ISSN: 1401-5617

An elementary theory of L^1C-sets.

Mats Erik Andersson

Research Reports in Mathematics
Number 1, 2000

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.matematik.su.se/reports/2000/1

Date of publication: February 11, 2000
1991 Mathematics Subject Classification: Primary 43A46

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.matematik.su.se
info@matematik.su.se

AN ELEMENTARY THEORY OF L^1C-SETS.

Mats Erik Andersson

Utkast avslutat den 11 februari i nådens år 2000.

A set $E \subseteq \mathbb{Z}$ is said to be a set of L^1-convergence, abbreviated to L^1C-set, if $\|s_n f - f\|_1 \to 0$ as $n \to \infty$ for every $f \in L^1_E$. Here $s_n f(\theta) = \sum_{-n}^{n} \widehat{f}(k) \exp(ik\theta)$ is a symmetric partial sum of f and $L^1_E = L^1_E(\mathbb{T})$ is the closed subspace of $L^1(\mathbb{T})$ consisting of E-spectral functions: $\widehat{f}(k) = 0$ for all $k \notin E$. Of course the circle group $\mathbb{T}$ is provided with the normalized Haar measure, denoted λ in case of need. In the following text the word polynomial always means trigonometric polynomial.

Using Banach–Steinhaus' theorem on uniform boundedness and also the denseness of E-spectral polynomials in L^1_E it is easily seen that E is an L^1C-set exactly when

$$\kappa_1(E) = \sup\Big\{ \frac{\|s_n f\|_1}{\|f\|_1} \,;\, f \in L^1_E \setminus \{0\},\, n \geqslant 0 \Big\}$$

is finite. Furthermore, we use the notion CL^1C-set to signify that E has the stronger property $\sup_{m\in\mathbb{Z}} \kappa_1(E+m) < \infty$. An alternative way of viewing the boundedness, is to realize that L^1_E is a homogeneous space in the sense of Katznelson and the techniques developed in [K], Chapters II and III, can be applied.

The intentions behind the present exposition is to display the close parallels of L^1C-sets to the following, already well established, two notions. The set $E \subseteq \mathbb{Z}$ is said to be a set of uniform convergence, abbreviated UC-set, in case

$$\kappa(E) = \sup\Big\{ \frac{\|s_n f\|_\infty}{\|f\|_\infty} \,;\, f \in C_E \setminus \{0\},\, n \geqslant 0 \Big\}$$

is finite, where C_E is the space of E-spectral continuous functions. Equivalently, $\|s_n f - f\|_\infty \to 0$ as $n \to \infty$ for every function f in C_E. The specialized notion of a CUC-set means that $\sup_{m\in\mathbb{Z}} \kappa(E+m) < \infty$. Finally, recall that E is a Sidon set in case there is a finite constant $c(E)$ such that for any $f \in C_E$ the inequality $\|f\|_{A(\mathbb{T})} \leqslant c(E)\,\|f\|_\infty$ obtains. Rudin [R], Theorem 2.1, lists equivalent characterizations.

To my knowledge the name L^1C-set was coined by J. Fournier in [F]. In an extended remark the author indicated that he himself and S. Hartman had gained insight into this notion. However, I have not been able to locate any further printed material in this direction. It will, not particularly surprising, turn out that all

1991 *Mathematics Subject Classification.* 43A46.
Key words and phrases. UC-set, $\Lambda(1)$-set.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

results about UC-sets found in [P], [T], and [ST] have counterparts for L^1C-sets. To a large extent the statements are verbatim to those in the old setting.

The inclusion relations in the following lemma were observed already in [F]. For convenience the proof is carried out. The five classes of sets $E \subseteq \mathbb{Z}$ are denoted in the obvious manner.

Lemma 1. Sidon $\subseteq$ UC $\subseteq \mathrm{L}^1$C *and* CUC $\subseteq \mathrm{CL}^1$C. *In addition the different set constants obey* $\kappa_1(E) \leqslant \kappa(E) \leqslant c(E)$.

Proof. It suffices to prove the inequalities and the second relation. Consider then a UC-set E. Pedemonte's Theorem 1 in [P] provides a sequence of measures $\mu_n \in M(\mathbb{T})$ for $n \geqslant 0$ with $\|\mu_n\| \leqslant \kappa(E)$ and $\widehat{\mu_n}\big|_E = \chi_{[-n,n]}\big|_E$, the latter denoting the characteristic function of the indicated interval. For any $f \in L^1_E$ we find $s_n f = \mu_n * f$, whence

$$\|s_n f\|_1 = \|\mu_n * f\|_1 \leqslant \|\mu_n\| \, \|f\|_1 \leqslant \kappa(E) \, \|f\|_1$$

and this establishes $\kappa_1(E) \leqslant \kappa(E)$.

Furthermore, the simple observation $\sup_m \kappa_1(E+m) \leqslant \sup_m \kappa(E+m)$ establishes the inclusion CUC $\subseteq \mathrm{CL}^1$C.

Consider finally a Sidon set E with Sidon constant $c(E)$. Rudin's characterization of Sidon sets in [R] provides to each $\rho \in \ell^\infty(E)$ a measure $\mu \in M(\mathbb{T})$ such that $\widehat{\mu}\big|_E = \rho$ and also $\|\mu\| \leqslant c(E) \, \|\rho\|_\infty$. In particular there are measures μ_n satisfying $\widehat{\mu_n}\big|_E = \chi_{[-n,n]}\big|_E$ and $\|\mu_n\| \leqslant c(E)$. By Pedemonte's theorem this is $\kappa(E) \leqslant c(E)$.

Later on some comments will be made on the fact that the inclusions are strict. Meanwhile, the next result will give an ample supply of examples. The statement is implicitly presented in [F]. For comparison it can be noted that a $\Lambda(1)$-set need not be a UC-set, even though it is a CL^1C-set. Fournier–Pigno [FP] provides a 4/3-Sidon set, which is $\Lambda(p)$ for all $p < \infty$ and hence is a CL^1C-set, but nonetheless is not a UC-set.

It must be recalled that Pedemonte found the existence of $\{\mu_n\}_0^\infty$, as above, to characterize UC-sets. Since, by Example 20, there are L^1C-sets which are not UC-sets, we conclude that in general the partial sums s_n on L^1_E, for L^1C-sets E, cannot be obtained as uniformly bounded convolution operators extending to $L^1(\mathbb{T})$.

Recall that E is said to be a $\Lambda(p)$-set in the case that for some $0 < q < p$ (and hence all) there is a constant $c_q = \Lambda(p, q, E)$ with the property $\|f\|_p \leqslant c_q \, \|f\|_q$ for all $f \in L^p_E$. This says that $L^q_E = L^p_E$ as metric spaces. For detailed information see [R] and [H2]. In particular, the notation $\Lambda(p, q, E)$ is explained in Hare's presentation.

Proposition 2. *Every* $\Lambda(1)$*-set is also a* CL^1C*-set.*

Proof. Let E be a $\Lambda(1)$-set. According to Hare's [H2] proof of Bachelis–Ebenstein's theorem, it is possible to choose a $p > 1$ that is independent of the finer structure of E and only depends on the structure constant $\Lambda(1, 1/2, E)$. The outcome is that E must in fact be a $\Lambda(p)$-set and hence there is some constant C such that $\|f\|_p \leqslant C \, \|f\|_1$ for all $f \in L^1_E$. Here C only depends on p and $\Lambda(1, 1/2, E)$.

Next, M. Riesz' theorem (cf. [Z], Theorem VII.2.4) provides for this particular p a further constant c_p such that $\|s_n g\|_p \leqslant c_p \, \|g\|_p$ for all $g \in L^p$ and all $n \geqslant 0$.

Thus we consider any $f \in L^1_{E+m}$, excepting the zero-function. The above preparations now prove the inequality

$$\frac{\|s_n f\|_1}{\|f\|_1} \leqslant \frac{\|s_n f\|_p}{\|f\|_1} \leqslant \frac{C\,\|s_n f\|_p}{\|f\|_p} \leqslant C\,c_p,$$

where the second inequality is due to the $\Lambda(p)$-property and its translation invariance. Since this calculation holds independently of the integer $m \in \mathbb{Z}$, the conclusion is $\sup_m \kappa_1(E+m) \leqslant C\,c_p$. This means that E in fact must be a CL^1C-set.

Remark. The point of the above proof is that given a collection $\{E_\alpha\}_{\alpha\in A}$ of $\Lambda(1)$-sets with uniformly bounded structure constants, that is $\sup_\alpha \Lambda(1, 1/2, E_\alpha) < \infty$, then also $\sup_{\alpha,m} \kappa_1(E_\alpha + m) < \infty$. In a loose manner of speaking: a uniform collection of $\Lambda(1)$-sets is also a uniform collection of CL^1C-sets. A possibly more general result than Proposition 2 will later appear as Theorem 8.

Two quick and entertaining results that appear via $\Lambda(p)$-sets are clearly worthwhile to record. The first is analytic, the second arithmetic in character.

Corollary 3. *If L^1_E is reflexive, then E is a* CL^1C*-set.*

Proof. According to [H2], Corollary, the reflexivity of L^1_E is equivalent to the $\Lambda(1)$-property of E. An application of the preceding proposition establishes the claim.

Corollary 4. *Suppose $E \subseteq \mathbb{Z}$ has the property that for all $a, s, t \in \mathbb{Z}$ the counting inequality $|E \cap \{a, a+s, a+t, a+s+t\}| \leqslant 3$ obtains. Then E is a* CL^1C*-set.*

Proof. The condition states that E does not contain parallelepipeds of dimension 2. Hence Hare's argument [H], top of page 153, says that E is a $\Lambda(4)$-set and hence also a $\Lambda(1)$-set as well as a CL^1C-set.

The possibility to distinguish between L^1C and CL^1C-sets is of course essential. An easy particular case where they coincide should be recorded first.

Lemma 5. *Every* L^1C*-set $E \subseteq \mathbb{N}$ is in fact a* CL^1C*-set.*

Proof. Let $f \in L^1_{E+m}$ be non-trivial. Every symmetric partial sum of f can be written as a difference between at most two expressions of the form $e^{im\theta} s_k\{e^{-im\theta} f(\theta)\}$, where the value of k changes. It follows that

$$\sup_{n\geqslant 0} \frac{\|s_n f\|_1}{\|f\|_1} \leqslant 2 \sup_{k\geqslant 0} \frac{\big\|s_k\big(e^{-im\theta} f(\theta)\big)\big\|_1}{\|f\|_1} \leqslant 2\,\kappa_1(E).$$

This certifies $\sup_m \kappa_1(E+m) \leqslant 2\kappa_1(E)$, so E is a CL^1C-set.

The just presented result is the counterpart of Lemma 6 in Travaglini [T]. It also displays rudiments of the way a CL^1C-set has stronger inner structure than L^1C-sets have. The decisive additional property required of the former class corresponds in essence to Soardi–Travaglini's Proposition 1 in [ST], where UC and CUC-sets are distinguished. This earlier result is preferably understood as the existence of a multiplier generated as convolution with a measure. In the new setting of CL^1C-sets, the multiplier still exists, but it no longer a priori arises from a measure.

Theorem 6. *An* $\mathrm{L}^1\mathrm{C}$*-set* E *is also a* $\mathrm{CL}^1\mathrm{C}$*-set if and only if the natural analytic projection* $P : L^1_E \to H^1_E \subseteq L^1_E$*, determined by* $e^{ikt} \mapsto \chi_{\mathbb{N}}(k)\, e^{ikt}$*, is bounded.*

Proof. Suppose E is an $\mathrm{L}^1\mathrm{C}$-set, that the natural projection $P : L^1_E \to H^1_E$ is bounded, denoting with $\|P\|$ its multiplier norm, and take $f \in L^1_E$ arbitrarily. Consider $s_n\big(e^{imt}f(t)\big)$. This can be written $e^{imt}\sum_{|k+m|\leqslant n}\widehat{f}(k)e^{ikt}$ and is hence expressible as a sum or difference between one or two expressions of the form $e^{imt}\, s_N(Pf)$ and $e^{imt}\, s_N(f - Pf)$. It follows that

$$\big\|s_n\big(e^{imt}f(t)\big)\big\|_1 \leqslant 2\,\kappa_1(E)\big\{\|Pf\|_{L^1_E}+\|f-Pf\|_{L^1_E}\big\} \leqslant 2\,\kappa_1(E)\big\{1+2\,\|P\|\big\}\,\|e^{imt}f\|_1.$$

Consequently $\sup_m \kappa_1(E+m) \leq \kappa_1(E)\,\big\{1 + 2\,\|P\|\big\}$, so E is a $\mathrm{CL}^1\mathrm{C}$-set.

Conversely we take a $\mathrm{CL}^1\mathrm{C}$-set E. We consider on L^1_E the new norm

$$|\!|\!|\, f \,|\!|\!| = \sup\, \Big\{\big\|e^{-imt}s_n\big(e^{imt}f(t)\big)\big\|_{L^1_E}\,;\, n \in \mathbb{N},\, m \in \mathbb{Z}\Big\}.$$

Immediately we find

$$\|f\|_{L^1_E} \leqslant \liminf_{n\to\infty}\,\|s_nf\|_{L^1_E} \leqslant |\!|\!|\, f \,|\!|\!|\,.$$

The property of being a $\mathrm{CL}^1\mathrm{C}$-set generates a finite ρ with $\kappa_1(E + m) \leq \rho$, for all $m \in \mathbb{Z}$. The inequality

$$\big\|e^{-imt}s_n\big(e^{imt}f(t)\big)\,\big\|_{L^1_E} \leqslant \kappa_1(E+m)\,\|e^{imt}f(t)\|_{L^1_{E+m}} \leqslant \rho\,\|f\|_{L^1_E}$$

now ensures $|\!|\!|\, f \,|\!|\!| \leqslant \rho\,\|f\|_{L^1_E}$ and hence $\|\;\|_{L^1_E}$ and $|\!|\!|\;|\!|\!|$ are equivalent norms.

Let now P denote the natural projection, initially defined at least for all E-polynomials. When g is a polynomial from L^1_E, there is an integer $M \geqslant 0$ such that $Pg(t) = e^{iMt}s_M\big(e^{-iMt}g(t)\big)$. This leads to the estimate

$$\|Pg\|_{L^1_E} = \big\|e^{iMt}s_M\big(e^{-iMt}g(t)\big)\big\|_{L^1_E} \leqslant |\!|\!|\, g \,|\!|\!| \leqslant \rho\,\|g\|_{L^1_E}.$$

Since the polynomial subspace is dense in L^1_E, the projection P may be extended to the whole of L^1_E with preserved norm, that is $\|P\| \leqslant \sup_m \kappa_1(E+m)$. This is the claimed boundedness and the proof is complete.

Example 7. It is not possible to remove the assumption that E be an $\mathrm{L}^1\mathrm{C}$-set in Theorem 6. This is because $E = \mathbb{N}$ trivially has the projection P bounded on $L^1_E = H^1$. However, $\mathbb{N}$ is no $\mathrm{L}^1\mathrm{C}$-set, much less a $\mathrm{CL}^1\mathrm{C}$-set. This can be seen from Theorem 13 below, or more constructively in Lemma 14.

The Zygmund class $L\log L$ consists of all $f \in L^1(\mathbb{T})$ such that $\int |f|\log^+|f|\,d\lambda$ is finite. We write $L^1_E \log L^1_E$ for $L^1_E \cap L\log L$.

Theorem 8. *Assume* $E \subseteq \mathbb{Z}$ *has the property* $L^1_E = L^1_E \log L^1_E$*. Then* E *is a* $\mathrm{CL}^1\mathrm{C}$*-set.*

Proof. According to Zygmund's theorem, see [Z], Theorem 2.8, page 254, there are two constants $A, B > 0$ such that $\|\tilde{f}\|_1 \leqslant A\int |f|\log^+|f|\,d\lambda + B$ for the conjugate function $\tilde{f}$ to any $f \in L\log L$.

Since $i\tilde{f} \sim \sum_{-\infty}^{\infty} \operatorname{sign}(n)\widehat{f}(n)\, e_n$, where $e_n(\theta) = \exp(in\theta)$, we conclude

$$2s_n f = 2\,\widehat{f}(0) + i\,e_{-n-1}[e_{n+1}f]\tilde{} - i\,e_{n+1}[e_{-n-1}f]\tilde{}.$$

Consequently the trivially bounded and linear mappings $s_n : L^1_E \to L^1_E$ have the property

$$\begin{aligned}\|s_n f\|_1 &\leqslant \|f\|_1 + \tfrac{1}{2}\big\|[e_{n+1}f]\tilde{}\big\|_1 + \tfrac{1}{2}\big\|[e_{-n-1}f]\tilde{}\big\|_1 \\ &\leqslant \|f\|_1 + B + A\int |f|\log^+|f|\,d\lambda.\end{aligned}$$

Hence $\sup_n \|s_n f\|_1 < \infty$ holds for every $f \in L^1_E$. According to Banach–Steinhaus' theorem we conclude that $\sup_n \|s_n\|_{L^1_E \to L^1_E} < \infty$, whence E at least is an L^{1}C-set.

With the same technique $\sup_n \|Ps_n\|_{L^1_E \to L^1_E} < \infty$ and a simple weak-$*$ argument provides therefore also $\|P\|_{L^1_E \to L^1_E} < \infty$. According to Theorem 6, E is thus even a CL^{1}C-set.

Now we can reprove Proposition 2, although with essentially the same idea.

Corollary 9. *Every* $\Lambda(1)$*-set is a* CL^{1}C*-set.*

Proof. To a given $\Lambda(1)$-set E there is $p > 1$ with $L^1_E = L^p_E$, according to Bachelis–Ebensteins' theorem. It follows that $L^1_E \supseteq L^1_E \log L^1_E \supseteq L^p_E = L^1_E$, so the preceding theorem can be applied to conclude the CL^{1}C-property of E.

Next, three results are listed with statements identical to the older cases of UC-sets, granted the obvious reformulation for L^{1}C-sets. The published proofs for UC-sets can be read verbatim, except the replacement of the inequality $\|f * \mu\|_\infty \leq \|f\|_\infty\,\|\mu\|$ for the now relevant form $\|f * \mu\|_1 \leq \|f\|_1\,\|\mu\|$. Therefore only the third result is here provided with an explicit proof.

Proposition 10 (Cf. [T], Theorem 3). *$A \subseteq \mathbb{N}$ is an* L^{1}C*-set, and hence* CL^{1}C*-set, if and only if there are $q > 1$ and $\overline{\kappa_1} \geqslant 1$ such that*

$$\sup_{n\geqslant 1} \kappa_1\Big(A \cap \big[N, \lceil qN \rceil\,\big]\Big) \leqslant \overline{\kappa_1}.$$

The ceiling function $\lceil x \rceil$ yields the least integer exceeding x.

Proposition 11 (Cf. [T], Theorem 2). *If $A \subseteq \mathbb{N}$, $B \subseteq \mathbb{Z}^-$ are* L^{1}C*-sets, and hence also* CL^{1}C*-sets, then $A \cup B$ is at least an* L^{1}C*-set.*

Proposition 12 (Cf. [ST], Proposition 2). *If there is a set $E \subseteq \mathbb{Z}$ which is an* L^{1}C*-set, but not a* CL^{1}C*-set, then the union of two* CL^{1}C*-sets need not even be an* L^{1}C*-set.*

Proof. Suppose E is an L^{1}C-set without being a CL^{1}C-set. Consider now the sets

$$E_n^+ = E\cap[0,n]\,,\ \ E_n^- = E\cap[-n,0[\,,\quad H^+ = \bigcup_{n=1}^{\infty}\big(E_n^+ + 2^{2n}\big),\ \ H^- = \bigcup_{n=1}^{\infty}\big(E_n^- + 2^{2n}\big).$$

According to Proposition 10 both H^+ and H^- are L^{1}C-sets. By Lemma 5 they are even CL^{1}C-sets. It is now claimed that $H^+ \cup H^-$ cannot be an L^{1}C-set.

Consider to this end a polynomial $g \in L^1_E$ and denote with P the analytic projection as before. One may suppose that $\operatorname{spec} g \subseteq [-N, N]$. Let now $g_N(\theta) = \exp\{i2^{2N}\theta\}\, g(\theta)$, whence $g_N \in L^1_{H^+ \cup H^-}$. Furthermore,

$$\begin{aligned} \|Pg\|_1 = \big\|e^{i2^{2N}\theta}\, Pg(\theta)\big\|_1 &= \big\|s_{2^{2N}+N}\, g_N - s_{2^{2N}-1}\, g_N\big\|_1 \\ &\leqslant 2\,\kappa_1\big(H^+ \cup H^-\big)\, \|g_N\|_1 = 2\,\kappa_1\big(H^+ \cup H^-\big)\, \|g\|_1. \end{aligned}$$

The freedom in choosing g implies $\|P\| = \|P\|_{L^1_E \to L^1_E} \leqslant 2\,\kappa_1(H^+ \cup H^-)$. Since E is not a CL^{1}C-set, Theorem 6 provides $\|P\| = \infty$, from which follows that $H^+ \cup H^-$ cannot be an L^{1}C-set. This completes the proof.

As was the case for UC-sets, we now know that the natural union problem stands and falls with the non-coincidence of the two classes L^{1}C and CL^{1}C.

Prior to the closer comparison of interrelation between UC and L^{1}C-sets, an arithmetic property of L^{1}C-sets should be brought to light. Since, by Proposition 2, every $\Lambda(1)$-set is a CL^{1}C-set, the following result can be seen as a generalization of Rudin's Theorem 4.1 in [R].

Theorem 13 (Cf. [P], Theorem 4). *There is a constant $b > 1$ such that no* L^{1}C*-set E can contain arithmetic progressions of length exceeding $2\, b^{\kappa_1(E)}$.*

We use a lemma to take care of the technical calculation:

Lemma 14. *Let K_n be the Fejér kernel. There is a constant $\gamma > 0$ such that*

$$\inf_{k \in \mathbb{Z}} \sup_{m \geq 0} \big\|s_m\big(e^{ik\theta} K_n(\theta)\big)\big\|_1 \geqslant \gamma\, \log(n+2).$$

Proof of Theorem 13. Assume the progression $\{k - nl, \ldots, k - l, k, k + l, \ldots, k + nl\}$ is contained in the L^{1}C-set E. Then $f(\theta) = e^{ik\theta}\, K_n(l\theta) \in L^1_E$ and $\|f\|_1 = 1$. According to Lemma 14 we find

$$\gamma\, \log(n+2) \leqslant \sup_{m \geqslant 0} \|s_m f\|_{L^1_E} \leqslant \kappa_1(E),$$

which is the estimate $n + 2 \leqslant \exp\{\kappa_1(E)/\gamma\}$. For $n \geq 1$ the length of the arithmetic progression above is $2n + 1$, so it is now obvious that this length is at most $2\, \exp\{\kappa_1(E)/\gamma\}$. Taking $b = \exp \gamma^{-1}$ the claimed statement has been achieved.

Proof of Lemma 14. We need to use the standard kernels as well as the conjugate versions, see [Z] for details. Known properties of the Dirichlet kernel and the relation $K_{2n+1} = \frac{1}{2n+2} D_{2n+1} + \frac{2n+1}{2n+2} K_{2n}$ together demonstrate

$$\sup_{m,k} \Big\|s_m\big(e^{ik\theta}\{K_{2n+1} - K_{2n}\}\big)\Big\|_1 = \mathcal{O}\left(\tfrac{\log n}{n}\right) \text{ as } n \to \infty.$$

Thus it suffices to show

$$\liminf_{n \to \infty} \inf_{k \geqslant 0} \sup_{m \geqslant 0} \frac{\big\|s_m(e^{ik\theta} K_{2n})\big\|_1}{\log(n+2)} > 0.$$

Denote $A_r = (K_r + i\tilde{K}_r)/2 = \sum_{0\leqslant j\leqslant r}(1-\frac{j}{r+1})e^{ij\theta}$. It is straightforward (a picture on the coefficient side is most illuminating) to obtain for $n, k \geqslant 0$ the decomposition

$$s_{n+k}(e^{ik\theta}K_{2n}) = \tfrac{n}{2n+1}e^{ik\theta}K_{n-1} + \tfrac{n+1}{2n+1}\Big(\sum_{j=M(k,n)}^{k+n} e^{ij\theta}\Big) - a_r e^{-i(n-k+r+1)\theta}A_r,$$

where $M(k,n) = max(k-2n, -n-k)$, $0 \leqslant r \leqslant n-1$, and $0 \leqslant a_r \leqslant n/(2n+1)$. An inequality obtains from this consideration.

$$\begin{aligned}\|s_{n+k}(e^{ik\theta}K_{2n})\|_1 &\geqslant \tfrac{n+1}{2n+1}\|D_n\|_1 - \tfrac{n}{2n+1} - \tfrac{n}{2n+1}\|A_{n-1}\|_1 \\ &\geqslant \tfrac{1}{2}\Big(\|D_n\|_1 - \|A_n\|_1 - 1\Big).\end{aligned}$$

However, $\|D_n\|_1 \sim 2\pi^{-2}\,\log(n+2)$ and $\|A_n\|_1 \sim (2\pi)^{-1}\,\log(n+2)$ together provide the estimate

$$\inf_{k\geq 0}\sup_{m\geq 0}\frac{\|s_m(e^{ik\theta}k_{2n})\|_1}{\log(n+2)} \geqslant \frac{\|D_n\|_1 - \|A_n\|_1 - 1}{2\log(n+2)} \sim \tfrac{1}{2}\left(\tfrac{2}{\pi^2} - \tfrac{1}{2\pi}\right).$$

This last constant is positive, whence the claim has been verified.

For completeness the asymptotics of $\|A_n\|_1$ should be derived. We have

$$\begin{aligned}\tfrac{1}{2}\|\tilde{K}_n\|_1 = \int_0^\pi \tilde{K}_n\,d\lambda \; &= \sum\Big\{\tfrac{1}{\pi j}\Big(1-\tfrac{j}{n+1}\Big)\,;\,1\leqslant j\leqslant n,\,j\text{ odd}\Big\} \\ &= \tfrac{1}{2\pi}\sum\nolimits_{j=1}^{n}\tfrac{1}{j} + \mathcal{O}(1).\end{aligned}$$

Since $\big|\,\|A_n\| - \|\tilde{K}_n\|_1/2\,\big| \leqslant \|K_n\|_1/2 = 1/2$, the claimed asymptotic relation holds.

To put the next few results into perspective, let us note that an L^{1}C-set is characterized by the property $\sup_n \|s_n\|_{L^1_E\to L^1_E} < \infty$, which a priori is weaker than the property of UC-sets exhibited below in Lemma 15. On the other hand, a CL^{1}C-set is recognized by the preceding property and simultaneously $\|P\|_{L^1_E\to L^1_E} < \infty$. Again these two together represent weaker demands than those Lemma 16 a priori puts on every CUC-set.

The following two auxiliary results are known, but will help to understand in what respect UC-sets are subject to harder restrictions. In particular, a significant strengthening of Lemma 15 was achieved in [DP], Theorem 5, to the effect that for any UC-set E one has $M_{E\cup\mathbb{Z}^-}(\mathbb{T}) \subseteq M_0(\mathbb{T})$. Of course $M_E(\mathbb{T})$ is here the vector subspace of E-spectral measures and $M_0(\mathbb{T})$ consists of the measures whose Fourier coefficients vanish at infinity.

Lemma 15. *For a* UC*-set E the norm estimate $\|s_n\|_{M_E\to L^1_E} \leqslant \kappa(E)$ holds for all n. In particular, $M_E(\mathbb{T}) \subseteq M_0(\mathbb{T})$.*

Proof. Let $\mu \in M_E(\mathbb{T})$. For each $f \in C(\mathbb{T})$ holds $\mu * f \in C_E(\mathbb{T})$, so

$$\|(s_n\mu) * f\|_\infty = \|s_n(\mu * f)\|_1 \leqslant \kappa(E)\,\|\mu * f\|_\infty \leqslant \kappa(E)\,\|\mu\|\,\|f\|_\infty.$$

It follows that

$$\|s_n\mu\|_{L^1_E} = \sup_{f\in C(\mathbb{T})}\frac{\|s_n\mu * f\|_\infty}{\|f\|_\infty} \leqslant \kappa(E)\,\|\mu\|_{M_E},$$

which is the first claim. According to Helson's theorem [He] any measure $\mu \in M(\mathbb{T})$ with $\sup_n \|s_n\mu\|_1 < \infty$ must belong to $M_0(\mathbb{T})$. This completes the last part of the claim.

Lemma 16. *Let P denote the analytic projection. If E is a* CUC*-set, then $P\mu \in M_E(\mathbb{T})$ for every $\mu \in M_E(\mathbb{T})$ and in addition $\|P\|_{M_E\to M_E} \leqslant \|P\|_{C_E\to C_E} < \infty$. In particular, $M_E(\mathbb{T}) = L^1_E(\mathbb{T})$.*

Proof. Let E be a CUC-set. Choose measures ν_n such that $\widehat{\nu_n}\big|_E = \chi_{[0,n]}\big|_E$ and $\|\nu_n\| \leqslant \kappa(E)$ for all $n \geqslant 0$. This is possible according Soardi–Travaglini's theorem. In particular, $\|\nu_n\| \leqslant \|P\|_{C_E\to C_E}$ is achievable.

For $\mu \in M_E(\mathbb{T})$ one deduces $\mu * \nu_n \in M_E(\mathbb{T})$ and $\|\mu * \nu_n\| \leqslant \|P\|_{C_E\to C_E}\,\|\mu\|_{M_E}$, where $(\mu * \nu_n)\widehat{\ }(k) = \widehat{\mu}(k)$ for all $k \in E$, $0 \leqslant k \leqslant n$, and $(\mu * \nu_n)\widehat{\ }(k) = 0$ otherwise. There is consequently a weak-$*$ accumulation point $\tilde{\mu} \in M_E(\mathbb{T})$ of the sequence $\{\mu * \nu_n\}_{n=0}^{\infty}$. Since

$$\widehat{\tilde{\mu}}(k) = \lim_{n\to\infty} (\mu * \nu_n)\widehat{\ }(k) = \begin{cases} \widehat{\mu}(k), & k \geqslant 0,\, k \in E,\\ 0, & \text{otherwise,}\end{cases}$$

it is clear that $P\mu = \tilde{\mu} \in M_E(\mathbb{T})$ and additionally $\|P\mu\|_{M_E} \leqslant \|P\|_{C_E\to\ C_E}\,\|\mu\|_{M_E}$, due to $\|\tilde{\mu}\|_{M_E} \leqslant \liminf_{n\to\infty} \|\mu * \nu_n\|_{M_E}$.

Furthermore, $\mu \in M_E(\mathbb{T})$ decomposes as $\mu = P\mu + (\mu - P\mu)$ with $\operatorname{spec} P\mu \subseteq \mathbb{N}$ and $\operatorname{spec}(\mu - P\mu) \subseteq \mathbb{Z}^-$. By F. and M. Riesz' theorem (cf. [K], page 89) both $P\mu$ and $\mu - P\mu$ are elements in $L^1_E(\mathbb{T})$ and so the same conclusion obtains for μ itself. This was the last claim.

Neither of the two conclusions of Lemma 15 and 16 characterize the two classes. In fact they are characteristic of L^1C and CL^1C-sets, as the following result clearly demonstrates.

Lemma 17. *Two norm estimates hold for* L^1C*-sets. Namely,*

$$\sup_n \|s_n\|_{M_E\to L^1_E} \leqslant 2\,\kappa_1(E) + 1 \quad \textit{and}$$

$$\|P\|_{M_E\to L^1_E} \leqslant \|P\|_{L^1_E\to L^1_E}\left[2\kappa_1(E) + 1\right]$$

Proof. Let $\mu \in M_E$ and $n \geq 1$. Denoting Fejér means as $\sigma_n\mu = K_n * \mu$ we have

$$\begin{aligned}\left[2s_n\,\sigma_{2n-1}\mu - \sigma_{n-1}\right]\widehat{\ }(k) &= \chi_{[-n,n]}(k)\left\{2\big(1 - \tfrac{|k|}{2n}\big) - \big(1 - \tfrac{|k|}{n}\big)\right\}\widehat{\mu}(k)\\ &= \widehat{\mu}(k)\chi_{[-n,n]}(k).\end{aligned}$$

This says $s_n\mu = 2s_n\sigma_{2n-1}\mu - \sigma_{n-1}\mu$, whence it follows that

$$\|s_n\mu\|_1 \leqslant 2\,\kappa_1(E)\,\|\sigma_{2n-1}\mu\|_1 + \|\mu\|_{M_E} \leqslant \big(2\,\kappa_1(E) + 1\big)\,\|\mu\|_{M_E}.$$

In other words $\|s_n\|_{M_E\to L^1_E} \leqslant 2\,\kappa_1(E) + 1$ for all n.

The same representation provides $Ps_n\mu = P(2\,s_n\sigma_{2n-1} - \sigma_{n-1})\mu$, from which the obvious weak-$*$ argument produces

$$\|P\mu\|_{L^1_E} \leqslant \liminf_{n\to\infty} \|P(2\,s_n\sigma_{2n-1} - \sigma_{n-1})\mu\|_1 \leqslant \|P\|_{L^1_E\to L^1_E}\left\{2\kappa_1(E) + 1\right\}\|\mu\|_{M_E}.$$

This completes the proof.

Recall that E is a Riesz set if $M_E(\mathbb{T}) = L^1_E(\mathbb{T})$. A Rajchman set has the property that $\mu \in M(\mathbb{T})$, $\widehat{\mu}\big|_{E^c} \in c_0(E^c)$ implies $\widehat{\mu}\big|_E \in c_o(E)$, that is $\mu \in M_0(\mathbb{T})$. According to [HP] this latter property is equivalent to the statement $M_E(\mathbb{T}) \subseteq M_0(\mathbb{T})$.

Corollary 18. *Every* CL^{1}C*-set is a Riesz set and every* L^{1}C*-set is a Rajchman set.*

Proof. For any L^{1}C-set E, we now know $\sup_n \|s_n\|_{M_E \to L^1_E} < \infty$. By Helson's theorem [He], every measure in $M_E(\mathbb{T})$ has Fourier coefficients vanishing at infinity. Thus E is a Rajchman set.

For a CL^{1}C-set E we additionally have $\|P\|_{M_E \to L^1_E} < \infty$ from Lemma 17. The same argument as in the proof of Lemma 16 demonstrates that also in the present case $M_E(\mathbb{T}) = L^1_E(\mathbb{T})$, so E is thus a Riesz set.

It can be observed that any UC-set which is not a Riesz set, must in fact be L^{1}C without being CL^{1}C and would thus resolve the union problem for L^{1}C-sets. The existence of UC-sets which are not Riesz sets has to my knowledge not been discussed in the literatur, nor will it be resolved in this exposition. The next result is mentioned in passing only, and is meant to stress that weaker facts than the $\Lambda(1)$-property, for which some $p > 1$ yields $L^1_E = L^p_E$, force sets to be Riesz sets.

Corollary 19. *Every $E \subseteq \mathbb{Z}$ such that $L^1_E = L^1_E \log L^1_E$ is a Riesz set.*

Proof. Every such E is CL^{1}C, according to Theorem 8. Hence Corollary 18 shows E to be a Riesz set.

Two already published examples due to Fournier and Fournier–Pigno clearly demonstrate that there are intricate structural properties distinguishing UC-sets from L^{1}C-sets. For ease of reading, the two parts will be presented separately. The reference in parentheses indicates to which of the above conclusions it acts as counterexample against reversability.

Example 20 (Cf. Lemma 15). *There is a Riesz E set which is* CL^{1}C *but not* UC. *In particular, $M_E(\mathbb{T}) = L^1_E(\mathbb{T})$, $\sup_n \|s_n\|_{M_E \to L^1_E} < \infty$, $\|P\|_{M_E \to L^1_E} < \infty$, but $\sup_n \|s_n\|_{C_E \to C_E} = \infty$.*

Proof. Fournier and Pigno construct in [FP] a 4/3-Sidon set E, which is non-UC but still $\Lambda(p)$ for all $p < \infty$. Since it is $\Lambda(1)$ we know that it must be CL^{1}C and by Corollary 18 or [R], Proposition 5.1(b), $M_E(\mathbb{T}) = L^1_E$. The two cases of boundedness in the statement thus express that E is CL^{1}C, whereas the non-UC-property provides the final lack of uniform bound.

Example 21 (Cf. Lemma 16). *There are* CL^{1}C*-sets which are* UC *but not* CUC*-sets. In particular, $M_E(\mathbb{T}) = L^1_E$, $\sup_n \|s_n\|_{M_E \to L^1_E} < \infty$, $\sup_n \|s_n\|_{C_E \to C_E} < \infty$, $\|P\|_{M_E \to L^1_E} < \infty$, but $\|P\|_{C_E \to C_E} = \infty$.*

Proof. We follow Fournier [F]. Let $H = \{h_1 < h_2 < \dots\} \subseteq \mathbb{N}$ be an infinite strongly Hadamard sequence. The set $H - H$ was in [F] proved to be UC but non-CUC. On the other hand, [B],Théorème 5, page 359, proves

$$H_2 = \big\{ \sum_{k=1}^{\infty} \varepsilon_k h_k \,;\, \varepsilon_k \in \{0, \pm 1\},\, \sum |\varepsilon_k| = 2 \big\}$$

to be a $\Lambda(p)$-set for every $p < \infty$. Hence H_2 and $H - H \subseteq H_2$ are both CL^{1}C-sets. Again Corollary 18 provides $M_E(\mathbb{T}) = L^1_E(\mathbb{T})$ and therefore the respective boundedness properties are simply the interpretation of membership in the three set classes mentioned in the statement.

The most obvious direction for further research is to resolve the natural union problem. Refering to Proposition 12 the next formulation is relevant.

Problem. *Construct an* $\mathrm{L}^1\mathrm{C}$*-set which is not* $\mathrm{CL}^1\mathrm{C}$*.*

As discussed above this problem would be resolved already if the next construction is possible.

Problem. *Construct a* UC*-set which is not a Riesz set.*

References

[B] Bonami, A., Étude des coefficients de Fourier des fonctions de $L^p(G)$, *Ann. Inst. Fourier, Grenoble* **20** (1970), no. 2, 335–402.

[DP] Dressler, R., and L. Pigno, Sets of uniform convergence and strong Riesz sets, *Math. Annalen* **211**, 227–231.

[F] Fournier, J., Two UC-sets whose union is not a UC-set, *Proc. AMS* **84** (1982), no. 1, 69–72.

[FP] ______ and L. Pigno, Analytic and arithmetic properties of thin sets, *Pacific J. of Math.* **105** (1983), no. 1, 115–141.

[H] Hare, K., Arithmetic properties of thin sets, *Pacific J. of Math.* **131** (1988), no. 1, 142–155.

[H2] ______, An elementary proof of a result on $\Lambda(p)$ sets, *Proc. AMS* **104** (1988), no. 3, 829–834.

[He] Helson, H., Proof of a conjecture of Steinhaus, *Proc. Nat. Acad. Sci. US.* **40** (1954), 205–206.

[HP] Host, B. and F. Parreau, Sur les measures dont la transformeé de Fourier Stieltjes ne tend pas vers zero à l'infini, *Colloq. Math.* **41** (1979), 285–289.

[K] Katznelson, Y., *An introduction to harmonic analysis*, Dover Publ., New York, 1976.

[P] Pedemonte, L., Sets of uniform convergence, *Colloq. Math.* **33** (1975), no. 1, 123–132.

[R] Rudin, W., Trigonometric series with gaps, *J. Math. Mech.* **9** (1960), no. 2, 203–227.

[ST] Soardi, P., and G. Travaglini, On sets of completely uniform convergence, *Colloq. Math.* **45** (1981), no. 2, 317–320.

[T] Travaglini, G., Some properties of UC-sets, *Bolletino Unione Mat. Ital.* **15B** (1978), 272–284.

[Z] Zygmund, A., *Trigonometric series I*, Cambridge University Press, 1959.

Mats Erik Andersson, Matematiska institutionen, Stockholms universitet, SE-106 91 Stockholm

E-mail address: matsa@matematik.su.se