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ON THE RESOLVENT PROPERTIES OF THE GENERATORS OF SOME C_0–GROUPS

ALEXANDER V. KISELEV

Abstract. The C_0–groups with polynomial growth in the Hilbert space are considered. The possibility of expressing the condition of the polynomial growth in terms of some integral estimates on the resolvent of the generator is investigated. In the general case of functional growth such necessary and sufficient conditions are given for one class of generators permitting functional model representation when the spectrum of the generator is known to be absolutely continuous.

1. Preliminaries

The group of operators $T(t)$ acting in the Hilbert space H is called [1] C_0–group, if

$$\lim_{t\to 0} T(t)x = x$$

for all $x \in H$. C_0–group is called a group with polynomial growth if, further, the following condition is satisfied:

$$\|T(t)\| \le M(1 + |t|^s)$$

for some $s > 0$ and $M < \infty$.

The linear operator L, defined by

$$Lx = \lim_{t\to 0} \frac{1}{it}(T(t)x - x), \quad x \in D(L),$$

is called generator of the corresponding C_0–group, where $D(L)$ is the set of elements $x \in H$, for which the limit in the latter expression exists in H.

In the book [1] it's shown, that the uniform boundedness of a C_0–group of operators in the Hilbert space is necessary and sufficient for the similarity of it's generator to a self-adjoint operator. Morethanthat, in [9, 12] it's proved that the validity of two conditions

$$\begin{aligned} \sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(L - k - i\varepsilon)^{-1}u\|^2 dk \le C\|u\|^2 \\ \sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(L^* - k - i\varepsilon)^{-1}u\|^2 dk \le C\|u\|^2, \end{aligned} \tag{1.1}$$

for all $u \in H$ is equivalent to the uniform boundedness of the C_0–group $\exp(iLt)$.

In [11] a number of results is proved that link the properties of the generator of a C_0–group with polynomial growth to the value s determining its growth. In particular, the following propositions hold.

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Theorem 1.1. *If L is densely defined closed operator in Hilbert space H, then for all $s > 0$ the following assertions are equivalent:*

(i) *There exists $M_1 \geq 0$ such that:*

$$\|(L-\lambda)^{-n}\| \leq M_1 \frac{1}{|\lambda|^n}\left(1+\left(\frac{n}{|\lambda|}\right)^s\right)$$

for all $\lambda \in i\mathbb{R} \setminus \{0\}$ and $n = 1, 2, \ldots$.

(ii) *L generates a C_0–group in H and there exists $M_2 \geq 0$ such that*

$$\|\exp(itL)\| \leq M_2(1+|t|^s)$$

for all $t \in \mathbb{R}$.

Theorem 1.2. *If L is densely defined closed operator in Hilbert space H, $\sigma(L) \subset \mathbb{R}$ and the following estimates hold for all $u \in H$*

$$\begin{aligned} \int_{\mathbb{R}} \|(L-k \pm i\varepsilon)^{-1}u\|^2 dk &\leq M\frac{1}{\varepsilon}\left(1+\frac{1}{\varepsilon^d}\right)\|u\|^2 \\ \int_{\mathbb{R}} \|(L^*-k \pm i\varepsilon)^{-1}u\|^2 dk &\leq M\frac{1}{\varepsilon}\left(1+\frac{1}{\varepsilon^d}\right)\|u\|^2, \end{aligned} \tag{1.2}$$

then the operator L generates a C_0–group in H and there exists a positive C such that

$$\|\exp(iLt)\| \leq C(1+|t|^d)$$

for all $t \in \mathbb{R}$.

Theorem 1.3. *If L generates a C_0–group with polynomial growth in the Hilbert space H, i. e. there exists such nonnegative $C < \infty$ that*

$$\|\exp(itL)\| \leq C(1+|t|^s)$$

for all $t \in \mathbb{R}$, then the estimates (1.2) hold for all $\varepsilon > 0$, $M = M(C,s) < \infty$ and $d = 2s$.

In the theorems 1.2, 1.3 there exists a gap between the necessary and sufficient conditions of polynomial growth. There is no such gap in the case of uniformly bounded C_0–groups in Hilbert space H, but, as we are going to prove in the next section, this gap originates quite naturally in the case considered and thus the conditions of the theorems 1.2 and 1.3 provide "almost" tight results.

2. The gap between necessary and sufficient conditions

In the present section we are going to prove the following result showing that the conditions of the theorem 1.3 are exact at least in the power scale.

Proposition 2.1. *For any $\gamma > 0$ and $\delta > 0$ there exist an operator L_δ in the Hilbert space H and vector $u \in H$, such that:*

1. *$\|\exp(iL_\delta t)\| \leq |t|^\gamma$ and $\|\exp(iL_\delta t_i)\| = |t_i|^\gamma$ for some subsequence t_i of real numbers, satisfying the condition $\lim_{i\to\infty} |t_i| = \infty$;*

2. *For all $\varepsilon > 0$, $\varepsilon \ll 1$ the following estimate holds:*

$$\int_{-\infty}^{\infty} \|(L_\delta - k + i\varepsilon)^{-1}u\|^2 dk \geq C(\delta,\gamma)\frac{1}{\varepsilon^{1+2\gamma-\delta}}.$$

Proof. We're going to prove the analogous result in the case of discrete C_0–group of operators in the Hilbert space H generated by bounded linear operator T. To this end let us choose l^2 as the Hilbert space H and construct the weighted shift operator T satisfying the required property.

First we require that $\|T\|^n = n$. Consider the sequence $\{a_i\}_{i=-\infty}^{\infty}$:

$$\{\underbrace{\ldots\ldots\ldots}_{\text{symmetrically defined}}, \underbrace{1}_{\text{0th cell}}, 1, \frac{1}{1}, \sqrt[2]{2}, \sqrt[2]{2}, \frac{1}{\sqrt[2]{2}}, \frac{1}{\sqrt[2]{2}}, \ldots, \underbrace{\sqrt[n]{n}}_{n \text{ times}}, \underbrace{\frac{1}{\sqrt[n]{n}}}_{n \text{ times}}, \ldots\} \tag{2.1}$$

Now let's define the operator T as follows:

$$(Tu)_{i+1} := a_i u_i.$$

Such operator can be represented in the form of the following infinite matrix:

$$T \sim \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \ldots & a_{-i} & \ddots & 0 & 0 & 0 & 0 & \ldots \\ \ldots & 0 & \ddots & 0 & 0 & 0 & 0 & \ldots \\ \ldots & 0 & 0 & a_0 & \ddots & 0 & 0 & \ldots \\ \ldots & 0 & 0 & 0 & \ddots & 0 & 0 & \ldots \\ \ldots & 0 & 0 & 0 & 0 & a_i & 0 & \ldots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.2}$$

In the case considered we can compute the norms of the powers of the operator T in the following way:

$$\|T^n\| = \max_{i_0} \prod_{i_0}^{i_0+n} a_i,$$

which, in turn, implies, that $\|T^n\| = n$. Hence we managed to introduce the operator T with required property.

Let's calculate the resolvent $(T-\lambda)^{-1}$ applied to the vector

$$u \equiv (\ldots, 0, \ldots, 0, \underbrace{1}_{\text{1st cell}}, 0, \ldots, 0, \ldots).$$

We immediately obtain:

$$(T-\lambda)^{-1}u = \left(\ldots, 0, -\frac{1}{\lambda}, -\frac{a_1}{\lambda}, -\frac{a_1 a_2}{\lambda^2}, \ldots, -\frac{a_1 \cdots a_n}{\lambda^{n+1}}, \ldots\right).$$

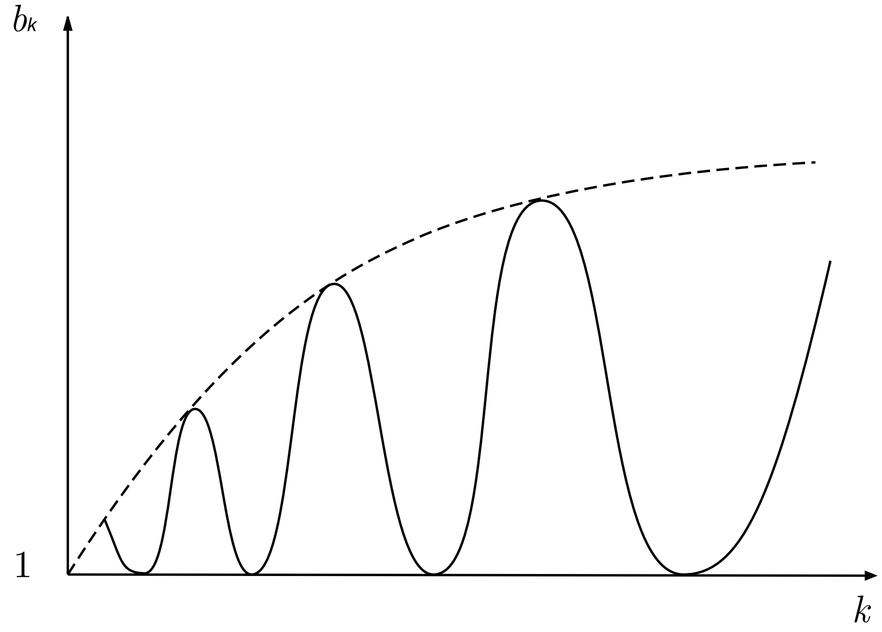

FIGURE 1. Illustration of $b_k \equiv b(k)$ behaviour.

We're going to estimate the following integral:

$$\int_0^{2\pi} \|(T-\lambda)^{-1}u\|^2 d\theta, \quad \text{where} \quad \lambda = |\lambda|e^{i\theta}.$$

It's clear that for our operator T and vector u

$$\int_0^{2\pi} \|(T-\lambda)^{-1}u\|^2 d\theta = 2\pi\|(T-\lambda)^{-1}u\|^2 =$$

$$= 2\pi\left(\frac{1}{|\lambda|^2} + \frac{a_1^2}{|\lambda|^4} + \cdots + \frac{a_1^2\cdots a_n^2}{|\lambda|^{2n+2}} + \cdots\right) =$$

$$= 2\pi\sum_{k=0}^{\infty}\frac{\prod_{i=0}^{k} a_i^2}{|\lambda|^{2k+2}} \quad (2.3)$$

Let's define $b_k \equiv \prod_{i=0}^{k} a_i$. Later we'll see that b_k behaves itself as shown on figure 1.

By the definition of b_k it's clear that the sequence of maximums of b_k is equal to the sequence $\{\|T^k\|\}_{k=1}^{\infty}$. Hence $b_k = 1, 2, 3, \ldots, n, \ldots$ represents the sequence of maximal values for b_k. Solving the equation $b_k \equiv b(k) = m$ with respect to k immediately leads to $k(m) = m^2 + 1$, hence $m = \sqrt{k-1}$, where m is the m-th member of the sequence of maximums for b_k (see figure 1).

Then for (2.3) we obtain the following estimate:

$$\int_0^{2\pi} \|(T-\lambda)^{-1}u\|^2 d\theta \geqslant \sum_{m=1}^{\infty} \frac{m^2}{|\lambda|^{2k(m)+2}} = \sum_{m=1}^{\infty} \frac{m^2}{|\lambda|^{2m^2+4}}.$$

Actually, we just left out all the non-maximal values of b_k in the latter estimate.

By [5], the last inequality implies that

$$\int_0^{2\pi} \|(T-\lambda)^{-1}u\|^2 d\theta \geqslant C \int_1^{\infty} \frac{x^2}{|\lambda|^{2x^2}} dx \sim \frac{1}{\epsilon^{3/2}}, \quad \text{where} \quad \epsilon = |\lambda| - 1.$$

Now let's modify the operator T in the following way. Consider the modification of the sequence (2.1) and the corresponding operator T (see (2.2)):

$$\left\{ \underbrace{\ldots\ldots\ldots}_{\text{symmetrically defined}}, \underbrace{1}_{\text{0th cell}}, 1, \frac{1}{1}, \sqrt[2]{2}, \sqrt[2]{2}, \frac{1}{\sqrt[2]{2}}, \frac{1}{\sqrt[2]{2}}, \underbrace{\sqrt[4]{4}}_{\text{4 times}}, \underbrace{\frac{1}{\sqrt[4]{4}}}_{\text{4 times}}, \underbrace{\sqrt[9]{9}}_{\text{9 times}}, \underbrace{\frac{1}{\sqrt[9]{9}}}_{\text{9 times}}, \ldots \right\} \tag{2.4}$$

In (2.4) we have in fact left only such elements

$$\left\{ \underbrace{n^{1/n}}_{\text{n times}}, \underbrace{\frac{1}{n^{1/n}}}_{\text{n times}} \right\}$$

of (2.1) that $\sqrt{n} \in \mathbb{N}$.

The sequence (2.4) can be further generalized. Let's now leave in (2.1) only such "parts"

$$\left\{ \underbrace{n^{1/n}}_{\text{n times}}, \underbrace{\frac{1}{n^{1/n}}}_{\text{n times}} \right\}$$

that $n^{1/j} \in \mathbb{N}$, $j = 1, 2, \ldots$. The case when $j = 1$ was discussed earlier; the case when $j = 2$ results in (2.4).

Clearly, for any j we still have $\|T^n\| \leqslant n$, moreover, there always exists such subsequence n_j of $n = 1, 2, \ldots$ that $\|T^{n_j}\| = n_j$. The equality (2.3) holds true too. Hence the behaviour of b_k in this case is essentially the same as shown on figure 1.

We again denote the maximums of b_k by m. For $j = 2$ we obtain $m = \{1, 2, 4, 9, 16, \ldots\}$. In the general case $m = \{k^j\}_{k=1}^{\infty}$. Now $k(m)$ (the values k for which $b_k = m$) cannot be calculated in the same simple manner as for $j = 1$ (for example, one can check that for $j = 2$ $k(m) = \frac{1}{3}\sqrt{m}(1+2m)$, and for $j = 3$ $k(m) = \frac{1}{2}m^{2/3}(1+m^{2/3})$, etc.), but for arbitrary j and large enough values of m we can obtain, that

$$k(m) \sim m^{\frac{j+1}{j}}, \quad m \gg 1.$$

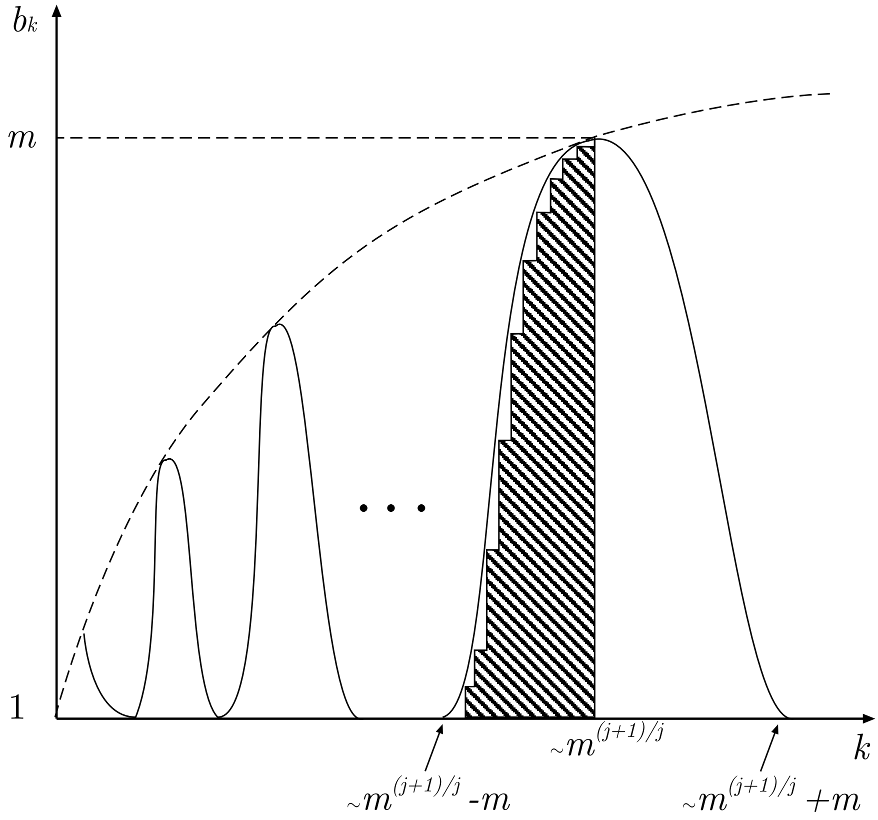

FIGURE 2. Detalization of $b_k(k)$ behaviour

Proceeding quite analogously to the case of $j = 1$, we obtain the following estimate for (2.3):

$$\int_0^{2\pi} \|(T-\lambda)^{-1}u\|^2 d\theta \geqslant \sum_{s=1}^{\infty} \frac{s^{2j}}{|\lambda|^{2s^{j+1}+2}} \asymp \int_0^{\infty} \frac{x^{2j}}{|\lambda|^{2x^{j+1}}} dx \underset{|\lambda|=1+\epsilon}{=} \frac{\Gamma(\frac{1+2j}{1+j})}{1+j} \frac{1}{\epsilon^{\frac{1+2j}{1+j}}}.$$

The latter implies that

$$\int_0^{2\pi} \|(T-\lambda)^{-1}u\|^2 d\theta \geqslant \text{Const} \frac{1}{\epsilon^{2-1/j}} \quad \text{when} \quad j \gg 1.$$

But we have not proved the required estimate

$$(2.3) \simeq \text{Const} \frac{1}{\epsilon^{3-\delta(j)}}, \quad \delta(j) \underset{j\to\infty}{\longrightarrow} 0 \quad \text{when} \quad j \gg 1.$$

yet. So let's analize the figure 1 once more (see figure 2).

Let's rewrite the part of the sum (2.3), corresponding to the m-th "hunch" of this graphic, in terms of the integral:[1]

$$2\int\limits_{m^{\frac{j+1}{j}}-m}^{m^{\frac{j+1}{j}}}(\sqrt[m]{m})^{2x}\frac{1}{|\lambda|^{2x+2}}dx=\frac{1}{\frac{\log m}{m}+\log|\lambda|}\left[\frac{m^2}{|\lambda|^{2m^{(j+1)/j}+2}}-\frac{1}{|\lambda|^{2m^{(j+1)/j}-2m+2}}\right]. \qquad (2.5)$$

Thus

$$\int\limits_0^{2\pi}\|(T-\lambda)^{-1}u\|^2d\theta\geqslant C\sum_{k=1}^{\infty}\frac{1}{\frac{j\log k}{k^j}+\log|\lambda|}\left[\frac{k^{2j}}{|\lambda|^{2k^{j+1}+2}}-\frac{1}{|\lambda|^{2k^{j+1}-2k^j+2}}\right].$$

The first term inside the square brackets here is absolutely the same we got before when summing the maximal points of b_k only.

The second term inside the square brackets is small in comparison with the first one when $k\gg 1$, hence

$$\int\limits_0^{2\pi}\|(T-\lambda)^{-1}u\|^2d\theta\geqslant\sum_{k=1}^{\infty}\frac{1}{\frac{j\log k}{k^j}+\log|\lambda|}\left[\frac{k^{2j}}{|\lambda|^{2k^{j+1}+2}}\right]\underset{\log|\lambda|=\log(1+\epsilon)\sim\epsilon}{\cong}\sum_{k=1}^{\infty}\frac{1}{\frac{j\log k}{k^j}+\epsilon}\left[\frac{k^{2j}}{|\lambda|^{2k^{j+1}}}\right].$$

Finally we obtain for every positive Const:

$$\int\limits_0^{2\pi}\|(T-\lambda)^{-1}u\|^2d\theta\geqslant C\int\limits_{\mathrm{Const}}^{\infty}\frac{1}{\frac{j\log x}{x^j}+\epsilon}\cdot\frac{x^{2j}}{|\lambda|^{2x^{j+1}}}dx\geq\tilde{C}\frac{1}{\epsilon}\frac{1}{\epsilon^{2-\delta(j)}}=\tilde{C}\frac{1}{\epsilon^{3-\delta(j)}}, \qquad (2.6)$$

where $\delta(j)\downarrow 0$ as $j\to\infty$.

In the latter estimate we utilized the following idea. For $d\geqslant\mathrm{Const}$

$$\int\limits_{\mathrm{Const}}^{\infty}\frac{1}{\frac{j\log x}{x^j}+\epsilon}\cdot\frac{x^{2j}}{|\lambda|^{2x^{j+1}}}dx\geqslant\int\limits_d^{\infty}\frac{1}{\frac{j\log x}{x^j}+\epsilon}\cdot\frac{x^{2j}}{|\lambda|^{2x^{j+1}}}dx. \qquad (2.7)$$

Let's choose such d that:

$$\frac{j\log k}{k^j}\leqslant\epsilon^{\alpha},\quad \alpha=1-\delta;\quad \delta>0\quad\text{when}\quad k\geqslant d.$$

This implies that $\frac{j\log k}{k^j}+\epsilon\leqslant 2\epsilon^{\alpha}$, since $\epsilon<\epsilon^{\alpha}$. Let d be determined from the following equality:

$$\frac{j\log d}{d^j}=\epsilon^{\alpha}$$

Then

$$\int\limits_d^{\infty}\frac{1}{\frac{j\log x}{x^j}+\epsilon}\cdot\frac{x^{2j}}{|\lambda|^{2x^{j+1}}}dx\geqslant\int\limits_d^{\infty}\frac{1}{2\epsilon^{\alpha}}\frac{k^{2j}}{|\lambda|^{2k^{j+1}}}dk=\frac{1}{2\epsilon^{\alpha}}\frac{2^{\frac{-1-2j}{1+j}}}{1+j}\frac{1}{\epsilon^{1+\frac{j}{1+j}}}\int\limits_{2\epsilon d^{1+j}}^{\infty}t^{\frac{1+2j}{1+j}-1}e^{-t}dt.$$

[1] It's quite simple to prove that the substitution of the sum by the integral for the large enough values of m can be made in this case.

It's left to prove that $\alpha = 1-\delta$ could be chosen arbitrary close to 1 and the latter integral is uniformly non-zero. Thus we need $\epsilon d^{1+j} \leqslant C$ uniformly. But in our case

$$\epsilon d^{j+1} = \frac{(j\log d)^{\delta/\alpha+1} d}{d^{\delta j/\alpha}} = (j\log d)^{\delta/\alpha+1} \cdot d^{1-\frac{\delta j}{\alpha}}.$$

Hence we have to satisfy the following inequality:

$$\delta > \frac{1}{j}.$$

Let $\delta = \frac{2}{j}$. Thus we immediately obtain:

$$\int\limits_{d}^{\infty} \frac{1}{\frac{j\log x}{x^j}+\epsilon} \cdot \frac{x^{2j}}{|\lambda|^{2x^{j+1}}} dx \geqslant \frac{1}{2\epsilon^{1-2/j}} \frac{1}{\epsilon^{1+\frac{j}{1+j}}} \frac{2^{\frac{-1-2j}{1+j}}}{1+j} \underbrace{\int\limits_{2C}^{\infty} t^{\frac{1+2j}{1+j}-1} e^{-t} dt}_{=C_0} \equiv \frac{C_0}{2(1+j)} \frac{1}{\epsilon^{2+\frac{j}{j+1}-\frac{2}{j}}} 2^{\frac{-1-2j}{1+j}}.$$

Since $2+\frac{j}{j+1}-\frac{2}{j}$ is arbitrary close to 3 when $j \gg 1$, we have proved the following proposition:

Lemma. *For any $\delta > 0$ there exists an operator T_δ acting in Hilbert space H and a vector u such that:*

1. $\|T_\delta^n\| \leqslant n$ *and* $\|T_\delta^{n_j}\| = n_j$ *for some subsequence* n_j *of natural numbers;*
2. $$\int\limits_0^{2\pi} \|(T_\delta - \lambda)^{-1}u\| d\theta \geqslant C(\delta) \frac{1}{(|\lambda|-1)^{3-\delta}},$$

   where $\lambda = |\lambda| e^{i\theta}$.

This proposition together with the proof given can be easily generalized to the following one:

Lemma. *For any $\delta > 0$ there exists an operator T_δ acting in Hilbert space H and a vector u such that:*

1. $\|T_\delta^n\| \leqslant n^\gamma$ *and* $\|T_\delta^{n_j}\| = n_j^\gamma$ *for some subsequence* n_j *of natural numbers;*
2. $$\int\limits_0^{2\pi} \|(T_\delta - \lambda)^{-1}u\| d\theta \geqslant C(\delta) \frac{1}{(|\lambda|-1)^{1+2\gamma-\delta}},$$

   where $\lambda = |\lambda| e^{i\theta}$.

In order to prove this proposition one has to consider the sequence $\{a_i\}_{i=-\infty}^{\infty}$ constructed analogously to (2.1) and (2.4) from the blocks

$$\left\{ \underbrace{(n^\gamma)^{1/n}}_{n \text{ times}}, \underbrace{\frac{1}{(n^\gamma)^{1/n}}}_{n \text{ times}} \right\}.$$

The rest of the proof described above remains intact, although slight modifications to the estimatory sums and integrals are to be made. For example, the integral in (2.5) has to be modified as follows:

$$\int\limits_{m^{\gamma\frac{j+1}{j}}-m}^{m^{\gamma\frac{j+1}{j}}} \left(\sqrt[m]{m^{\gamma}}\right)^{2x} \frac{1}{|\lambda|^{2x+2}} dx.$$

The definition of m here is changed accordingly: in the general situation it denotes the maximal values of $\left(\prod_{i=0}^{k} a_i\right)^{\gamma}$ rather then the maximal values of just $\prod_{i=0}^{k} a_i$.

To finish the proof of the Proposition 2.1 we now apply the Caley transformation to the operator T_δ as in [9, 11] which leads to the result claimed. □

3. C_0–groups with functional growth generated by the operators with purely absolutely continuous spectrum

In the present section we are going to prove the results analogous to the ones from [11] for the arbitrary C_0–group with functional growth acting in the Hilbert space H in the case when the generator of such C_0–group admits functional model representation [1] and its spectrum is absolutely continuous. To this end we first provide the necessary background, following [1, 10, 8, 6, 7].

3.1. **Functional model approach.** We are going to consider a class of operators of the form [8] $L = A + iV$, where A is a selfadjoint operator in H defined on the domain $D(A)$ and the perturbation V admits the factorization $V = \frac{\alpha J \alpha}{2}$, where α is a nonnegative selfadjoint operator in H and J is a unitary operator in $E \equiv \overline{R(\alpha)}$. This factorization corresponds to the polar decomposition of the operator V. In order that the expression $A + iV$ be meaningful, we impose the condition that V be (A)-bounded with the relative bound less then 1, i.e. $D(A) \subset D(V)$ and for some a and b ($a < 1$) the condition

$$\|Vu\| \le a\|Au\| + b\|u\|, \quad u \in D(A)$$

is satisfied, see [4]. Then the operator L is well-defined on the domain $D(L) = D(A)$.

Alongside with the operator L we are going to consider the maximal dissipative operator $L^{\|} = A + i\frac{\alpha^2}{2}$ and the one adjoint to it, $L^{-\|} \equiv L^{\|*} = A - i\frac{\alpha^2}{2}$. Since the functional model for the dissipative operator $L^{\|}$ will be used below, we require that $L^{\|}$ be completely non-selfadjoint, i.e. that it has no reducing selfadjoint parts. This requirement is not restrictive in our case due to the Proposition 1 in [8].

Now we are going to briefly describe the construction of the selfadjoint dilatation of the completely nonselfadjoint dissipative operator $L^{\|}$, following [1, 10], see also [8].

The characteristic function $S(\lambda)$ of the operator $L^{\|}$ is the contractive, analytic operator-valued function acting in the Hilbert space E, defined for $Im\lambda > 0$ by

$$S(\lambda) = I + i\alpha(L^{-\|} - \lambda)^{-1}\alpha, \quad Im\lambda > 0. \tag{3.1}$$

In the case of unbounded α the characteristic function is first defined by the latter expression on the manifold $E \cap D(\alpha)$ and then extended by continuity to the whole space E.

The formula (3.1) makes it possible to consider $S(\lambda)$ for $Im\lambda < 0$ with $S(\overline{\lambda}) = (S^*(\lambda))^{-1}$. Finally, $S(\lambda)$ possesses boundary values on the real axis in the strong sense: $S(k) \equiv S(k + i0),\ k \in \mathbb{R}$ (see [1]).

Consider the model space $\mathcal{H} = L_2\left(\begin{smallmatrix} I & S^* \\ S & I \end{smallmatrix}\right)$, which is defined in [10] as the Hilbert space of two-component vector-functions $(\tilde{g}, g)$ on the axis $(\tilde{g}(k), g(k) \in E, k \in \mathbb{R})$ with metric

$$\left(\begin{pmatrix} \tilde{g} \\ g \end{pmatrix}, \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}\right) = \int_{-\infty}^{\infty} \left(\begin{pmatrix} I & S^*(k) \\ S(k) & I \end{pmatrix} \begin{pmatrix} \tilde{g}(k) \\ g(k) \end{pmatrix}, \begin{pmatrix} \tilde{g}(k) \\ g(k) \end{pmatrix}\right)_{E \oplus E} dk.$$

It is assumed here that the set of two-component functions has been factored by the set of elements with the norm equal to zero.

Let's define the following orthogonal subspaces in $\mathcal{H}$:

$$D_- \equiv \begin{pmatrix} 0 \\ H_2^-(E) \end{pmatrix}, \ D_+ \equiv \begin{pmatrix} H_2^+(E) \\ 0 \end{pmatrix}, \ K \equiv \mathcal{H} \ominus (D_- \oplus D_+),$$

where $H_2^{+(-)}(E)$ denotes the Hardy class of analytic functions f in the upper (lower) half plane with the values in the Hilbert space E.

The subspace K can be described as $K = \{(\tilde{g}, g) \in \mathcal{H} : \ \tilde{g} + S^* g \in H_2^-(E), S\tilde{g} + g \in H_2^+(E)\}$. Let P_K be the orthogonal projection of $\mathcal{H}$ onto K:

$$P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = \begin{pmatrix} \tilde{g} - P_+(\tilde{g} + S^* g) \\ g - P_-(S\tilde{g} + g) \end{pmatrix},$$

where $P_\pm$ are orthogonal projections of $L_2(E)$ onto $H_2^\pm(E)$.

The following result holds [1, 10]: the operator $(L^{\parallel} - \lambda_0)^{-1}$ is unitary equivalent to the operator $P_K(k - \lambda_0)^{-1}|_K$ for all $\lambda_0, Im\lambda_0 < 0$. In effect this means, that the operator of multiplication by k serves as the minimal $(clos_{Im\lambda \neq 0}(k - \lambda)^{-1} K = \mathcal{H})$ selfadjoint dilatation [1] of the operator $L^{\parallel}$.

The characteristic function of the operator L is defined by the following expression:

$$\Theta(\lambda) \equiv I + iJ\alpha(L^* - \lambda)^{-1}\alpha, \qquad Im\lambda \neq 0,$$

and is a meromorphic, J-contractive $(\Theta^*(\lambda)J\Theta(\lambda) \leq J, \quad Im\lambda > 0)$ operator-function [2]. The characteristic function $\Theta(\lambda)$ admits factorization in the form of the ratio of two bounded analytic operator-functions (in the corresponding half-planes $Im\lambda < 0$, $Im\lambda > 0$) triangular with respect to the decomposition of the space E into the orthogonal sum

$$E = (\mathcal{X}_+ E) \oplus (\mathcal{X}_- E), \quad \mathcal{X}_\pm \equiv \frac{I \pm J}{2}.$$

Following [7], we define the subspaces $\hat{N}_\pm$ in $\mathcal{H}$ as follows:

$$\hat{N}_\pm \equiv \left\{ \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} : \quad \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathcal{H}, \quad \mathcal{X}_-(\tilde{g} + S^* g) + \mathcal{X}_+(S\tilde{g} + g) = 0 \right\}$$

and introduce the following designation:

$$N_\pm = clos P_K \hat{N}_\pm.$$

Then, as it is shown in [8], one gets for $Im\lambda < 0$ ($Im\lambda > 0$) and $(\tilde{g}, g) \in \hat{N}_{-(+)}$, respectively:

$$(L-\lambda)^{-1} P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = P_K \frac{1}{k-\lambda} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}.$$

The absolutely continuous and singular subspaces of the nonselfadjoint operator L were defined in [6]: let[2] $N \equiv \hat{N}_+ \cap \hat{N}_-$, $\tilde{N}_\pm \equiv P_K \hat{N}_\pm$, $\tilde{N}_e \equiv \tilde{N}_+ \cap \tilde{N}_-$, then

$$\begin{gathered} N_e \equiv clos\left(\tilde{N}_e\right) = clos P_K N \\ N_i \equiv K \ominus N_e(L^*). \end{gathered} \tag{3.2}$$

We call operator L an "operator with purely absolutely continuous spectrum" if $N_e = H$, i.e. $P_K N$ is dense in K.

3.2. **Characterization of the C_0–group with functional growth in terms of its generator.** Consider arbitrary nonnegative continuous function on the right semiaxis, $f \in C(\mathbb{R}_+)$.

We call a C_0–group $T(t)$ in Hilbert space H, satisfying the following condition

$$\|T(t)\| \le M f(|t|) \tag{3.3}$$

for all $t \in \mathbb{R}$, C_0–group *with functional growth* of operators in the Hilbert space H.

Let the generator L of a group $T(t)$ with functional growth belong to the class described in section 3.1 (i. e. it admits a model representation described in the latter section of the present work). Let's further restrict ourselves to the case when $\sigma(L) \subset \mathbb{R}$.

We are primarily interested in obtaining necessary (respectively, sufficient) conditions of (3.3) in the form of a pair of integral estimates for the resolvent of the operator L generating the group $T(t)$:

$$\begin{gathered} \int_{\mathbb{R}} \|(L-k-i\varepsilon)^{-1}u\|^2 dk \le C\frac{1}{\varepsilon} g(\varepsilon)\|u\|^2 \\ \int_{\mathbb{R}} \|(L^*-k-i\varepsilon)^{-1}u\|^2 dk \le C\frac{1}{\varepsilon} g(\varepsilon)\|u\|^2, \end{gathered} \tag{3.4}$$

where $g \in C(\mathbb{R}_+)$ is some continuous nonnegative function of $\varepsilon > 0$, considered for all $u \in H$.

One can prove the following results:

Theorem 3.1. *Let the spectrum of L be absolutely continuous. Then for every nonnegative function $g \in C(\mathbb{R}_+)$, satisfying the condition $g(\varepsilon) \le M < \infty$ when $\varepsilon \gg 1$, the estimates* (3.4) *suffice for the operator L to be a generator of C_0–group in Hilbert space H with*

$$\|\exp(iLT)\| \le M f(|t|)$$

for all $t \in \mathbb{R}$ and function $f(t)$, defined for $t > 0$ by $f(t) \equiv g(1/t)$.

[2]The linear set N is called a set of "smooth" vectors of the operator L (see [8])

Proof. Let $t > 0$. Then

$$e^{itk}e^{-\varepsilon t} = -\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} e^{ixt}\left[(k-x+i\varepsilon)^{-1} - (k-x-i\varepsilon)^{-1}\right]dx$$

for any $\varepsilon > 0$.

Then in the conditions of the Theorem one immediately obtains

$$e^{itL}e^{-\varepsilon t}u = -\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} e^{ixt}\left[(L-x+i\varepsilon)^{-1} - (L-x-i\varepsilon)^{-1}\right]udx \tag{3.5}$$

on the set of "smooth" vectors $\tilde{N}_e$ of L [8] (provided, that the latter integral exists). Then

$$\frac{\varepsilon}{\pi}\left|\int\limits_{\mathbb{R}} e^{ixt}(\left[(L-x+i\varepsilon)^{-1}(L-x-i\varepsilon)^{-1}\right]u,v)dx\right| \leq$$

$$\leq \frac{\varepsilon}{\pi}\left|\int\limits_{\mathbb{R}} \|(L-x-i\varepsilon)^{-1}u\|\cdot\|(L^*-x-i\varepsilon)^{-1}v\|dx\right| \leq$$

$$\leq \frac{\varepsilon}{\pi}\left(\int\limits_{\mathbb{R}} \|(L-x-i\varepsilon)^{-1}u\|^2dx\right)^{1/2}\left(\int\limits_{\mathbb{R}} \|(L^*-x-i\varepsilon)^{-1}u\|^2dx\right)^{1/2} \leq \tilde{C}\|u\|\|v\|g(\varepsilon)$$

for all $u, v \in \tilde{N}_e$, which justifies the formula (3.5) and for all u, v from a linear manifold dense in H the following estimate is obtained:

$$\left|(\exp(iLt)e^{-\varepsilon t}u,v)\right| \leq M\|u\|\|v\|g(\varepsilon).$$

Having chosen $\varepsilon \equiv 1/t$, we get

$$|(\exp(iLt)u,v)| \leq M\|u\|\|v\|g(1/t)$$

for all $u, v \in H$ and $t > 0$.

The boundedness of $\exp(iLt)$ when $t < 0$ can be shown in an analogous fashion. □

Theorem 3.2. *Let the spectrum of L be absolutely continuous. Let the Laplace-type transform of nonnegative $f \in C(\mathbb{R}_+)$*

$$g'(\varepsilon) \equiv \int\limits_{0}^{\infty} e^{-2\varepsilon t}(f(t))^2dt$$

is finite for all $\varepsilon \in (0,\infty)$, admitting the estimate $g'(\varepsilon) \leq C$ for $\varepsilon \gg 1$. Then the condition (3.4) with $g(\varepsilon) = \varepsilon g'(\varepsilon)$ is necessary for the operator L to generate a functionally bounded C_0–group,

$$\|\exp(iLt)\| \leq Mf(|t|) \quad \textit{for all } t \in \mathbb{R}.$$

Proof. For all $z \in \mathbb{C}_+$, $k \in \mathbb{R}$

$$\frac{1}{k-z} = i\int\limits_{0}^{\infty} e^{izt}e^{-itk}dt.$$

Then in the conditions of the Theorem we get for all $u \in \tilde{N}_e$ and $z \in \mathbb{C}_+$:

$$(L-z)^{-1} P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = i \int_0^\infty dt e^{izt} P_K e^{-itk} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = i \int_0^\infty dt e^{izt} \exp(-iLt) P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix},$$

provided that the latter integral exists. The Plancherel theorem applied to the right hand side of the latter identity gives

$$\int_{\mathbb{R}} \|(L-k-i\varepsilon)^{-1} u\|^2 dk = \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ikt} [e^{-\varepsilon t} e^{-iLt} u] dt, \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ikt} [e^{-\varepsilon t} e^{-iLt} u] dt \right) dk =$$

$$= 2\pi \int_0^\infty e^{-2\varepsilon t} \|\exp(-iLt)\|^2 dt \le C \int_0^\infty e^{-2\varepsilon t} (f(t))^2 \|u\|^2 dt = C\|u\|^2 \frac{1}{\varepsilon} g(\varepsilon).$$

for all $u \in \tilde{N}_e$, i. e. for the linear manifold dense in H.

The corresponding estimate for $(L-k+i\varepsilon)^{-1}$ is obtained based on the following identity:

$$\|\exp(iLt)\| = \|(\exp(iLt))^*\| = \|\exp(-iL^*t)\|.$$

□

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