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Abstract

We study the existence of homoclinic orbits for first order time-dependent Hamiltonian systems $\dot{z} = \mathcal{J}H_z(z,t)$, where $H(z,t)$ depends periodically on t and $H_z(z,t)$ is asymptotically linear in z as $|z| \to \infty$. We also consider an asymptotically linear Schrödinger equation in $\mathbf{R}^N$.

1 Introduction

In this paper we shall be concerned with the existence of homoclinic orbits of the Hamiltonian system

$$(H) \qquad \dot{z} = \mathcal{J}H_z(z,t),$$

where $z = (p,q) \in \mathbf{R}^N \times \mathbf{R}^N = \mathbf{R}^{2N}$ and

$$\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

is the standard symplectic matrix. We assume that the Hamiltonian H is 1-periodic in t, $H \in C(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$, $H_z \in C(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R}^{2N})$ and H_z is asymptotically linear as $|z| \to \infty$. Recall that a solution z of (H) is said to be homoclinic (to 0) if $z \not\equiv 0$ and $z(t) \to 0$ as $|t| \to \infty$. In recent years several authors studied homoclinic orbits for Hamiltonian system via critical point theory. In particular, second order systems were considered in [1], [2], [4]-[6], [13]-[15], and those of first order in [3], [7]-[9], [17], [18], [20]. We emphasize that in all these papers the nonlinear term was assumed to be superlinear at infinity. To the best of our knowledge, the existence of homoclinics for first order Hamiltonian systems has not been previously studied by variational methods.

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Let $H(z,t) = \frac{1}{2}Az \cdot z + G(z,t)$ and assume without loss of generality that $H(0,t) \equiv 0$. Suppose $\sigma(\mathcal{J}A) \cap i\mathbf{R} = \emptyset$ (σ denotes the spectrum). Let μ_1 be the smallest positive, μ_{-1} the largest negative μ such that $\sigma(\mathcal{J}(A + \mu I)) \cap i\mathbf{R} \neq \emptyset$ and set $\mu_0 := \min\{\mu_1, -\mu_{-1}\}$. We introduce the following assumptions:

($\mathbf{H_1}$) A is a constant symmetric $2N \times 2N$-matrix such that $\sigma(\mathcal{J}A) \cap i\mathbf{R} = \emptyset$;

($\mathbf{H_2}$) G is 1-periodic in t, $G(z,t) \geq 0$ for all z, t and $G_z(z,t)/|z| \to 0$ uniformly in t as $z \to 0$;

($\mathbf{H_3}$) $G(z,t) = \frac{1}{2}A_\infty(t)z \cdot z + F(z,t)$, where $F_z(z,t)/|z| \to 0$ uniformly in t as $|z| \to \infty$ and $A_\infty(t)z \cdot z \geq \mu z \cdot z$ for some $\mu > \mu_1$;

($\mathbf{H_4}$) $\frac{1}{2}G_z(z,t) \cdot z - G(z,t) \geq 0$ for all z, t;

($\mathbf{H_5}$) There exists $\delta \in (0, \mu_0)$ such that if $|G_z(z,t)| \geq (\mu_0 - \delta)|z|$, then $\frac{1}{2}G_z(z,t) \cdot z - G(z,t) \geq \delta$.

In Section 3 we shall make some comments on these assumptions. The main result of this paper is the following

Theorem 1.1 *Assume $(H_1) - (H_5)$. Then system (H) has at least one homoclinic orbit.*

It follows from (H_2) and (H_3) that $|G_z(z,t)| \leq c|z|$ for some $c > 0$ and all z, t. Therefore

$$\Phi(z) := \frac{1}{2}\int_{\mathbf{R}} (-\mathcal{J}\dot{z} - Az) \cdot z dt - \int_{\mathbf{R}} G(z,t) dt \tag{1.1}$$

is continuously differentiable in the Sobolev space $H^{\frac{1}{2}}(\mathbf{R}, \mathbf{R}^{2N})$ and critical points $z \neq 0$ of Φ correspond to homoclinic solutions of (H) (see e.g. [19, Section 10]). It will be shown later that Φ has the so-called linking geometry; therefore it follows from [11] that there exists a Palais-Smale sequence (z_n) with $\Phi(z_n) \to c > 0$. However, it is not clear whether this sequence is bounded and therefore it cannot be used in order to construct a critical point $z \neq 0$. To circumvent this difficulty we adapt a method due to Jeanjean [10]. More precisely, we consider a family $(\Phi_\lambda)_{1 \leq \lambda \leq 2}$ of functionals such that $\Phi_1 = \Phi$ and show that for almost all $\lambda \in [1,2]$ there exists a bounded Palais-Smale sequence. This we do in Section 2, and in Section 3 we use the above result in order to prove Theorem 1.1. Finally, in Section 4 we consider an asymptotically linear Schrödinger equation.

2 Abstract Result

Let E^- be a closed separable subspace of a Hilbert space E and let $E^+ := (E^-)^\perp$. For $u \in E$ we shall write $u = u^+ + u^-$, where $u^\pm \in E^\pm$. On E we define the norm

$$\|u\|_\tau := \max\{\|u^+\|, \sum_{k=1}^{\infty} \frac{1}{2^k}|\langle u^-, e_k\rangle|\},$$

where (e_k) is a total orthonormal sequence in E^-. The topology generated by $\|\cdot\|_\tau$ will be called the τ-topology.

Recall from [11] that a homotopy $h = I - g : [0,1] \times A \to E$, where $A \subset E$, is called admissible if:
(i) h is τ-continuous, i.e. $h(s_n, u_n) \xrightarrow{\tau} h(s, u)$ whenever $s_n \to s$ and $u_n \xrightarrow{\tau} u$;
(ii) g is τ-locally finite-dimensional, i.e. for each $(s, u) \in [0,1] \times A$ there exists a neighborhood U of (s, u) in the product topology of $[0,1]$ and (E, τ) such that $g(U \cap ([0,1] \times A))$ is contained in a finite-dimensional subspace of E.
Admissible maps are defined similarly. Recall also that admissible maps and homotopies are necessarily continuous and on bounded subsets of E the τ-topology coincides with the product topology of E^-_{weak} and E^+_{strong}.

Let $\Phi \in C^1(E, \mathbf{R})$, $R > r > 0$ and $u_0 \in E^+$ with $\|u_0\| = 1$ be given and define

$$M := \{u = u^- + \rho u_0 : \|u\| \leq R,\ \rho \geq 0\}, \quad N := \{u \in E^+ : \|u\| = r\},$$

$$\Gamma := \{h \in C([0,1] \times M, E) : h \text{ is admissible, } h(0,u) = u \text{ and } s \mapsto \Phi(h(s,u)) \text{ is nonincreasing}\}.$$

The boundary of M in $\mathbf{R}u_0 \oplus E^-$ will be denoted by ∂M.

Theorem 2.1 *Let $E = E^+ \oplus E^-$ be a Hilbert space with E^- separable and orthogonal to E^+. Suppose that*
(i) $\psi \in C^1(E, \mathbf{R})$, $\psi \geq 0$, ψ is weakly sequentially lower semicontinuous and ψ' is weakly sequentially continuous;
(ii) $\Phi_\lambda(u) := \frac{1}{2}\|u^+\|^2 - \lambda(\frac{1}{2}\|u^-\|^2 + \psi(u)) = A(u) - \lambda B(u)$, $1 \leq \lambda \leq 2$;
(iii) there exist $R > r > 0$, $b > 0$ and $u_0 \in E^+$, $\|u_0\| = 1$, such that $\Phi_\lambda|_N \geq b > 0 \geq \sup_{\partial M} \Phi_\lambda$ for all $\lambda \in [1,2]$.
Then for almost every $\lambda \in [1,2]$ there exists a bounded sequence (u_n) such that $\Phi'_\lambda(u_n) \to 0$ and $\Phi_\lambda(u_n) \to c_\lambda \geq b$, where

$$c_\lambda := \inf_{h \in \Gamma} \sup_{u \in M} \Phi_\lambda(h(1,u)).$$

This theorem should be compared with Theorem 1.1 in [10], where a similar result was proved for functionals having the mountain pass geometry. Note also that it follows from Theorem 3.4 in [11] and Corollary 6.11 in [21] that for any λ a (not necessarily bounded) sequence (u_n) as above exists. Although no variational characterization of c_λ was given in [11, 21], it is easy to obtain such characterization by a slight modification of the arguments there.

The conclusion of Theorem 2.1 is a direct consequence of Lemma 2.3 below.

Since $B(u) \geq 0$, $\lambda \mapsto c_\lambda$ is nonincreasing. Therefore $c'_\lambda = \frac{dc_\lambda}{d\lambda}$ exists for almost every $\lambda \in [1,2]$. Let $\lambda \in (1,2]$ be an arbitrary (fixed) value where c'_λ exists and let $(\lambda_n) \subset [1,2]$ be a strictly increasing sequence such that $\lambda_n \to \lambda$.

Lemma 2.2 *There exists a sequence $h_n \in \Gamma$ and $k = k(c'_\lambda) > 0$ such that for almost all n:*
(i) If $\Phi_\lambda(h_n(1,u)) \geq c_\lambda - (\lambda - \lambda_n)$, then $\|h_n(1,u)\| \leq k$.
(ii) $\sup_{u \in M} \Phi_\lambda(h_n(1,u)) \leq c_\lambda + (2 - c'_\lambda)(\lambda - \lambda_n)$.

Proof Our argument is a straightforward modification of the one in Proposition 2.1 of [10]. We include it for the reader's convenience.

By the definition of c_{λ_n}, there exists $h_n \in \Gamma$ such that

$$\sup_{u \in M} \Phi_{\lambda_n}(h_n(1,u)) \leq c_{\lambda_n} + (\lambda - \lambda_n). \tag{2.1}$$

(i) If $\Phi_\lambda(h_n(1,u)) \geq c_\lambda - (\lambda - \lambda_n)$ for some $u \in M$, then

$$\frac{\Phi_{\lambda_n}(h_n(1,u)) - \Phi_\lambda(h_n(1,u))}{\lambda - \lambda_n} \leq \frac{c_{\lambda_n} - c_\lambda}{\lambda - \lambda_n} + 2.$$

Since c'_λ exists, there is $n(\lambda)$ such that if $n \geq n(\lambda)$, then

$$-c'_\lambda - 1 \leq \frac{c_{\lambda_n} - c_\lambda}{\lambda - \lambda_n} \leq -c'_\lambda + 1. \tag{2.2}$$

Therefore, for $n \geq n(\lambda)$,

$$B(h_n(1,u)) = \frac{\Phi_{\lambda_n}(h_n(1,u)) - \Phi_\lambda(h_n(1,u))}{\lambda - \lambda_n} \leq -c'_\lambda + 3$$

and

$$\begin{aligned} A(h_n(1,u)) &= \Phi_{\lambda_n}(h_n(1,u)) + \lambda_n B(h_n(1,u)) \\ &\leq c_{\lambda_n} + (\lambda - \lambda_n) + \lambda_n(-c'_\lambda + 3). \end{aligned}$$

Note that the right-hand side above is bounded independently of n. Since $\psi \geq 0$, then either $A(h_n(1,u))$ or $B(h_n(1,u))$ tends to infinity as $\|h_n(1,u)\| \to \infty$, and it follows that $\|h_n(1,u)\| \leq k$ for some k and all $u \in M$, $n \geq n(\lambda)$.

(ii) Since $\Phi_{\lambda_n}(v) \geq \Phi_\lambda(v)$ for any $v \in E$, it is easy to see from (2.1) and (2.2) that

$$\begin{aligned} \Phi_\lambda(h_n(1,u)) &\leq \Phi_{\lambda_n}(h_n(1,u)) \leq c_\lambda + (c_{\lambda_n} - c_\lambda) + (\lambda - \lambda_n) \\ &\leq c_\lambda + (2 - c'_\lambda)(\lambda - \lambda_n). \end{aligned}$$

□

Lemma 2.3 *(i)* $c_\lambda \geq b$ *for all* λ.
(ii) Let $k = k(c'_\lambda)$ *be the constant of Lemma 2.2. Then there exists a sequence* (u_n) *such that* $\|u_n\| \leq k + 4$ *for all* n, $\Phi'_\lambda(u_n) \to 0$ *and* $\Phi_\lambda(u_n) \to c_\lambda$.

Proof (i) The proof can be easily deduced from the argument on p. 456 in [11] or from the proof of Theorem 6.10 in [21]. Therefore we only sketch it briefly. Let $G : [0,1] \times M \to \mathbf{R}u_0 \oplus E^-$ be given by

$$G(s,u) := g(s,u)^- + (\|g(s,u)^+\| - r)u_0,$$

where $g \in \Gamma$ and $g(s,u) = g(s,u)^+ + g(s,u)^- \in E^+ \oplus E^-$. G is an admissible homotopy and $G(s,u) = 0$ if and only if $g(s,u) \in N$. Since $\Phi_\lambda|_{\partial M} \leq 0$ and $\Phi_\lambda(u) \geq \Phi_\lambda(g(s,u)) \geq b$ whenever

$g(s,u) \in N$, $0 \notin G([0,1] \times \partial M)$. Hence $\deg(G(s,\cdot), M, 0)$, where deg denotes the degree of [11], is well-defined and

$$\deg(G(1,\cdot), M, 0) = \deg(G(0,\cdot), M, 0) = 1.$$

Therefore $G(1,\bar{u}) = 0$ for some $\bar{u}$, so $g(1,\bar{u}) \in N$ and $\Phi(g(1,\bar{u})) \geq b$.

(ii) If the conclusion is not true, there exists $\varepsilon > 0$ such that $\|\Phi'_\lambda(u)\| \geq \varepsilon$ for all u with $\|u\| \leq k+4$ and $|\Phi_\lambda(u) - c_\lambda| \leq \varepsilon$. In order to obtain a contradiction we shall construct a certain deformation by modifying an argument which may be found on pp. 454-455 of [11] and in Lemmas 6.7, 6.8 of [21]. Choose g_n satisfying the conclusions of Lemma 2.2 with n so large that $(2 - c'_\lambda)(\lambda - \lambda_n) \leq \varepsilon$ and $\varepsilon_0 := \lambda - \lambda_n \leq \varepsilon$. For $u \in F := \{u \in E : \|u\| \leq k+4,\ c_\lambda - \varepsilon_0 \leq \Phi_\lambda(u) \leq c_\lambda + \varepsilon\}$ we set

$$w(u) := \frac{2\Phi'_\lambda(u)}{\|\Phi'_\lambda(u)\|^2}.$$

Since on bounded sets $v_n \xrightarrow{\tau} v$ if and only if $v_n^+ \to v^+$ and $v_n^- \rightharpoonup v^-$, it follows from the weak sequential continuity of Φ'_λ that the map $v \mapsto \langle \Phi'_\lambda(v), w(u)\rangle$ is τ-continuous on F (i.e., $\langle \Phi'_\lambda(v_n), w(u)\rangle \to \langle \Phi'_\lambda(v), w(u)\rangle$ as $v_n \xrightarrow{\tau} v$). Hence there exists a τ-open neighborhood U_u of u such that

$$\langle \Phi'_\lambda(v), w(u)\rangle > 1$$

for all $v \in U_u \cap F$. Let $U_0 := \Phi_\lambda^{-1}(-\infty, c_\lambda - \varepsilon_0)$. Since Φ_λ is τ-upper semicontinuous (by the weak lower semicontinuity of B, cf. [11, Remark 2.1(iv)]), U_0 is τ-open, the family $(U_u)_{u\in F} \cup U_0$ is a τ-open covering of $F \cup U_0$, and we can find a τ-locally finite τ-open refinement $(N_j)_{j\in J}$ with a corresponding τ-Lipschitz continuous partition of unity $(\lambda_j)_{j\in J}$. For each j we can either find $u \in F$ such that $N_j \subset U_u$, or if such u does not exist, then we have $N_j \subset U_0$. In the first case we set $w_j = w(u)$, in the second $w_j = 0$. Let $N := \bigcup_{j\in J} N_j$; then N is τ-open and $N \supset F \cup U_0$. Define

$$V(u) := \sum_{j\in J} \lambda_j(u) w_j$$

and consider the initial value problem

$$\frac{d\eta}{ds} = -V(\eta), \quad \eta(0,u) = u$$

for all u with $\|u\| \leq k$ and $c_\lambda - \varepsilon_0 \leq \Phi_\lambda(u) \leq c_\lambda + \varepsilon$. According to [11, 21], V is τ-locally and locally Lipschitz continuous. So in particular, for each u as above there exists a unique solution $\eta(\cdot,u)$. Since w_j is either 0 or $\|w_j\| = \|w(u)\| = 2/\|\Phi'_\lambda(u)\| \leq 2/\varepsilon$, V is bounded and $\eta(\cdot,u)$ exists as long as it does not approach the boundary of N. Furthermore, $\langle \Phi'_\lambda(u), V(u)\rangle \geq 0$ for all $u \in N$ and $\langle \Phi'_\lambda(u), V(u)\rangle > 1$ whenever $u \in F$. Therefore

$$\|\eta(s,u) - u\| = \| \int_0^s \frac{d\eta}{dt} dt \| \leq \int_0^s \|V(\eta(t,u))\| dt \leq \frac{2s}{\varepsilon},$$

$s \mapsto \Phi_\lambda(\eta(s,u))$ is nonincreasing and if $\Phi_\lambda(\eta(s,u)) \geq c_\lambda - \varepsilon_0$, then

$$\begin{aligned} \Phi_\lambda(\eta(s,u)) - \Phi_\lambda(u) &= \int_0^s \frac{d}{dt}\Phi_\lambda(\eta(t,u))dt \\ &= -\int_0^s \langle \Phi'_\lambda(\eta(t,u)), V(\eta(t,u))\rangle dt \ < \ -s. \end{aligned}$$

It follows that if $\|u\| \leq k$ and $c_\lambda - \varepsilon_0 \leq \Phi_\lambda(u) \leq c_\lambda + \varepsilon$, then $\|\eta(s,u)\| \leq k+4$ whenever $0 \leq s \leq 2\varepsilon$. So $\eta(s,u)$ exists for $0 \leq s \leq 2\varepsilon$ and $\Phi_\lambda(\eta(2\varepsilon,u)) < c_\lambda - \varepsilon_0$. According to Proposition 2.2 in [11] or Lemma 6.8 in [21], η is an admissible homotopy.

Now we complete the proof of (ii) by setting

$$g(s,u) := \begin{cases} g_n(2s,u), & 0 \leq s \leq \frac{1}{2} \\ \eta(4\varepsilon s - 2\varepsilon, g_n(1,u)), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Then $g \in \Gamma$ and $\Phi_\lambda(g(1,u)) \leq c_\lambda - \varepsilon_0$ for all $u \in M$, a contradiction to the definition of c_λ. □

3 Proof of Theorem 1.1

Throughout this section we assume that the hypotheses $(H_1)-(H_5)$ are satisfied even though some lemmas below remain valid under weaker conditions.

Let $\mathcal{L}z := -\mathcal{J}\dot{z} - Az$ and denote the inner product in $L^2(\mathbf{R},\mathbf{R}^{2N})$ by $(\cdot,\cdot)$. Since $\sigma(\mathcal{J}A)\cap i\mathbf{R} = \emptyset$, it follows from the results of Sections 8 and 10 of [19] that if $E := \mathcal{D}(|\mathcal{L}|^{\frac{1}{2}})$ ($\mathcal{D}$ denotes the domain), then E is a Hilbert space with inner product

$$\langle z,v\rangle_{\mathcal{D}} := (z,v) + (|\mathcal{L}|^{\frac{1}{2}}z, |\mathcal{L}|^{\frac{1}{2}}v)$$

and $E = H^{\frac{1}{2}}(\mathbf{R},\mathbf{R}^{2N})$. Moreover, to $\mathcal{L}$ there corresponds a bounded selfadjoint operator $L : E \to E$ such that

$$\langle Lz,v\rangle_{\mathcal{D}} = \int_{\mathbf{R}} (-\mathcal{J}\dot{z} - Az)\cdot v dt,$$

$E = E^+ \oplus E^-$, where $E^\pm$ are L-invariant and $\langle z^+, z^-\rangle_{\mathcal{D}} = (z^+,z^-) = 0$ whenever $z^\pm \in E^\pm$. Also, $\langle Lz,z\rangle_{\mathcal{D}}$ is positive definite on E^+ and negative definite on E^-. We introduce a new inner product in E by setting $\langle z,v\rangle := \langle Lz^+,v^+\rangle_{\mathcal{D}} - \langle Lz^-,v^-\rangle_{\mathcal{D}}$. Then $\langle Lz,z\rangle_{\mathcal{D}} = \|z^+\|^2 - \|z^-\|^2$, where $\|\cdot\|$ is the norm corresponding to $\langle\cdot,\cdot\rangle$. It is easy to see from [19, Corollary 10.2] and the definitions of μ_0, $\mu_{\pm 1}$ that

$$\|z^+\|^2 \geq \mu_1(z^+,z^+), \quad \|z^-\|^2 \geq -\mu_{-1}(z^-,z^-) \quad \text{and} \quad \|z\|^2 \geq \mu_0(z,z). \tag{3.1}$$

Let

$$\psi(z) := \int_{\mathbf{R}} G(z,t)dt.$$

Clearly, $\psi \geq 0$ and it follows from Fatou's lemma that ψ is weakly sequentially lower semicontinuous. Since $|G_z(z,t)| \leq c|z|$ and $z_n \rightharpoonup z$ implies $z_n \to z$ in $L^2_{loc}(\mathbf{R},\mathbf{R}^{2N})$, it is easily seen that ψ' is weakly sequentially continuous. So (i) of Theorem 2.1 is satisfied. Set

$$\Phi_\lambda(z) := \frac{1}{2}\|z^+\|^2 - \lambda(\frac{1}{2}\|z^-\|^2 + \psi(z)), \quad 1 \leq \lambda \leq 2.$$

Then $\Phi_1 \equiv \Phi$ (cf. (1.1)).

Remark 3.1 *(i)* Let $\{E_\mu : \mu \in \mathbf{R}\}$ be the resolution of identity corresponding to $\mathcal{L}$. Then E_0 is the orthogonal projector of E onto E^- and $E_\mu(E) \supset E^-$ whenever $\mu \geq 0$. If μ is as in (H_3), then

$\mu > \mu_1$ and since μ_1 is in the spectrum of $\mathcal{L}$ [19, Corollary 10.2], it follows that $E_\mu(E) \neq E^-$ and there exists $z_0 \in E^+$, $\|z_0\| = 1$, such that

$$\int_{\mathbf{R}} (-\mathcal{J}\dot{z}_0 - Az_0 - \mu z_0) \cdot z_0 dt = 1 - \mu(z_0, z_0) < 0. \tag{3.2}$$

(ii) Hypothesis (H_2) implies that $H_z(z,t) = Az + o(|z|)$ as $z \to 0$, where A is independent of t. In general, $A = A(t)$; however, as was observed in [3], in many cases one can get rid of t-dependence of A by a suitable 1-periodic symplectic change of variables. If this is not possible, then the assumption $\sigma(\mathcal{J}A) \cap i\mathbf{R} = \emptyset$ in (H_1) should be replaced by the one that 0 is in a gap (μ_{-1}, μ_1) of the spectrum of $\mathcal{L} = -\mathcal{J}\frac{d}{dt} - A(t)$, and in (H_3) the constant μ should be larger than μ_1. Also (H_5) should be changed accordingly. Note that by a result in [8] the spectrum of $\mathcal{L}$ is completely continuous and is the union of disjoint closed intervals.

(iii) Assuming (H_1)–(H_4), a sufficient condition for (H_5) to be satisfied is that $s \mapsto s^{-1}G_z(sz,t) \cdot z$ is nondecreasing for all $s > 0$. Indeed, suppose $|z| = 1$. Then

$$\frac{1}{2}G_z(sz,t) \cdot sz - G(sz,t) = \int_0^s \left(\frac{G_z(sz,t) \cdot z}{s} - \frac{G_z(\sigma z,t) \cdot z}{\sigma}\right) \sigma d\sigma$$

and the integrand is nonnegative. Since $s^{-1}G_z(sz,t) \cdot z \to 0$ uniformly in t as $s \to 0$, we either have $G_z(\sigma z,t) \cdot z = G(\sigma z,t) = 0$ and hence $G_z(\sigma z,t) = 0$ for $0 \le \sigma \le s$ (recall $G \ge 0$), or $G_z(sz,t) \neq 0$ and the left-hand side above is positive. Moreover, since $s^{-1}G_z(sz,t) \cdot z \ge \frac{1}{2}\mu_1$ for all $s \ge s_0$ (s_0 independent of z and t), the integrand is positive and bounded away from 0 for small σ and large s. Hence the conclusion. Let us also note that if G is twice differentiable with respect to z, then $s \mapsto s^{-1}G_z(sz,t) \cdot z$ is nondecreasing if and only if $G_{zz}(z,t)z \cdot z \ge G_z(z,t) \cdot z$ for all z, t.

Choose now $z_0 \in E^+$ as in Remark 3.1(i) and let

$$N := \{z \in E^+ : \|z\| = r\} \quad \text{and} \quad M := \{z = z^- + \rho z_0 : \|z\| \le R,\ \rho \ge 0\},$$

$R > r > 0$ to be determined.

Lemma 3.2 *There exist $r > 0$ and $b > 0$ (independent of λ) such that $\Phi_\lambda|_N \ge b$.*

Proof Choose $p > 2$. By (H_2) and (H_3), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$G(z,t) \le \varepsilon|z|^2 + C_\varepsilon|z|^p.$$

Hence

$$\psi(z) = \int_{\mathbf{R}} G(z,t)dt \le \varepsilon\|z\|_2^2 + C_\varepsilon\|z\|_p^p \le c(\varepsilon\|z\|^2 + C_\varepsilon\|z\|^p),$$

where c is independent of ε ($\|\cdot\|_s$ denotes the usual norm in $L^s(\mathbf{R}, \mathbf{R}^{2N})$). Since ε was chosen arbitrarily, it follows that $\psi(z) = o(\|z\|^2)$ as $z \to 0$ and there are $r > 0$, $b > 0$ (independent of λ) such that $\Phi_\lambda(z) = \frac{1}{2}\|z\|^2 - \lambda\psi(z) \ge b > 0$ for $z \in N$. □

Lemma 3.3 *There exists $R > r$ (R independent of λ) such that $\Phi_\lambda|_{\partial M} \le 0$.*

Proof Since $G(z,t)\ge 0$ according to (H_2), we have

$$\Phi(z^-) = -\frac{1}{2}\|z^-\|^2 - \int_{\mathbf{R}} G(z^-,t)dt \le 0.$$

Noting that $\Phi_\lambda(z)\le\Phi(z)$ for any $z\in E$, it suffices to prove that $\Phi|_{\partial M}\le 0$ whenever R is large enough. If this is not true, then there exist $z_n=\rho_n z_0+z_n^-$, $\|z_n\|\to\infty$, such that

$$\frac{\Phi(z_n)}{\|z_n\|^2} = \frac{1}{2}\delta_n^2 - \frac{1}{2}\|v_n^-\|^2 - \int_{\mathbf{R}} \frac{G(z_n,t)}{\|z_n\|^2}dt \ge 0, \tag{3.3}$$

where $\delta_n=\frac{\rho_n}{\|z_n\|}$ and $v_n^-=\frac{z_n^-}{\|z_n\|}$. Therefore $\delta_n\ge\|v_n^-\|$. Since $\delta_n^2+\|v_n^-\|^2=1$, $\delta_n\to\delta>0$ and $v_n^-\rightharpoonup v^-$ weakly in E after passing to a subsequence. Set $v=\delta z_0+v^-$. Since $(z_0,v^-)=0$, it follows from (H_3) and (3.2) that

$$\delta^2-\|v^-\|^2-\int_{\mathbf{R}} A_\infty(t)v\cdot v dt \le \delta^2-\|v^-\|^2-\mu\delta^2(z_0,z_0)-\mu(v^-,v^-)<0.$$

Therefore there exists a bounded interval I such that

$$\delta^2-\|v^-\|^2-\int_I A_\infty(t)v\cdot v dt<0.$$

On the other hand, by (3.3),

$$\begin{aligned} 0 &\le \frac{1}{2}\delta_n^2-\frac{1}{2}\|v_n^-\|^2-\int_I \frac{G(z_n,t)}{\|z_n\|^2}dt \\ &= \frac{1}{2}\delta_n^2-\frac{1}{2}\|v_n^-\|^2-\frac{1}{2}\int_I A_\infty(t)v_n\cdot v_n dt-\int_I \frac{F(z_n,t)}{\|z_n\|^2}dt, \end{aligned}$$

where $v_n=\delta_n z_0+v_n^-$. Since $v_n\rightharpoonup\delta z_0+v^-=v$ in E, $v_n\to v$ in $L^2(I,\mathbf{R}^{2N})$. By (H_2) and (H_3) it is easy to check that $|F(z,t)|\le c|z|^2$ for all $z\in\mathbf{R}^{2N}$. Since $F(z,t)/|z|^2\to 0$ as $|z|\to\infty$, it follows from Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty}\int_I \frac{F(z_n,t)}{\|z_n\|^2}dt = \lim_{n\to\infty}\int_I \frac{F(z_n,t)}{|z_n|^2}|v_n|^2 dt = 0,$$

and therefore

$$\delta^2-\|v^-\|^2-\int_I A_\infty(t)v\cdot v dt \ge 0,$$

a contradiction. Consequently, there exists $R>0$ such that $\Phi_\lambda(z)\le\Phi(z)\le 0$ for $z\in\partial M$. □

Combining Lemmas 3.2, 3.3 and Theorem 2.1 we obtain

Corollary 3.4 *For almost every $\lambda\in[1,2]$ there exists a bounded sequence $(z_n)\subset E$ such that $\Phi_\lambda'(z_n)\to 0$ and $\Phi_\lambda(z_n)\to c_\lambda$.*

Remark 3.5 Let $(z_n)\subset E$ be a bounded sequence. Then, up to a subsequence, either

(i) $\lim_{n\to\infty}\sup_{y\in\mathbf{R}}\int_{y-R}^{y+R}|z_n|^2dt=0$ for all $0<R<\infty$, or

(ii) there exist $\alpha>0, R>0$ and $y_n\in\mathbf{R}$ such that $\lim_{n\to\infty}\int_{y_n-R}^{y_n+R}|z_n|^2dt\ge\alpha>0$.

In the first case we shall say that (z_n) is vanishing, in the second that it is nonvanishing.

Lemma 3.6 *For any bounded vanishing sequence $(z_n) \subset E$, we have*

$$\lim_{n\to\infty} \int_{\mathbf{R}} G(z_n, t)dt = \lim_{n\to\infty} \int_{\mathbf{R}} G_z(z_n, t) \cdot z_n^{\pm} dt = 0.$$

Proof Since (z_n) is vanishing, by the concentration-compactness lemma of P.L. Lions [12, 21], $z_n \to 0$ in L^s for all $2 < s < \infty$ (usually this lemma is stated for $z \in H^1$; however, a simple modification of the argument of Lemma 1.21 in [21] shows that the conclusion remains valid in $H^{\frac{1}{2}}$). On the other hand, by assumptions (H_2) and (H_3), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|G_z(z,t)| \le \varepsilon|z| + C_\varepsilon |z|^{p-1}, \tag{3.4}$$

where $p > 2$. Hence

$$\int_{\mathbf{R}} G(z_n, t)dt \le c(\varepsilon\|z_n\|^2 + C_\varepsilon \|z_n\|_p^p),$$
$$\int_{\mathbf{R}} |G_z(z_n, t)|\, |z_n^{\pm}| dt \le c(\varepsilon\|z_n\|\, \|z_n^{\pm}\| + C_\varepsilon \|z_n\|_p^{p-1} \|z_n^{\pm}\|_p)$$

(c independent of ε), and the conclusion follows. □

Lemma 3.7 *Let $\lambda \in [1, 2]$ be fixed. If a bounded sequence $(v_n) \subset E$ satisfies*

$$0 < \lim_{n\to\infty} \Phi_\lambda(v_n) \le c_\lambda \quad \textit{and} \quad \lim_{n\to\infty} \Phi'_\lambda(v_n) = 0,$$

then there exist $y_n \in \mathbf{Z}$ such that, up to a subsequence, $u_n(t) := v_n(t + y_n)$ satisfies

$$u_n \rightharpoonup u_\lambda \neq 0, \quad \Phi_\lambda(u_\lambda) \le c_\lambda \quad \textit{and} \quad \Phi'_\lambda(u_\lambda) = 0.$$

Proof Since $\langle \Phi'_\lambda(v_n), v_n\rangle = 0$,

$$\lim_{n\to\infty} \lambda \int_{\mathbf{R}} (\frac{1}{2} G_z(v_n, t) \cdot v_n - G(v_n, t))dt = \lim_{n\to\infty} \Phi_\lambda(v_n) > 0,$$

and it follows from Lemma 3.6 that (v_n) is nonvanishing, that is, there exist $\alpha > 0$, $R > 0$ and $\bar{y}_n \in \mathbf{R}$ such that

$$\lim_{n\to\infty} \int_{\bar{y}_n - R}^{\bar{y}_n + R} |v_n|^2 dt \ge \alpha > 0.$$

Hence we may find $y_n \in \mathbf{Z}$ such that, setting $u_n(t) := v_n(t + y_n)$,

$$\lim_{n\to\infty} \int_{-2R}^{2R} |u_n|^2 dt \ge \alpha > 0. \tag{3.5}$$

Since $G(z, t)$ is 1-periodic in t, (u_n) is still bounded,

$$0 < \lim_{n\to\infty} \Phi_\lambda(u_n) \le c_\lambda \quad \text{and} \quad \lim_{n\to\infty} \Phi'_\lambda(u_n) = 0. \tag{3.6}$$

Therefore, up to a subsequence, $u_n \rightharpoonup u_\lambda$ and $u_n \to u_\lambda$ a.e. in $\mathbf{R}$ for some $u_\lambda \in E$. Since $u_n \to u_\lambda$ in $L^2_{loc}(\mathbf{R}, \mathbf{R}^{2N})$, it follows from (3.5) that $u_\lambda \neq 0$. Recall ψ' is weakly sequentially continuous. Therefore $\Phi'_\lambda(u_n) \rightharpoonup \Phi'_\lambda(u_\lambda)$ and by (3.6), $\Phi'_\lambda(u_\lambda) = 0$.

Finally, by (H_4) and Fatou's lemma,

$$\begin{aligned} c_\lambda &\geq \lim_{n\to\infty}(\Phi_\lambda(u_n) - \frac{1}{2}\langle\Phi'_\lambda(u_n), u_n\rangle) \\ &= \lim_{n\to\infty}\lambda\int_{\mathbf{R}}(\frac{1}{2}G_z(u_n,t)\cdot u_n - G(u_n,t))dt \\ &\geq \lambda\int_{\mathbf{R}}(\frac{1}{2}G_z(u_\lambda,t)\cdot u_\lambda - G(u_\lambda,t))dt \ = \ \Phi_\lambda(u_\lambda). \end{aligned}$$

□

Corollary 3.8 *If the sequence (v_n) in Lemma 3.7 is nonvanishing, then the hypothesis $\lim_{n\to\infty}\Phi_\lambda(v_n)$ > 0 may be omitted.*

Lemma 3.9 *There exists a sequence $(\lambda_n) \subset [1,2]$ and $(z_n) \subset E\setminus\{0\}$ such that*

$$\lambda_n \to 1, \quad \Phi_{\lambda_n}(z_n) \leq c_{\lambda_n} \quad \text{and} \quad \Phi'_{\lambda_n}(z_n) = 0.$$

Proof This is a straightforward consequence of Corollary 3.4 and Lemma 3.7. □

Lemma 3.10 *The sequence (z_n) obtained in Lemma 3.9 is bounded.*

Proof We modify an argument of [10]. Assume $\|z_n\| \to \infty$ and set $w_n = z_n/\|z_n\|$. Then we can assume that, up to a subsequence, $w_n \rightharpoonup w$. We shall show that (w_n) is neither vanishing nor nonvanishing thereby obtaining a contradiction.

Step 1. Nonvanishing of (w_n) is impossible.
If (w_n) is nonvanishing, we proceed as in the proof of Lemma 3.7 and find $\alpha > 0$, $R > 0$ and $y_n \in \mathbf{Z}$ such that if $\tilde{w}_n(t) := w_n(t+y_n)$, then

$$\int_{-2R}^{2R}|\tilde{w}_n(t)|^2dt \geq \alpha \quad \text{for almost all } n.$$

Moreover, since $\Phi'_{\lambda_n}(z_n) = \Phi'_{\lambda_n}(\tilde{z}_n) = 0$, where $\tilde{z}_n(t) = z_n(t+y_n)$, for any $\phi \in C_0^\infty(\mathbf{R},\mathbf{R}^{2N})$ we have

$$\langle\tilde{w}_n^+,\phi\rangle - \lambda_n\langle\tilde{w}_n^-,\phi\rangle - \lambda_n\int_{\mathbf{R}}A_\infty(t)\tilde{w}_n\cdot\phi\,dt - \lambda_n\int_{\mathbf{R}}\frac{F_z(\tilde{z}_n,t)\cdot\phi}{|\tilde{z}_n|}|w_n|dt = 0. \tag{3.7}$$

Since $\|\tilde{w}_n\| = \|w_n\| = 1$, up to a subsequence, $\tilde{w}_n \rightharpoonup \tilde{w}$ in E, $\tilde{w}_n \to \tilde{w}$ in $L^2_{loc}(\mathbf{R},\mathbf{R}^{2N})$ and $\tilde{w}_n(t) \to \tilde{w}(t)$ a. e. in $\mathbf{R}$. In particular, $\tilde{w} \neq 0$. Since $|F_z(z,t)| \leq c|z|$ for all z,t, it follows from (H_3) and Lebesgue's dominated convergence theorem that passing to the limit in (3.7) gives

$$\langle\tilde{w}^+,\phi\rangle - \langle\tilde{w}^-,\phi\rangle - \int_{\mathbf{R}}A_\infty(t)\tilde{w}\cdot\phi\,dt = 0,$$

that is, equation $\dot{z} = \mathcal{J}(A + A_\infty(t))z$ has a nontrivial solution in E, which contradicts the already mentioned fact that the spectrum of the operator $-(\mathcal{J}\frac{d}{dt} + (A + A_\infty(t))$ is continuous (cf. [8]). Therefore nonvanishing of (w_n) is impossible.

Step 2. Also vanishing of (w_n) is impossible.
By contradiction, suppose that (w_n) is vanishing. Since $\Phi'_{\lambda_n}(z_n) = 0$, we have

$$\langle \Phi'_{\lambda_n}(z_n), z_n^+ \rangle = \|z_n^+\|^2 - \lambda_n \int_{\mathbf{R}} G_z(z_n,t) \cdot z_n^+ dt = 0,$$
$$\langle \Phi'_{\lambda_n}(z_n), z_n^- \rangle = -\lambda_n \|z_n^-\|^2 - \lambda_n \int_{\mathbf{R}} G_z(z_n,t) \cdot z_n^- dt = 0.$$

Since $\|w_n\|^2 = \|w_n^+\|^2 + \|w_n^-\|^2 = 1$,

$$\int_{\mathbf{R}} \frac{G_z(z_n,t) \cdot (\lambda_n w_n^+ - w_n^-)}{\|z_n\|} dt = 1.$$

Setting

$$\Omega_n := \{t \in \mathbf{R} : \frac{|G_z(z_n,t)|}{|z_n|} \le \mu_0 - \delta\},$$

we obtain using Hölder's inequality, the relation $(w^+, w^-) = 0$ and (3.1) that

$$\begin{aligned} \int_{\Omega_n} \frac{G_z(z_n,t) \cdot (\lambda_n w_n^+ - w_n^-)}{\|z_n\|} dt &\le (\mu_0 - \delta) \int_{\Omega_n} |w_n|\, |\lambda_n w_n^+ - w_n^-| dt \\ &\le (\mu_0 - \delta)\lambda_n \|w_n\|_2^2 \le \frac{(\mu_0 - \delta)\lambda_n}{\mu_0} < 1 \end{aligned}$$

for almost all n. Hence

$$\lim_{n\to\infty} \int_{\mathbf{R}\setminus\Omega_n} \frac{G_z(z_n,t) \cdot (\lambda_n w_n^+ - w_n^-)}{\|z_n\|} dt > 0, \tag{3.8}$$

and since $|G_z(z,t)| \le c|z|$, it follows that

$$\int_{\mathbf{R}\setminus\Omega_n} \frac{G_z(z_n,t) \cdot (\lambda_n w_n^+ - w_n^-)}{\|z_n\|} dt \le \tilde{c} \int_{\mathbf{R}\setminus\Omega_n} |w_n|^2 dt \le \tilde{c} \text{ meas}(\mathbf{R} \setminus \Omega_n)^{(p-2)/p} \|w_n\|_p^{2/p}$$

for some $\tilde{c} > 0$. Since (w_n) is vanishing, $w_n \to 0$ in $L^p(\mathbf{R}, \mathbf{R}^{2N})$ and we obtain from (3.8) that $\text{meas}(\mathbf{R} \setminus \Omega_n) \to \infty$ as $n \to \infty$. So by (H_4) and (H_5),

$$\int_{\mathbf{R}} (\frac{1}{2} G_z(z_n,t) \cdot z_n - G(z_n,t)) dt \ge \int_{\mathbf{R}\setminus\Omega_n} (\frac{1}{2} G_z(z_n,t) \cdot z_n - G(z_n,t)) dt \ge \int_{\mathbf{R}\setminus\Omega_n} \delta dt \to \infty.$$

However, recalling that $\Phi_{\lambda_n}(z_n) \le c_{\lambda_n}$ and $\langle \Phi'_{\lambda_n}(z_n), z_n \rangle = 0$, we obtain

$$\int_{\mathbf{R}} (\frac{1}{2} G_z(z_n,t) \cdot z_n - G(z_n,t)) dt \le \frac{c_{\lambda_n}}{\lambda_n},$$

a contradiction because the right-hand side above is bounded. □

Proof of Theorem 1.1 We have shown that there exist $\lambda_n \to 1$ and a bounded sequence (z_n) such that $\Phi_{\lambda_n}(z_n) \le c_{\lambda_n}$ and $\Phi'_{\lambda_n}(z_n) = 0$. Therefore

$$\Phi'(z_n) = \Phi'_{\lambda_n}(z_n) + (\lambda_n - 1)(z_n^- + \psi'(z_n)) = (\lambda_n - 1)(z_n^- + \psi'(z_n)) \to 0.$$

Since $\langle \Phi'_{\lambda_n}(z_n), z_n^\pm \rangle = 0$, we obtain using (3.4) that

$$\|z_n^+\|^2 = \lambda_n \int_{\mathbf{R}} G_z(z_n,t) \cdot z_n^+ dt \le \frac{1}{4}\|z_n\|^2 + C\|z_n\|^p,$$
$$\|z_n^-\|^2 = -\int_{\mathbf{R}} G_z(z_n,t) \cdot z_n^- dt \le \frac{1}{4}\|z_n\|^2 + C\|z_n\|^p,$$

where $p > 2$. Hence $\|z_n\|^2 \le \frac{1}{2}\|z_n\|^2 + 2C\|z_n\|^p$ and $\|z_n\| \ge c$ for some $c > 0$. If (z_n) is vanishing, it follows from Lemma 3.6 that the middle terms above tend to 0; therefore $z_n \to 0$. Hence (z_n) is nonvanishing. According to Corollary 3.8 there exist $y_n \in \mathbf{Z}$ such that if $\tilde{z}_n(t) := z_n(t + y_n)$, then $\tilde{z}_n \rightharpoonup \tilde{z} \neq 0$ and $\Phi'(\tilde{z}) = 0$. This completes the proof. □

4 Asymptotically linear Schrödinger equation

In this section we consider the Schrödinger equation

$$(S) \qquad -\Delta u + V(x)u = f(x,u),$$

where $x \in \mathbf{R}^N$, $V \in C(\mathbf{R}^N, \mathbf{R})$ and $f \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$. Suppose that 0 is not in the spectrum of $-\Delta + V$ in $L^2(\mathbf{R}^N)$ (denoted $0 \notin \sigma(-\Delta + V)$). Let μ_1 be the smallest positive and μ_{-1} the largest negative μ such that $0 \in \sigma(-\Delta + V - \mu)$ and as in Section 1, set $\mu_0 := \min\{\mu_1, -\mu_{-1}\}$. Furthermore, denote $F(x,u) = \int_0^u f(x,s)ds$. It is well-known that if V is periodic in each of the x-variables, then the spectrum of $-\Delta + V$ (in L^2) is bounded below but not above and consists of disjoint closed intervals [16, Theorem XIII.100]. Similarly as in Section 1, we introduce the following hypotheses:

($\mathbf{S_1}$) V is 1-periodic in x_j for $j = 1, \dots, N$, and $0 \notin \sigma(-\Delta + V)$;

($\mathbf{S_2}$) f is 1-periodic in x_j for $j = 1, \dots, N$, $F(x,u) \ge 0$ for all x, u and $f(x,u)/u \to 0$ uniformly in x as $u \to 0$;

($\mathbf{S_3}$) $f(x,u) = V_\infty(x)u + g(x,u)$, where $g(x,u)/u \to 0$ uniformly in x as $|u| \to \infty$ and $V_\infty(x) \ge \mu$ for some $\mu > \mu_1$;

($\mathbf{S_4}$) $\frac{1}{2}uf(x,u) - F(x,u) \ge 0$ for all x, u;

($\mathbf{S_5}$) There exists $\delta \in (0, \mu_0)$ such that if $f(x,u)/u \ge \mu_0 - \delta$, then $\frac{1}{2}uf(x,u) - F(x,u) \ge \delta$.

Theorem 4.1 *If the hypotheses $(S_1) - (S_5)$ are satisfied, then (S) has a solution $u \neq 0$ such that $u(x) \to 0$ as $|x| \to \infty$.*

It is well-known (see e.g. [11, 19, 21]) that the functional

$$\Phi(u) := \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \int_{\mathbf{R}^N} F(x,u)dx$$

is of class C^1 in the Sobolev space $E := H^1(\mathbf{R}^N)$ and critical points of Φ correspond to solutions u of (S) such that $u(x) \to 0$ as $|x| \to \infty$. If $\sigma(-\Delta + V) \cap (-\infty, 0) \neq 0$, then $E = E^+ \oplus E^-$, where $E^\pm$ are infinite-dimensional, and the proof of Theorem 4.1 follows by repeating the arguments of Section 3. Note only that in Lemma 3.7 we now have $y_n \in \mathbf{Z}^N$. If $\sigma(-\Delta + V) \subset (0, \infty)$, then $E^- = \{0\}$, $\mu_{-1} = -\infty$ and Φ has the mountain pass geometry. Theorem 4.1 remains valid in this case, and it is in fact already contained in Theorem 1.2 of [10].

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