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Amoebas, Monge-Ampère measures and triangulations of the Newton polytope

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1 Introduction

Suppose f is an entire function in the complex plane with zeros $a_1, a_2, a_3, \dots$ ordered so that $|a_1| \le |a_2| \le \dots$. Assume for simplicity that $f(0) \ne 0$. The classical Jensen formula states that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{it})|dt = \log|f(0)| + \sum_{k=1}^{N} \log\frac{r}{|a_k|}$$

where N is the largest index such that $|a_N| < r$. If this expression is considered as a function N_f of $\log r$ there is a strong connection between this function and the zeros of f. Thus, it follows immediately that N_f is a piecewise linear convex function whose gradient is equal to the number of zeros of f inside the disc $\{|z| < r\}$. The second derivative of N_f, in the sense of distributions, is a sum of point masses at $\log|a_k|$, $k = 1, 2, 3, \dots$. In this paper we consider a certain generalisation of the function occuring in the Jensen formula to holomorphic functions of several variables.

Let Ω be a convex open set in $\mathbf{R}^n$ and let f be a holomorphic function defined in $\mathrm{Log}^{-1}(\Omega)$, where $\mathrm{Log} : (\mathbf{C} \setminus \{0\})^n \longrightarrow \mathbf{R}^n$ is the mapping $(z_1, \dots, z_n) \mapsto (\log|z_1|, \dots, \log|z_n|)$. In [8] Ronkin considers the function N_f defined in Ω by the integral

$$N_f(x) = \frac{1}{(2\pi i)^n}\int_{\mathrm{Log}^{-1}(x)} \frac{\log|f(z_1, \dots, z_n)|\, dz_1 \dots dz_n}{z_1 \dots z_n}. \tag{1}$$

As we will see, the function N_f retains some of its properties from the one-variable case, while others are lost or attain a new form. For example, N_f is a convex function, but it is no longer piecewise linear. In section 2 below we consider the consequences of approximating N_f by a piecewise linear function. In section 3 we investigate the relation between local properties of N_f and the hypersurface $f^{-1}(0)$.

The properties of the function N_f are closely related to the amoeba of f. The amoeba of f, which we denote $\mathcal{A}_f$, is defined to be the image in Ω of the hypersurface $f^{-1}(0)$ under the map Log. The term amoeba was first used by Gelfand, Kapranov and Zelevinsky in the case where f is a polynomial.

Suppose E is a connected component of the amoeba complement $\Omega \setminus \mathcal{A}_f$. It is not difficult to show that all such components are convex. For example, $\mathrm{Log}^{-1}(E)$ is the intersection of $\mathrm{Log}^{-1}(\Omega)$ with the domain of convergence of a certain Laurent series expansion of $1/f$, and domains of convergence of Laurent series are always logarithmically convex. In [3] Forsberg, Passare and Tsikh defined the order of such a component to be the vector $\nu = (\nu_1, \ldots, \nu_n)$ given by the formula

$$\nu_j = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x)} \frac{\partial f}{\partial z_j} \frac{z_j dz_1 \ldots dz_n}{f(z) z_1 \ldots z_n}, \quad x \in E. \tag{2}$$

Here x may be any point in E. They proved, for the case when f is a polynomial, that the order is an integer vector, that is $\nu \in \mathbf{Z}^n$, that ν is in the Newton polytope of f, and that two distinct components always have different orders. These conclusions remain true, with essentially the same proofs, in the more general setting considered here.

Ronkin proved a theorem, which in the language of amoebas amounts to the followin statement.

Theorem. *Let f be a holomorphic function as above. Then N_f is a convex function. If $U \subset \Omega$ is a connected open set, then the restriction of N_f to U is affine linear if and only if U does not intersect the amoeba of f. If x is in the complement of the amoeba, then* $\mathrm{grad}\, N_f(x)$ *is equal to the order of the complement component containing x.*

Sketch of proof. The convexity of N_f follows from a general theorem because $\log|f|$ is plurisubharmonic, see for example [9], Corollary 1 on p. 84. Differentiation with respect to x_j under the integral sign in the definition (1) of N_f yields precisely the real part of the integral (2) defining the order. However, the integral (2) is always real valued and this shows immediately that N_f is affine linear in each connected component of $\Omega \setminus \mathcal{A}_f$. The fact that N_f is not linear on any open set intersecting the amoeba of f can be proved in several ways. It follows, for instance, from the results in section 3.

2 Triangulations and polyhedral subdivisions

In this section we will consider an approximation to Ronkin's function N_f by a piecewise linear function. This will lead to a polyhedral complex approximating the amoeba of f.

First we establish a more general construction. Recall the concept of a coherent triangulation from [4]. We will here use a slight generalization of this idea.

Let A be a possibly infinite subset of the lattice $\mathbf{Z}^n$, and let there be given for each $\alpha \in A$ a real number c_α. Assume also that

$$S(x) = \max_{\alpha \in A} (c_\alpha + \langle \alpha, x \rangle) \tag{3}$$

is finite for all x in a convex domain Ω and that Ω is the maximal domain with this property. We call Ω the domain of convergence of S. Here and in what follows, we identify $\mathbf{R}^n$ with its dual by the standard scalar product. Consider the convex hull of the set

$$\{(\alpha, t) \in A \times \mathbf{R};\, t \leq c_\alpha\} \subset \mathbf{R}^n \times \mathbf{R}$$

which we denote by G. It is an unbounded polyhedron which is mapped onto the convex hull of A under the projection $\mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^n$ onto the first factor. The bounded faces of G are mapped to polyhedra with vertices among the points of A, and they are easily seen to constitute a polyhedral subdivision of the convex hull of A, which we denote by T. For generic choices of the coefficients c_α, this will actually be a triangulation, which is called the coherent (or regular) triangulation corresponding to the c_α.

It is also possible to construct a polyhedral subdivision of Ω from these data. By a polyhedron in Ω we mean a convex subset of Ω which is locally the intersection of a finite number of closed halfspaces. For each cell σ of T, we define

$$\sigma^* = \{x \in \Omega; S(x) = c_\alpha + \langle \alpha, x \rangle \text{ for all vertices } \alpha \text{ of } \sigma\}.$$

Clearly, σ^* is a polyhedron in Ω. Denote the collection of all such σ^* by T^*.

Theorem 1. *T^* is a polyhedral subdivision of Ω which is dual to T in the following sense:*

(i) If σ is a k-dimensional cell of T, then σ^* is an $(n-k)$-dimensional cell of T^* which is orthogonal to σ.

(ii) τ is a face of σ if and only if σ^* is a face of τ^*.

Proof. Let $G^* = \{(x,t); S(x) \leq t\} \subset \Omega \times \mathbf{R}$. This is a polyhedron whose faces project onto Ω under the projection $\pi : \Omega \times \mathbf{R} \longrightarrow \Omega$ to produce a polyhedral subdivision. We shall prove that this subdivision coincides with T^*. Assume that σ is a cell in T. For every $\alpha \in A$, $\{(x,t) \in G^*; t = c_\alpha + \langle \alpha, x \rangle\}$ is a face of G^*. The intersection of all such faces as α ranges over all vertices of σ projects onto σ^*. Conversely, let $\tilde{\sigma}$ be a face of G^*. Take a point (x_0, t_0) in the relative interior of $\tilde{\sigma}$, and let σ be the convex hull of $A_{\tilde{\sigma}} = \{\alpha \in A; S(x_0) = c_\alpha + \langle \alpha, x_0 \rangle\}$. Since $c_\alpha \leq S(x_0) - \langle \alpha, x_0 \rangle$ for all $\alpha \in A$, with equality precisely if $\alpha \in A_{\tilde{\sigma}}$, it follows that σ is the projection of a face of G^*, hence a cell of T. Since $S(x)$ is linear in $\pi(\tilde{\sigma})$ and $S(x) \geq c_\alpha + \langle \alpha, x \rangle$ for all α it follows that $S(x) = c_\alpha + \langle \alpha, x \rangle$ for all $x \in \pi(\tilde{\sigma})$ and all $\alpha \in A_{\tilde{\sigma}}$, hence $\pi(\tilde{\sigma}) \subset \sigma^*$. On the other hand it is not difficult to verify that $\sigma^* \subset \pi(\tilde{\sigma})$. This shows that the polyhedral subdivision obtained from the faces of G^* coincides with T^*.

If σ is a k-dimensional cell in T and α_1, α_2 are vertices of σ, then

$$\langle \alpha_1 - \alpha_2, x \rangle = (S(x) - c_{\alpha_1}) - (S(x) - c_{\alpha_2}) = c_{\alpha_2} - c_{\alpha_1}$$

for all $x \in \sigma^*$. This shows that $\dim \sigma^* \leq n-k$. On the other hand, there is some $x_0 \in \Omega$ such that $c_\alpha + \langle \alpha, x_0 \rangle \leq S(x_0)$, with equality if α is a vertex of σ, and strict inequality if α is outside σ. The same holds with x_0 replaced by x for all x near x_0 in the $(n-k)$-dimensional plane through x_0 orthogonal to σ. This completes the proof that $\dim \sigma^* = n-k$ and also that σ^* is orthogonal to σ.

Finally, if τ is a face of σ, it is immediate from the definition that σ^* is a subset of τ^*, hence it is a face since T^* is a polyhedral subdivision. The converse follows in a similar way if we observe that σ is spanned by $\{\alpha \in A; c_\alpha + \langle \alpha, x \rangle = S(x)$ for all $x \in \sigma^*\}$. ■

Assume now that f is a holomorphic function in $\mathrm{Log}^{-1}(\Omega)$ where Ω is a convex open set in $\mathbf{R}^n$, and write $A = \{\alpha \in \mathbf{Z}^n; \mathbf{R}^n \setminus \mathcal{A}_f$ has a component of order $\alpha\}$. It may happen that A is empty, but we will assume that this is not the case. The most interesting situation arises when A has plenty of points, for example when the convex hull of A coincides with the Newton polyhedron of f. This will always happen if f is a Laurent polynomial. Let the numbers c_α used in the definition (3) of $S(x)$ be given by

$$c_\alpha = N_f(x) - \langle \alpha, x \rangle, \quad x \in E_\alpha. \tag{4}$$

Recall that the gradient of N_f in E_α is equal to α, so the definition does not depend on x. By the following proposition the domain of convergence of $S(x)$ contains Ω, so we have a triangulation T of the convex hull of A and a polyhedral subdivision T^* of Ω. We call the $(n-1)$-skeleton of T^* the spine of the amoeba $\mathcal{A}_f$ and denote it by $\mathcal{S}_f$.

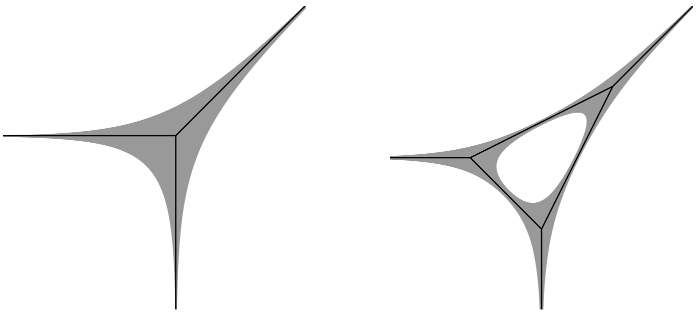

Figure 1: *Amoebas of the polynomials* $1+z_1+z_2$ *and* $1+z_1^3+z_2^3-6z_1z_2$ *(shaded) together with their spines (solid).*

Proposition 1. *With notations as in the construction above the following hold,*

(i) $S(x) \leq N_f(x)$ *with equality for* $x \in \Omega \setminus \mathcal{A}_f$.

(ii) *For every* $\alpha \in A$, $\{\alpha\}$*is a vertex of* T *and the component of order* α *in* $\Omega \setminus \mathcal{A}_f$ *is contained in* $\{\alpha\}^*$.

(iii) *The spine of the amoeba of* f *is contained in the amoeba. Every component of* $\Omega \setminus \mathcal{S}_f$ *contains a unique component of* $\Omega \setminus \mathcal{A}_f$.

Proof. The linear function $c_\alpha + \langle \alpha, x\rangle$ coincides with N_f on an open set, hence $c_\alpha + \langle \alpha, x\rangle \le N_f(x)$ for all x. Taking the supremum over all α in A proves part (i). If E_α denotes the component of $\Omega \setminus \mathcal{A}_f$ of order α, then

$$c_\alpha + \langle \alpha, x\rangle \le S(x) = N_f(x) = c_\alpha + \langle \alpha, x\rangle, \text{ for } x \in E_\alpha,$$

which shows that $E_\alpha \subset \{\alpha\}^*$. Since $\operatorname{grad} N_f = \beta$ in E_β we have a strict inequality $c_\alpha + \langle \alpha, x\rangle < N_f(x) = S(x)$ for $x \in E_\beta$ and all $\beta \neq \alpha$, which implies that $E_\alpha = \{\alpha\}^* \setminus \mathcal{A}_f$. If $x \in \mathcal{S}_f$ then there exist $\alpha, \beta \in A$ such that $x \in \{\alpha\}^* \cap \{\beta\}^* \subset \Omega \setminus \bigcup_{\gamma\neq\alpha} E_\gamma \setminus \bigcup_{\gamma\neq\beta} E_\gamma = \mathcal{A}_f$. Finally we note that the connected components of $\Omega \setminus \mathcal{S}_f$ are precisely the sets $\{\alpha\}^*$. ■

Let us now turn to the question of how to compute the coefficients c_α corresponding to a given Laurent polynomial f. If Γ is a face of the Newton polytope of $f(z) = \sum_{\alpha\in C} w_\alpha z^\alpha$, where C denotes a finite subset of $\mathbf{Z}^n$, let $f|_\Gamma$ denote the truncation of f to Γ, that is $f|_\Gamma(z) = \sum_{\alpha\in\Gamma} w_\alpha z^\alpha$. It is well known that when $\alpha \in \Gamma \cap \mathbf{Z}^n$, the complement of the amoeba of f has a component of order α precisely if the complement of the amoeba of $f|_\Gamma$ does (see [3] Prop. 2.6 and also [4]).

Proposition 2. *Let f be a Laurent polynomial and let Γ be a face of the Newton polytope of f. If $\alpha \in \Gamma$ and the complement of $\mathcal{A}_f$ has a component of order α, then $c_\alpha(f) = c_\alpha(f|_\Gamma)$. In particular, if α is a vertex of the Newton polytope of f, then $c_\alpha = \log |w_\alpha|$.*

Proof. Take an outward normal v to the Newton polytope at Γ. If $\alpha \in \Gamma$ and x is in the component of $\mathbf{R}^n \setminus \mathcal{A}_f$ of order α, it is known that $x + tv$ is also in that component for all $t > 0$. Therefore

$$c_\alpha(f) - c_\alpha(f|_\Gamma) = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x+tv)} \log \left| \frac{f(z)}{f|_\Gamma(z)} \right| \frac{dz_1 \ldots dz_n}{z_1 \ldots z_n}.$$

and here the integrand clearly tends to 0 when $t \to \infty$. ■

To further describe the dependence of the coefficients c_α on the polynomial $f(z) = \sum_{\alpha\in C} w_\alpha z^\alpha$ it is useful to introduce the functions

$$\Phi_\alpha(w) = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x)} \frac{\log(f(z)/z^\alpha)\, dz_1 \ldots dz_n}{z_1 \ldots z_n}, \tag{5}$$

where x is in the component of order α. This means that $c_\alpha = \operatorname{Re} \Phi_\alpha$. Notice that Φ_α is a holomorphic function in the coefficients w with values in $\mathbf{C}/2\pi i\mathbf{Z}$, defined whenever the complement of the amoeba of f has a component of order α.

Example. Suppose $f(z) = (z + a_1) \ldots (z + a_N) = w_0 + \ldots + w_{N-1} z^{N-1} + z^N$ is a polynomial in one variable. Let $0 \le k \le N$ and assume that $0 < |a_1| <$

$\ldots < |a_k| < r < |a_{k+1}| < \ldots < |a_N|$. Then one finds that

$$\begin{aligned}
\Phi_k(w) &= \frac{1}{2\pi i}\int_{|z|=r} \frac{\log(f(z)/z^k)dz}{z} \\
&= \sum_{j=1}^{k}\int_{|z|=r} \frac{\log((z+a_j)/z)dz}{z} + \sum_{j=k+1}^{N}\int_{|z|=r} \frac{\log(z+a_j)dz}{z} \\
&= \sum_{j=k+1}^{N} \log a_j = \log(a_{k+1}\ldots a_N).
\end{aligned}$$

In this case, we observe that $\Phi_k(w)$ may be continued as a finitely branched holomorphic function, whose branches correspond to various permutations of the roots of f. Incidentally, the sum of all branches of $\exp\Phi_k(w)$ is equal to w_k.

We now return to the general case and show that the functions Φ_α very nearly satisfy a system of GKZ hypergeometric equations. We remark that the functions Φ_α, or rather $\partial\Phi_\alpha/\partial w_\alpha$ were used in [2] in the study of constant terms in powers of Laurent polynomials. The main result was obtained by showing, essentially, that the second term in the representation (6) is non-constant along every complex line parallel to the w_α axis, provided that α is not a vertex of the Newton polytope of f.

Theorem 2. *Let $f(z) = \sum_{\gamma\in C} w_\gamma z^\gamma$ be a Laurent polynomial with C being a fixed finite subset of $\mathbf{Z}^n$. Then the holomorphic functions Φ_α have the power series expansion*

$$\Phi_\alpha(w) = \log w_\alpha + \sum_{k\in K_\alpha} \frac{(-k_\alpha - 1)!}{\prod_{\beta\neq\alpha} k_\beta!}(-1)^{k_\alpha-1} w^k, \tag{6}$$

where

$$K_\alpha = \{k \in \mathbf{Z}^C; k_\alpha < 0, k_\beta \geq 0 \text{ if } \beta\neq\alpha, \sum_\gamma k_\gamma = 0, \sum_\gamma \gamma k_\gamma = 0\}. \tag{7}$$

The series converges for example when $|w_\alpha| > \sum_{\beta\neq\alpha}|w_\beta|$. Moreover, Φ_α satisfies the differential equations

$$(\partial^u - \partial^v)\Phi_\alpha = 0 \quad \text{if} \quad \sum_\gamma (u_\gamma - v_\gamma) = 0 \quad \text{and} \quad \sum_\gamma \gamma(u_\gamma - v_\gamma) = 0 \tag{8}$$

and

$$\sum_\gamma w_\gamma \partial_\gamma \Phi_\alpha = 1 \tag{9}$$

$$\sum_\gamma \gamma w_\gamma \partial_\gamma \Phi_\alpha = \alpha. \tag{10}$$

Here we have used the notation $\partial_\gamma = \partial/\partial w_\gamma$ and ∂^u for $u \in \mathbf{N}^C$ is the obvious multiindex notation for a higher partial derivative.

Proof. Use the power series expansion of the logarithm function to write

$$\log(f(z)/z^\alpha) = \log w_\alpha + \sum_{m\geq 1} \frac{(-1)^{m-1}}{m} \left(\sum_{\beta\neq\alpha} \frac{w_\beta z^\beta}{w_\alpha z^\alpha} \right)^m .$$

Now

$$\sum_{m\geq 1} \frac{(-1)^{m-1}}{m} \left(\sum_{\beta\neq\alpha} \frac{w_\beta z^\beta}{w_\alpha z^\alpha} \right)^m = \sum_{m\geq 1} \sum_{\sum k_\beta = m} \frac{(-1)^{m-1}}{m} \frac{m!}{\prod k_\beta!} \frac{\prod w_\beta^{k_\beta} z^{k_\beta \beta}}{w_\alpha^m z^{m\alpha}}$$
$$= \sum_{L_\alpha} (-1)^{k_\alpha - 1} \frac{(-k_\alpha - 1)!}{\prod k_\beta!} w^k z^{\sum k_\gamma \gamma}.$$

Here all sums and products indexed by β are taken over $\beta \in C \setminus \{\alpha\}$ while γ ranges over all of C and $L_\alpha = \{k \in \mathbf{Z}^C; k_\alpha < 0, k_\beta \geq 0, \sum k_\gamma = 0\}$. The constant terms in this expression, considered as monomials in the z variables, are precisely those corresponding to the set K_α, and we have proved (6).

To verify the differential equations, we differentate under the sign of integration defining Φ_α. By a simple computation,

$$\partial^u \log(f(z)/z^\alpha) = -(\sum u_\gamma - 1)!\, z^{\sum u_\gamma \gamma} (-f(z))^{-\sum u_\gamma}$$

which depends only on $\sum u_\gamma$ and $\sum u_\gamma \gamma$. Also,

$$\sum w_\gamma \partial_\gamma \log(f(z)/z^\alpha) = \sum w_\gamma z^\gamma / f(z) = 1.$$

This verifies (8) and (9). Finally,

$$\sum \gamma_j w_\gamma \partial_\gamma \log(f(z)/z^\alpha) = \sum \gamma_j w_\gamma z^\gamma / f(z) = \frac{z_j \partial f / \partial z_j}{f(z)}.$$

Comparing this to the definition (2) of the order of a component proves the relation (10). ■

Example. Let us consider the polynomial $f(z) = 1 + z_1^{n+1} + \ldots + z_n^{n+1} + a z_1 \ldots z_n$ in n variables. The Newton polytope of this polynomial is a simplex and the terms correspond to the vertices and the barycenter of the Newton polytope. The complement of the amoeba of f has at least $n+1$ components corresponding to the vertices of the Newton polytope, and in addition there may be a component corresponding to the single interior point $(1, \ldots, 1)$. Let K_n denote the set of $a \in \mathbf{C}$ such that $\mathbf{R}^n \setminus \mathcal{A}_f$ does not have a component of order $(1, \ldots, 1)$.

Proposition 3. *Let $f(z) = 1 + z_1^{n+1} + \ldots + z_n^{n+1} + a z_1 \ldots z_n$. Then $\mathbf{R}^n \setminus \mathcal{A}_f$ has a component of order $(1, \ldots, 1)$ if and only if $\mathcal{A}_f$ does not contain the origin. Hence $\mathbf{R}^n \setminus \mathcal{A}_f$ has such a component precisely if*

$$a \notin K_n = \{-t_0 - \ldots - t_n; |t_0| = \ldots = |t_n| = 1, t_0 \ldots t_n = 1\}.$$

Proof. Let M be an invertible $n \times n$ matrix with integer entries and let $f(z) = \sum_\alpha a_\alpha z^\alpha$ be any Laurent polynomial. Define a new Laurent polynomial $Mf = \sum_\alpha a_\alpha z^{M\alpha}$ where a multiindex α is regarded as a column vector. It is not difficult to show that the linear mapping $x \mapsto M^T x$ (where T denotes transpose) takes the amoeba of Mf onto the amoeba of f.

Now let f be the special polynomial in the proposition and consider M which map the Newton polytope of f onto a translate of itself. The set of such M can be identified with the symmetric group S_{n+1} via its action on the vertices. Then Mf coincides with f up to an invertible factor, hence they have the same amoeba. In particular, M^T maps the component of order $(1, \dots, 1)$ in the complement of $\mathcal{A}_f$ (if it exists) onto itself. If x is any point in that component, then the convex hull of the points $M^T x$, where M ranges over S_{n+1}, contains the origin. Since every component in the complement of the amoeba is convex, it follows that the component of order $(1, \dots, 1)$ contains the origin. Conversely, if some component E contains the origin, then $M^T E$ also contains the origin and hence coincides with E. This implies that E has order $(1, \dots, 1)$.

Now, the amoeba of f contains the origin precisely if

$$a = -(1 + z_1^{n+1} + \dots + z_n^{n+1})/z_1 \dots z_n$$

with $|z_1| = \dots = |z_n| = 1$, or equivalently,

$$-a \in \{t_0 + \dots + t_n; |t_0| = \dots = |t_n| = 1, t_0 \dots t_n = 1\}. \blacksquare$$

We depict the sets K_n for a few small values of n. Note that the cusps on the boundary correspond to polynomials with singular hypersurfaces.

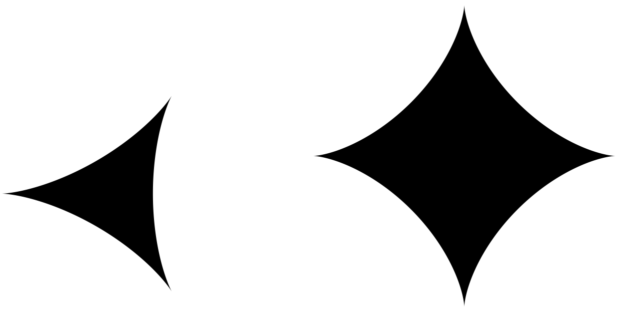

Figure 2: *The sets K_n for $n = 2$ and 3.*

Let us in particular consider the case $n = 2$. The coefficients c_α are equal to 0 when α is a vertex of the Newton polytope while

$$c_{(1,1)} = \log|a| + \operatorname{Re} \sum_{k>0} \frac{(3k-1)!}{(k!)^3} (-1)^{k-1} a^{-3k}$$

when $|a| > 3$, by Theorem 2. In fact it can easily be checked that the series converges even when $|a| = 3$. What happens with the spine when a approaches the boundary of K_2 from the outside? For example, when $a \to -3$ a numerical computation of the power series shows that $c_{(1,1)}$ converges to a limit with the approximate value 0.9693. This means that the complement of the spine has a rather large component which suddenly disappears when a enters K_2.

3 Associated Monge-Ampère measures

Let f be a holomorphic function in $\mathrm{Log}^{-1}(\Omega)$ where Ω is a convex domain in $\mathbf{R}^n$. We have seen that N_f is a convex function which is linear in each component of $\Omega \setminus \mathcal{A}_f$. It might be interesting to measure how much this function deviates from being linear at points in the amoeba.

If u is a smooth convex function, its Hessian $\mathrm{Hess}(u)$ is a positive semidefinite matrix which in a certain sense determines how convex u is near a given point. In particular, the determinant of the Hessian is a nonnegative function. The product of this function with ordinary Lebesgue measure is known as the real Monge-Ampère measure of u, and we will denote by Mu. In fact, the Monge-Ampère operator can be extended to all convex functions. In general, if u is a non-smooth convex function, Mu will be a positive measure, see [7]. We will denote the Monge-Ampère measure of N_f by μ_f. Hence μ_f is a positive measure supported on the amoeba of f. The following result motivates why it might be interesting to consider the Monge-Ampère measure of the Ronkin function.

Theorem 3. *If f is a Laurent polynomial, then the total mass of μ_f is equal to the volume of the Newton polytope of f.*

Proof. This is an almost immediate consequence of the definition. From the results in [7], it is known that when E is a Borel set $Mu(E)$ is equal to the Lebesgue measure of the set of all ξ such that $u(x) - \langle \xi, x\rangle$ attains its global minimum in E. Let $u = N_f$ and $E = \mathbf{R}^n$, and denote this set by F. Also let G be the set of all $\xi \in \mathbf{R}^n$ such that $N_f(x) - \langle \xi, x\rangle$ is bounded from below on $\mathbf{R}^n$. It is easy to see that F is contained in G and that the interior of G is contained in F. Hence the claim will follow if we prove that G is equal to the Newton polytope $\mathcal{N}$ of f.

Now, if ξ is in $\mathcal{N}$ then $N_f(x) - \langle \xi, x\rangle \geq S(x) - \langle \xi, x\rangle$ where $S(x)$ is defined by (3) and (4), and this latter function is certainly bounded from below. Conversely, if ξ is outside $\mathcal{N}$, take $v \in \mathbf{R}^n$ so that $\langle \xi, v\rangle > \sup_{\zeta\in\mathcal{N}}\langle \zeta, v\rangle$ and a vertex α of $\mathcal{N}$ where this supremum is attained. If x belongs to the complement component of order α, then $x + tv$ is also in that component for all $t > 0$. Hence $N_f(x + tv) - \langle \xi, x + tv\rangle = c_\alpha - \langle \xi - \alpha, x + tv\rangle \to -\infty$ as $t \to \infty$, so ξ is not in G. ■

It is interesting to consider a generalisation of the Monge-Ampère operator. Notice that on smooth functions u, the operator M is actually the restriction to the diagonal of a symmetric multilinear operator $\tilde{M}$ on n functions $u_1, \ldots, u_n$. Conversely, $\tilde{M}$ can be recovered from M by the polarization formula

$$\tilde{M}(u_1, \ldots, u_n) = \frac{1}{n!}\sum_{k=1}^{n} \sum_{1\leq j_1<\ldots<j_k\leq n} (-1)^{n-k} M(u_{j_1} + \ldots + u_{j_k}). \qquad (11)$$

This expression still makes sense if $u_1, \ldots, u_n$ are arbitrary convex functions, and by approximating $u_1, \ldots, u_n$ with smooth convex functions it follows that

$\tilde{M}(u_1, \ldots, u_n)$ is a positive measure which depends multilinearly on the arguments u_j. We call $\tilde{M}$ the mixed real Monge-Ampère operator.

The real Monge-Ampère operator is related to its complex counterpart as follows. Suppose $u_1, \ldots, u_n$ are smooth convex functions on the domain Ω. Let $U_1, \ldots, U_n$ be plurisubharmonic functions on $\operatorname{Log}^{-1}(\Omega)$ defined by $U_j(z) = u_j(\operatorname{Log} z)$. Then

$$n! \int_E \tilde{M}(u_1, \ldots, u_n) = \int_{\operatorname{Log}^{-1}(E)} dd^c U_1 \wedge \ldots \wedge dd^c U_n \tag{12}$$

where $d^c = (\partial - \bar{\partial})/2\pi i$. This remains true if one of the functions U_j, say U_1, is allowed to be an arbitrary smooth plurisubharmonic function in $\operatorname{Log}^{-1}(\Omega)$, u_1 being defined by

$$u_1(x) = \frac{1}{(2\pi i)^n} \int_{\operatorname{Log}^{-1}(x)} \frac{U_1(z) dz_1 \ldots dz_n}{z_1 \ldots z_n}.$$

More generally, if $U_1, \ldots, U_n$ are arbitrary smooth plurisubharmonic functions on $\operatorname{Log}^{-1}(\Omega)$, and

$$u_j(x) = \frac{1}{(2\pi i)^n} \int_{\operatorname{Log}^{-1}(x)} \frac{U_j(z) dz_1 \ldots dz_n}{z_1 \ldots z_n} \tag{13}$$

then

$$n! \int_E \tilde{M}(u_1, \ldots, u_n) = \int_{T^{n^2}} \int_{\operatorname{Log}^{-1}(E)} dd^c U_1(t^{(1)} z) \wedge \ldots \wedge dd^c U_n(t^{(n)} z) d\lambda(t). \tag{14}$$

Here T^{n^2} denotes the real n^2-dimensional torus $\{t = (t_j^{(k)}); |t_j^{(k)}| = 1, j, k = 1, \ldots n\}$ equipped with the usual normalized Haar measure λ, and each $t^{(k)} = (t_1^{(k)}, \ldots, t_n^{(k)})$ acts on $\mathbf{C}^n$ by componentwise multiplication. These formulas can be checked by direct computation. A proof of (12) when $u_1 = \ldots = u_n$ can be found in [6]. The general case follows by polarization since both sides are multilinear. Formula (14) follows by reversing the order of integration on the right. Since the inner integral is constant along certain n dimensional submanifolds of T^{n^2} it is actually possible to omit some of the variables in the outer integration. This also proves the generalised version of (12) with U_1 an arbitrary smooth plurisubharmonic function.

Theorem 4. *Let E be any Borel set in Ω, and let Δ denote the Laplace operator. Then*

$$(n-1)! \int_E \Delta N_f = \int_{\operatorname{Log}^{-1}(E) \cap f^{-1}(0)} \omega^{n-1}$$

where $\omega = (|z_1|^{-2} d\bar{z}_1 \wedge dz_1 + \ldots + |z_n|^{-2} d\bar{z}_n \wedge dz_n)/2\pi i$.

Proof. If u is any convex function, then $\Delta u = n\tilde{M}(u, |x|^2, \ldots, |x|^2)$ and $\omega = dd^c |\operatorname{Log} z|^2$. Since $dd^c \log |f|$ is equal to the current of integration along $f^{-1}(0)$ it follows from (12) that

$$\begin{aligned} n! \int_E \Delta N_f &= n \int_{\operatorname{Log}^{-1}(E)} dd^c \log |f| \wedge \omega^{n-1} \\ &= n \int_{\operatorname{Log}^{-1}(E) \cap f^{-1}(0)} \omega^{n-1}. \blacksquare \end{aligned}$$

The following theorem can be thought of as a local analog of Bernstein's theorem [1] relating the number of solutions to a system of polynomial equations to the mixed volume of their Newton polytopes.

Theorem 5. *Let $f_1, \dots, f_n$ be holomorphic functions in* $\operatorname{Log}^{-1}(\Omega)$ *and let $E \subset \Omega$ be a Borel set. Then $n!\tilde{M}(N_{f_1}, \dots, N_{f_n})(E)$ is equal to the average number of solutions in* $\operatorname{Log}^{-1}(E)$ *to the system of equations*

$$f_j(t_1^{(j)} z_1, \dots, t_n^{(j)} z_n) = 0 \quad j = 1, \dots, n \tag{15}$$

as $t = (t_k^{(j)})$ ranges over the torus $\{t; |t_k^{(j)}| = 1, j, k = 1, \dots, n\}$.

Proof. If U_j are smooth plurisubharmonic functions which converge to $\log|f_j|$, then u_j defined by (13) converge to N_{f_j}. By the general properties of the real Monge-Ampère operator this implies that $\tilde{M}(u_1, \dots, u_n)$ converges weakly to $\tilde{M}(N_{f_1}, \dots, N_{f_n})$. Also $dd^c U_1(t^{(1)}z) \wedge \dots \wedge dd^c U_n(t^{(n)}z)$ converges weakly to the sum of point masses at the solutions of $f_1(t^{(1)}z) = \dots = f_n(t^{(n)}z) = 0$. Hence the theorem follows by passing to the limit in (14) if we only show that

$$\int_{\operatorname{Log}^{-1}(E)} dd^c U_1(t^{(1)}z) \wedge \dots \wedge dd^c U_n(t^{(n)}z)$$

remains uniformly bounded as $U_j \to \log|f_j|$. Here we may assume that E is compact and that U_j is of the form $U_j = \psi(|f_j|)$ where ψ will converge to log. Let $f_t(z) = (f_1(t^{(1)}z), \dots, f_n(t^{(n)}z))$. Then $f_t(z)$ is a holomorphic function in z and t defined for z in a neighbourhood of $\operatorname{Log}^{-1}(E)$ and t in a complex neighbourhood of T^{n^2}. Using compactness arguments it is not difficult to show that there exists a constant C such that the number of solutions in $\operatorname{Log}^{-1}(E)$ to the equation $f_t(z) = w$ is bounded above by C for almost all $t \in T^{n^2}$ and $w \in \mathbf{C}^n$. Since $\eta = dd^c\psi(|w_1|) \wedge \dots \wedge \psi(|w_n|)$ induces a positive measure on $\mathbf{C}^n$ with total mass 1, it follows that

$$0 \le \int_E f_t^* \eta \le C$$

and this completes the proof. ∎

If E is a compact component of $\mathcal{A}_{f_1} \cap \dots \cap \mathcal{A}_{f_n}$, then for topological reasons, the number of solutions to (15) in $\operatorname{Log}^{-1}(E)$ does not depend on t. Hence we obtain the following corollary.

Corollary 1. *Suppose K is a compact component of $\mathcal{A}_{f_1} \cap \dots \cap \mathcal{A}_{f_n}$. Then $n!\tilde{M}(N_{f_1}, \dots, N_{f_n})(K)$ is a positive integer, which is equal to the number of solutions of the system $f_1(z) = \dots = f_n(z) = 0$ in* $\operatorname{Log}^{-1}(K)$.

Let us now see what the measure μ_f looks like in some specific cases. First, if $n = 1$, the amoeba is a discrete point set and μ_f is a sum of point masses. More precisely, $\mu_f(\{x\})$ is equal to the number of zeros of f on the circle $\log|z| = x$. In the case of two variables there is an interesting estimate on the Monge-Ampère measure.

Theorem 6. *Let f be a holomorphic function in two variables defined on a circular domain* $\operatorname{Log}^{-1}(\Omega)$. *Then μ_f is greater than or equal to π^{-2} times Lebesgue measure on the amoeba of f.*

Proof. We prove the inequality in a neighbourhood of a point $x \in \mathcal{A}_f$ where $\mathrm{Log}^{-1}(x)$ intersects $f^{-1}(0)$ transversely in a finite number of points. Since this is true for almost all x it will establish the inequality.

Write $\log z_j = x_j + iy_j$ for $j = 1, 2$ and assume that the hypersurface $f^{-1}(0)$ is given locally as the union of graphs $y = \phi_k(x)$. We shall express the Hessian of N_f in terms of the functions ϕ_k.

Differentiating the integral (1) defining N_f with respect to x_1 we obtain

$$\begin{aligned}\frac{\partial N_f}{\partial x_1} &= \mathrm{Re}\,\frac{1}{(2\pi i)^2}\int_{\mathrm{Log}^{-1}(x)} \frac{\partial f/\partial z_1\, dz_1\, dz_2}{f(z) z_2} \\ &= \frac{1}{2\pi i}\int_{\log|z_2|=x_2} n(f(\cdot, z_2), x_1)\frac{dz_2}{z_2} \\ &= \frac{1}{2\pi}\int_0^{2\pi} n(f(\cdot, e^{x_2+iy_2}), x_1)dy_2.\end{aligned}$$

Here $n(f(\cdot, z_2), x_1)$ is the number of zeros minus the number of poles of the function $z_1 \mapsto f(z_1, z_2)$ inside the disc $\{\log|z_1| < x_1\}$ provided that it is meromorphic in that domain. In general, $n(f(\cdot, z_2), x_1) - n(f(\cdot, z_2, x_1')$ is equal to the number of zeros in the annulus $\{x_1' < \log|z_1| < x_1\}$ when $x_1' < x_1$. The integrand in the last integral is a piecewise constant function with a jump of magnitude 1 at $y_2 = \phi_{k,2}(x)$. It follows that the gradient of $\partial N_f/\partial x_1$ is given by a sum of terms $\pm(2\pi)^{-1}\,\mathrm{grad}\,\phi_{k,2}$. The correct sign of each term can be found by observing that $n(f(\cdot, e^{x_2+iy_2}), x_1)$ is increasing as a function of x_1, hence all the terms contributing to $\partial^2 N_f/\partial x_1^2$ should be positive. A similar computation applies to $\partial N_f/\partial x_2$.

Assume now that $f^{-1}(0)$ is given locally by an equation

$$a\log z_1 + b\log z_2 + \text{ higher terms } = \text{ constant}.$$

Solving for y in this equation yields that

$$\frac{\partial \phi_k}{\partial x} = \frac{1}{\mathrm{Im}(a\bar{b})}\begin{pmatrix}\mathrm{Re}(a\bar{b}) & |b|^2 \\ -|a|^2 & -\mathrm{Re}(a\bar{b})\end{pmatrix}.$$

The crucial observation here is that

$$\pm\begin{pmatrix}\partial\phi_{k,2}/\partial x_1 & \partial\phi_{k,2}/\partial x_2 \\ -\partial\phi_{k,1}/\partial x_1 & -\partial\phi_{k,1}/\partial x_2\end{pmatrix} \tag{16}$$

is a positive definite matrix with determinant 1, and that $2\pi\,\mathrm{Hess}(N_f)$ is a sum of such matrices.

The inequality now follows from the following lemma since $\mathrm{Log}^{-1}(x)$ intersects $f^{-1}(0)$ in at least two points for generic x in $\mathcal{A}_f$. ■

Lemma. *If A_1, A_2 are 2×2 positive definite matrices, then* $\det(A_1 + A_2) \geq \det A_1 + \det A_2 + 2\sqrt{\det A_1 \det A_2}$. *Equality holds if and only if A_1 and A_2 are real multiples of one another.*

Proof. To see this, write $A_j = \left(\begin{smallmatrix} a_j & b_j \\ b_j & c_j\end{smallmatrix}\right)$ and apply the Cauchy-Schwarz inequality to the vectors $(b_j, \sqrt{a_j c_j - b_j^2})$ to obtain $b_1 b_2 + \sqrt{\det A_1 \det A_2} \leq \sqrt{a_1 a_2 c_1 c_2}$.

Then it follows that $\det(A_1 + A_2) - \det A_1 - \det A_2 = a_1c_2 + c_1a_2 - 2b_1b_2 \geq 2\sqrt{a_1a_2c_1c_2} - 2b_1b_2 \geq 2\sqrt{\det A_1 \det A_2}$. The conditions for equality are that $(b_1^2, a_1c_1 - b_1^2)$ is proportional to $(b_2^2, a_2c_2 - b_2^2)$ and that (a_1, c_1) is proportional to (a_2, c_2) which clearly is equaivalent to A_1 being proportional to A_2. ■

As an immediate consequence of Theorem 3 and Theorem 6 we have the following estimate.

Corollary 2. *Let f be a Laurent polynomial in two variables. Then the area of the amoeba of f is not greater than π^2 times the area of the Newton polytope of f.*

On the contrary, when $n \geq 3$ the volume of the amoeba of a polynomial is almost always infinite.

Example. As an illustration of the last theorem we consider the polynomials $f(z_1, z_2) = f_a(z_1, z_2) = a + z_1 + z_2 + z_1z_2$ where a is assumed to be a real number.

We want to compute the number of points in $\mathrm{Log}^{-1}(x) \cap f^{-1}(0)$ for a given point $x \in \mathbf{R}^n$. For points z in this set it must hold that $|a + z_1| = |1 + z_1|e^{x_2}$. Conversely, if this equation holds, then $(z_1, (a + z_1)/(1 + z_1))$ is in this set. If $\theta = \arg z_1$, the equation can be rewritten $2e_1^x(e^{2x_2} - a)\cos\theta = a^2 + e^{2x_1} - e^{2x_2} - e^{2x_1+2x_2}$. From this it follows immediately that the amoeba of f is defined by the inequality $4e^{2x_1}(e^{2x_2} - a)^2 \geq (a^2 + e^{2x_1} - e^{2x_2} - e^{2x_1+2x_2})^2$. Moreover, the following can be deduced when we assume that $a \neq 1$. If x is in the interior of the amoeba, then $\mathrm{Log}^{-1}(x) \cap f^{-1}(0)$ has precisely two points. If x is in the boundary of the amoeba, and not equal to $(\log a, \log a)/2$ when a is positive, then $\mathrm{Log}^{-1}(x) \cap f^{-1}(0)$ has exactly one point. If $a > 0$ and $x = (\log a, \log a)/2$, then $\mathrm{Log}^{-1}(x) \cap f^{-1}(0)$ contains a real curve.

By the preceding theorem, μ_f is greater than or equal to π^{-2} times the Lebesgue measure on the amoeba of f. Assume now that $a < 0$. For x in the interior of $\mathcal{A}_f$, $2\pi\,\mathrm{Hess}(N_f)$ is a sum of two matrices of the form (16). Since f has real coefficients it follows that $\phi_1 = -\phi_2$, hence the two matrices are equal. This means that all inequalities in the proof of Theorem 6 actually become equalities. It follows also from Theorem 5 that μ_f has no mass on the boundary of the amoeba. Hence the area of the amoeba is equal to π^2.

When a is positive and not equal to 1, the same conciderations hold away from the special point $(\log a, \log a)/2$. On the other hand, the amoeba of f_a is strictly smaller than the amoeba of f_{-a}. The remaining mass of μ_{f_a}, which must have the same total mass as $\mu_{f_{-a}}$ resides as a point mass at $(\log a, \log a)/2$.

In the particular case $a = 1$, there is a factorization $f(z) = (z_1 + 1)(z_2 + 1)$ and the amoeba consists of the two lines $\{x_1 = 0\}$ and $\{x_2 = 0\}$. The Monge-Ampère measure μ_f then degenerates into a single point mass at the intersection point of the two lines.

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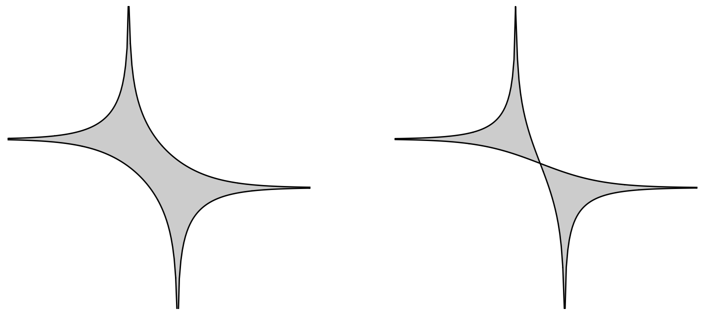

Figure 3: The amoebas of $a + z_1 + z_2 + z_1 z_2$ for $a = -5$ and $a = 5$.

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