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SOME BANACH SPACES WITH UNIFORMLY NORM-BOUNDED PARTIAL SUMS

Mats Erik Andersson

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Abstract. Spaces of measures and functions on the unit circle, such that their partial Fourier sums are uniformly bounded in L^1-norm are introduced and studied. They are made into Banach spaces by a suitable norm. The emerging five spaces are demonstrated to be strict subsets of $M(\mathbb{T})$, L^1 and H^1 respectively. Non-boundedness under conjugation is discussed.

Contents

Convergence properties of the partial sums of a given formal Fourier series are central to important parts of analysis. In particular, the pointwise uniform convergence has been studied extensively. Here another viewpoint will be taken. Given a formal Fourier series and its partial sums

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta} \quad \text{and} \quad s_N = \sum_{n=-N}^{N} c_n e^{in\theta},$$

we single out those with the property that

$$\sup_{N\geqslant 0} \|s_N\|_{L^1} < \infty.$$

An elementary argument with weak-$*$ compactness produces a measure μ in $M(\mathbb{T})$ such that the formal series in fact must be the Fourier–Stieltjes series of μ; correspondingly we henceforth denote the partial sums by $s_N\mu$.

We first introduce a norm $\|\ \|$ on subsets of $M(\mathbb{T})$ by the prescription

$$\|\mu\| = \sup_{N\geqslant 0} \|s_N\mu\|_1.$$

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From now on we indiscriminately write $\|\mu\|_1$ and $\|f d\theta\|_1 = \|f\|_1$ for the variation norm of general finite measures μ and of the absolutely continuous measures identified with functions $f \in L^1(\mathbb{T})$. We use normalised Haar measure. In addition, $\|f\|_1$ will also denote the H^1-norm of the boundary function whenever f is an element of $H^1(\mathbb{T})$. Since the new norm just introduced will be used extensively, it carries no index or markings whatsoever.

In the subsequent sections we will find reasons to study the following spaces, *a priori* linear subspaces of $M(\mathbb{T})$.

$$\begin{aligned} M_b &= \{\mu \in M(\mathbb{T})\,;\, \|\mu\| < \infty\} \\ L_b^1 &= \big\{f \in L^1\,;\, \|f\| < \infty\big\} \\ H_b^1 &= \big\{f \in H^1\,;\, \|f\| < \infty\big\} \\ L_{b0}^1 &= \big\{f \in L^1\,;\, \|f - s_N f\|_1 \to 0 \text{ as } N \to \infty\big\} \\ H_{b0}^1 &= \big\{f \in H^1\,;\, \|f - s_N f\|_1 \to 0 \text{ as } N \to \infty\big\} \end{aligned}$$

They will all turn out to be Banach spaces with respect to $\|\ \|$ and in addition to be convolution ideals of $M(\mathbb{T})$. Clearly L_{b0}^1 and H_{b0}^1 are linear subspaces of L_b^1 and H_b^1 respectively.

The weak-$*$ argument alluded to above immediately yields an important property of the new norm.

Observation. $\|\mu\|_1 \leqslant \|\mu\|$ *for all* $\mu \in M_b$.

In the course of this exposition we will be able to deduce the following inclusion relations as linear spaces.

Proposition. $M(\mathbb{T}) \supsetneqq M_b \supsetneqq L_b^1$, $L^1 \supsetneqq L_b^1 \supsetneqq L_{b0}^1$, *and* $H^1 \supsetneqq H_b^1 \supsetneqq H_{b0}^1$.

Some words on earlier references and scientific credit is appropriate before proceeding to the main body of this manuscript; the precise statements alluded to will appear elsewhere in the text. Some explicit properties of M_b are presented in Zygmund's book [Z] and more information will be extracted from the very same recource. Among the named contributions in [Z] we find important results of Helson [H] and Salem–Zygmund. Two illuminating examples that M_b contains singular measures were established by M. Weiss [We] and Katznelson [K2] respectively. Two qualitative improvements on Helson's theorem were deduced by Pigno, Smith [PS] and Fournier [F]. Finally, the most substantial treatment of M_b and L_b^1 known to me, although of somewhat differing intentions, is the exposition by Wojtaszczyk [Wo]. It overlaps to some degree with my text.

Finally, some comments on the contents of this *excercitio academicum.* Its purpose is to display as much as is known to me on the above five spaces and efforts to condense the material were not judged desirable, mainly due to the wish of presenting a document upon which later research of a deeper character may be founded. Throughout the text there will appear small examples of a fairly trivial character just to illustrate aspects of the spaces in question. They are probably neither very original nor unknown, but they improve intuition on this new norm and its properties. Section 1 deals with the basic properties of M_b and some technical tools for calculations inside the five spaces. Decay properties of the Fourier coefficients are treated in the second section. Section 3 concentrates on non-inclusions between H^1, H_b^1, and H_{b0}^1. To end the exposition, section 4 displays that harmonic conjugation is not bounded on any one of the spaces L_b^1 and L_{b0}^1.

1. Basic properties and classical results

Immediate steps must be taken to secure completeness under the new norm.

Proposition 1.1. *$M_b \supseteq L^1_b \supseteq H^1_b$ are closed subspaces of $M(\mathbb{T})$ and they are also convolution ideals there.*

Proof. The first claim will follow if the spaces are Banach space with respect to $\| \ \|$, thanks to the observation above. Consider a formal absolutely convergent series $\sum \mu_n$ in M_b under its own norm. A forteriori $\sum \|\mu_n\|_1 \leqslant \sum \|\mu\| < \infty$, whence $\sum \mu_n$ converges inside $M(\mathbb{T})$ to some measure $\mu \in M(\mathbb{T})$. Obviously $s_N\mu = \sum s_N\mu_n$ is also absolutely convergent and independently of N

$$\|s_N\mu\|_1 \leqslant \sum \|s_N\mu_n\|_1 \leqslant \sum \|\mu_n\|.$$

In consequence, $\|\mu\| \leqslant \sum \|\mu_n\|$ and $\mu \in M_b$. This means that M_b indeed is a closed subspace of $M(\mathbb{T})$. An identical argument works for L^1_b and H^1_b also.

Consider finally $\nu \in M(\mathbb{T})$, as well as $\mu \in M_b$ or $f \in L^1_b$ and H^1_b respectively. Since

$$\|s_N(\mu * \nu)\|_1 = \|\nu * s_N\mu\|_1 \leqslant \|\nu\|_1\|s_N\mu\|_1 \leqslant \|\nu\|_1\|\mu\|,$$

we find $\|\mu * \nu\| \leqslant \|\mu\| \, \|\nu\|_1$ and $\mu * \nu \in M_b$.

Since L^1 and H^1 are convolution ideals in $M(\mathbb{T})$, the above argument applied to $\mu = f dt$ supplies the verification of ideal structure also for these funcion spaces. □

Remark. It is appropriate to extract from the above proof the operator inequality $\|\nu * \mu\| \leqslant \|\nu\|_1\|\mu\|$ for the action of $M(\mathbb{T})$ on M_b. In particular, the Fejér means satisfy $\|\sigma_n\mu\| \leqslant \|\mu\|$. This could equally well be derived from $\|s_N\mu\| \leqslant \|\mu\|$, which holds by definition of the norm. It is appropriate to compare this to the property that their respective integral kernels have $\|K_N\|(\log N)^{-1}$ and $\|D_N\|(\log N)^{-1}$ both tending to $4\pi^{-2}$ with increasing N. This will be verified later on.

Proposition 1.2. *L^1_{b0} and H^1_{b0} are Banach spaces under $\| \ \|$ and in addition closed convolution ideals of $M(\mathbb{T})$.*

Proof. Consider a sequence $\{f_k\}_1^\infty \subseteq L^1_{b0}$ such that $\sum \|f_k\| < \infty$. The previous proposition provides $f = \sum f_k$ as an element in L^1_b. We must prove $\|s_N f - f\|_1 \to 0$.

To an arbitrary $\varepsilon > 0$ we choose N such that $\sum_{N+1}^\infty \|f_k\| < \varepsilon/4$. Since $\|s_m f_k - f_k\|_1 \to 0$ as $m \to \infty$ for each individual $k \in [1, N]$, there is an M with the property that

$$m > M \quad \text{implies} \quad \max_{1\leqslant k\leqslant N} \|s_m f_k - f_k\|_1 < \varepsilon/2N.$$

In consequence, the condition $m > M$ also implies

$$\begin{aligned}
\|s_m f - f\|_1 &= \Big\|s_m\Big(\sum\nolimits_1^\infty f_k\Big) - \sum\nolimits_1^\infty f_k\Big\|_1 \\
&\leqslant \sum\nolimits_{k=1}^N \|s_m f_k - f_k\|_1 + \Big\| s_m\Big(\sum\nolimits_{N+1}^\infty f_k\Big)\Big\|_1 + \Big\|\sum\nolimits_{N+1}^\infty f_k\Big\|_1 \\
&< N \cdot \frac{\varepsilon}{2N} + 2\sum\nolimits_{k=N+1}^\infty \|f_k\| < \varepsilon.
\end{aligned}$$

Hence f is in fact an element of L^1_{b0} and this proves L^1_{b0} to be a Banach space. Since H^1_{b0} is the intersection of L^1_{b0} with H^1_b, the same conclusion holds for this space.

As regards to the ideal structure we consider any $f \in L^1_{b0}$ and $\nu \in M(\mathbb{T})$. Immediately we deduce $f * \nu \in M_b$. However, also

$$\|s_N(f*\nu) - f*\nu\|_1 = \|\nu * (s_N f - f)\|_1 \leqslant \|\nu\|_1 \|s_N f - f\|_1 \to 0 \quad \text{as} \quad N \to \infty,$$

since $f \in L^1_{b0}$. This proves L^1_{b0} to be a convolution ideal in $M(\mathbb{T})$ and the claim for H^1_{b0} follows as before.

The next observation is proved in [Z], pages 148–149, but an alternative demonstration follows from Helson's theorem in the next section.

Observation 1.3. *M_b consists of continuous measures alone.*

Most of the functions with strong integrability properties are elements in L^1_{b0}. For all $1 < p < \infty$ we know that (see [Z], page 266)

$$\|s_N f\|_1 \leqslant \|s_N f\|_p \leqslant C_p \|f\|_p, \qquad \|\tilde{s}_N f\|_1 \leqslant \|\tilde{s}_N f\|_p \leqslant C_p \|f\|_p,$$

and in addition

$$\|s_n f - f\|_1 \leqslant \|s_n f - f\|_p \to 0, \qquad \|\tilde{s}_n f - \tilde{f}\|_1 \leqslant \|\tilde{s}_n f - \tilde{f}\|_p \to 0.$$

This means that $\cup_{p>1} L^p \subseteq L^1_{b0}$ and $\cup_{p>1} H^p \subseteq H^1_{b0}$. Even more elementary we find $L^2 \hookrightarrow L^1_{b0}$ of unit norm, since Parseval's formula provides $\|s_N f\|_1 \leqslant \|s_N f\|_2 \leqslant \|f\|_2$ as well as $\|s_N f - f\|_1 \leqslant \|s_N f - f\|_2 \to 0$ with increasing N.

Theorem 1.4. *M_b contains singular measures and consequently $M_b \supsetneq L^1_b$:*

(1) *(Weiss* [We]*) There is a Riesz product $\mu \in M_{\mathrm{sing}}(\mathbb{T}) \cap M_b$,*
(2) *(Katznelson* [K]*) There is a singular Riesz product in M_b such that for all N the positivity $s_N\mu \geqslant 0$ obtains.*

We shall repeatedly have need for the Dirichlet and Fejér kernels and their conjugates. To set our standards, dictated by $s_N f = D_N * f$ with normalised Haar measure, the definitions are repeated here for convenience.

$$D_n(\theta) = \sum_{k=-n}^{n} e^{ik\theta} = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta},$$

$$K_n(\theta) = \sum_{k=-n}^{n} \left(1 - \tfrac{|k|}{n+1}\right) e^{ik\theta} = \frac{1}{n+1} \left\{ \frac{\sin\frac{n+1}{2}\theta}{\sin\frac{1}{2}\theta} \right\}^2,$$

$$\tilde{D}_n(\theta) = \sum_{k=-n}^{n} \operatorname{sign} k \, e^{ik\theta}, \quad \text{and} \quad \tilde{K}_n(\theta) = \sum_{k=-n}^{n} \operatorname{sign} k \left(1 - \tfrac{|k|}{n+1}\right) e^{ik\theta}.$$

Observe the convention $\operatorname{sign} 0 = 0$ and that $K_{-1} = 0$ etcetera, including $\tilde{D}_0 = \tilde{K}_0 = 0$. In addition, we will at several instances have need of the standard results $\|D_n\|_1 = 4\pi^{-2}\log(n+1) + \mathcal{O}(1)$ and $\|\tilde{D}_n\|_1 = 2\pi^{-1}\log(n+1) + \mathcal{O}(1)$.

A substantial amount of relevant examples of L^1-functions will be constructed based on the next classical result.

Lemma 1.5. *Let $\{a_n\}_{n=0}^{\infty}$ be a positive and convex sequence. Then $\sum_{-\infty}^{\infty} a_{|n|}e^{in\theta}$ represents a positive function f in L^1. In addition,*

$$\lim_{n\to\infty} \|f - s_n f\|_1 = \lim_{n\to\infty} 4\pi^{-2} a_n log\, n.$$

It follows that $f \in M_b$ if and only if $a_n \log n$ stays bounded.

The argument is standard and may be found in [Z], pages 183–185. A slightly stronger statement will be derived presently. Let us at this point single out a special and well known function. The preceding Lemma essentially gives the properties mentioned below.

Definition. The following function belongs to L^1_b.

$$\ell(\theta) = \sum_{n\geqslant 2} \frac{2\cos n\theta}{\log n} = \sum_{|n|\geqslant 2} \frac{e^{in\theta}}{\log |n|}.$$

In addition, $\ell(\theta) + (4/\log 2 - 2/\log 3)(1 + \cos\theta) \geqslant 1/\log 2$ everywhere.

An application of the lemma to ℓ really gives slightly more information:

Observation 1.6. $L^1_b \supsetneq L^1_{b0}$.

The additional elements in L^1_b produced by the Lemma are relatively few, as the following refined argument displays.

Proposition 1.7. *Let $\{a_n\}_{n=0}^{\infty}$ be positive and convex. If $f \sim \sum_{-\infty}^{\infty} a_{|n|}e^{in\theta}$ belongs to M_b, then $A = \lim_{n\to\infty} a_n \log n$ exists finitely and $f \in A\ell + L^1_{b0}$.*

Proof. Recall that any sequence with the specified properties necessarily also has $n\Delta a_n \to 0$ and that $\sum_{n\geqslant 0}(n+1)\Delta^2 a_n$ becomes a positive convergent series, summing to $a_0 - \lim a_n$. Here the difference notation $\Delta a_n = a_n - a_{n+1}$ and $\Delta^2 a_n = a_n - 2a_{n+1} + a_{n+2}$ is used. In particular, $\Delta a_n \log n \to 0$ as $n \to \infty$.

Considering next the equality

$$\Delta a_n \log n = a_n \log n - a_{n+1} \log(n+1) + a_{n+1} \log \tfrac{n+1}{n},$$

the known properties of $\{a_n\}$ tells us that the convergence $a_n \to 0$ is equivalent to the existence of $\lim_{n\to\infty} a_n \log n$ in the extended interval $[0, \infty]$.

The last lemma says further that $f \in M_b$ implies boundedness of $a_n \log n$, which in turn forces $a_n \to 0$ with increasing index and that $A = \lim_{n\to\infty} a_n \log n$ exists as a finite positive number.

Two expansions are readily established by two summations by parts:

$$s_N f = \sum_{n=0}^{N-2} (n+1)\Delta^2 a_n K_n + N\Delta a_{N-1} K_{N-1} + a_N D_N \quad \text{and}$$

$$f = \sum_{n=0}^{\infty} (n+1)\Delta^2 a_n K_n,$$

the infinite series being convergent in L^1.

Take now $g = A\ell$ and denote its Fourier coefficients with $b_n = A/\log|n|$ for $|n| \geqslant 2$ and $b_0 = b_{\pm 1} = 0$. The expansions of g similar to those of f are immediate. Except for $\Delta^2 b_1 < 0$ the same kind of positivity and convergence hold for g as for f.

Observe that the ingredients were arranged so as to give

$$(a_n - b_n)\log n = a_n \log n - A \to 0 \quad \text{as} \quad n \to \infty.$$

It is clear that in L^1 the difference $(f-g) - s_N(f-g)$ is given by

$$-(a_N - b_N)D_N + \sum\nolimits_{n=N-1}^{\infty}(n+1)\{\Delta^2 a_n - \Delta^2 b_n\}K_n - N\{\Delta a_n - \Delta b_n\}K_{N-1}.$$

We know that the series and the last term converge to zero in L^1 as N increases. Hence we deduce that

$$\|(f-g) - s_N(f-g)\|_1 = 4\pi^{-2}|a_N - b_N|\log N + o(1) \to 0 \quad \text{as} \quad N \to \infty.$$

The conclusion is $f - A\ell = f - g \in L^1_{b0}$, and the claim has been fully established.

Remark. It is not true that L^1_b/L^1_{b0} is of dimension 1. This follows from an example in section 3.

Recall next the standard notation μ_τ and f_τ for $\mu_\tau(E) = \mu(E-\tau)$ and $f_\tau(\theta) = f(\theta - \tau)$. Since the norm $\|\,\|_1$ is translation invariant and $s_N\mu_\tau = (s_N\mu)_\tau$ we conclude the following result.

Observation 1.8. *M_b is translation invariant with all norms preserved after translation.*

This means that the property of being a homogeneous space (see Katznelson [K] for a treatment) for any of our five spaces is equivalent to the validity of $\|\mu_\tau - \mu\| \to 0$ as $\tau \to 0$.

Theorem 1.9. *L^1_{b0} is a homogeneous space, whereas L^1_b is not. In particular, there are $g \in L^1_b \setminus L^1_{b0}$ with $\|g_\tau - g\| \not\to 0$, as $\tau \to 0$.*

Proof. It obvious that as an operator $\|s_N\|_{L^1_b \to L^1_b} = 1$, all N. The assumption that L^1_b were a homogeneous space, would lead to $\|f - s_N f\| \to 0$, as $N \to \infty$, according to [K], Theorem II:1.1. However, the last lemma demonstrates that $\ell \in L^1_b$, in spite of the fact that $\|\ell - s_N\ell\|_1 \to 4\pi^{-2}$, whence $\liminf_N \|\ell - s_N\ell\| \geqslant 4\pi^{-2}$. It follows that L^1_b impossibly can be a homogeneous space in its own norm and hence that there are elements $g \in L^1_b$ with $\|g - g_\tau\| \not\to 0$.

On the other hand it is claimed that for any $f \in L^1_{b0}$ one always has $\|f - f_\tau\| \to 0$ as $\tau \to 0$. To that end one takes an arbitrary $\varepsilon > 0$ and choose a corresponding M with $\|s_m f - f\|_1 < \varepsilon/3$ for all $m \geqslant M$, thanks to $f \in L^1_{b0}$. Since L^1 is a homogeneous space in its own norm, $f \in L^1$ and since $s_0 f, \dots, s_M f$ are continuous, there is a $\delta > 0$ with the property

$$|\tau| < \delta,\ 0 \leqslant k \leqslant M \quad \text{implies} \quad \|f - f_\tau\|_1,\ \|s_k f - (s_k f)_\tau\|_1 < \varepsilon/3.$$

The present proof needs $\|f - f_\tau\| = \sup_{N \geqslant 0}\|s_N f - (s_N f)_\tau\|_1 < \varepsilon$ for all $|\tau| < \delta$.

For the indices $0 \leqslant N \leqslant M$ the choice of δ provides $\|s_N f - (s_N f)_\tau\|_1 < \varepsilon/3$ if only $|\tau| < \delta$. The remaining case $N \geqslant M$ follows from the next inequality, valid as soon as $|\tau| < \delta$.

$$\begin{aligned}\|s_N f - (s_N f)_\tau\|_1 &= \|s_N f - f + f - f_\tau + (f - s_N)_\tau\|_1 \\ &\leqslant \|s_N f - f\|_1 + \|f - f_\tau\|_1 + \|(f - s_N f)_\tau\|_1 \\ &= 2\|s_N f - f\|_1 + \|f - f_\tau\|_1 < \varepsilon.\end{aligned}$$

Hence $|\tau| < \delta$ implies $\|f - f_\tau\| \leqslant \varepsilon$, which means that L^1_{b0} indeed is a homogeneous space in its own norm.

Corollary 1.10. *The closed linear hull in L^1_b of the polynomial subspace is exactly the space L^1_{b0}.*

Proof. Every polynomial is trivially an element of L^1_{b0}, whence the norm-closure must be a subspace of L^1_{b0}. But L^1_{b0} is a homogeneous space, so [K], Theorem I:2.11, shows that for the Fejér means $\sigma_N f$ the convergence $\|\sigma_N f - f\| \to 0$, as $N \to \infty$, obtains for all $f \in L^1_{b0}$. Hence the polynomials are dense in L^1_{b0}.

At this stage we can formally strengthen the definition of L^1_{b0} simply by interpreting the previous result as producing three different characterisations of L^1_{b0}.

Lemma 1.11. *In L^1_{b0} the convergence $\|s_N f - f\| \to 0$ obtains as $N \to \infty$ for each individual $f \in L^1_{b0}$. The same holds in H^1_{b0}.*

Proof. L^1_{b0} is a homogeneous space with $\sup_N \|s_N\|_{L^1_{b0} \to L^1_{b0}} = 1$. The norm convergence of $s_N f$ to f follows from [K], Theorem II:1.1. Finally, $H^1_{b0} \subseteq L^1_{b0}$ takes care of the last claim.

Theorem 1.12.

$$\begin{aligned}L^1_{b0} &= \{f \in L^1_b\,;\ \|s_N f - f\| \to 0 \textit{ as } N \to \infty\} \\ &= \{f \in L^1_b\,;\ \|s_N f - f\|_1 \to 0 \textit{ as } N \to \infty\} \\ &= \{f \in L^1_b\,;\ \tau \mapsto f_\tau \textit{ is cont. in } L^1_b\textit{-norm}\}.\end{aligned}$$

Proof. Write X_1, X_2, and X_3 for the three braced spaces in the given order. By definition $L^1_{b0} = X_2$. By the argument used in the preceding Theorem we have the implications (the first is trivial)

$$\|s_N f - f\| \to 0 \quad \Rightarrow \quad \|s_N f - f\|_1 \to 0 \quad \Rightarrow \quad \tau \mapsto f_\tau \text{ is cont. and } L^1_b\text{-valued.}$$

In consequence, $X_1 \subseteq X_2 \subseteq X_3$. The last Lemma yields $L^1_{b0} \subseteq X_1 \subseteq X_2 = L^1_{b0}$, whence $X_1 = X_2$.

Consider now an element $f \in X_3$, such that the L^1_b-valued map $\tau \mapsto f_\tau$ is norm-continuous. Due to the relation $\|\ \|_{L^1} \leqslant \|\ \|_{L^1_b}$, the Fejér kernel K_n ensures that $\int K_n(\tau) f_\tau \, d\tau/2\pi = K_n * f$ in L^1_b as well as L^1. The continuity of $\tau \mapsto f_\tau$ provides the convergence $\|K_n * f - f\| \to 0$ as $\tau \to 0$. Since $K_n * f$ is a polynomial, it follows that

$$X_3 \subseteq \overline{\{\text{polyn.}\}}^{L^1_b} = L^1_{b0} = X_2,$$

according to the last Corollary. Hence $L^1_{b0} \subseteq X_3 \subseteq L^1_{b0}$ and the proof is complete.

Example. We now know that the element $\ell(\theta) = \sum_{n \geqslant 2} \frac{2 \cos n\theta}{\log n} \in L^1_b \setminus L^1_{b0}$ has the property $\|\ell - \ell_\tau\| \not\to 0$ as $\tau \to 0$, in spite of the fact that $\|\ell - \ell_\tau\|_1 \to 0$; this latter limit since $\ell \in L^1$.

On the other hand, a result of Doss [D] demonstrates that $\limsup_{\tau \to 0} \|\mu - \mu_\tau\|_1 = 2\|\nu\|_1$, where $\mu = g d\theta + \nu$, $\nu \perp d\theta$. M_b contains singular measures, so $\|\mu - \mu_\tau\| \not\to 0$ occurs in M_b. The preceding paragraph shows that it also occurs in L^1_b.

It is convenient at this point to introduce a subset of M_b as follows.

$$M_b^+ = \{\mu \in M_b \, ; \, s_N\mu \geqslant 0, \text{ all } N \geqslant 0\}.$$

Obviously, M_b^+ is a nonempty positive cone as a subset of M_b. By Katznelson's example [K2] there are non-trivial singular measures in this cone. On the other hand, M_b^+ contains also many harmonic functions.

A theorem by Sidon [S] and Armitage [A] demonstrates that any function g harmonic in the unit disk and non-negative on $\mathbb{T}$ has the property that every partial sum of $g(e^{i\theta}/2)$ is non-negative, and hence that this latter function belongs to M_b^+. The theorem alluded to also says that this radius $r = 1/2$ is the largest possible with this property. In conclusion, we have established $M_b^+ \cap M_{\text{sing}}(\mathbb{T}) \neq \emptyset$ and $M_b^+ \cap L^1_{b0} \neq \emptyset$.

To round off this section, some simple observations on computational properties of the present norm will be provided.

Lemma 1.13. *For every $\mu \in M(\mathbb{T})$ and $N \geqslant 1$ holds $\|s_0\mu\|_1 = |\hat{\mu}(0)| \leqslant \|s_N\mu\|_1$. Consequently, $\|\mu\| = \sup_{N \geqslant 1} \|s_n\mu\|_1$.*

Proof. Trivially, $\|s_0\mu\|_1 = \left|\int d\mu\right| = \left|\int s_N\mu \, d\theta/2\pi\right| \leqslant \|s_N\mu\|_1$. The supremum defining $\|\mu\|$ and involving $N \geqslant 0$ may hence be reduced to $N \geqslant 1$ without loss of information.

Proposition 1.14. *The equality $\|\mu\| = \|s_0\mu\|_1$ obtains for $\mu \in M_b$ if and only if μ belongs to $e^{i\alpha}M_b^+$ for some α. If this is the case, then $\|s_N\mu\|_1 = \|\mu\|$ for all N.*

Remark. This result is the counterpart to the well-known characterisation that $f \in L^1$ has $\|f\|_1 = |\hat{f}(0)|$ if and only if $f(e^{i\theta}) = e^{i\alpha}|f(e^{i\theta})|$ almost everywhere. Here α is uniquely determined for $f \not\equiv 0$ by $e^{i\alpha}\hat{f}(0) > 0$.

Proof. Consider first $\mu = e^{i\alpha}\nu$, $\nu \in M_b^+$. Then $|\mu| = \nu = e^{-i\alpha}\mu$. Hence $s_N\mu = e^{i\alpha}s_N\nu$ for all $N \geqslant 0$ and, in addition,

$$\|s_N\mu\|_1 = \|s_N\nu\|_1 = \int s_N\nu \, d\theta/2\pi = \hat{\nu}(0) = e^{-i\alpha}\hat{\mu}(0) = |\hat{\mu}(0)| = \|s_0\mu\|_1.$$

From this follows $\|\mu\| = \|s_0\mu\|_1 = \|s_N\mu\|_1$ for all N.

Suppose on the other hand that $\|\mu\| = \|s_0\mu\|$, $\mu \neq 0$. Since by the last Lemma every $N \geqslant 0$ gives $\|s_N\mu\|_1 \geqslant |\hat{\mu}(0)| = \|s_0\mu\|_1$, we must deduce the equality $\|s_N\mu\| = |\hat{\mu}(0)| = |\widehat{s_N\mu}(0)|$ for all N. The Fourier coefficient $\widehat{s_N\mu}(0)$ does not depend on N, so the remark above provides a single α ($\mu \neq 0$ makes some $s_N\mu$ non-trivial) such that $e^{i\alpha}\hat{\mu}(0) > 0$ and $s_N\mu = e^{i\alpha}|s_N\mu|$ for all N. Now the measure $\nu = e^{-i\alpha}\mu$ has the property $s_N\nu \geqslant 0$ for all N, whence indeed $\mu \in e^{i\alpha}M_b^+$.

Observation 1.15. $\|\mu\| > \overline{\lim}_{N\to\infty} \|s_N\mu\|_1$ *does occur.*

Consider for this purpose the Fejér element $k_2(\theta) = \frac{3}{2} + 2\cos\theta + \cos 2\theta$. Then $k_2(\theta) = \frac{1}{2}(\sin\frac{3}{2}\theta/\sin\frac{1}{2})^2 \geqslant 0$ and every $N \geqslant 2$ has $\|s_N k_2\|_1 = \frac{3}{2} = \|s_0 k_2\|_1$. On the other hand, $s_1 k_2(\theta) = \frac{3}{2} + 2\cos\theta$ assumes both signs, whence $\|s_1 k_2\|_1 > \frac{3}{2}$, which is to be interpreted as $\|k_2\| = \|s_1 k_2\|_1 > \frac{3}{2} = \lim_{N\to\infty} \|s_N k_2\|_1$.

This last observation is just a scratch on the surface, as the next result shows. An even more explicit version will be discussed in the next section.

It is convenient to introduce a norm by the prescription $|\!|\!|\,\mu\,|\!|\!| = \overline{\lim}_{N\to\infty} \|s_N\mu\|_1$. That this defines a norm, finite exactly on M_b, is elementary. In fact, one quickly establishes $\|\mu\|_1 \leqslant |\!|\!|\,\mu\,|\!|\!| \leqslant \|\mu\|$. That this new norm is not complete on L^1_{b0} is the content of the next result.

Proposition 1.16. $\sup\left\{ \dfrac{\|f\|}{|\!|\!|\,f\,|\!|\!|} \,;\, f \in L^1_{b0} \setminus \{0\} \right\} = \infty.$

Proof. Since $\|s_N f - f\|_1 \to 0$ obtains for all $f \in L^1_{b0}$, as $N \to \infty$, we find $\|s_N f\|_1 \to \|f\|_1$ and $|\!|\!|\,f\,|\!|\!| = \|f\|_1$ in L^1_{b0}.

That the supremum in the statement would be finite, is the same as claiming the equivalence $\|\;\| \sim |\!|\!|\;|\!|\!|$ on L^1_{b0}. This would mean that $(L^1_{b0}, |\!|\!|\;|\!|\!|) = (L^1_{b0}, \|\;\|_1)$, by the previous paragraph, would be a complete normed space. This can be refuted as follows.

Take an element $f \in L^1 \setminus L^1_{b0}$ and polynomials p_n such that $p_n \to f$ in L^1. Since $\{p_n\}_1^\infty \subseteq L^1_{b0}$ and $|\!|\!|\,p_n - p_m\,|\!|\!| = \|p_n - p_m\|_1 \to 0$ as $n, m \to \infty$, we deduce that $\{p_n\}_1^\infty$ is a Cauchy sequence also in $(L^1_{b0}, |\!|\!|\;|\!|\!|)$. However, from L^1-$\lim p_n = f$ and $\|\;\|_1 \leqslant |\!|\!|\;|\!|\!|$, it follows that the only possible point of convergence, that is $f \in L^1 \setminus L^1_{b0}$, is such that the Cauchy sequence $\{p_n\}_1^\infty$ cannot be convergent in $(L^1_{b0}, |\!|\!|\;|\!|\!|)$.

2. Coefficient properties.

The demands put on a measure by the requirement of having norm-bounded partial sums do not put restraints on individual Fourier coefficients, since addition of $ae^{in\theta}$ just changes the norm, without disturbing the membership of M_b. The asymptotics of the coefficients, however, are heavily influenced by this demand. The following statement was first conjectured by Steinhaus and subsequently proved by Helson.

Helson's Theorem [H]**.** *For any* $\mu \in M_b$, $\lim_{|n|\to\infty} |\hat{\mu}(n)| = 0$ *obtains.*

The argument used in the proof was later named as the Helson translation lemma.

As an immediate corollary we see that

$$\mu \in M_b \quad \text{implies} \quad \frac{1}{2N+1} \sum_{n=-N}^{N} |\hat{\mu}(n)|^2 \to 0 \text{ as } N \to \infty.$$

By Wiener's theorem it follows that M_b contains only continuous measures, a result achieved also by other means in [Z], as mentioned in the first section.

A different way of capturing the decay of the Fourier coefficients, in a mean value sense this time, is provided by the following result. Its proof is based on Hardy's inequality for functions in H^1. Essentially the same technique will be employed in the next section to get one useful strengthened form, valid for analytic functions.

Salem–Zygmund's Theorem [Z]. *If $\mu \in M_b$, then for a finite constant C and all n*

$$\frac{\log n}{n} \sum\nolimits_{k=-n}^{n} |\hat{\mu}(k)| \leqslant C\|\mu\|.$$

To my knowledge only two efforts have been made to give a quantitavely strengthened version of Helson's theorem. The earliest result was obtained by Pigno–Smith [PS] and then later Fournier [F] established a statement using another method. Each of the two papers exhibit a function $L(b)$ yielding the conclusion below, but as neither seems optimal, Fournier remarks on this, they will not be repeated here.

Theorem 2.1. ([PS] and [F]) *There exists an integer-valued function $L(b)$ defined on $]\,0,1[$, such that in cardinality*

$$\big|\{n \in \mathbb{Z}\,;\, |\hat{\mu}(n)| \geqslant b\,\|\mu\|\}\big| \leqslant L(b),$$

for every measure μ belonging to M_b.

It seems that the reason for the discrepancies between $M(\mathbb{T})$ and M_b is due to the fact that the Dirichlet and Fejér kernels share essentially the same behaviour in norm inside L^1_{b0}, but of course differ significantly in L^1. This is the content of the next few paragraphs.

It is evident that $\|D_N\|$ coincides with

$$\sup_{0\leqslant n\leqslant N} \|D_n\|_1 = 4\pi^{-2}\log N + \mathcal{O}(1).$$

As a general tool to estimate $\|s_N\mu\|_1$ we also have the inequality, obtained by realising the partial sum as convolution with the corresponding Dirichlet kernel,

$$\frac{\|s_M\mu\|_1}{\|\mu\|_1} \leqslant \|D_M\|_1 = 4\pi^{-2}\log M + \mathcal{O}(1).$$

An application of this with $\mu = K_N\,d\theta$, and taking maximum of all M between 0 and N, provides the upper bound in the next claim.

Proposition 2.2.

$$\|K_N\| \begin{cases} \leqslant 4\pi^{-2}\log N + \mathcal{O}(1), \\ \geqslant 4\pi^{-2}\log N - \mathcal{O}(\log\log N). \end{cases}$$

Proof. For $0 < M \leqslant N$ and $N \geqslant 2$, we have

$$\begin{aligned} s_M K_N &= \sum_{n=-M}^{M} \Big(1 - \tfrac{|n|}{N+1}\Big)\, e^{in\theta} = \Big(1 - \tfrac{M}{N+1}\Big)\, D_M + \sum_{n-M+1}^{M-1} \Big(\tfrac{M-|n|}{N+1}\Big)\, e^{in\theta} \\ &= \Big(1 - \tfrac{M}{N+1}\Big)\, D_M + \tfrac{M}{N+1}\, K_{M-1}. \end{aligned}$$

Since $\big\|\frac{M}{N+1}\,K_{M-1}\big\|_1 = \frac{M}{N+1}$, it follows that

$$\|s_M K_N\|_1 = \big\|\big(1 - \tfrac{M}{N+1}\big) D_M\big\|_1 + \mathcal{O}(1) = 4\pi^{-2}\big(1 - \tfrac{M}{N+1}\big)\log M + \mathcal{O}(1).$$

A choice of M as the integer closest to $(N+1)/\log N$ provides a lower bound for $\|K_N\|$ as

$$4\pi^{-2}\,\frac{\log N-1}{\log N}\,\log\frac{N+1}{\log N}+\mathcal{O}(1)=4\pi^{-2}\log N-\mathcal{O}(\log\log N).$$

The proof is complete.

This last proposition provides us with explicit functions to describe the distance from L^1_b to L^1 using the natural identity as embedding. Writing $d_N = \sup\{\|p\|/\|p\|_1\,;\, p \text{ polynomial of degree } N\}$ we find from the proposition and its immediately preceding displayed formula, that for suitable constants $c_2 > c_1 > 0$

$$c_1\log(N+1)\leqslant d_N\leqslant c_2\log(N+1).$$

This quantifies the last proposition in section 1.

3. The analytic spaces.

When considering norm-bounded partial sums inside the Hardy space H^1, the added analyticity and its consequences causes the previous results for L^1_b to have counterparts of a more elusive character. They simply need stronger methods than do L^1_b or L^1_{b0}. It is at this point also worthwhile to recall the set inclusion $H^1_{b0}\supseteq \cup_{p>1}H^p$ from the first section.

Lemma 3.1. *Let $\{a_n\}_{n=0}^{N+1}$ be a positive and convex sequence with the added property that $a_{N+1}=0<a_N$. The analytic polynomial $A(z)=\sum_{n=0}^{N}a_nz^n$ has an expansion*

$$\begin{aligned}2A(e^{i\theta}) &= a_0+\sum_{n=0}^{N-1}(n+1)\Delta^2a_nK_n(\theta)+(N+1)a_NK_N(\theta)\\ &\quad +i\sum_{n=1}^{N-1}(n+1)\Delta^2a_n\tilde{K}_n(\theta)+i(N+1)a_N\tilde{K}_N(\theta).\end{aligned}$$

In particular, $\operatorname{Re}A(e^{i\theta})\geqslant 0$ *and* $\operatorname{sign}(\theta)\operatorname{Im}A(e^{i\theta})\geqslant 0$ *for all* $|\theta|\leqslant\pi$.

Proof. Starting from $A(e^{i\theta})=\frac12a_0+\frac12a_0+\sum_{n=1}^{N+1}a_n\cos n\theta+i\sum_{n=1}^{N+1}a_n\sin n\theta$, two repeated summations by parts give the claimed expansion, just bearing in mind that $A_{N+1}=0$ yields $\Delta a_N=a_N$. Due to $K_n\geqslant 0$ and $\operatorname{sign}(\theta)\tilde{K}_n(\theta)\geqslant 0$ for all $n\geqslant 0$, the claimed inequalities follow from $\Delta^2a_n\geqslant 0$, which is due to the convexity.

Proposition 3.2.

i) $\left\|1+\frac{N}{N+1}z+\frac{N-1}{N+1}z^2+\cdots+\frac{1}{N+1}z^N\right\|_1=2\pi^{-1}\log(N+1)+\mathcal{O}(1)$.

ii) *The function $A_N(z)=1+2z+3z^2+\cdots+Nz^{N-1}+(N-1)z^N+\cdots+z^{2N-2}$ has the properties that $\|A_N\|_1=N$ and $\|s_MA_N\|_1=2\pi^{-1}M\log M+\mathcal{O}(1)$ for all $0<M\leqslant N-1$. In particular, these M give*

$$\frac{\|s_MA_N\|_1}{\|A_N\|_1}=2\pi^{-1}\frac{M}{N}\log M+\mathcal{O}(1)\quad\textit{and}\quad\frac{\|A_N\|}{\|A_N\|_1}=2\pi^{-1}\log N+\mathcal{O}(1).$$

Proof. The Lemma can be applied with $a_n = 1 - \frac{n}{N+1}$, $0 \leqslant n \leqslant N+1$, and gives

$$1 + \tfrac{N}{N+1} e^{i\theta} + \tfrac{N-1}{N+1} e^{2i\theta} + \cdots + \tfrac{1}{N+1} e^{iN\theta} = \tfrac{1}{2} a_0 + \tfrac{1}{2} K_N(\theta) + \tfrac{i}{2} \tilde{K}_N(\theta).$$

Since $\tilde{D}_n^*(\theta) = \tilde{D}_n(\theta) - \sin n\theta$, has $\operatorname{sign} \cdot \tilde{D}_n^* \geqslant 0$ and $\|\tilde{D}_N^*\|_1 = 2\pi^{-1} \log n + \mathcal{O}(1)$, the representation

$$\tilde{K}_n = \tfrac{1}{n+1} \sum_{\nu=1}^{n} \tilde{D}_n = \tfrac{1}{n+1} \sum_{\nu=1}^{n} \tilde{D}_n^* + \mathcal{O}(1)$$

shows that

$$\begin{aligned} \|\tilde{K}_n\|_1 &= 2\pi^{-1} \tfrac{1}{n+1} \sum_{\nu=1}^{n} \log n + \mathcal{O}(1) = 2\pi^{-1} \log n!^{1/n+1} + \mathcal{O}(1) \\ &= 2\pi^{-1} \log n + \mathcal{O}(1). \end{aligned}$$

Due to $\|K_n\|_1 = 1$ the first claim follows.

Observing $e^{i(1-N)\theta} A_N(e^{i\theta}) = N K_{N-1}(\theta)$ we deduce $\|A_N\|_1 = N$. Furthermore, we see that for $0 < M \leqslant N-1$

$$e^{iM\theta} s_M A_N(e^{-i\theta}) = (M+1) + M e^{i\theta} + \cdots + e^{iM\theta},$$

whence an application of $i)$ verifies the norm

$$\|s_M A_N\|_1 = 2\pi^{-1} M \log M + \mathcal{O}(M).$$

We need to consider also $N-1 \leqslant M \leqslant 2N-3$ in order to decide on $\|A_N\|$, since $s_{2N-2} A_N = A_N$. For such M a little reflection on the different partial sums shows that $\|s_M A_N\|_1 = \|A_N - s_{2N-3-M} A_N\|_1$, which essentially recovers the partial sums already calculated in norm. We may therefore deduce

$$\frac{\|A_N\|}{\|A_N\|_1} = \max_{1 \leqslant M < N} \frac{\|s_M A_N\|_1}{\|A_N\|_1} + \mathcal{O}(1) = 2\pi^{-1} \log N + \mathcal{O}(1)$$

and the claim has been demonstrated.

In the first section the classical construction of elements in L_b^1 based on convex sequences was discussed. There is also a counterpart for the analytic spaces, whose at first surprising strength is due to the extra demands posed by analyticity.

Proposition 3.3. *Let a_n decrease to zero and consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$, a function analytic in the interior of the unit disk. The following properties occur or fail simultaneously.*

(1) $f \in H_{b0}^1$,
(2) $f \in H^1$,
(3) $\operatorname{Im} f(e^{i\theta}) = \sum_{n \geqslant 1} a_n \sin n\theta \in L^1$,
(4) $\sum_{n \geqslant 2} \Delta a_n \log n < \infty$, *and*
(5) $\sum_{n \geqslant 1} n^{-1} a_n < \infty$.

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3) is obvious. Assume therefore $\operatorname{Im} f \in L^1$. We aim at proving $\operatorname{Re} f$, $\operatorname{Im} f \in L^1_{b0}$ from this assumption.

According to Zygmund [Z], Theorem V:1.14, $a_n \searrow 0$ makes $\operatorname{Im} f \in L^1$ equivalent to $\sum \Delta a_N \log n < \infty$ and in case of their truth, even $\|\operatorname{Im} f - s_N \operatorname{Im} f\|_1 \to 0$. In particular, (3)$\Leftrightarrow$(4) and their validity implies $\operatorname{Im} f \in L^1_{b0}$.

Furthermore, the argument after formula [Z], V:(1.16), gives property (5) as equivalent to (4), given the present decreasing sequence. In the proof of that equivalence, it was remarked that $a_n \log n \to 0$ is a consequence.

A summation by parts argument provides $2s_N \operatorname{Re} f = a_0 + \sum_{n=0}^{N-1} \Delta a_n D_n + a_N D_N$, from which the expression

$$\operatorname{Re} f - s_N \operatorname{Re} f = -\tfrac{1}{2} a_N D_N + \tfrac{1}{2} \sum_{n=N}^{\infty} \Delta a_n D_n$$

is useful to us. Taking norms we find

$$\|\operatorname{Re} f - s_N \operatorname{Re} f\|_1 \leqslant C \left\{ a_N \log N + \sum_{n=N}^{\infty} \Delta a_n \log n \right\} \to 0 \quad \text{as} \quad N \to \infty.$$

Hence first $\operatorname{Re} f \in L^1_{b0}$ and then $f \in L^1_{b0}$ follow. This shows that the only remaining implication (3)$\Rightarrow$(1) holds true.

Remark. $\operatorname{Re} f$ is always continuous on $\mathbb{T} \setminus \{0\}$ and has a generalised Riemann integral on $\mathbb{T}$; cf. [Z], V:1.8, yielding this from the assumption $a_n \searrow 0$ only. Consequently, $\operatorname{Re} f(e^{i\theta})d\theta$ is a Radon measure on $\mathbb{T}$, but not necessarily a measure in Lebesgue's sense. This influences H^1_b only indirectly, since $\operatorname{Im} f$ determines the membership $f \in H^1_b$.

The same construction as above can be modified to produce a class of elements in H^1_b as follows. The preceding Proposition shows that the condition cannot be relaxed unless substantially altered in character.

Proposition 3.4. *Let the real number α and the sequence $\{a_n\}_{n=0}^{\infty} \subset \mathbb{C}$ be such that $a_n \to 0$ and $\sum_{n=2}^{\infty} |a_n - e^{i\alpha} a_{n+1}| \log n < \infty$. Then $\sum_{n=0}^{\infty} a_n z^n \in H^1_{b0}$.*

Proof. The general case can be reduced in the following manner. Write $b_n = e^{in\alpha} a_n$. Then $\sum |b_n - b_{n+1}| \log n = \sum |a_n - e^{i\alpha} a_{n+1}| \log n$, so the simple identity

$$\sum a_n z^n = \sum b_n w^n \Big|_{w=e^{-i\alpha}}$$

reduces general α to the case of $\alpha = 0$.

Take now the stated conditions on $\{a_n\}_0^{\infty}$ for $\alpha = 0$. In particular, $\sum_{n=3}^{\infty} |\Delta a_n| \leqslant \sum |\Delta a_n| \log n < \infty$, whence $a_N = \sum_{n=N}^{\infty} \Delta a_n$ for all N. Multiplying we get $a_N \log N = \sum_N^{\infty} \Delta a_n \log N$, from which follows

$$|a_N| \log N \leqslant \sum_{n=N}^{\infty} |\Delta a_n| \log n \to 0 \quad \text{as} \quad N \to \infty.$$

Next a summation by parts yields

$$\sum_{n=1}^{N} 2a_n \sin n\theta = \sum_{n=1}^{N-1} \Delta a_n \tilde{D}_n(\theta) + a_N \tilde{D}_N(\theta), \qquad \sum_{n=1}^{\infty} 2a_n \sin n\theta = \sum_{n=1}^{\infty} \Delta a_n \tilde{D}_n(\theta),$$

the latter equality being momentarily understood pointwise. In L^1-norm as $N \to \infty$

$$\Big\| \sum\nolimits_{n=N+1}^{\infty} a_n \sin n\theta \Big\|_1 \leqslant \tfrac{1}{2}|a_N| \, \big\|\tilde{D}_N\big\|_1 + \tfrac{1}{2} \sum\nolimits_{n=N}^{\infty} |\Delta a_n| \, \big\|\tilde{D}_n\big\|_1$$
$$\leqslant C \Big\{ |a_N| \log N + \sum\nolimits_{n=N}^{\infty} |\Delta a_n| \log N \Big\} \to 0$$

Consequently $\sum_{n=1}^{\infty} a_n \sin n\theta \in L^1_{b0}$. Furthermore, a similar argument based on

$$a_0 + \sum_{n=0}^{N} 2a_n \cos n\theta = \sum_{n=0}^{N-1} \Delta a_n D_n(\theta) + a_N D_N(\theta)$$

provides $\sum_{n=0}^{\infty} a_n \cos n\theta \in L^1_{b0}$, adding up to the conclusion $\sum_{n=0}^{\infty} a_n z^n \in H^1_{b0}$.

We need also to understand how certain elementary multiplication operators behave on H^1_b, the main reason being a desire to understand how coefficient shifts affect the norm.

Proposition 3.5. *Consider $f \in H^1_b$ and $n \geqslant 0$. Then $\|z^n f\| = \|f\|$ always, whereas $\|\overline{z} f\| \leqslant \|f\|$ and $\|\overline{z}^n f\| \leqslant 2\|f\|$. In addition, $\|f\| = |\hat{f}(0)| = |f(0)|$ if and only if f is constant.*

Remark. Observe that in general $\overline{z}^n f(z) \in L^1_b$ when $f \in H^1_b$. It will be seen in the next sextion that the uniform boundedness for these multiplication operators is particular to the subspace H^1_b of L^1_b, it breaks down for L^1_{b0}.

Proof. Since $s_N(z^n f) = z^n s_{N-n} f$ for $N \geqslant n$, while $= 0$ otherwise, it is obvious that $\|z^n f\| = \sup_{M \geqslant 0} \|s_M f\|_1 = \|f\|$.

Furthermore, $s_N(\overline{z} f) = \overline{z} s_{N+1} f$, for $N \geqslant 1$. An earlier lemma, telling that only partial sums with $N \geqslant 1$ are needed, then demonstrates

$$\|\overline{z} f\| = \sup_{n \geqslant 1} \|s_N(\overline{z} f)\|_1 = \sup_{N \geqslant 2} \|s_N f\|_1 \leqslant \|f\|.$$

On the other hand, $n \geqslant 2$ and $f \in H^1_b$ give $\overline{z}^n f(z) = a_0 \overline{z}^n + \cdots + a_n + a_{n+1} z + \ldots$, whence

$$s_N(\overline{z}^n f) = \begin{cases} \overline{z}^n s_{n+N} f & \text{if} \quad N \geqslant n, \\ \overline{z}^n \left(s_{n+N} f - s_{n-N-1} f \right) & \text{if} \quad 0 \leqslant N < n. \end{cases}$$

Hence $\|s_N(\overline{z}^n f)\|_1 \leqslant 2\|f\|$ for each N, and $\|\overline{z}^n f\| \leqslant 2\|f\|$ follows.

Finally, since every function in the Nevanlinna class $\mathcal{N}$ which has a positive boundary function, must in fact be constant and since $H^1_b \subseteq \mathcal{N}$, we observe that $M_b^+ \cap \mathcal{N}$ consists of positive constant functions alone. The Proposition that characterised the property $\|\mu\| = \|s_0 \mu\|_1$ in M_b, now proves the claim as to when $\|f\| = |\hat{f}(0)|$ takes place inside H^1_b.

Example. Considerations of $k(z) = 1 + 2z + z^2$ gives an obvious case when $\|k\| = \|s_1\|_1 > \sup_{N \neq 1} \|s_N k\|_1 = 2 = \|\overline{z}\, k(z)\|$, and hence an example that $\overline{z} f$ can have smaller norm than $f \in H^1_b$.

Now we are ready to derive a counterpart of Salem–Zygmund's Theorem that is able to capture more of the properties intrinsic to H^1_b. The result will be useful in showing that the Hardy space H^1 is strictly larger that H^1_b.

Proposition 3.6. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function in H_b^1. Then for $N, M \geqslant 0$ the inequality*

$$\left\| \sum_{n=0}^{N} a_{M+n} z^n \right\|_1 \leqslant 2 \|f\|$$

obtains. In particular, all of the functions $\sum_{n=0}^{\infty} a_{M+n} z^n$, $\sum_{n=0}^{N} a_{M+n} z^n$, and $\sum_{n=0}^{N} a_{M-n} z^n$ in H_b^1, have norms not exceeding $2\|f\|$. For the latter expression $0 \leqslant N \leqslant M$ is assumed.

Proof. $s_{M+N}f - s_{M-1}f = \sum_{n=0}^{N} a_{M+n} z^{n+M}$ ensures the inequality

$$\left\| \sum_{n=0}^{N} a_{M+n} z^n \right\|_1 = \|s_{M+N}f - s_{M-1}f\|_1 \leqslant 2\|f\|.$$

This holds true for all $N, M \geqslant 0$.

For each of the three functions in the second part of the statement, every partial sum has L^1-norm at most $2\|f\|$ by the first claim. It follows that their norms in H_b^1 are at most the stated quantity.

Theorem 3.7. *For every function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in H_b^1, the following two inequalities hold for every $M \geqslant 0$.*

$$\sum_{n=0}^{\infty} \frac{|a_{M+n}|}{n+1} \leqslant 2\pi \|f\| \quad \textit{and} \quad \sum_{n=0}^{M} \frac{|a_{M-n}|}{n+1} \leqslant 2\pi \|f\|.$$

Proof. Hardy's inequality $\sum_{n=0}^{\infty} \frac{|b_n|}{n+1} \leqslant \pi \left\| \sum_{n=0}^{\infty} b_n z^n \right\|_1$ can be applied to the norm estimates from the previous Proposition:

$$\left\| \sum_{n=0}^{\infty} a_{M+n} z^n \right\|_1 \leqslant 2\|f\| \quad \text{and} \quad \left\| \sum_{n=0}^{M} a_{M-n} z^n \right\|_1 \leqslant 2\|f\|.$$

The claimed inequalities are results of this application.

For convenience, let us introduce an analytic variant of the Fejér kernel.

$$F_N(z) = \sum_{n=0}^{2N} \left(1 - \tfrac{|n-N|}{N+1}\right) z^n.$$

Somewhat abusively one can formally write this as $F_N(z) = z^N K_N(z)$. Obviously $\|F_N\|_1 = 1$ for all N.

Corollary 3.8. $H^1 \supsetneq H_b^1$.

Proof. We first fix a sequence $\{a_k\}_{k=0}^{\infty}$ according to $a_k > 0$, $\sum_{k=0}^{\infty} a_k = 1$, and study the function

$$f(z) = \sum_{k=0}^{\infty} a_k F_{N_k}(z),$$

where the integers N_k increase without bound. We readily observe

$$\|f\|_1 \leqslant \sum_{k=0}^{\infty} a_k \|F_{N_k}\|_1 = \sum_{k=0}^{\infty} a_k = 1,$$

whence in particular $f \in H^1$. Every Taylor coefficient of f is easily seen to be positive; write $f(z) = \sum b_n z^n$. From each F_{N_k} we get a contribution

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{b_{N_k+n}}{n+1} &\geqslant a_k \sum_{n=0}^{N_k} \frac{1}{n+1}\left(1-\frac{n}{N_k+1}\right) \\ &= a_k \left\{ \sum_{n=0}^{N_k} \frac{1}{n+1} - \frac{1}{N_k+1} \sum_{n=0}^{N_k} \frac{n}{n+1} \right\} \\ &= a_k \left\{ \log N_k + \mathcal{O}(1) \right\}.\end{aligned}$$

By the preceding Theorem we conclude that $\|f\| \geqslant C a_k \log N_k + \mathcal{O}(1)$ for every k. A choice of increase for $\{N_k\}_0^\infty$ rapid enough, more specifically $a_k \log N_k \to \infty$, shows that the function f can be made to satisfy $f \in H^1 \setminus H_b^1$.

Lemma 3.9. *Consider complex numbers a_k such that $\sum |a_k| < \infty$, and also natural numbers N_k, M_k with the property $M_k + 2N_k + 1 \leqslant M_{k+1}$. Write $f(z) = \sum_{k=0}^\infty a_k z^{M_k} F_{N_k}(z)$. Then $f \in H^1$. Furthermore, $f \in H_b^1 \Leftrightarrow \sup |a_k| \log N_k < \infty$ and $f \in H_{b0}^1 \Leftrightarrow |a_k| \log N_k \to 0$ as $k \to \infty$.*

Proof. Since $\|F_N\|_1 = 1$, $\|f\|_1 \leqslant \sum |a_k| < \infty$ shows $f \in H^1$.

The demand on $M_k + 2N_k + 1 \leqslant M_{k+1}$ ensures that at most one of the members $a_k z^{M_k} F_{N_k}(z)$ gives a contribution to particular Taylor coefficient. In particular, any index N such that $M_{l-1} + 2N_{l-1} + 1 \leqslant N < M_l$ gives

$$f(z) - s_N f(z) = \sum\nolimits_{k=l}^{\infty} a_k z^{M_k} F_{N_k}(z),$$

and $\|f - s_N f\|_1 = \mathcal{O}\big(\sum_{k=l}^\infty |a_k|\big)$ follows for these special values of N.

Should on the other hand $M_l \leqslant N < M_l + 2N_l + 1$ be the case, then $f - s_N F f$ would coincide with

$$a_l z^{M_l} \left\{ F_{N_l}(z) - s_{N-M_l} F_{N_l}(z) \right\} + \sum\nolimits_{k=l+1}^{\infty} a_k z^{M_k} F_{N_k}(z)$$

and for this case emerges

$$\|f - s_N f\|_1 = |a_l| \left\| F_{N_l} - s_{N-M_l} f \right\|_1 + \mathcal{O}\big(\sum\nolimits_{k=l+1}^{\infty} |a_k|\big).$$

From earlier computation we know

$$\max_{0 \leqslant n \leqslant M} \|F_M - s_n F_M\|_1 = 2\pi^{-1} \log M + \mathcal{O}(1),$$

whence we may for f deduce the property

$$\sup_{N \geqslant M_l} \|f - s_N f\|_1 = \sup_{k \geqslant l} 2\pi^{-1} |a_k| \log N_k + o(1).$$

Immediately we get $f \in H_{b0}^1$ if and only if $|a_k| \log N_k \to 0$. Using the triangle inequality $\big| \|f\|_1 - \|s_N f\|_1 \big| \leqslant \|f - s_N f\|_1$, it is a simple matter to verify the membership $f \in H_b^1$ as claimed.

Now we are in a position give the last example to prove that all the five spaces studied in this text are indeed distinct spaces.

Proposition 3.10. $H^1_b \setminus H^1_{b0} \neq \emptyset$.

Proof. Choose in the previous Lemma the indices N_k according to $|a_k| \log N_k \to 1$. The function so constructed belongs to H^1_b, but $\|f - s_{n_j} f\|_1 \to 2\pi^{-1}$ for a suitable integer sequence n_j tending to infinity. It follows that $f \notin H^1_{b0}$.

A surely well-known corollary, to which I was unable to find a reference, follows immediately from this example.

Corollary 3.11. *The partial sums of a function in H^1 need not converge to the function itself in norm.*

Proof. Any function in $H^1 \setminus H^1_{b0} \supset H^1_b \setminus H^1_{b0} \neq \emptyset$ makes a good counter example.

Remark. We see that, not even the stronger demand that the partial sums be bounded suffices to get convergence in L^1-norm.

Observation 3.12. $L^1_b \setminus \cup_{c \in \mathbb{C}} \big(c\ell + L^1_{b0}\big) \neq \emptyset$, that is $\dim L^1_b / L^1_{b0} \geqslant 2$.

This was claimed in the first section and can now be seen as follows. Take any $c \in \mathbb{C}$ and construct any f as in the last Lemma above. Then

$$(f - c\ell) - s_{M_k - 1}(f - c\ell) = \sum_{j \geqslant k} a_j z^{M_j} F_{N_j}(z) + c(s_{M_k-1}\ell - \ell).$$

The series converges to zero in L^1, whence

$$\limsup_{N \to \infty} \big\|(f - c\ell) - s_N(f - c\ell)\big\|_1 \geqslant |c| \lim_{N \to \infty} \|s_n \ell - \ell\|_1 = 4\pi^{-2}|c|.$$

Consequently the membership $f - c\ell \in L^1_{b0}$ implies $c = 0$ and $f \in L^1_{b0}$. A choice of indices M_k and N_k such that $f \in L^1_b \setminus L^1_{b0}$ is thus sufficient to produce the desired example to the effect that L^1_b is large enough.

Observation 3.13. *The Taylor coefficients of a function in H^1_{b0} can tend to zero arbitrarily slowly.*

Take $\delta_n \searrow 0$ arbitrary and consider a fixed positive decreasing sequence $\{a_k\}_{k=0}^{\infty}$ such that $\sum a_k = 1$. Take $N_k = 1$ for all k and determine the integers M_k according to *i)* $M_0 = 0$, *ii)* $M_{k+1} \geqslant M_k + 3$, and *iii)* $\delta_k^{-1} a_{M_k} \to \infty$.

Write $f(z) = \sum_{k=0}^{\infty} a_k z^{M_k} F_1(z)$. The last Lemma shows that $f \in H^1_{b0}$. Since $M_n \geqslant 3n \geqslant n$, the limiting behaviour ${\delta_k}^{-1} \hat{f}(M_k + 1) = {\delta_k}^{-1} a_{M_k} \to \infty$ shows that the Taylor coefficients of f tend to zero at a slower rate than δ_k does.

Of course the statement can be interpreted in L^1_{b0} also, saying that the Fourier coefficients can have arbitrarily slow decay.

4. Behaviour under harmonic conjugation.

Having found that the introduced spaces are distinct from L^1 as well as H^1, one should wonder how L^1_b and L^1_{b0} behave with respect to harmonic conjugation. It turns out that neither space is closed in this sense. As a consequence, neither L^1_b nor L^1_{b0} has a bounded projection onto H^1.

Recall the classical result that $\sum_{n \geqslant 2} \frac{2 \sin n\theta}{\log n}$ is not a Fourier–Stieltjes series, cf. [Z], page 186, and that this is the formal Fourier series conjugate to $\ell(\theta) = \sum_{n \geqslant 2} \frac{2 \cos n\theta}{\log n}$ in L^1_b. We deduce a first negative property.

Observation 4.1. *L^1_b is not closed with respect to harmonic conjugation. More specifically, the harmonic conjugates need not even be integrable.*

Recall at this point a result from the preceding section. Denote the multiplication operator $f(\theta) \mapsto e^{i\theta} f(\theta)$ by T_z. Itself and all powers T_z^n, $n \in \mathbb{Z}$, are well defined in M_b. In fact, for each $k \in \mathbb{Z}$, $|\hat{\mu}(k)| \leqslant \|\mu\|_1 \leqslant \|\mu\|$ so

$$\|s_N T_z^n \mu\|_1 \leqslant \|s_{N+|n|} f\|_1 + \sum_{k=N+1}^{N+|n|} \max\big(|\hat{\mu}(k)|, |\hat{\mu}(-k)|\big) \leqslant \big(|n|+1\big)\|\mu\|,$$

which is $\| T_z^n \|_{M_b \to M_b} \leqslant |n|+1$. In comparison, we know that $\| T_z^n \|_{H^1_b \to L^1_b} \leqslant 2$ for all n.

Since L^1_{b0} is a homogeneous space where the partial summation operators are uniformly bounded, the uniform boundedness of T_z^n on L^1_{b0} is equivalent to boundedness of the conjugation operator.

Proposition. *If some projection $L^1_{b0} \to H^1_{b0}$ is bounded, then also the natural projection $\sum_{n=-\infty}^{\infty} a_n e^{in\theta} \mapsto \sum_{n=0}^{\infty} a_n z^n$ is bounded $L^1_{b0} \to H^1_{b0}$.*

Proof (Rudin's proof of Newman's theorem; cf. [Ho]).

Suppose $P : L^1_{b0} \to H^1_{b0}$ is bounded and the restriction of P to H^1_{b0} is the identity. Both L^1_{b0} and H^1_{b0} being homogeneous spaces, the mapping described by $\tilde{P}f = \int_{-\pi}^{\pi} [Pf_\theta]_{-\theta}\, d\theta/2\pi$ is well-defined and continuous $L^1_{b0} \to H^1_{b0}$. In addition, $\|\tilde{P}\| \leqslant \|P\|$. Here the translation invariance $\|f_\theta\| = \|f\|$ and the continuity $\theta \to f_\theta$ entered.

For $n \geqslant 0$ we have $\big[P[e^{inx}]_\theta\big]_{-\theta} = \big[e^{-in\theta} P e^{inx}\big]_{-\theta} = e^{-in\theta} e^{in(x+\theta)} = e^{inx}$, while $n < 0$ gives $Pe^{inx} = a_0 + a_1 e^{ix} + a_2 e^{2ix} + \cdots \in H^1_{b0}$ and hence

$$\big[P(e^{inx})_\theta\big]_{-\theta} = a_0 e^{-in\theta} + a_1 e^{-i(n-1)\theta} + a_2 e^{-i(n-2)\theta} + \dots.$$

All the multiples of θ are here non-zero, whence we conclude

$$\tilde{P}e^{inx} = \begin{cases} e^{inx}, & n \geqslant 0, \\ 0, & n < 0. \end{cases}$$

This shows that $\tilde{P}$ is the natural projection, and its boundedness is contained in $\|\tilde{P}\| \leqslant \|P\| < \infty$.

Corollary 4.3. *If any projection $L^1_{b0} \to H^1$ is bounded, the same holds for the natural projection.*

The proof is identical, since H^1 is a homogeneous space in its own norm.

Proposition 4.4. *Not all conjugate functions of elements in L^1_{b0} are members of L^1.*

Proof. The sequence $\{a_n = (\log n \, \log\log n)^{-1}\}_{n=3}^{\infty}$ is positive, decreasing and convex. Since $a_n \log n \to 0$ as $n \to \infty$, the function $f(\theta) = \sum_{|n|\geqslant 3} a_{|n|} e^{in\theta}$ is member of L^1_{b0}.

The conjugation operator is of weak $(1,1)$-type, so the analytic function $F(z) = \sum_{n\geqslant 3} a_n z^n$ is an element in $\cup_{p<1} H^p$. However, according to Hardy's inequality, F does not belong to H^1, to wit we have $\sum_{n\geqslant 3} n^{-1} a_n = \sum_{n\geqslant 3} (n \log n \, \log\log n)^{-1}$ diverging to infinity. It follows that the harmonic conjugate $\tilde{f}$ of f is not integrable.

Corollary 4.5. $\sup_{n\in\mathbb{Z}} \| T_z^n \|_{L^1_{b0}\to L^1_{b0}} = \infty$.

Proof. L^1_{b0} is a homogeneous space that has norm convergence of the partial sums for each function in the space. Were in fact the supremum finite, then [K], Theorem II:1.4, would demonstrate the closedness of L^1_{b0} under conjugation. This is not the case by the last proposition, so the claim follows.

Corollary 4.6. *There are no bounded projections from L^1_{b0} into either H^1, H^1_b, or H^1_{b0}.*

Proof. For H^1 and H^1_{b0} the conclusion follows immediately from the last few results. On the other hand we readily see

$$\|Pf\|_{H^1} \leqslant \|Pf\|_{H^1_b} \leqslant \|P\|_{L^1_{b0}\to H^1_b} \|f\|_{L^1_{b0}},$$

that is to say $\|P\|_{L^1_{b0}\to H^1} \leqslant \|P\|_{L^1_{b0}\to H^1_b}$. Consequently, any bounded projection $L^1_{b0} \to H^1_b$ is simultaneously bounded $L^1_{b0} \to H^1$. We already showed that no projection of the latter kind exists.

To round off we may also deduce that L^1_{b0} is fairly large.

Observation 4.7. $L^1_{b0} \supsetneq \bigcup_{p>1} L^p$.

The simple reason is that each L^p with $p > 1$ is invariant with respect to harmonic conjugation.

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