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Infinite Dimensional Cohomology Groups and Periodic Solutions of Asymptotically Linear Hamiltonian Systems

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Abstract: In this paper we study the existence of nontrivial 2π-periodic solutions of asymptotically linear Hamiltonian systems. We consider the case of resonance both at zero and at infinity, and we permit time-dependent asymptotic matrices. Our main tools are an infinite dimensional cohomology theory and a corresponding Morse theory recently constructed by W. Kryszewski and the first author. We develop a method to compute the new critical groups.

Key words and phrases: Hamiltonian, filtration, $\mathcal{E}$-cohomology, critical groups, $\mathcal{E}$-Morse index, Morse inequalities.

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1. INTRODUCTION

We consider the existence of nontrivial 2π-periodic solutions of asymptotically linear Hamiltonian systems

$$\dot{z} = JH'(z,t), \qquad z \in \mathbf{R}^{2N}, \tag{S}$$

where

$$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

is the standard symplectic matrix, $H \in C^2(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ is 2π-periodic in t, H' denotes the gradient of H with respect to the first $2N$ variables and there exist $s > 0, c > 0$ such that

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(H$_0$) $|H_{zz}(z,t)| \le c(1+|z|^s)$ for all $(z,t) \in \mathbf{R}^{2N} \times \mathbf{R}$.

In what follows we assume that there exist two symmetric $2N \times 2N$-matrices $A(t)$ and $A_0(t)$ with continuous and 2π-periodic entries such that

$$H(z,t) = \frac{1}{2}A(t)z \cdot z + G(z,t), \tag{1.1}$$

where $G'(z,t) = o(|z|)$ uniformly in t as $|z| \to \infty$ and

$$H(z,t) = \frac{1}{2}A_0(t)z \cdot z + G_0(z,t), \tag{1.2}$$

with $G_0'(z,t) = o(|z|)$ uniformly in t as $|z| \to 0$. We denote by $\cdot$ and $|*|$ the usual inner product and norm in $\mathbf{R}^{2N}$. The Hamiltonian system (S) satisfying (1.1) and (1.2) is called asymptotically linear both at infinity and at zero. Moreover, it is called nonresonant at infinity if 1 is not a Floquet multiplier of the linear system $\dot{z} = JA(t)z$; nonresonance at 0 is defined in a similar way by replacing $A(t)$ with $A_0(t)$.

Before introducing our assumptions on $H(z,t)$ and stating the main results, let us recall some earlier work on asymptotically linear Hamiltonian systems. The case of (S) nonresonant at infinity was considered in [2, 3] under the additional assumptions that H_{zz} is bounded and A, A_0 are time-independent; in [4] H_{zz} was bounded and (S) was also nonresonant at zero. In [5] A, A_0 were time-independent and in [6] H_{zz} was bounded. For (S) resonant at infinity it was assumed in [7] that $A(t)$ is a constant matrix, [8, 9] considered the strongly resonant case and [14] studied (S) under the assumption that $A(t), A_0(t)$ are so-called finitely degenerate, which is a strong condition. Moreover, no results on the existence of multiple solutions were obtained in [7-9, 14]. Recently Kryszewski and the first author [1] constructed an infinite dimensional cohomology theory and a Morse theory corresponding to it. These theories were applied to the study of Hamiltonian systems and wave equations. In particular, the case of (S) resonant at infinity was studied in [1] under the hypotheses that $G'(z,t)$ is bounded and $G(z,t) \to \infty$ (or $-\infty$) uniformly in t as $|z| \to \infty$. This was done by computing the new critical groups (the $\mathcal{E}$-cohomology groups) at zero and at infinity. However, in the case of resonance at 0, [1] contained no detailed computation of critical groups there; it was only shown that the groups at zero and at infinity were different under certain assumptions.

The purpose of the present paper is to develop a method to compute the $\mathcal{E}$-cohomology groups both at infinity and at zero when resonance occurs at infinity and at zero simultaneously. We admit H such that $G'(z,t)$ and $G_0'(z,t)$ are unbounded and $G(z,t), G_0(z,t)$ may change sign. Under rather weak conditions we obtain at least two nontrivial solutions for (S).

In order to state our assumptions, we introduce a control function $h_\infty : \mathbf{R}^+ \to \mathbf{R}^+$ such that $h_\infty(t)$ is increasing in t and

$$1 \le \frac{t h_\infty(t)}{H_\infty(t)} \le \alpha < 2, \quad h_\infty(s+t) \le m(h_\infty(s) + h_\infty(t)) \quad \text{for any } s, t \in \mathbf{R}^+,$$

where $H_\infty(t) = \int_0^t h_\infty(s)ds$ and α, m are constants. Evidently, $h_\infty(t) = t^\sigma$ with $0 < \sigma < 1$ is a simple example. Now we assume

$(\mathbf{H}_1)$ $|G'(z,t)| \le c(1+h_\infty(|z|))$ for all $z \in \mathbf{R}^{2N}$ and $t \in \mathbf{R}$;

$(\mathbf{H}_2^\pm)$ $\liminf_{|z|\to\infty} \dfrac{\pm G(z,\,t)}{H_\infty(|z|)} := a^\pm(t) \succeq 0$ uniformly for $t \in \mathbf{R}$.

Here and in the sequel the letter c will be repeatedly used to denote various positive constants whose exact value is irrelevant. For a function a we write $a(t) \succeq 0$ if $a(t) \ge 0$ and strict inequality holds on a set of positive measure.

Since different behavior of H at zero and infinity plays an important role in the existence of nontrivial 2π-periodic solutions of (S), we need some hypotheses on G_0 around zero.

Let $h_0 : \mathbf{R}^+ \to \mathbf{R}^+$ be a control function (for G_0) such that

$$2 < \beta \le \frac{t h_0(t)}{H_0(t)} \le \gamma \quad \text{for } t \text{ small}, \tag{1.3}$$

where $H_0(t) = \int_0^t h_0(s)ds$, and β, γ are constants. Obviously, $h_0(t) = t^\delta$ with $\delta > 1$ satisfies (1.3). Moreover, although h_0 is defined only for small $t > 0$, we may assume without loss of generality that it has been extended so that (1.3) holds for all $t \in \mathbf{R}^+$. We suppose that

$(\mathbf{H}_3)$ $|G_0'(z,t)| \le c h_0(|z|)$ for $|z|$ small;

$(\mathbf{H}_4^\pm)$ $\liminf_{|z|\to 0} \dfrac{\pm G_0'(z,\,t)\cdot z}{H_0(|z|)} := b^\pm(t) \succeq 0$ uniformly for $t \in \mathbf{R}$.

Remark 1.1. It is easy to see that $c_1 t \le H_\infty(t) \le c_2 t^\alpha$ for large t and $H_0(t) \le ct^\beta$ for small $t > 0$. Moreover, if $z = z_1 + z_2 \in L^\alpha([0,2\pi], \mathbf{R}^{2N})$ and $w \in L^\alpha([0,2\pi], \mathbf{R}^{2N})$, then

$$\begin{aligned}
&|\int_0^{2\pi} G'(z,t)\cdot w dt| \\
&\le c\int_0^{2\pi} (1 + h_\infty(|z_1|) + h_\infty(|z_2|))|w|dt \\
&\le c\int_0^{2\pi} (1 + |z_1|^{\alpha-1} + h_\infty(|z_2|))|w|dt \\
&\le c(1 + \|z_1\|_\alpha^{\alpha-1})\|w\|_\alpha + c\int_0^{2\pi} h_\infty(|z_2|)|w|dt,
\end{aligned}$$

and similarly, $|\int_0^{2\pi} G_0'(z,t)\cdot w dt| \le c\|z\|_\beta^{\beta-1}\|w\|_\beta$ ($\|\cdot\|_\alpha$ denotes the usual norm in $L^\alpha([0,2\pi], \mathbf{R}^{2N})$).

Remark 1.2. (H_1) and (H_3) imply that $G'(z,t) = o(|z|)$ uniformly in t as $|z| \to \infty$ and $G_0'(z,t) = o(|z|)$ uniformly in t as $|z| \to 0$. However, (H_1) does not imply that $|G'(z,t)|$ is bounded. Since $a^\pm(t)$ and $b^\pm(t)$ may be zero on a set of positive measure, $G(z,t)$ and $G_0'(z,t)\cdot z$ may not be of constant sign; moreover, $G(z,t)$ may be bounded on a subset of positive measure. So our results will extend different conclusions contained in [1] (and [7-10, 14]). In [1] it was assumed that $G'(z,t)$ is bounded and $G(z,t) \to \infty$ (or $-\infty$) uniformly in t as $|z| \to \infty$.

In order to state our main result, we shall need the notion of $\mathcal{E}$-Morse index which was introduced in [1] and will be recalled in Section 2. It is a kind of relative Morse index for the quadratic form

$\int_0^{2\pi}(-J\dot{z}-Az)\cdot zdt$, where $A=A(t)$ is a symmetric $2N\times 2N$-matrix. Denote this index by $j^-(A)$, the nullity of this quadratic form by $j^0(A)$ and let $j^+(A)=-j^-(A)-j^0(A)$. If we denote the Maslov-type index (cf. [3, 4, 6]) of A by (j,n), then $j=j^-(A)$ and $n=j^0(A)$ (cf. Remark 7.2 of [1]). Now we state the main results.

Theorem 1.1. *Suppose that $H\in C^2(\mathbf{R}^{2N}\times\mathbf{R},\mathbf{R})$ satisfies (H_0), (H_1) and one of the conditions $(H_2^{\pm})$. Then (S) has a nontrivial 2π-periodic solution in each of the following two cases:*

(i) (H_2^-) *and* $j^-(A)\notin[j^-(A_0),\ j^-(A_0)+j^0(A_0)]$;

(ii) (H_2^+) *and* $j^+(A)\notin[j^+(A_0),\ j^+(A_0)+j^0(A_0)]$.

Theorem 1.2. *Suppose that $H\in C^1(\mathbf{R}^{2N}\times\mathbf{R},\mathbf{R})$ satisfies (H_1) and (H_3). Then (S) has a nontrivial 2π-periodic solution in each of the following four cases:*

(i) (H_2^+), (H_4^+) *and* $j^-(A)+j^0(A)\neq j^-(A_0)+j^0(A_0)$;
(ii) (H_2^+), (H_4^-) *and* $j^-(A)+j^0(A)\neq j^-(A_0)$;
(iii) (H_2^-), (H_4^+) *and* $j^-(A)\neq j^-(A_0)+j^0(A_0)$;
(iv) (H_2^-), (H_4^-) *and* $j^-(A)\neq j^-(A_0)$.

If the difference between the $\mathcal{E}$-Morse indices at zero and at infinity is large enough, we obtain the following results on the existence of multiple solutions.

Theorem 1.3. *Suppose that $H\in C^2(\mathbf{R}^{\mathbf{2N}}\times\mathbf{R},\mathbf{R})$ satisfies (H_0), (H_1) and (H_3). Then (S) has at least two nontrivial 2π-periodic solutions in each of the following four cases:*

(i) (H_2^+), (H_4^+) *and* $|j^+(A)-j^+(A_0)|\geq 2N$;
(ii) (H_2^+), (H_4^-) *and* $|j^+(A)+j^-(A_0)|\geq 2N$;
(iii) (H_2^-), (H_4^+) *and* $|j^-(A)+j^+(A_0)|\geq 2N$;
(iv) (H_2^-), (H_4^-) *and* $|j^-(A)-j^-(A_0)|\geq 2N$.

Corollary 1.1. *Suppose that $H\in C^2(\mathbf{R}^{2N}\times\mathbf{R},\mathbf{R})$ satisfies (H_0), (H_1), one of the conditions $(H_2^{\pm})$ and $A(t)=A_0(t)\equiv 0$ (hence $H(z,t)=G(z,t)=G_0(z,t)$). Furthermore, let $H'(z,t)=o(|z|)$ uniformly in t for $|z|\to 0$. Then (S) has at least two nontrivial 2π-periodic solutions in each of the following two cases:*

(i) (H_2^+) *and either there exists a $\delta>0$ such that $H(z,t)\leq 0$ whenever $|z|<\delta$ or (H_3), (H_4^-) are satisfied;*

(ii) (H_2^-) *and either there exists a $\delta>0$ such that $H(z,t)\geq 0$ whenever $|z|<\delta$ or (H_3), (H_4^+) are satisfied.*

Remark 1.3. Theorem 1.1 extends Theorem 7.5 in [1] where G' was assumed to be bounded and $G(z,t)\to\infty$ (or $-\infty$) uniformly in t as $|z|\to\infty$. Theorem 1.2 is a new result. Theorem 1.3 extends Theorem 7.8 in [1] where 0 was nondegenerate ($j^0(A_0)=0$, i.e., (S) is nonresonant at zero), G' was bounded and $G(z,t)\to\infty$ (or $-\infty$) uniformly in t as $|z|\to\infty$. Corollary 1.1 is a generalization of Corollary 7.9 of [1].

2. PRELIMINARIES

In this section we recall some basic facts about the infinite dimensional cohomology theory and Morse theory of [1].

Assume that E is a real Hilbert space and there is a filtration $(E_n)_{n=1}^{\infty}$ of E, i.e., an increasing sequence of closed subspaces of E such that $E = cl(\bigcup_{n=1}^{\infty} E_n)$($cl$ denotes the closure). Suppose that a sequence $(d_n)_{n=1}^{\infty}$ of nonnegative integers is given and let $\mathcal{E} = \{E_n, d_n\}_{n=1}^{\infty}$. If (X, A) is a closed pair of subsets of E, then for any integer q we define the *q-th* $\mathcal{E}$-cohomology group of (X, A) with coefficients in $\mathcal{F}$ by the formula

$$H^q_{\mathcal{E}}(X, A) := [(H^{q+d_n}(X \cap E_n, A \cap E_n))_{n=1}^{\infty}],$$

where $[(\xi_n)_{n=1}^{\infty}]$ is the equivalence class of sequences $(\xi'_n)_{n=1}^{\infty}$ such that $\xi'_n = \xi_n$ for almost all n (cf. [1]). When $\mathcal{F}$ is a field, $H^*_{\mathcal{E}}(X, A)$ is a (graded) vector space over $\mathcal{F}$. We shall use the symbol $[\mathcal{G}]$ to denote the group $[(\mathcal{G}_n)_{n=1}^{\infty}]$ if $\mathcal{G}_n = \mathcal{G}$ for almost all n.

Let $\Phi \in C^1(E, \mathbf{R})$ be a functional satisfying the $(PS)^*$-condition with respect to $\mathcal{E}$, that is, whenever a sequence $(y_j)_{j=1}^{\infty}$ is such that $\Phi(y_j)$ is bounded, $y_j \in E_{n_j}$ for some $n_j, n_j \to \infty$ and $P_{n_j}\nabla\Phi(y_j) \to 0$ as $j \to \infty$, then $(y_j)_{j=1}^{\infty}$ has a convergent subsequence. Here P_{n_j} denotes the orthogonal projector of E onto E_{n_j}. If p is an isolated critical point of Φ, then there exists an admissible pair (W, W^-) for Φ and p (i.e., a kind of Gromoll-Meyer pair with filtration, see Definition 2.3 and Proposition 2.6 of [1]) and the *q-th* critical group ($q \in \mathbf{Z}$) of Φ at p with respect to $\mathcal{E}$ can be defined by

$$C^q_{\mathcal{E}}(\Phi, p) := H^q_{\mathcal{E}}(W, W^-).$$

It was proved in [1] that the critical groups $C^*_{\mathcal{E}}(\Phi, p)$ are well-defined and have a certain continuity property (see Propositions 2.7 and 2.8 of [1]).

If the critical set $K = K(\Phi)$ is compact, then there exists an admissible pair (W, W^-) for Φ and K (cf. Lemma 2.13 of [1]). The critical groups of (Φ, K) given by

$$C^q_{\mathcal{E}}(\Phi, K) := H^q_{\mathcal{E}}(W, W^-)$$

are well-defined and have a continuity property (cf. Propositions 2.12 and 2.14 of [1]). Further properties of critical groups and $\mathcal{E}$-cohomology groups, including the Morse inequalities, may be found in [1].

For an arbitrary linear self-adjoint operator L, denote the Morse index of L by $M^-(L)$. Suppose that L is a Fredholm operator of index 0 and $Q_n : R(L) \to R(L) \cap E_n$ is the orthogonal projector of $R(L)$ onto $R(L) \cap E_n$. Define the $\mathcal{E}$-Morse index $M^-_{\mathcal{E}}(L)$ of L by the formula

$$M^-_{\mathcal{E}}(L) := \lim_{n\to\infty}(M^-(Q_n L|_{R(L)\cap E_n}) - d_n).$$

Although this limit does not exist in general, it exists for operators L associated with (S) provided the sequence (d_n) is chosen properly.

Now we turn to the asymptotically linear Hamiltonian system (S). Let $E := H^{\frac{1}{2}}(S^1, \mathbf{R}^{2N})$ be the Sobolev space of 2π-periodic $\mathbf{R}^{2N}$- valued functions

$$z(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt), \quad a_0, a_k, b_k \in \mathbf{R}^{2N},$$

such that $\sum_{k=1}^{\infty} k(|a_k|^2+|b_k|^2)<\infty$. Then E is a Hilbert space with a norm $\|\cdot\|$ induced by the inner product $\langle\cdot,\cdot\rangle$ given by

$$\langle z,z'\rangle := 2\pi a_0\cdot a_0' + \pi\sum_{k=1}^{\infty} k(a_k\cdot a_k' + b_k\cdot b_k').$$

Set

$$F_k := \{a_k\cos kt + b_k\sin kt : a_k,b_k\in \mathbf{R}^{2N}\},\quad k\geq 0,$$

and

$$E_n := \bigoplus_{k=0}^{n} F_k \equiv \{z\in E : z(t) = a_0 + \sum_{k=1}^{n}(a_k\cos kt + b_k\sin kt)\}.$$

Then $(E_n)_{n=1}^{\infty}$ is a filtration of E. Denote $\mathcal{E}=\{E_n,d_n\}$ with $d_n := N(1+2n) = \frac{1}{2}\dim E_n$.

Suppose that $B(t)$ is a symmetric $2N\times 2N$-matrix with continuous 2π-periodic entries. Then the operator B given by the formula

$$\langle Bz,w\rangle := \int_0^{2\pi} B(t)z\cdot w dt$$

is compact. According to Proposition 5.2 of [1] (see also the argument following Proposition 7.1 there), the operator L_B given by

$$\langle L_B z,w\rangle := \int_0^{2\pi}(-J\dot z - B(t)z)\cdot w dt \tag{2.1}$$

is A-proper and $M^-_{\mathcal{E}}(L_B)$ is well-defined and finite.

Denote

$$\begin{aligned} j^-(B) &:= M^-_{\mathcal{E}}(L_B),\\ j^+(B) &:= M^+_{\mathcal{E}}(L_B) := M^-_{\mathcal{E}}(-L_B),\\ j^0(B) &:= M^0(L_B) := \dim\ker(L_B). \end{aligned} \tag{2.2}$$

Then $j^-(B)+j^+(B)+j^0(B)=0$ (cf. p. 3214 of [1]). Since $M^0(L_B)$ is in fact the number of linearly independent 2π-periodic solutions of the linear system $\dot z = JB(t)z$, $0\leq M^0(L_B)\leq 2N$.

It is well known (cf. [11]) that under condition (H_1) $z(t)$ is a 2π-periodic solution of (S) if and only if it is a critical point of the C^1-functional

$$\begin{aligned} \Phi(z) &= \tfrac{1}{2}\int_0^{2\pi}(-J\dot z - A(t)z)\cdot z dt - \int_0^{2\pi} G(z,t)dt := \tfrac{1}{2}\langle Lz,z\rangle - \varphi(z)\\ &= \tfrac{1}{2}\int_0^{2\pi}(-J\dot z - A_0(t)z)\cdot z dt - \int_0^{2\pi} G_0(z,t)dt := \tfrac{1}{2}\langle L_0 z,z\rangle - \varphi_0(z). \end{aligned}$$

Moreover, $\Phi\in C^2(E,\mathbf{R})$ if (H_0) is satisfied. By (1.1), (1.2) and [1, 5, 11] (or by Remark 1.1), $\nabla\varphi(z) = o(\|z\|)$ as $\|z\|\to\infty$ and $\nabla\varphi_0(z) = o(\|z\|)$ as $\|z\|\to 0$. In particular, (S) has the trivial solution $z=0$.

3. COMPUTATION OF CRITICAL GROUPS

Let $L := L_B$ and $L_0 := L_{B_0}$ (cf. (2.1)) and introduce a new filtration $\mathcal{E}' := \{E_n', d_n\}_{n=1}^{\infty}$, where $E_n' := (R(L)\cap E_n)\oplus\ker(L)$ and $d_n = N(1+2n)$ as before. Then L, L_0 are A-proper with respect to $\mathcal{E}'$ (because they are with respect to $\mathcal{E}$) and

$$M^-_{\mathcal{E}'}(L) = M^-_{\mathcal{E}}(L)\equiv j^-(A)\quad\text{and}\quad M^-_{\mathcal{E}'}(L_0) = M^-_{\mathcal{E}}(L_0)\equiv j^-(A_0).$$

(see the proof of Theorem 7.5 of [1]). In this section we will compute the critical groups $C^q_{\mathcal{E}'}(\Phi,0)$ and $C^q_{\mathcal{E}'}(\Phi,K(\Phi))$. For this aim, we first show how conditions (H_1) and $(H_2^{\pm})$ imply $(PS)^*$ with respect to $\mathcal{E}'$.

Lemma 3.1. *Suppose that $(H_2^{\pm})$ holds. Then*

$$\liminf_{\substack{\|z\|\to\infty \\ z\in\ker(L)}} \frac{\pm\int_0^{2\pi} G(z,t)dt}{H_\infty(\|z\|)} > 0.$$

Proof. Since $\dim\ker(L)<\infty$, the norm $\|\cdot\|$ and the L^∞-norm are equivalent on $\ker(L)$. Moreover, if $z\in\ker(L)$ and $z(t_0)=0$ for some t_0, then $z\equiv 0$. Therefore $\delta\|z\|\le|z(t)|\le c\|z\|$ for some $\delta,c>0$ and all t. Since h_∞ is increasing and $h_\infty(s+t)\le m(h_\infty(s)+h_\infty(t))$, it is easy to see that $c_1h_\infty(\|z\|)\le h_\infty(|z|)\le c_2h_\infty(\|z\|)$ and therefore $c_3H_\infty(\|z\|)\le H_\infty(|z|)\le c_4H_\infty(\|z\|)$ for a suitable choice of constants. Hence it follows from $(H_2^{\pm})$ that for any $\varepsilon>0$ and $\|z\|>R=R(\varepsilon)$,

$$\begin{aligned}
&\pm\int_0^{2\pi}\frac{G(z,t)}{H_\infty(|z|)}\cdot\frac{H_\infty(|z|)}{H_\infty(\|z\|)}dt\\
&\ge\int_0^{2\pi}(a^{\pm}(t)-\varepsilon)\frac{H_\infty(|z|)}{H_\infty(\|z\|)}dt\\
&\ge c_3\int_0^{2\pi}a^{\pm}(t)dt-2\pi\varepsilon c_4.
\end{aligned}$$

Since $a^{\pm}(t)\succeq 0$ and ε is arbitrary, the conclusion follows. □

Lemma 3.2. *Assume (H_1) and $(H_2^{\pm})$. Then Φ satisfies $(PS)^*$ with respect to $\mathcal{E}'$. Moreover, under these hypotheses $\Phi|_{E_n'}$ satisfies the usual (PS)-condition for each n.*

Proof. We only consider the case where (H_2^-) holds, the other one is similar. Let (z_j) be a $(PS)^*$-sequence, i.e., $z_j\in E'_{n_j},\Phi(z_j)$ is bounded, $P'_{n_j}\nabla\Phi(z_j)\to 0$ and $n_j\to\infty$ as $j\to\infty$ (P'_n is the orthogonal projector onto E'_n). By Theorem 4.5 in [1], we may find $c>0$ and $n_0>0$ such that $\|P'_nLz\|\ge c\|z\|$ for all $z\in R(L)\cap E_n$ and $n\ge n_0$. For $z\in E'_n$, write $z=w+z^0\in R(L)\cap E_n\oplus\ker(L)$. Then $P'_{n_j}\nabla\Phi(z_j)=P'_{n_j}Lw_j-P'_{n_j}\nabla\varphi(z_j)\to 0$.

Since

$$\int_0^{2\pi}h_\infty(|z^0|)|y|dt\le c\int_0^{2\pi}h_\infty(\|z^0\|)|y|dt\le ch_\infty(\|z^0\|)\|y\|$$

(cf. the proof of Lemma 3.1), we obtain by Remark 1.1 and the Sobolev embedding theorem that

$$c\|w_j\|\le\|P'_{n_j}Lw_j\|\le c(1+\|w_j\|^{\alpha-1}+h_\infty(\|z_j^0\|)).$$

Therefore $\|w_j\|\le c(1+h_\infty(\|z_j^0\|))$. Moreover, by Remark 1.1 again and by the mean value theorem,

$$\begin{aligned}
\Phi(z_j) &\ge -c\|w_j\|^2-\varphi(z_j)+\varphi(z_j^0)-\varphi(z_j^0)\\
&= -c\|w_j\|^2-\int_0^{2\pi}(G(z_j,t)-G(z_j^0,t))dt-\varphi(z_j^0)\\
&\ge -c\|w_j\|^2-c(1+\|w_j\|^{\alpha-1}+h_\infty(\|z_j^0\|))\|w_j\|-\varphi(z_j^0)\\
&\ge -c(1+h_\infty^2(\|z_j^0\|))-\varphi(z_j^0).
\end{aligned}$$

If $\|z_j^0\| \to \infty$, then it follows from Lemma 3.1 that

$$\begin{aligned} \frac{\Phi(z_j)}{h_\infty^2(\|z_j^0\|)} &\geq -c - \frac{\varphi(z_j^0)}{h_\infty^2(\|z_j^0\|)} \\ &= -c + \frac{-\varphi(z_j^0)}{H_\infty(\|z_j^0\|)} \cdot \frac{H_\infty(\|z_j^0\|)}{h_\infty^2(\|z_j^0\|)} \\ &\to \infty \end{aligned}$$

as $j \to \infty$ because $\frac{H_\infty^2(t)}{h_\infty^2(t)} \cdot \frac{1}{H_\infty(t)} \geq ct^{2-\alpha} \to \infty$ whenever $t \to \infty$. This contradicts the boundedness of $\Phi(z_j)$. It follows that $\|z_j^0\|$ and hence $\|z_j\|$ is bounded. Recalling the compactness of $\nabla\varphi$, we see that (z_j) has a convergent subsequence. □

In order to compute $C_{\mathcal{E}'}^q(\Phi, 0)$, we first prove the following auxiliary results.

Lemma 3.3. *Suppose that* (H_3) *and* $(H_4^\pm)$ *hold. Then for any sequence* $(z_n) \in E$ *such that* $z_n = z_n^0 + w_n$, *where* $z_n^0 \in \ker(L_0), w_n \in (\ker(L_0))^\perp$, $\|z_n\| \to 0$ *and* $\frac{\|z_n^0\|}{\|z_n\|} \to 1$, *we have*

$$\liminf_{n\to\infty} \frac{\pm \int_0^{2\pi} G_0'(z_n, t) \cdot z_n dt}{H_0(\|z_n\|)} > 0.$$

Proof. First, by the definition of h_0, it is easy to check that

$$(\frac{s}{t})^\beta \leq \frac{H_0(s)}{H_0(t)} \leq (\frac{s}{t})^\gamma \quad \text{for } s \geq t > 0 \quad \text{and} \;\; s, t \text{ small.} \tag{3.1}$$

Since h_0 may be extended in such a way that (1.3) holds for all $t > 0$, we may assume that also the above inequality holds for all $t > 0$.

Let $z = w + z^0 \in (\ker L_0)^\perp \oplus \ker L_0$. Since $w \in L^2([0, 2\pi], \mathbf{R}^{2N})$, for each $\varepsilon_1 > 0$ there exists $R(\varepsilon_1) > 0$, independent of w and such that

$$\text{meas}\{t \in [0, 2\pi] : |w(t)| > R(\varepsilon_1)\|w\|\} < \varepsilon_1.$$

Set

$$\Omega_n = \{t \in [0, 2\pi] : |w_n(t)| \leq R(\varepsilon_1)\|w_n\|\};$$

then $\text{meas}([0, 2\pi] \setminus \Omega_n) < \varepsilon_1$. As $\int_0^{2\pi} b^\pm(t)dt > 0$, we may choose ε_1 so small that

$$\int_{\Omega_n} b^\pm(t)dt \geq \frac{1}{2}\int_0^{2\pi} b^\pm(t)dt > 0.$$

Since $\ker L_0$ is finite dimensional, we may assume

$$|z_n(t)| \leq c(R(\varepsilon_1) + c)\|z_n\| \quad \text{whenever } t \in \Omega_n.$$

For any $\varepsilon_2 > 0$, by $(H_4^\pm)$, we have that

$$\frac{\pm G_0'(z_n, t) \cdot z_n}{H_0(|z_n|)} \geq b^\pm(t) - \varepsilon_2$$

whenever $t \in \Omega_n$ and n is large enough. Since H_0 is increasing, $H_0(|z_n|) \geq H_0(\|z_n\|)$ for $|z_n| \geq \|z_n\|$. On the other hand, recalling that $\dfrac{\|z_n^0\|}{\|z_n\|} \to 1$, we obtain

$$\frac{|z_n(t)|}{\|z_n\|} \geq \frac{|z_n^0(t)| - |w_n(t)|}{\|z_n\|} \geq \frac{\delta\|z_n^0\| - R(\varepsilon_1)\|w_n\|}{\|z_n\|} \to \delta$$

as $t \in \Omega_n$ and $n \to \infty$, where δ is as in the proof of Lemma 3.1. This and (3.1) imply

$$\frac{H_0(|z_n|)}{H_0(\|z_n\|)} \geq (\frac{\delta}{2})^\gamma \quad \text{for } t \in \Omega_n,\ |z_n(t)| \leq \|z_n\| \text{ and } n \text{ large enough.}$$

Since it is easy to check by (3.1) that

$$|\int_0^{2\pi} \frac{H_0(|z_n|)}{H_0(\|z_n\|)} dt| \leq c_1$$

for some $c_1 > 0$, it follows, for n large enough, that

$$\begin{aligned}
&\int_{\Omega_n} \frac{\pm G_0'(z_n, t) \cdot z_n}{H_0(\|z_n\|)} dt \\
&\geq \int_{\Omega_n} (b^\pm(t) - \varepsilon_2) \frac{H_0(|z_n|)}{H_0(\|z_n\|)} dt \\
&\geq c_2 \int_{\Omega_n} b^\pm(t) dt - c_1\varepsilon_2 \\
&\geq c_3 \int_0^{2\pi} b^\pm(t) dt - c_1\varepsilon_2 \\
&= c_4 - c_1\varepsilon_2,
\end{aligned} \tag{3.2}$$

where the constants c_i are independent of $\varepsilon_1, \varepsilon_2$. On the other hand, we may assume without loss of generality that (H_3) holds for all z. Indeed, suppose that (H_3) is satisfied whenever $|z| \leq \delta_0$. Since h_0 may be extended so that (1.3) holds for all t, then by (1.3) and (3.1) it is easy to check that

$$\frac{\beta}{\gamma}(\frac{s}{t})^{\beta-1} \leq \frac{h_0(s)}{h_0(t)} \leq \frac{\gamma}{\beta}(\frac{s}{t})^{\gamma-1} \quad \text{for all } s \geq t > 0.$$

It follows that $h_0(t) \geq ct^{\beta-1}$ for $t > \delta_0$. Hence by the asymptotic linearity of $H'(z,t)$,

$$|G_0'(z,t)| \leq c|z| \leq \tilde{c}h_0(|z|) \qquad \text{for some } \tilde{c} > 0 \text{ and all } |z| > \delta_0. \tag{3.3}$$

Using (H_3), which now holds for all z, we see that

$$\frac{|\pm G_0'(z_n, t) \cdot z_n|}{H_0(|z_n|)} \leq \frac{ch_0(|z_n|)|z_n|}{H_0(|z_n|)} \leq c.$$

Since $\operatorname{meas}([0, 2\pi] \setminus \Omega_n) < \varepsilon_1$, it follows that

$$\begin{aligned}
&|\int_{[0,2\pi]\setminus\Omega_n} \frac{\pm G_0'(z_n, t) \cdot z_n}{H_0(\|z_n\|)} dt| \\
&\leq c \int_{[0,2\pi]\setminus\Omega_n} \frac{H_0(|z_n|)}{H_0(\|z_n\|)} dt \\
&\leq c\varepsilon_1^{\frac{1}{2}} (\int_0^{2\pi} \frac{H_0^2(|z_n|)}{H_0^2(\|z_n\|)} dt)^{\frac{1}{2}}.
\end{aligned}$$

If $|z_n| \le \|z_n\|$, then $\dfrac{H_0(|z_n|)}{H_0(\|z_n\|)} \le 1$. Otherwise, by (3.1),

$$\frac{H_0(|z_n|)}{H_0(\|z_n\|)} \le (\frac{|z_n|}{\|z_n\|})^\gamma.$$

Using this and the Sobolev embedding of E into $L^{2\gamma}([0,2\pi], \mathbf{R}^{2N})$, we obtain that

$$|\int_{[0,\,2\pi]\backslash\Omega_n} \frac{\pm G_0(z_n,t)\cdot z_n}{H_0(\|z_n\|)}dt| \le c\varepsilon_1^{\frac{1}{2}} \tag{3.4}$$

for n large enough. Combining (3.2), (3.4) and letting n be large enough, we have

$$\int_0^{2\pi} \frac{\pm G_0(z_n,t)}{H_0(\|z_n\|)}dt \ge c_4 - c_1\varepsilon_2 - c\varepsilon_1^{\frac{1}{2}} > 0$$

since c, c_1, c_4 are independent of $\varepsilon_1, \varepsilon_2$ and $\varepsilon_1, \varepsilon_2$ may be chosen arbitrarily small. □

Lemma 3.4. *Assume* (H_3), $(H_4^\pm)$ *and set*

$$\begin{aligned}\mathcal{D}(\rho,\theta) \quad := \quad & \{z \in E : z = z^0 + w \in \ker(L_0) \oplus (\ker(L_0))^\perp, \\ & 0 < \|z\| \le \rho \text{ and } \|w\| \le \theta\|z\|\}.\end{aligned}$$

Then there exist $\rho > 0$ *and* $\theta \in (0,1)$ *such that*

$$\pm\langle\nabla\Phi(z), z^0\rangle < 0 \qquad \text{for all } z \in \mathcal{D}(\rho,\theta).$$

Proof. Assume by contradiction that for any n there exists $z_n = z_n^0 + w_n \in \ker(L_0)\oplus(\ker(L_0))^\perp$ such that $0 < \|z_n\| < \frac{1}{n}$, $\|w_n\| \le \frac{1}{n}\|z_n\|$ but $\pm\langle\nabla\Phi(z_n), z_n^0\rangle \ge 0$. This implies that $\|z_n\| \to 0$, $\dfrac{\|z_n^0\|}{\|z_n\|} \to 1$ as $n \to \infty$ and

$$-\int_0^{2\pi} \pm G_0'(z_n,t)\cdot z_n^0 dt = -\langle \pm\varphi_0(z_n), z_n^0\rangle = \pm\langle\nabla\Phi(z_n), z_n^0\rangle \ge 0;$$

it follows that

$$\limsup_{n\to\infty} \frac{\int_0^{2\pi} \pm G_0'(z_n,t)\cdot z_n^0 dt}{h_0(\|z_n\|)\|z_n\|} \le 0.$$

By (3.1) and the definition of h_0,

$$\frac{h_0(|z_n|)}{h_0(\|z_n\|)} \le c\max\{(\frac{|z_n|}{\|z_n\|})^{\beta-1}, (\frac{|z_n|}{\|z_n\|})^{\gamma-1}\}.$$

Therefore, using (H_3) and (3.3), we obtain

$$\begin{aligned}& |\frac{\int_0^{2\pi} \pm G_0'(z_n,t)\cdot w_n dt}{h_0(\|z_n\|)\|z_n\|}| \\ & \le c(\int_0^{2\pi} \frac{h_0^2(|z_n|)}{h_0^2(\|z_n\|)}dt)^{\frac{1}{2}}(\int_0^{2\pi} \frac{|w_n|^2}{\|z_n\|^2}dt)^{\frac{1}{2}} \\ & \le c\frac{\|w_n\|}{\|z_n\|} \\ & \to 0\end{aligned}$$

as $n \to \infty$. Finally, in view of Lemma 3.3,

$$\liminf_{n\to\infty} \frac{\int_0^{2\pi} \pm G_0'(z_n,t)\cdot z_n^0 dt}{h_0(\|z_n\|)\|z_n\|} = \liminf_{n\to\infty} \frac{\int_0^{2\pi} \pm G_0'(z_n,t)\cdot z_n dt}{h_0(\|z_n\|)\|z_n\|} > 0.$$

This contradicts the preceding estimate about the upper limit. $\square$

Using the above lemmas we can now compute the critical groups $C^q_{\mathcal{E}'}(\Phi,0)$ by making a perturbation and using the continuity property of $C^q_{\mathcal{E}'}(\Phi,0)$.

Lemma 3.5. *Assume* (H_3) *and* (H_4^+) *(or* (H_4^-)*). Then*
(i) (H_4^+) *implies that* $C^q_{\mathcal{E}'}(\Phi,0) = [\mathcal{F}]$ *for* $q = j^-(A_0) + j^0(A_0)$ *and [0] otherwise;*
(ii) (H_4^-) *implies that* $C^q_{\mathcal{E}'}(\Phi,0) = [\mathcal{F}]$ *for* $q = j^-(A_0)$ *and [0] otherwise.*

Proof. (i) For any $\lambda \in [0,1]$ and $z = z^0 + w \in \ker(L_0) \oplus (\ker(L_0))^\perp = E$ we consider the following perturbation of Φ:

$$\Phi_\lambda(z) := \Phi(z) - \frac{1}{2}\lambda\|z^0\|^2 = \frac{1}{2}\langle L_0 z - \lambda z^0, z\rangle - \varphi_0(z).$$

We claim that there exists a neighborhood $\mathcal{N}$ of 0 such that 0 is the unique critical point of Φ_λ in $\mathcal{N}$ for any $\lambda \in [0,1]$. In fact, if $z \in \mathcal{D}(\rho,\theta)$, then by Lemma 3.4 $z^0 \neq 0$ and

$$\langle \nabla\Phi_\lambda(z), z^0\rangle = \langle \nabla\Phi(z), z^0\rangle - \lambda\langle z^0, z^0\rangle < 0.$$

If $z \in \{z \in E : 0 < \|z\| \le \rho\} \setminus \mathcal{D}(\rho,\theta)$, then $\|w\| > \theta\|z\|$. Let $w = w^+ + w^-$; then there exists a constant c such that $\pm\langle L_0 w^\pm, w^\pm\rangle \ge c\|w^\pm\|^2$. Therefore

$$\begin{aligned}
&\langle \nabla\Phi_\lambda(z), w^+ - w^-\rangle \\
&= \langle L_0 w, w^+ - w^-\rangle - \langle \nabla\varphi_0(z), w^+ - w^-\rangle \\
&\ge \|w^+ + w^-\|^2 (c - \frac{\|\nabla\varphi_0(z)\|}{\|w^+ + w^-\|}) \\
&\ge \|w^+ + w^-\|^2 (c - \frac{\|\nabla\varphi_0(z)\|}{\theta\|z\|}) \\
&> 0
\end{aligned}$$

for sufficiently small ρ and $\|z\| \le \rho$. The above arguments imply that 0 is the only critical point of Φ_λ in $\mathcal{N} := \{z : \|z\| \le \rho\}$ for all $\lambda \in [0,1]$. Since $\|P_n' L_0 w\| \ge c\|w\|$ whenever $w \in R(L_0) \cap E_n'$ and n is large enough, it is easy to see that Φ_λ satisfies $(PS)^*$ in $\mathcal{N}$. Moreover, $\sup_{\mathcal{N}} |\Phi_\lambda| < \infty$ and the mapping $\lambda \longmapsto \nabla\Phi_\lambda$ is continuous uniformly in $z \in \mathcal{N}$. By Corollary 2.9 of [1], $C^*_{\mathcal{E}'}(\Phi_\lambda,0)$ is independent of $\lambda \in [0,1]$. Therefore

$$C^*_{\mathcal{E}'}(\Phi,0) = C^*_{\mathcal{E}'}(\Phi_1,0).$$

On the other hand, since $\ker L_0$ is finite dimensional and L_0 is A-proper, it is easy to check that the operator $\bar{L}_0$ defined by $\bar{L}_0 z = L_0 z - z^0$ is invertible and A-proper.

$M^-(P_n'\bar{L}_0 z|_{E_n'})$ is the Morse index of the quadratic form

$$\langle \bar{L}_0 z, z\rangle = \langle L_0 w, w\rangle - \langle z^0, z^0\rangle, \qquad z \in E_n',$$

and according to Theorem 4.5 in [1], this form is nondegenerate for almost all n. By Lemma 4.2 of [1], $E_n' = R(L_0) \cap E_n' \oplus P_n' \ker(L_0)$; therefore $z = w + z^0 = \tilde{w} + \tilde{z}^0 \in R(L_0) \cap E_n' \oplus P_n' \ker(L_0)$ and

$w-\tilde{w}=\tilde{z}^0-z^0$. Since $P'_n y\to y$ uniformly for y on bounded subsets of $\ker(L_0)$ and $w-\tilde{w}\in R(L_0)$, it follows that

$$\sup\{\|w-\tilde{w}\| : z=w+z^0=\tilde{w}+\tilde{z}^0\in E'_n, \|z\|=1\}\to 0 \qquad \text{as } n\to\infty.$$

So for n large, $M^-(P'_n\bar{L}_0 z|_{E'_n})$ is the sum of the Morse indices of the form $\langle L_0\tilde{w},\tilde{w}\rangle, \tilde{w}\in R(L_0)\cap E'_n$ and $-\langle \tilde{z}^0,\tilde{z}^0\rangle, \tilde{z}^0\in\ker(L_0)$. Hence, according to the definition of $\mathcal{E}'$-Morse index, we have

$$M^-_{\mathcal{E}'}(\bar{L}_0)=M^-_{\mathcal{E}'}(L_0)+\dim\ker(L_0)\equiv j^-(A_0)+j^0(A_0),$$

and by Theorem 5.3 of [1],

$$C^q_{\mathcal{E}'}(\Phi_1,0)=[\mathcal{F}] \quad \text{for } q=j^-(A_0)+j^0(A_0) \text{ and } [0] \text{ otherwise.}$$

(ii) The proof is analogous with $\Phi_\lambda(z):=\frac{1}{2}\langle L_0 z+\lambda z^0,z\rangle-\varphi_0(z)$. □

Next we turn to the computation of the critical groups $C^q_{\mathcal{E}'}(\Phi,K(\Phi))$.

Lemma 3.6. *Suppose that (H_1) and one of the conditions $(H_2^\pm)$ hold and $K=K(\Phi)$ is finite. Then*
(i) (H_2^+) implies that $C^q_{\mathcal{E}'}(\Phi,K)=[\mathcal{F}]$ for $q=j^-(A)+j^0(A)$ and [0] otherwise;
(ii) (H_2^-) implies that $C^q_{\mathcal{E}'}(\Phi,K)=[\mathcal{F}]$ for $q=j^-(A)$ and [0] otherwise.

Proof. (i) Let $E'_n=(R(L)\cap E_n)\oplus\ker(L)=E_n^+\oplus E_n^-\oplus\ker(L)$ be the decomposition corresponding to the positive, the negative and the zero part of the operator L on E'_n. Then there exist $c^*>0$ and $n_0>0$ such that $\pm\langle Lz^\pm,z^\pm\rangle\ge c^*\|z^\pm\|^2$ for all $z^\pm\in E_n^\pm$, $n\ge n_0$. Consider the following set:

$$\mathcal{U}_n:=\{z=z^++z^-+z^0\in E'_n : \|z^+\|^2-\frac{c^*}{8\|L\|}\|z^-\|^2-\frac{\lambda H^2_\infty(\|z^0\|)}{1+\|z^0\|^2}\le M\},$$

where $z^\pm\in E_n^\pm, z^0\in\ker(L)$; the constants $\lambda>0, M>0$ will be determined later. An outer normal vector to $\partial\mathcal{U}_n$ (the boundary of $\mathcal{U}_n$) is

$$\nu_n=\nu_n(z)=z^+-dz^--\frac{\lambda}{2}p'(\|z^0\|)\frac{z^0}{\|z^0\|},$$

where $d=\dfrac{c^*}{8\|L\|}$ and $p(t)=\dfrac{H^2_\infty(t)}{1+t^2}$. We claim that $\Phi|_{E'_n}$ has no critical point in $E'_n\setminus\mathcal{U}_n$. In fact, by Remark 1.1, it is easy to check that

$$\|\nabla\varphi(z)\|\le c(1+\|z^+\|^{\alpha-1}+\|z^-\|^{\alpha-1}+h_\infty(\|z^0\|)) \text{ for } z\in E.$$

Therefore, for ε small enough and $n\ge n_0$,

$$\begin{aligned}
&\langle\nabla\Phi(z),\nu_n\rangle\\
&=\langle Lz^+,z^+\rangle-d\langle Lz^-,z^-\rangle-\langle\nabla\varphi(z),\nu_n\rangle\\
&\ge c^*\|z^+\|^2+dc^*\|z^-\|^2-\\
&\quad c_1(1+h_\infty(\|z^0\|)+\|z^+\|^{\alpha-1}+\|z^-\|^{\alpha-1})(\|z^+\|+d\|z^-\|+\lambda|p'(\|z^0\|)|)\\
&\ge\frac{1}{2}c^*\|z^+\|^2-\frac{d}{2}c^*\|z^-\|^2-c_1\varepsilon\lambda^2|p'(\|z^0\|)|^2-c_1\varepsilon^{-1}h^2_\infty(\|z^0\|)-c_2.
\end{aligned}$$

Here we have used the inequalities $xy \le \varepsilon^{-1}x^2 + \varepsilon y^2$ and $xy^{\alpha-1} \le x^2 + \varepsilon y^2 + c$ which hold for all $x, y \ge 0$, $\varepsilon > 0$ and an appropriate $c = c(\varepsilon)$. By the definition of h_∞, we see that

$$|p'(t)|^2 \le \frac{4H_\infty^4(t)}{(1+t^2)^4}(\frac{\alpha}{t}(1+t^2)+t)^2, \quad h_\infty^2(t) \le \frac{4H_\infty^2(t)}{1+t^2} + c$$

for $t > 0$. Let $\lambda > \frac{10c_1}{\varepsilon c^*}$. Since $\dfrac{H_\infty(t)}{1+t^2} \to 0$ as $t \to \infty$, it is easy to verify that

$$c_1\varepsilon\lambda^2|p'(\|z^0\|)|^2 + c_1\varepsilon^{-1}h_\infty^2(\|z^0\|) \le \frac{\lambda c^*}{2}\frac{H_\infty^2(\|z^0\|)}{1+\|z^0\|^2} + c.$$

Therefore

$$\begin{aligned}\langle \nabla\Phi(z), \nu_n\rangle &\ge \frac{c^*}{2}(\|z^+\| - d\|z^-\|^2 - \lambda p(\|z^0\|)) - c \\ &\ge \frac{c^*}{2}M - c \\ &> 0\end{aligned}$$

for an appropriate M. So $\Phi|_{E_n'}$ has no critical point outside $\mathcal{U}_n$ and on $\partial\mathcal{U}_n$. It is easy to construct a pseudogradient vector field V on E_n' such that $\langle V(z), \nu_n(z)\rangle > 0$ on $\partial\mathcal{U}_n$. This implies that the flow of $-V$ points into $\mathcal{U}_n$ on $\partial\mathcal{U}_n$.

Next we show that on $\mathcal{U}_n$

$$\Phi(z) \to -\infty \quad \text{if and only if} \quad \|z^0 + z^-\| \to \infty \tag{3.5}$$

and the convergence is uniform with respect to the choice of $n \ge n_0$. Indeed, if $z \in \mathcal{U}_n$, then $\|z^+\|^2 \le M + d\|z^-\|^2 + \lambda p(\|z^0\|)$, and since $p(t) \le c(1 + h_\infty^2(t))$, it follows using the mean value theorem as in the proof of Lemma 3.2 that

$$\begin{aligned}\Phi(z) &= \frac{1}{2}(\langle Lz^+, z^+\rangle + \langle Lz^-, z^-\rangle) - \varphi(z) \\ &\le \frac{1}{2}\|L\|\|z^+\|^2 - \frac{1}{2}c^*\|z^-\|^2 - \varphi(z^0) + \varphi(z^0) - \varphi(z) \\ &\le \frac{1}{2}\|L\|\|z^+\|^2 - \frac{1}{2}c^*\|z^-\|^2 - \varphi(z^0) \\ &\quad + c(1 + h_\infty(\|z^0\|) + \|z^+\|^{\alpha-1} + \|z^-\|^{\alpha-1})\|z^+ + z^-\| \\ &\quad \le \|L\|\|z^+\|^2 - \frac{1}{4}c^*\|z^-\|^2 + ch_\infty^2(\|z^0\|) - \varphi(z^0) + c \\ &\le (-\frac{1}{4}c^* + d\|L\|)\|z^-\|^2 + \|L\|\lambda p(\|z^0\|) + ch_\infty^2(\|z^0\|) + \|L\|M - \varphi(z^0) + c \\ &\le -\frac{c^*}{8}\|z^-\|^2 + ch_\infty^2(\|z^0\|) - \varphi(z^0) + c.\end{aligned}$$

In view of the definition of h_∞ and Lemma 3.1, we have that

$$\lim_{t\to\infty}\frac{h_\infty^2(t)}{H_\infty(t)} \le \lim_{t\to\infty} ct^{\alpha-2} = 0 \quad \text{and} \quad \liminf_{\|z^0\|\to\infty}\frac{\varphi(z^0)}{H_\infty(\|z^0\|)} > 0;$$

consequently,

$$\lim_{\|z^0\|\to\infty}\frac{\varphi(z^0)}{h_\infty^2(\|z^0\|)} = \infty,$$

and $\Phi(z) \to -\infty$ uniformly in n as $\|z^- + z^0\| \to \infty$.

On the other hand, if $z \in \mathcal{U}_n$ and $\|z^0 + z^-\| \le c$, then $\|z^+\| \le \tilde{c}$ for an appropriate $\tilde{c} > 0$; hence $\Phi(z) \to -\infty$ implies that $\|z^0 + z^-\| \to \infty$.

Now we adapt an argument of Lemma 7.6 in [1]. Choose $a > 0$ such that $K = K(\Phi) \subset \{z \in E : |\Phi(z)| < a\}$. By (3.5), there exists $R_2 = R_2(a)$ (R_2 independent of n) such that

$$D_2 := \{z \in \mathcal{U}_n : \|z^- + z^0\| \ge R_2\} \subset \mathcal{U}_n \cap \Phi^{-a}.$$

Using (3.5) again, we first find $b > a$ with the property that $\Phi^{-b} \cap \mathcal{U}_n \subset D_2$, and then $R_1 > R_2$ such that

$$D_1 := \{z \in \mathcal{U}_n : \|z^0 + z^-\| \ge R_1\} \subset \Phi^{-b} \cap \mathcal{U}_n.$$

Define $\xi : [0,1] \times D_2 \to D_1$ as follows:

$$\xi(t,z) = \begin{cases} z & \text{if } \|z^- + z^0\| \ge R_1, \\ z^+ + \frac{z^- + z^0}{\|z^- + z^0\|}(tR_1 + (1-t)\|z^- + z^0\|) & \text{if } \|z^- + z^0\| \le R_1. \end{cases}$$

It is easy to see that ξ is a strong deformation retraction of D_2 onto D_1 (since $p' > 0$, ξ does not leave $\mathcal{U}_n$). By $(PS)^*$, $K(\Phi|_{E_n'}) \subset \mathcal{U}_n \setminus \Phi^{-1}([-b,-a])$ for $n \ge n_0$ (possibly after choosing a larger n_0). Therefore, using the flow of $-V$, it is easy to construct a strong deformation retraction η of $\Phi^{-a} \cap \mathcal{U}_n$ onto $\Phi^{-b} \cap \mathcal{U}_n$. Let $\xi * \eta$ denote the deformation η followed by ξ. Then $\xi * \eta$ is a strong deformation retraction of $\Phi^{-a} \cap \mathcal{U}_n$ onto D_1. Applying the flow of $-V$ again, we obtain a strong deformation retraction of $\Phi^a \cap E_n'$ onto $(\Phi^{-a} \cap E_n') \cup \mathcal{U}_n$. Finally, by the above-mentioned properties and the strong excision (cf. Property 1.2 of [1]), we have that for $n \ge n_0$,

$$\begin{aligned} H^q(\Phi^a \cap E_n', \Phi^{-a} \cap E_n') &\cong H^q((\Phi^{-a} \cap E_n') \cup \mathcal{U}_n, \Phi^{-a} \cap E_n') \\ &\cong H^q(\mathcal{U}_n, \Phi^{-a} \cap \mathcal{U}_n) \quad (\textit{excision}) \\ &\cong H^q(\mathcal{U}_n, D_1) \\ &\cong \begin{cases} \mathcal{F} & \text{if } q = j^-(A) + j^0(A) + d_n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since the excision property implies that

$$H^q_{\mathcal{E}'}(\Phi^a, \Phi^{-a}) \cong H^q_{\mathcal{E}'}(\Phi^{-1}([-a,a]), \Phi^{-1}(-a))$$

and $(\Phi^{-1}([-a,a]), \Phi^{-1}(-a))$ is an admissible pair for Φ and K (cf. Proposition 2.5 of [1]), the conclusion of case (i) follows from the definition of $C^*_{\mathcal{E}'}(\Phi, K(\Phi))$.

(ii) Set

$$\mathcal{V}_n := \{z \in E_n' : \|z^-\|^2 - \frac{c^*}{8\|L\|}\|z^+\|^2 - \frac{\lambda H^2_\infty(\|z^0\|)}{1 + \|z^0\|^2} \le M\}.$$

Then an outer normal vector to $\partial \mathcal{V}_n$ is

$$\nu_n = \nu_n(z) = z^- - \frac{c^*}{8\|L\|}z^+ - \frac{\lambda}{2}p'(\|z^0\|)\frac{z^0}{\|z^0\|}, \quad \text{where } p(t) = \frac{H^2_\infty(t)}{1+t^2}.$$

By an argument similar to that in case (i), there exist λ and M such that

$$\begin{aligned} \langle \nabla\Phi(z), \nu_n \rangle &\le -\tfrac{c^*}{2}(\|z^-\|^2 - \tfrac{c^*}{8\|L\|}\|z^+\|^2 - \lambda p(\|z^0\|)) + c \\ &\le -\tfrac{c^*}{2}M + c \\ &< 0, \end{aligned}$$

where c is independent of $n \geq n_0$. It follows that $\Phi|_{E_n'}$ has no critical point in $E_n' \setminus \mathcal{V}_n$ and there exists a pseudogradient vector field V such that the flow of $-V$ points outwards on $\partial\mathcal{V}_n$. Furthermore,

$$\|z^-\|^2 \leq \frac{c^*}{8\|L\|}\|z^+\|^2 + \frac{\lambda H_\infty^2(\|z^0\|)}{1+\|z^0\|^2} + M \quad \text{for } z \in \mathcal{V}_n;$$

consequently,

$$\begin{aligned}
\Phi(z) &= \frac{1}{2}\langle Lz^+, z^+\rangle + \frac{1}{2}\langle Lz^-, z^-\rangle - \varphi(z) \\
&\geq \frac{1}{2}c^*\|z^+\|^2 - \frac{1}{2}\|L\|\|z^-\|^2 - \varphi(z^0) \\
&\quad -c(1 + h_\infty(\|z^0\|) + \|z^+\|^{\alpha-1} + \|z^-\|^{\alpha-1})\|z^+ + z^-\| \\
&\geq \frac{c^*}{8}\|z^+\|^2 - ch_\infty^2(\|z^0\|) - \varphi(z^0) - c.
\end{aligned}$$

Since by Lemma 3.1,

$$\lim_{\|z^0\|\to\infty} \frac{-\varphi(z^0)}{h_\infty^2(\|z^0\|)} = \infty,$$

it follows that $\Phi(z) \to \infty$ uniformly in n as $\|z^+ + z^0\| \to \infty$. As in case (i) we also see that the reverse implication is true.

It follows that we can find $a > 0$ such that $K = K(\Phi) \subset \{z \in E : |\Phi(z)| < a\}$ and $\Phi^{-a} \cap E_n' \subset \overline{E_n' \setminus \mathcal{V}_n}$. Since $\Phi^a \cap \mathcal{V}_n$ is a bounded set, we find $R_0 > 0$ such that

$$\Phi^a \cap \mathcal{V}_n \subset D := \{z \in \mathcal{V}_n : \|z^+ + z^0\| \leq R_0\}.$$

Since also D is bounded, there exists $b > a$ such that $D \subset \Phi^b \cap \mathcal{V}_n$. Similarly as in the proof of Lemma 7.6 in [1], we find a strong deformation retraction ξ of E_n' onto $D \cup \partial\mathcal{V}_n$ (we can e.g. use the flow of $-\nu_n$ to deform E_n' onto $\mathcal{V}_n$ and that of ν_n to deform $\mathcal{V}_n$ onto $D \cup \partial\mathcal{V}_n$). By $(PS)^*$, we may assume that $K(\Phi|_{E_n'}) \subset \mathcal{V}_n \setminus \Phi^{-1}[a,b]$ for $n \geq n_0$, so the flow of $-V$ provides a strong deformation retraction of $\overline{E_n' \setminus \mathcal{V}_n}$ onto $\Phi^{-a} \cap E_n'$. Moreover, the flow of $-V$ induces a strong deformation retraction η of $\overline{(E_n' \setminus \mathcal{V}_n)} \cup D$ onto $\Phi^a \cap E_n'$. Now it is easy to see that the mapping $\eta * \xi$ is a strong deformation retraction of E_n' onto $\Phi^a \cap E_n'$. Therefore

$$\begin{aligned}
H^q(\Phi^a \cap E_n', \Phi^{-a} \cap E_n') &\cong H^q(E_n', \Phi^{-a} \cap E_n') \\
&\cong H^q(E_n', \overline{E_n' \setminus \mathcal{V}_n}) \\
&\cong \begin{cases} \mathcal{F} & \text{if } q = j^-(A) + d_n, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Now by the same argument as in case (i) we get the conclusion. □

Remark 3.1. For the computation of the usual relative homology groups, see [12, 13, 15]. We emphasize that the results of [12, 13, 15] cannot be used directly to deal with strongly indefinite functionals.

4. PROOFS OF THE MAIN RESULTS

Based on the computations of the critical groups $C^*_{\mathcal{E}'}(\Phi, 0)$ and $C^*_{\mathcal{E}'}(\Phi, K)$, we can prove the main results of Section 1.

Proof of Theorem 1.1.

(i) By Lemma 3.6, (H_2^-) implies that $C^q_{\mathcal{E}'}(\Phi, K) = [\mathcal{F}]$ for $q = j^-(A)$ and $[0]$ otherwise. On the other hand, if 0 is the only critical point of Φ, then $C^q_{\mathcal{E}'}(\Phi, K) = C^q_{\mathcal{E}'}(\Phi, 0)$. It follows from the shifting theorem (cf. Theorem 5.4 of [1]) that $C^q_{\mathcal{E}'}(\Phi, 0) = [C^{q-j^-(A_0)}(\tilde{\varphi}_0, 0)]$, where $\tilde{\varphi}_0$ is defined on a subset of $\ker(L_0)$. Since $\dim\ker(L_0) = j^0(A_0)$, $C^q_{\mathcal{E}'}(\Phi, 0) = [0]$ whenever $q \notin [j^-(A_0), j^-(A_0) + j^0(A_0)]$. So by our assumption, $C^{j^-(A)}_{\mathcal{E}'}(\Phi, 0) = [0] \neq C^{j^-(A)}_{\mathcal{E}'}(\Phi, K)$, a contradiction.

(ii) Since $j^-(A) + j^0(A) + j^+(A) = 0$, the conclusion follows from Lemma 3.6-(i) and a similar argument. □

Proof of Theorem 1.2. It follows from Lemmas 3.5 and 3.6 that $C^q_{\mathcal{E}'}(\Phi, 0) \neq C^q_{\mathcal{E}'}(\Phi, K)$ for some q, hence $K \neq \{0\}$.

Proof of Theorem 1.3. We only prove the case (i) as an example. The other cases are similar. Since

$$(H_2^+) \text{ implies that } C^q_{\mathcal{E}'}(\Phi, K) = \begin{cases} [\mathcal{F}] & \text{for } q = j^-(A) + j^0(A), \\ [0] & \text{otherwise,} \end{cases}$$

and

$$(H_4^+) \text{ implies that } C^q_{\mathcal{E}'}(\Phi, 0) = \begin{cases} [\mathcal{F}] & \text{for } q = j^-(A_0) + j^0(A_0), \\ [0] & \text{otherwise,} \end{cases}$$

there exists a nonzero critical point z_0. Suppose there are no other ones, then by Theorem 5.4 of [1], $C^q_{\mathcal{E}'}(\Phi, z_0) = [C^{q-r_0}(\tilde{\varphi}_0, 0)]$ for some $r_0 \in \mathbf{Z}$ and some functional $\tilde{\varphi}_0$ defined on a space Z with $\dim Z \leq 2N$. In this case the Morse inequalities read

$$t^{j^-(A_0)+j^0(A_0)} + \sum_{i=0}^{2N-2} b_i t^{\alpha+i} = t^{j^-(A)+j^0(A)} + (1+t)Q(t),$$

where $b_i \in [\mathbf{Z}]$ and $\alpha \in \mathbf{Z}$. That the sum on the left-hand side above contains at most $2N-1$ nonzero terms follows from the fact that if $C^0(\tilde{\varphi}_0, 0) \neq 0$, then $\tilde{\varphi}_0$ has a local minimum at 0 and $C^p(\tilde{\varphi}_0, 0) = 0$ for $p \neq 0$, and if $C^{2N}(\tilde{\varphi}_0, 0) \neq 0$, then $\tilde{\varphi}_0$ has a local minimum there and $C^p(\tilde{\varphi}_0, 0) = 0$ for $p \neq 2N$. By comparing the exponents, we can find i and j such that $\alpha + i = j^-(A) + j^0(A)$ and $\alpha + j = j^-(A_0) + j^0(A_0) \pm 1$, where $i, j \in \{0, 1, \cdots, 2N-2\}$. So $|j^+(A) - j^+(A_0)| = |j^-(A) + j^0(A) - j^-(A_0) - j^0(A_0)| = |i - j \pm 1| \leq 2N-1$, a contradiction. □

Proof of Corollary 1.1. We only prove case (ii). Since $A = A_0 \equiv 0$, $j^-(0) = -N$ and $j^0(0) = 2N$ (cf. Proposition 7.1 of [1]). Consequently, by Lemma 3.6, $C^q_{\mathcal{E}'}(\Phi, K) = [\mathcal{F}]$ if $q = N$ and $[0]$ otherwise. On the other hand, by Corollary 5.5 of [1] and Lemma 3.5, $C^q_{\mathcal{E}'}(\Phi, 0) = [\mathcal{F}]$ if $q = -N$ and $[0]$ otherwise. If Φ has only one nontrivial critical point, then by the Morse inequalities,

$$t^{-N} + \sum_{i=0}^{2N-2} b_i t^{\alpha+i} = t^N + (1+t)Q(t),$$

and similarly as in the proof of Theorem 1.3, we get a contradiction. □

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