one section each.

Analysis of Weiss' construction.

The result of Weiss' deals with classical Riesz products and lacunary index sets. Recall that the original Riesz product concerns expressions

$$\prod_{k=1}^{\infty} (1 + a_k \cos n_k\theta),$$

converging weak-$*$ in $M(\mathbb{T})$. Here $-1 \leqslant a_k \leqslant 1$ and the integers $0 < n_1 < n_2 < \dots$ are lacunary in the sense $n_{k+1}/n_k \geqslant 3$. This is the setting for Weiss' contribution. General facts of relevance to Riesz products are found in the monograph by Graham and McGehee [G-M]. Later on, a generalised Riesz product will be described.

1991 *Mathematics Subject Classification.* Primary 42A55 Secondary 42A16.
Key words and phrases. Riesz product, radical.

Theorem [W]. *Let the real parameters $\{a_k\}$ of a Riesz product satisfy the condition*

$$(\dagger) \qquad |a_k|\big(|a_1| + \cdots + |a_k|\big) = \mathcal{O}(1).$$

Then the resulting measure belongs to M_b.

This result was originally applied with $a_k = 1/\sqrt{k}$, which clearly gives $\sum_1^k a_j \sim 2\sqrt{k}$ and $\sum_1^k a_j^2 \sim \log k$. As is well known [G-M, Thm. 7.2.1], the divergence expressed by the second relation makes the Riesz product a singular measure and it belongs to M_b by Weiss' theorem.

It should be observed that the stronger decay $|a_k|\big(|a_1| + \cdots + |a_k|\big) = o(1)$ does not improve the conclusion as far as producing an absolutely continuous measure. This can be seen when studying $b_k = 1/\sqrt{k \log k}$, upon which $\sum_1^k b_j \sim 2\sqrt{k}/\sqrt{\log k}$ and $\sum_1^k b_j^2 \sim \log\log k$. Hence the resulting measure is still singular.

Proposition 1. *Let the measure ν be constructed according to the preceding theorem. Then $\nu * \nu \in L^2(\mathbb{T})$ and $s_N(\nu * \nu) \to \nu * \nu$ in L^2 as well as in L^1. In contrast, $s_N \nu$ does not converge in $M(\mathbb{T})$, should $\sum a_k^2 = \infty$ take place.*

The last statement is clear, since the L^1-functions $s_N \nu$ tend weak-$*$ to the singular measure ν, whence no convergence in norm is possible. It thus suffices to demonstrate $\nu * \nu \in L^2$, from which the remaining claim follows. A simple case of real analysis prepares for this.

Lemma 2. *Consider sequences $\{x_k\}_1^\infty$ of positive numbers, such that for all indices $k \geqslant 1$, the inequality $x_k(x_1 + \cdots + x_k) \leqslant 1$ holds. For any such sequence and $p > 2$, the series $\sum x_k^p$ converges.*

Observe first that the already mentioned example $x_k = 1/\sqrt{2k}$ shows the condition $p > 2$ to be best possible.

Put now $X(k) = [x_1 + \cdots + x_k]^2$. Clearly the two identities

$$X(k) + x_1^2 + \cdots + x_k^2 = 2x_1x_1 + 2x_2(x_1 + x_2) + \cdots + 2x_k(x_1 + \cdots + x_k),$$

$$X(k) - X(k-1) = 2x_k(x_1 + \cdots + x_{k-1} + \tfrac{1}{2}x_k),$$

and the assumption on $\{x_k\}_1^\infty$ together imply

$$X(k) < 2k \qquad \text{and} \qquad 0 < X(k) - X(k-1) < 2.$$

Let also $\gamma = 1 + 2/x_1^2$. Then $k \geqslant 2$ provides

$$\frac{X(k)}{X(k-1)} = 1 + \frac{X(k) - X(k-1)}{X(k-1)} \qquad \text{so} \qquad \gamma^{-1}X(k) < X(k-1) < X(k).$$

By the mean value theorem applied to the square root function

$$\begin{aligned} \sum_{k=2}^n x_k^p &= \sum_{k=2}^n \big[\sqrt{X(k)} - \sqrt{X(k-1)}\big]^p \\ &< \sum_{k=2}^n \left[\frac{X(k) - X(k-1)}{2\sqrt{X(k-1)}}\right]^p \\ &< \sum_{k=2}^n \frac{\gamma^p}{2} \, \frac{X(k) - X(k-1)}{X(k)^{p/2}}. \end{aligned}$$

On the other hand, applications of the mean value theorem to $t \mapsto t^{-p/2}$ produces $\xi_k \in [X(k-1), X(k)]$, yielding

$$\begin{aligned} X(k-1)^{1-\frac{p}{2}} - X(k)^{1-\frac{p}{2}} &= \left(\tfrac{p}{2}-1\right)\frac{X(k)-X(k-1)}{\xi_k^{p/2}} \\ &\geqslant \left(\tfrac{p}{2}-1\right)\frac{X(k)-X(k-1)}{X(k)^{p/2}}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{k=2}^{n} x_k^p &< \gamma^p (p-2)^{-1} \sum_{k=2}^{n} \left[X(k-1)^{(2-p)/2} - X(k)^{(2-p)/2}\right] \\ &< \gamma^p (p-2)^{-1} x_1^{2-p}. \end{aligned}$$

This gives the claimed convergence.

Now the proposition can be completed. On grounds of the standard procedure in constructing Riesz products, it is clear by identifying Fourier coefficients, that if ν is constructed with the parameters $a_k \in [-1,1]$, then $\nu * \nu$ is also a Riesz product based on the same independent set, but with parameters $\{a^2/2\}_1^\infty$. By assumption there is $\rho > 0$, such that $\{\rho |a_k|\}$ satisfies the condition of the lemma, so one may conclude the convergence of $\sum a_k^4$. By known theory, this now says that $\nu * \nu \in L^2$. The proof of the proposition has been completed.

Generalised Riesz products.

Both papers [K] and [B-H] consider measures arising from weak-$*$ convergence of products of a common kind

$$\nu = \underset{m\to\infty}{*\text{-lim}} \prod_{j=1}^{m} \left(1 - P_j\right).$$

Here each P_j is a real valued polynomial with a particular condition on its spectrum. Write R_m for the m:th partial product and $P_j = \sum_{k\neq 0} p_{j,k} e_k$. Then it is demanded that for each m, the spectra of

$$e_k R_m, \quad \text{for all } k \text{ with } p_{j,k} \neq 0,$$

be pairwise disjoint. In particular, it follows that

$\hat{\nu}(k)$ is a finite product (over j) with at most one factor from each of the non-zero elements of $\{p_{j,k}\}_{k\neq 0}$.

Introduce now distance functions

$$\sigma_p(\mu) = \|\hat{\mu}\|_{\ell^p}^p = \sum_k |\hat{\mu}(k)|^p$$

for $p \geqslant 1$ and measures $\mu \in M(\mathbb{T})$. In particular, $\sigma_1(f) = \|f\|_{A(\mathbb{T})}$ and $\sigma_2(f)^{1/2} = \|f\|_{L^2}$. Thus, by the spectral condition, the partial products obey

$$\begin{aligned} \sigma_p(R_m) &= \sigma_p\big([1-P_m]R_{m-1}\big) \\ &= \sigma(R_{m-1}) + \sum_{k\neq 0} |p_{m,k}|^p \, \sigma_p(R_{m-1}) \\ &= \sigma(R_{m-1})\left[1 + \sigma_p(P_m)\right], \end{aligned}$$

which by induction gives

$$\sigma_p(R_m) = \prod_{j=1}^{m} \big[1 + \sigma_p(P_j)\big].$$

Observe finally that

$$\nu * \nu = \underset{m\to\infty}{*\text{-lim}}\ R_m * R_m = \underset{m\to\infty}{*\text{-lim}} \prod_{j=1}^{m} \big(1 + P_j * P_j\big)$$

and

$$\sigma_2(R_m * R_m) = \prod_{j=1}^{m} \big[1 + \sigma_2(P_j * P_j)\big] = \prod_{j=1}^{m} \big[1 + \sigma_4(P_j)\big].$$

Thus it follows that $\nu * \nu \in L^2$ as soon as $\prod_{j=1}^{\infty}[1 + \sigma_4(P_j)] < \infty$. This will be achieved for Katznelson's construction as well as for Brown's and Hewitt's.

Analysis of Katznelson's measure.

In [K] the above-mentioned polynomials take the form

$$P_j(\theta) = \mathrm{Re}\ \gamma N_j^{-1/2} \sum\nolimits_{k=1}^{N_j} e^{in\log n} e^{in\lambda_j\theta},$$

where γ is a constant, the integers λ_j increase fast enough to provide spectral disjointness, and N_j satisfies (see [K], relation (4))

$$2^{2(j+2)} \Big\| \prod\nolimits_1^{j-1} (1 - P_j) \Big\|_{A(\mathbb{T})}^2 < N_j.$$

In particular, $N_j > 2^{2(j+2)}$. Furthermore, it is clear that for a constant A

$$\sigma_4(P_j) = 2N_j \cdot \big(2^{-1}\gamma N_j^{-1/2}\big)^4 < A \cdot 4^{-j}.$$

Thus

$$\|\nu * \nu\|_{L^2}^2 = \lim_{m\to\infty} \sigma_2(R_m * R_m) \leqslant \prod_{j=1}^{\infty} (1 + A \cdot 4^{-j}) < \infty,$$

which is the claimed property $\nu * \nu \in L^2$.

It could be recalled that the singularity of ν is based on the value $\|P_j\|_2 = \sqrt{\sigma_2(P_j)} = \gamma/\sqrt{2}$ for all $j \geqslant 1$.

The measures of Brown and Hewitt.

In a sense the construction of Brown and Hewitt builds on the idea of Katznelson, so this last analysis must by necessity resemble the previous calculation.

This time the polynomials P_j are of a more intricate nature. Among other things the non-zero coefficients $|p_{j,k}|$, for $k \geqslant 1$, of P_j, form a non-increasing finite sequence. In addition, $|p_{j,k}| \leqslant \omega(|k|)$, where $\{\omega(n)\}_{n=0}^{\infty}$ is a given *admissible* sequence. The details must be extracted from [B-H]. For the present purpose, it is important that $\{\omega(n)\}_{n=0}^{\infty}$ is positive, non-increasing, and tends to zero.

In the inductive procedure of constructing P_{m+1}, one has to satisfy (see [B-H], relation (7.2.6))

$$\omega(l)\,\|R_m\|_{A(\mathbb{T})} < 2^{-m-2}$$

by choosing l large enough. Thus one finds

$$\omega(l) < 2^{-m-2}\sigma_1(R_m)^{-1} = 2^{-m-2}\prod_{j=1}^{m}\big[1+\sigma_1(P_j)\big]^{-1} \leqslant 2^{-m-2}.$$

The construction then proceeds to build P_j from frequencies of order at least l, i.e., if $p_{m+1,k} \neq 0$, then $|k| \geqslant l$. Thus

$$|p_{m+1,k}| \leqslant 2^{-m-2}, \quad \text{all } k.$$

On the other hand, in the central result [B-H, Thm. 6.3], the inequality (6.3.10) is equivalently providing positive constants α and β with

$$\alpha \leqslant \|P_j\|_2^2 = \sigma_2(P_j) \leqslant \beta, \quad \text{all } j \geqslant 1.$$

Hence there is an estimate for all $m \geqslant 2$

$$\sigma_4(P_m) = \sum_k |p_{m,k}|^4 \leqslant 2^{-2(m+1)} \sum_k |p_{m,k}|^2 \leqslant \beta \cdot 2^{-2m-2}.$$

The partial products can now be estimated as

$$\|R_m * R_m\|_2^2 = \sigma_2(R_m * R_m) = \prod_{j=1}^{m}\big[1+\sigma_4(P_j)\big]$$
$$\leqslant \prod_{j=1}^{m}\big(1+\beta\cdot 2^{-2j-2}\big).$$

Letting $m \to \infty$, it is clear that $R_m * R_m$ converge in L^2, so in fact $\nu * \nu \in L^2(\mathbb{T})$, where as before $\nu = *\text{-}\lim_{m\to\infty} R_m$. This was the intended property.

Concluding remarks.

It is clear that not every measure in $M_0(\mathbb{T})$ belongs to Rad L^1. A particular example is the Riesz product

$$\mu = \mathop{*\text{-}\lim}_{m\to\infty} \prod_{k=2}^{m}\big[1+(\log k)^{-1}\cos 3^k\theta\big].$$

This measure is very far from being in M_b. There is even a result of Salem–Zygmund':

Theorem [Z, page 287]**.** *If $\mu \in M_b$, then*

$$\frac{\log n}{n}\sum_{k=-n}^{n} |\hat{\mu}(k)| = \mathcal{O}(1).$$

This decay of Fourier coefficients is in fact obtained for the function $\ell \in L^1$ defined by $\ell \sim \sum_{n=2}^{\infty} \frac{\cos n\theta}{\log n}$. Many more elements in $M_b \cap L^1$ may be constructed using Polya's concavity theorem.

There seem to be available only the above three constructions, for the purpose of producing singular measures in $M_b \cap M_s(\mathbb{T})$. Based on the just presented calculations, it is thus natural to ask the following question.

Problem. *Is $M_b \setminus \operatorname{Rad} L^1$ non-empty?*

Surrendering something to the radical, it seems more reasonable to come to terms with the next question. For $k = 2$ examples are plentiful, as has just been demonstrated.

Problem. *Are there other $k \geqslant 3$, such that some singular measure $\mu \in M_b$ has its k-fold self-convolution $\mu^k \in L^1(\mathbb{T})$, but still is such that $\mu^{k-1} \in M_b \setminus L^1(\mathbb{T})$?*

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