ISSN: 1401-5617

Spaces with uniformly norm-bounded partial sums II

Mats Erik Andersson

Research Reports in Mathematics
Number 11, 1999

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.matematik.su.se/reports/1999/11

Date of publication: September 22, 1999
1991 Mathematics Subject Classification: Primary 46E37 Secondary 42A32, 26J99.

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.matematik.su.se
info@matematik.su.se

SPACES WITH UNIFORMLY NORM-BOUNDED PARTIAL SUMS II

Mats Erik Andersson

Utkast av den 21 september i nådens år 1999.

Abstract. An investigation on consequences of norm-boundedness for partial sums is continued. This time the richness in structure of L^1_b in relation to L^1_{b0} is made apparent. Next, a rudimentary theory for the dual spaces is initiated. In particular, ${L^1_{b0}}^*$ and ${H^1_{b0}}^*$ can be realised as convolution algebras consisting of functions integrable to any finite order on the unit circle.

Contents

This report is a direct continuation of a previous presentation with similar aims [An]. References will be made at will to that paper and the numbering of results will be continued here. Rather few steps will be taken to recollect the material from the preceding report.

5. Structural properties of L^1_b.

The interrelation between L^1_b and L^1_{b0} can be understood from the general theory of Segal algebras.

Definition. Given a commutative Banach algebra $(A, \|\,\|_A)$, a subalgebra $B \subseteq A$ is an A-Segal algebra provided (i) the natural injection of B into A as well as multiplication from $A \times B$ to B are continuous mappings, and (ii) B is a dense ideal of A and is also a Banach algebra in its own norm $\|\,\|_B$.

The relative completion of B with respect to A is defined to be the space and norm

$$\widetilde{B}^A = \bigcup_{r>0} \overline{S_B(r)}^A, \qquad \|\!|\, f \,|\!\| = \inf\{r\,;\, f \in \overline{S_B(r)}^A\}.$$

Here $\overline{E}^A$ denotes completion in A-norm and $S_B(r) = \{f \in B\,;\, \|f\|_B \leqslant r\}$ is a closed ball.

1991 *Mathematics Subject Classification.* 46E27, 42A32, 46J99.
Key words and phrases. Segal algebra, dual spaces..

Typeset by $\mathcal{AMS}$-TEX

It was established by Burnham [B] that in case A has bounded approximate units, then $\widetilde{B}^A$ is in its turn an A-Segal algebra. In our case it turns out that L^1_b is the relative completion of L^1_{b0} with respect to L^1 and the new norm agrees with the one we use on L^1_{b0}. These two spaces were briefly mentioned as examples in [B] and [Wa].

It is a consequence of general results, see [B], Corollary 15, and the Module factorisation theorem of Hewitt, that $L^1 * L^1_b = L^1_{b0}$. Even in our special case, the general methods of Burnham give useful information on the position of L^1_{b0} inside L^1_b. Specifically, L^1_b is the unique, largest ideal of L^1, wherein L^1_{b0} becomes a closed ideal.

Now it is even possible to identify $L^1 * M_b$ with almost no additional effort. For integrable functions $f \in L^1$ we consider the approximation number $E_N(f) = \inf \|f - p_n\|_1$, where infimum is taken over all polynomials p_N of degree at most N. It is a standard fact that $E_N(f) = o(1)$.

Theorem 5.1. $L^1 * M_b = L^1_{b0}$.

Proof. Let $f \in L^1$ and $\mu \in M_b$. For each polynomial p_N of degree at most N an identity obtains:

$$f * \mu - s_N(f * \mu) = (f - p_N) * (\mu - s_N\mu).$$

It follows that

$$\|f * \mu - s_N(f * \mu)\|_1 \leqslant E_N(f)\, \|\mu - s_N\mu\|_1 \leqslant 2\, \|\mu\|\, E_N(f),$$

which is $o(1)$ as $N \to \infty$. Consequently $f * \mu \in L^1_{b0}$ and then $L^1 * M_b \subseteq L^1_{b0}$. The set equality follows from the known identity $L^1 * L^1_b = L^1_{b0}$.

It is rewarding to express the construction of the space M_b in a form displaying the connection to approximation numbers. For $\mu \in M(\mathbb{T})$ we extend the notion of approximation numbers beyond L^1 simply by denoting $E_N(\mu) = \inf \|\mu - p_N\, dm\|_1$. It is clear that $E_N(\mu)$ is decreasing to the limit $\|\mu_s\|_1$, where μ_s is the singular part of μ. In particular, the property $\mu \in M_b \cap M_{\mathrm{s}}(\mathbb{T})$ holds true when $\|\mu - s_N\mu\|_1$ and $E_N(\mu)$ both are of order $\mathcal{O}(1)$. In a sense this means that the particular approximation $s_N\mu$ is close to being optimal among all polynomials of this degree.

Proposition 5.2. *Let $\{a_n\}_{n=-\infty}^{\infty}$ be even and convex on $[0, \infty[$, with $a_{|n|} \to 0$. For each $\mu \in M(\mathbb{T})$ two characterisations appear:*

i) $\mu * a \in L^1_b \quad \Leftrightarrow \quad \mu * a \in M_b \quad \Leftrightarrow \quad a_N\, \|s_N\mu\|_1 = \mathcal{O}(1)$,
ii) $\mu * a \in L^1_{b0} \quad \Leftrightarrow \quad a_N\, \|s_N\mu\|_1 = o(1)$,

where a is the L^1-function $a(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$.

Proof. It is well known that $a = \sum_{n=0}^{\infty} b_n\, K_n$, $s_N a = a_N\, D_N + \sum_{n=0}^{N-1} b_n\, K_n$, and $s_N a - a = a_N\, D_N - \sum_{n=N}^{\infty} b_n\, K_n$. Here $b_n \geqslant 0$ and $\sum_{n=0}^{\infty} b_n = a_0$.

Convolving with μ we have the representations

$$s_N(\mu * a) = a_N\, s_N\mu + \sum_{n=0}^{N-1} b_n\, K_n * \mu, \quad s_N(\mu * a) - \mu * a = a_N\, s_N\mu - \sum_{n=N}^{\infty} b_n\, K_n * \mu.$$

One observes two variation bounds: $\| \sum_{n=0}^{N-1} b_n\, K_n * \mu\|_1 \leqslant \|\mu\|_1 \sum b_n = a_0\, \|\mu\|_1$, uniformly in N, and $\| \sum_N^\infty b_n\, K_n * \mu\|_1 \leqslant \|\mu\|_1 \sum_{n=N}^\infty b_n \to 0$ as $N \to \infty$. Hence

$$\|s_N(\mu * a)\|_1 = a_N\, \|s_N\mu\|_1 + \mathcal{O}(1), \quad \|s_N(\mu * a) - \mu * a\,\|_1 = a_N\, \|s_N\mu\|_1 + o(1).$$

All claims follow from these when one takes into account that trivially $a \in L^1$ provides $\mu * a \in L^1$.

From the proof we note a useful result for future reference.

Observation 5.3. *With assumptions as in Proposition 5.2,*

$$\|s_N(\mu * a)\|_1 = a_N\, \|s_N\mu\|_1 + \mathcal{O}(1) \quad \textit{and} \quad \|s_N(\mu * a) - \mu * a\,\|_1 = a_N \|s_N\mu\|_1 + o(1).$$

Recall next the special element $\ell(\theta) = \sum_{n \geqslant 2} (\log n)^{-1} \cos nx$ in $L^1_b \setminus L^1_{b0}$.

Corollary 5.4. *For each Riesz product ν, parameters satisfying $\sum_{j=1}^k |a_j| = o(k)$, the membership $\nu * \ell \in L^1_{b0}$ obtains. In particular, any Riesz product in $M_0(\mathbb{T})$ enjoys this property.*

Proof. Weiss' theorem [We] yields for the Riesz product $\nu = \prod_{j=1}^\infty (1 + a_j \cos n_j\theta)$ the inequality

$$\|s_N\nu\|_1 \leqslant C|a_k|\big(|a_1| + \cdots + |a_k|\big), \quad N_k \leqslant N < N_{k+1}, \quad N_k = n_1 + \cdots + n_k.$$

The lacunarity produces $N_k \geqslant n_1(1 + \rho + \cdots + \rho^{k-1})$ and hence $k \leqslant C \log N_k$. It follows that $\|s_N\nu\|_1 = C|a_k|\, o(\log N)$ for N in $[N_k, N_{k+1}[$. By the previous proposition the convolution $\nu * \ell$ is an element of L^1_{b0}. It is obvious that $M_0(\mathbb{T})$ is a sufficient assumption.

A small digression from the main subject is motivated at this point, since it illustrates the optimality of Helson's theorem and hence relates to the present material.

Proposition. *Let ρ be a positive, unbounded, and increasing function on $\mathbb{N}$. Then there is a probability measure ν such that $\widehat{\nu}(n) = 1/2$ for infinitely many n, yet $\|s_N\nu\|_1 \leqslant \rho(N)$ from some point $N \geqslant N_1$ onwards.*

Proof. Let A be the positive constant such that Weiss' theorem ensures $\|s_N\nu\|_1 \leqslant A\,|a_k|\big(|a_1| + \cdots + |a_k|\big)$ when $N_k \leqslant N < N_{k+1}$; see the preceding proof for notation.

Define $n'_k = \min\{n \geqslant 1\,;\, \rho(n) \geqslant Ak\}$ and choose $n_1 = n'_1$, $n_k = \max\{3n_{k-1}, n'_k\}$ for $k \geqslant 2$. We claim that the Riesz product based on $\{n_k\}$ and $a_k \equiv 1$ has the desired property. Since $n_k \geqslant 3n_{k-1}$, the classical construction as well as Weiss' theorem apply. According to the latter we have for $N_k \leqslant N < N_{k+1}$

$$\|s_N\nu\|_1 \leqslant Ak \leqslant \rho(n_k) \leqslant \rho(N_k) \leqslant \rho(N).$$

This is the claim for $N \geqslant N_1 = n_1$, once we observe $\widehat{\nu}(n_k) = 1/2$ for all k.

As mentioned above, this demonstrates that Helson's theorem is sharp with respect to the assumption $\|s_n\mu\|_1 = \mathcal{O}(1)$ producing $\mu \in M_0(\mathbb{T})$.

Closely related to absorption properties of the function ℓ, in the sense of collapsing measures into elements of L^1_{b0} via convolution, is the space M_ℓ of measures displayed in the next result.

Theorem 5.5. *$M_\ell = \{\mu \in M(\mathbb{T})\,;\, \|s_N\mu\|_1 = o(\log N)\}$ is a closed subspace of $M(\mathbb{T})$ as well as an ideal. Equivalently, $M_\ell = \{\mu \in M(\mathbb{T})\,;\, \mu * \ell \in L^1_{b0}\}$. This space of measures has the property $M_{\mathrm{ac}}(\mathbb{T}) \subset M_\ell \subseteq M_{\mathrm{c}}(\mathbb{T})$.*

Proof. The two descriptions of M_ℓ as a set are identical by Proposition 5.2. Convolution with ℓ is a continuous mapping from $M(\mathbb{T})$ to L^1_b. Since M_ℓ is the inverse image of the closed subspace L^1_{b0} of L^1_b under this mapping, the closedness of M_ℓ in $M(\mathbb{T})$ follows immediately.

Furthermore, $\mu \in M_\ell$ and $\nu \in M(\mathbb{T})$ imply $\|s_N(\mu * \nu)\|_1 \leqslant \|s_N\mu\|_1\|\nu\|_1 = o(\log N)$, whence M_ℓ becomes an ideal.

The subsequent Lemma 5.6, fully independent of the present proof, shows that every $\mu \in M_\ell$ is void of discrete part, that is $M_\ell \subseteq M_{\mathrm{c}}(\mathbb{T})$. Finally, $\ell * L^1 \subseteq L^1_b * L^1 = L^1_{b0}$ demonstrates $L^1 \cong M_{\mathrm{ac}}(\mathbb{T}) \subset M_\ell$.

Remark. Naturally one can prove closedness directly from only the assumption $\|s_N\mu_n\|_1 = o(\log N)$ for each term in an absolutely convergent series $\sum \mu_n$, without any reference to either L^1_{b0} or L^1_b whatsoever. By Corollary 5.4 M_ℓ is easily seen to contain elements which are not in $M_0(\mathbb{T})$.

To be able to construct some subspaces of L^1_b the next result is instrumental.

Lemma 5.6. *Let $\mu \in M(\mathbb{T})$ and denote its discrete part by μ_d. Then holds*

$$\liminf_{N\to\infty} \frac{\|s_N\mu\|_1}{\log N} \geqslant \frac{4}{\pi^2}\,\|\mu_d\|_1.$$

Proof. Take an $\varepsilon > 0$ less that $\|\mu_d\|_1$. There is a finite set $E \subseteq \mathbb{T}$ such that $\|\mu_d - \mu_\varepsilon\|_1 < \varepsilon$, so in particular $\|\mu_\varepsilon\|_1 \geqslant \|\mu_d\|_1 - \varepsilon$. Here the measure μ_ε is defined by $\mu_\varepsilon(G) = \mu_d(G \cap E)$.

Since E is finite, there is a symmetric interval $J \subseteq \mathbb{T}$ around zero such that (i) $x + 2J$ are disjoint for $x \in E$ and (ii) $|\mu - \mu_\varepsilon|(E + 2J) < \varepsilon$. Write $V = E + J$ and $U = E + 2J$. Finally, there is N_ε such that $N \geqslant N_\varepsilon$ implies $\int_{\mathbb{T}\setminus J} |D_N|\,dm < \varepsilon\,\|D_N\|_1$.

Denote $D_N^\varepsilon = D_N\,\chi_J$, which for $N \geqslant N_\varepsilon$ provides $(1-\varepsilon)\|D_N\|_1 < \|D_N^\varepsilon\|_1 < \|D_N\|_1$. The convolution product $\mu_\varepsilon * D_N^\varepsilon(\theta) = \sum_{x\in E} \mu(\{x\})\,D_N^\varepsilon(\theta - x)$ has terms of disjoint support, whence $\|\mu_\varepsilon * D_N^\varepsilon\|_1 = \|\mu_\varepsilon\|_1\,\|D_N^\varepsilon\|_1$.

For any $\nu \in M(\mathbb{T})$ one has an estimate

$$\begin{aligned}
\int_V |\nu * D_N^\varepsilon(\theta)|\,dm(\theta) &\leqslant \sum_{x\in E} \int_{x+J}\int_{\theta+J} |D_N(\theta - t)|\,d|\nu|(t)dm(\theta)\\
&= \sum_{x\in E} \int_J \int_{x+\theta+J} |D_N(x+\theta-t)|\,d|\nu|(t)dm(\theta)\\
&\leqslant \sum_{x\in E} \int_J \int_{x+2J} |D_N(x+\theta-t)|\,d|\nu|(t)dm(\theta)\\
&= \sum_{x\in E} \int_{x+2J} \int_J |D_N(x+\theta-t)|\,dm(\theta)d|\nu|(t)\\
&\leqslant \|D_N\|_1\,|\nu|(U).
\end{aligned}$$

In particular, $\big|\, \|\chi_V \cdot \mu * D_N^\varepsilon\|_1 - \|\chi_V \cdot \mu_\varepsilon * D_N^\varepsilon\|_1 \,\big| \leqslant \|D_N\|_1 \,|\, \mu - \mu_\varepsilon|(U) < \varepsilon \,\|D_N\|_1$, which is used presently. These facts taken together say that for $N \geqslant N_\varepsilon$ one has

$$\begin{aligned}\|\mu * D_N\|_1 &\geqslant \|\mu * D_N^\varepsilon\|_1 - \varepsilon\,\|D_N\|_1\,\|\mu\|_1\\ &\geqslant \|\chi_V \cdot \mu * D_N^\varepsilon\|_1 - \varepsilon\,\|D_N\|_1\,\|\mu\|_1\\ &\geqslant \|\chi_V \cdot \mu_\varepsilon * D_N^\varepsilon\|_1 - \varepsilon\,\|D_N\|_1 - \varepsilon\,\|D_N\|_1\,\|\mu\|_1\\ &= \|\mu_\varepsilon\|_1\,\|D_N\|_1 - \varepsilon\,\|D_N\|_1(1 + \|\mu\|_1).\end{aligned}$$

Here the last equality is due to $\chi_V \cdot \mu_\varepsilon * D_N^\varepsilon = \mu_\varepsilon * D_N^\varepsilon$. Consequently

$$\liminf_{N\to\infty} \frac{\|\mu * D_N\|_1}{\|D_N\|_1} \geqslant \big(\|\mu_d\|_1 - \varepsilon\big) - \varepsilon(1 + \|\mu\|_1).$$

The freedom in choosing ε demonstrates that the limit inferior cannot fall below the value $\|\mu_d\|_1$, which is exactly the claimed statement.

With only slightly different techniques it is possible to prove that for discrete measures the statement above is true with the full limit replacing limes inferior. The routine demonstration is left for the reader to fill in.

Lemma 5.7. *Let $\mu \in M_{\mathrm{d}}(\mathbb{T})$. Then* $\displaystyle\lim_{N\to\infty} \frac{\|s_N\mu\|_1}{\log N} = \frac{4}{\pi^2}\,\|\mu_d\|_1$.

Now we are in a position to produce a varied selection of subspaces inside L_b^1, as well as strictly between L_b^1 and L_{b0}^1. The result may be considered as an explicit version of Burnham's general observation [B], Theorem 13.

Proposition 5.8. *Let $\mathcal{N}$ be a closed subalgebra of $M_{\mathrm{d}}(\mathbb{T})$ with variation norm. Then $\{\mu * \ell\,;\, \mu \in \mathcal{N}\}$ is a closed subspace of L_b^1 and $\{\mu * \ell + f\,;\, \mu \in \mathcal{N},\ f \in L_{b0}^1\}$ is an ideal of L^1 and M_b, closed in the norm of the latter and intermediate to L_{b0}^1 and L_b^1.*

Proof. Assume that $\sum_{n=1}^\infty \|\mu_n * \ell + f_n\| < \infty$ with $\mu_n \in \mathcal{N}$ and $f_n \in L_{b0}^1$. For each $n \geqslant 1$, since $s_N f_n \to f_n$ in L^1, Lemma 5.7 and Proposition 5.2 show that

$$\begin{aligned}\|\mu_n\|_1 &= \lim_{N\to\infty} \tfrac{\pi^2}{4}\,\|s_N(\mu_n * \ell + f_n) - \mu_n * \ell - f_n\|_1\\ &\leqslant \tfrac{\pi^2}{2}\,\|\mu_n * \ell + f_n\|.\end{aligned}$$

It now follows that

$$\sum_{n=1}^\infty \|\mu_n\|_1 \leqslant \tfrac{\pi^2}{2} \sum_{n=1}^\infty \|\mu_n * \ell + f_n\| < \infty$$

and hence $\mu = \sum \mu_n$ converges in $\mathcal{N}$. Next,

$$\begin{aligned}\|f_n\| &\leqslant \|\mu_n * \ell + f_n\| + \|\mu_n * \ell\| \leqslant \|\mu_n * \ell + f_n\| + \|\mu_n\|_1\,\|\ell\|\\ &\leqslant \big(1 + \tfrac{\pi^2}{2}\,\|\ell\|\big)\,\|\mu_n * \ell + f_n\|,\end{aligned}$$

from which $f = \sum f_n$ converges absolutely in L_{b0}^1. These two convergence results taken together demonstrate that

$$\mu * \ell + f = \sum_{n=1}^\infty \big[\, \mu_n * \ell + f_n \,\big]$$

converges in L^1_b. It follows that the two kinds of subspaces considered indeed are closed.

The second space is an ideal in L^1 due to a simple inclusion calculation:

$$L^1 * (\mu * \ell + f) \subseteq \mu * L^1_{b0} + L^1_{b0} \subseteq L^1_{b0}.$$

Finally, Theorem 5.1 gives $M_b * (\mu * \ell + f) \subseteq L^1_{b0}$ making the algebra an ideal also in M_b.

Theorem 5.9. *The closed ideal* $\{\mu*\ell+f \,;\, \mu \in M_{\mathrm{d}}(\mathbb{T}),\ f \in L^1_{b0}\}$ *is strictly smaller than* L^1_b.

Remark. Of course this is the least translation invariant ideal of L^1_b containing L^1_{b0} and ℓ. The only translation invariant spaces produced by Proposition 5.8 appears for $\mathcal{N} = M_{\mathrm{d}}(\mathbb{T})$ and $\emptyset$.

Proof. Consider with the aid of Lemma 3.9 the element $g = \sum_{k=0}^{\infty} a_k z^{M_k} F_{N_k}$ in $H^1_b \setminus H^1_{b0} \subset L^1_b$. We intend to show that $g = \mu * \ell + f$ for $\mu \in M_{\mathrm{d}}(\mathbb{T})$ and $f \in L^1_{b0}$ is an impossibility.

Assuming the contrapositive, we observe

$$\|g - \mu * \ell - s_N(g - \mu * \ell)\| = \|f - s_N f\| \to 0 \text{ as } N \to \infty$$

and

$$g - \mu * \ell - s_{M_k - 1}(g - \mu * \ell) = \sum_{j \geqslant k} a_j\, z^{M_j}\, F_{N_j} + s_{M_k-1}(\mu * \ell) - \mu * \ell.$$

From this follows

$$\|s_{M_k-1}(\mu * \ell) - \mu * \ell\|_1 \to 0, \text{ as } k \to \infty,$$

which says $\|s_{M_k-1}\mu\|_1 / \log(M_k - 1) = o(1)$ by Proposition 5.2. Next, Lemma 5.6 implies $\mu = 0$, and hence $g = f \in L^1_{b0}$. This contradiction proves the claim.

We finish this section with a discussion of multipliers acting on M_b. Recall that a convolution measure algebra B is called an L-algebra if $\mu \in B$ and $\nu \ll \mu$ implies $\nu \in B$. Since they contain the constant functions but not the whole of L^1 and H^1 respectively, it is clear that neither M_b, L^1_b, H^1_b, L^1_{b0}, nor H^1_{b0} are L-algebras. The question arises what space of functions ensures that $f\,d\mu$ belongs to the same space among these as μ itself does? It is useful to recall that $M_0(\mathbb{T})$, of which M_b is an ideal, in fact is an L-algebra.

Lemma 5.10. $\Big\| \sum_{n=0}^{N} \dfrac{z^n}{\log(n+2)} \Big\|_1 \sim C \log\log N$ *as* $N \to \infty$.

Proof. Let $a_n = [\log(n+2)]^{-1} - [\log(N+3)]^{-1}$. Then $\Delta a_n \sim \frac{1}{n \log^2 n}$, $a_N = \Delta a_N$, and $\Delta^2 a_n \sim \frac{1}{n^2 \log^2 n}$.

Lemma 3.1 and the fact that $\tilde{K}_n$ has sign independent of n show

$$\Big\| \sum_{n=0}^{N} \frac{z^n}{\log(n+2)} \Big\|_1 = \mathcal{O}(1) + C \sum_{n=2}^{N} \frac{1}{n \log n} \sim C \log\log N.$$

This verifies the claim.

The multiplication operator T^k_z has been defined by $T^k_z \mu = e^{ik\theta}\, d\mu$. We see from section 3 that T^k_z has operator norm 1 as acting $H^1_b \mapsto H^1_b$ for $k \geqslant 0$ and norm at most 2 as a mapping $H^1_b \mapsto L^1_b$ for general $k \in \mathbb{Z}$.

Lemma 5.11. (1) $\|T_z^k\|_{M_b \to M_b} \leqslant C \log(|k|+2)$.
(2) $\|T_z^k\|_{L^1_{b0} \to L^1_{b0}} \geqslant C \log\log(|k|+3)$.

Proof. Write for simplicity $e_k(\theta) = e^{ik\theta}$. Obviously we may suppose $k \geqslant 1$ for the sake of proving (1). It is clear that

$$s_N\, T_z^k \mu = \sum_{n=-N}^{N} \widehat{\mu}(n-k) e_n = e_k \sum_{n=-N-k}^{N-k} \widehat{\mu}(n) e_n.$$

In case $0 \leqslant N < k$, it follows that

$$\|s_N\, T_z^k \mu\|_1 = \|(e_{-k} D_N) * \mu\|_1 \leqslant C \log(N+2)\|\mu\|_1 \leqslant C \log(k+2)\|\mu\|,$$

whereas $N \geqslant k$ yields $s_N\, T_z^k \mu = e_k \left\{ s_{N+k}\mu - (e_N D_k) * \mu + e_{N-k} * \mu \right\}$, whence

$$\|s_N\, T_z^k \mu\|_1 \leqslant \|\mu\| + C \log(k+2)\|\mu\|_1 \leqslant C \log(k+2)\|\mu\|.$$

Taken together they demonstrate $\|T_z^k \mu\| \leqslant C \log(|k|+2)\|\mu\|$, which is claim (1).

On the other hand, the functions $g_N(\theta) = \sum_0^N \frac{\cos k\theta}{\log(|k|+2)} \in L^1_{b0}$ have uniformly bounded norms. According to Lemma 5.10 one finds

$$\|T_z^k g_{|k|}\| \geqslant \|s_{|k|}(T_z^k g_{|k|})\|_1 \sim C \log\log(|k|+3).$$

This is the last claim.

We next consider Beurling algebras A_ω as consisting of expansions $\sum_{-\infty}^{\infty} a_n e^{in\theta}$ with finite norm $\|f\|_\omega = \sum |a_n|\,\omega_n$, where $\{\omega_n\}_{-\infty}^{\infty}$ is a positive sequence, submultiplicative for both positive and negative indices.

We say that A_ω is a multiplier space on M_b in case $f \in A_\omega$ and $\mu \in M_b$ imply $f\,d\mu \in M_b$ and that for some constant independent of f and μ the norm inequality $\|f\,d\mu\| \leqslant C\,\|f\|_\omega\,\|\mu\|$ obtains.

Theorem 5.12. *The following statements each imply the subsequent ones in the order stated.*
(1) $\inf_n \dfrac{\omega_n}{\log(|n|+2)} > 0$.
(2) *A_ω is a multiplier space on M_b.*
(3) $\inf_n \dfrac{\omega_n}{\log\log(|n|+3)} > 0$.

Remark. It is clear that this gap at the moment prevents the characterisation of the weights ensuring A_ω to be a multiplier space. From the proof it will become clear that any improvement of Lemma 5.11 will in an immediate manner improve on the present result.

Proof. A brief moment of reflection gives the presentation

$$f\,d\mu = \sum_{k=-\infty}^{\infty} a_k\, T_z^k \mu, \quad \text{for} \quad f(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}.$$

Taking norms in M_b of the first series demonstrates, by Lemma 5.11,

$$\|f\,d\mu\| \leqslant C \sum_{k=-\infty}^{\infty} |a_k| \log(|k|+2)\,\|\mu\|.$$

Now the statement (1) clearly implies (2) for a constant only depending on ω.

Take on the other hand, according to Lemma 5.11, a function $g_k \in L^1_{b0}$ with $\|T_z^k g_k\| \geqslant C \log\log(|k|+3)\,\|g_k\|$, where C is a universal constant and $g_k \not\equiv 0$. Assuming (2) to hold, the algebra A_ω by necessity has the property

$$\log\log(|k|+3)\,\|g_k\| \leqslant C\,\|e^{ik\theta} g_k\| \leqslant C\,\|e^{ik\theta}\|_\omega\,\|g_k\| = C\,\omega_k\,\|g_k\|.$$

From this inequality, the statement (3) follows immediately.

For the analytic spaces one has sharper results. The method of proof is the same as in Theorem 5.12, but incorporating knowledge of $\|T_z^k\|_{H^1_b \to L^1_b} \leqslant 2$, for all $k \in \mathbb{Z}$, and $\|T_z^k\|_{H^1_b \to H^1_b} = 1$, for $k \geqslant 0$, respectively.

Theorem 5.13. (1) *$\|\rho f\| \leqslant 2\,\|\rho\|_{A(\mathbb{T})}\,\|f\|$ for all $\rho \in A(\mathbb{T})$ and $f \in H^1_b$. In other words, $A(\mathbb{T})$ is a multiplier space from H^1_b into L^1_b.*

(2) *The analytic algebra $A^+ = A(\mathbb{T}) \cap H^1$ is a multiplier space on H^1_b, and in fact $\|\rho f\| \leqslant \|\rho\|_{A^+}\,\|f\|$ for all $\rho \in A^+$ and $f \in H^1_b$.*

6. The space dual to L^1_{b0}.

We let $\|\,\|_*$ denote the norm dual to $\|\,\|$. When $\phi \in L^\infty$ we have $|\int f\phi\,dm| \leqslant \|f\|_1\,\|\phi\|_\infty \leqslant \|\phi\|_\infty\,\|f\|$, whence $\phi \in {L^1_{b0}}^*$ with $\|\phi\|_* \leqslant \|\phi\|_\infty$. The continuity of ϕ in its action on L^1_{b0} is a consequence of the obvious

$$\Big|\int f\phi\,dm - \int s_N f\,\phi\,dm\Big| \leqslant \|f - s_N f\|_1 \|\phi\|_\infty \to 0,$$

since $s_N f$ converges to f in L^1_{b0}.

In general we consider $\phi \in {L^1_{b0}}^*$, $f \in L^1_{b0}$ and denote the dual pairing by $\langle f, \phi\rangle$. Based on $s_N f \to f$ in L^1_{b0} we have

$$\langle f, \phi\rangle = \lim_{N\to\infty} \langle s_N f, \phi\rangle = \lim_{N\to\infty} \sum_{n=-N}^{N} \widehat{f}(n)\,\langle e^{in\theta}, \phi\rangle.$$

Each $e^{in\theta} \in L^1_{b0}$, so the notation $\phi_n = \langle e^{-in\theta}, \phi\rangle$ indicates a formal Fourier series $\phi \sim \sum \phi_n\,e^{in\theta}$, which will in time turn out to arise from an integrable function. Clearly the partial sum $s_n\phi$ allows us a formulation for the L^1_{b0}–${L^1_{b0}}^*$-duality completely determining ϕ, using the density of the polynomial subspace:

$$\langle f, \phi\rangle = \lim_{N\to\infty} \langle f, s_N\phi\rangle, \qquad \begin{cases} s_N\phi(\theta) = \sum_{n=-N}^{N} \phi_n e^{in\theta}, \\ \phi_n = \langle e^{-in\theta}, \phi\rangle. \end{cases}$$

Here we define $\langle f, \psi\rangle = \int f\psi\,dm$ for $\psi \in L^\infty$. For polynomial ψ the notation is consistent.

Notation. For $\phi \in {L^1_{b0}}^*$ the relation $\phi \sim \sum_{-\infty}^{\infty} \phi_n e^{in\theta}$ means simply that $\langle f, \phi\rangle = \lim_{N\to\infty}\langle f, s_N\phi\rangle$ for all $f \in L^1_{b0}$.

There is an expansion at our disposal:

$$\langle f, s_N\phi\rangle = \int f(\theta)\Big\{\sum\nolimits_{-N}^{N} \phi_n e^{in\theta}\Big\}\, dm(\theta) = \sum_{n=-N}^{N} \widehat{f}(n)\phi_{-n}.$$

Obviously $\langle s_N f, \phi\rangle = \langle f, s_N\phi\rangle$, so $|\langle f, s_N\phi\rangle| \leqslant \|\phi\|_*\, \|s_N f\| \leqslant \|\phi\|_* \|f\|$, from which follows $s_N\phi \in {L^1_{b0}}^*$ with $\|s_N\phi\|_* \leqslant \|\phi\|_*$.

Conversely, one considers a formal series $\sum_{-\infty}^{\infty} \phi_n e^{in\theta}$ such that for a constant C

$$\Big|\sum\nolimits_{n=-N}^{N} \widehat{f}(n)\phi_{-n}\Big| \leqslant C\, \|s_N f\|, \quad \text{all } f \in L^1_{b0},\ N \geqslant 0.$$

The pairing $\langle f, \phi\rangle = \sum_{-\infty}^{\infty} \widehat{f}(n)\phi_{-n}$ is linear and well defined for polynomials and is such that $|\langle f, \phi\rangle| \leqslant C\, \|f\|$. The density of polynomials in L^1_{b0} shows that $\langle f, \phi\rangle = \lim_{N\to\infty}\langle s_N f, \phi\rangle$ determines a unique functional $\phi \in {L^1_{b0}}^*$ with $\|\phi\|_* \leqslant C$. The continuity obtains from

$$\big|\langle s_N f, \phi\rangle - \langle s_M f, \phi\rangle\big| = \big|\langle s_N f - s_M f, \phi\rangle\big| \leqslant C\, \|s_N f - s_M f\| \to 0, \quad N, M \to \infty.$$

We summarize our observations, just remarking that the identification of $\|\phi\|_*$ from all possible C is standard functional analysis.

Proposition 6.1. *${L^1_{b0}}^*$ consists of formal expansions $\phi \sim \sum_{-\infty}^{\infty} \phi_n e^{in\theta}$ with the following properties. The dual pairing is written $\langle f, \phi\rangle$ and for the partial sums $s_N\phi(\theta) = \sum_{-N}^{N} \phi_n e^{in\theta}$, where $\phi_n = \langle e^{-in\theta}, \phi\rangle$, the following hold.*

(1) $\langle f, s_N\phi\rangle = \int f\, s_N\phi\, dm = \sum_{n=-N}^{N} \widehat{f}(n)\phi_{-n}$.

(2) *For some constant C and all $f \in L^1_{b0}$, $N \geqslant 0$*

$$\big|\langle f, s_N\phi\rangle\big| \leqslant C\, \|s_N f\|, \qquad \langle f, s_N f\rangle = \langle s_N f, \phi\rangle.$$

The minimal constant C coincides with $\|\phi\|_*$. In particular, $\|s_N\phi\|_* \leqslant \|\phi\|_*$.

Two useful representations of ${L^1_{b0}}^*$ are important and will provide much information. The next result is the first method, whose proof will be delayed somewhat in order to display immediate consequences thereof. In Theorem 6.6 the second technique of representation is displayed.

Theorem 6.2. *For each $\phi \in {L^1_{b0}}^*$ there exists a function element $g_\phi \in \bigcap_{p\geqslant 1} L^p$, determined almost everywhere, such that $\langle f, \phi\rangle = \langle f, g_\phi\rangle$ for all $f \in L^1_{b0}$. In addition, $\|g_\phi\|_p \leqslant \|\phi\|_*$ for all $1 \leqslant p \leqslant 2$, and in general $\|g_\phi\|_p \leqslant C \cdot p\, \|\phi\|_*$.*

Since $\phi_n = \widehat{g_\phi}(n)$, this tells us that the formal series in fact corresponds to a proper Fourier expansion.

Corollary 6.3. *The maximal ideal space ΔL^1_{b0} is isomorphic to $\mathbb{Z}$.*

Proof. We need to determine all multiplicative functionals on L^1_{b0}. Let the non-trivial $\phi \in {L^1_{b0}}^*$ be multiplicative and consider its representative g_ϕ.

For all polynomials p and q we find

$$\int p(\alpha)q(\beta)\, g_\phi(\alpha)g_\phi(\beta)\, dm(\alpha)dm(\beta) = \langle p, \phi\rangle\langle q, \phi\rangle$$
$$= \langle p * q, \phi\rangle = \int p * q(\theta)\, g_\phi(\theta)\, dm(\theta)$$
$$= \int p(\alpha)q(\beta)\, g_\phi(\alpha+\beta)\, dm(\alpha)dm(\beta).$$

It is a routine use of approximation arguments to deduce that this implies

$$(*) \qquad g_\phi(\alpha+\beta) = g_\phi(\alpha)g_\phi(\beta)$$

almost everywhere in Haar measure on $\mathbb{T}\times\mathbb{T}$. Hence g_ϕ is a measurable homomorphism of $\mathbb{T}$ into $\mathbb{C}$.

Next it is claimed that g_ϕ is non-zero almost everywhere. In the contrary case there is an α such that $g_\phi(\alpha)$ is zero and $(*)$ holds for almost every β. For all such points one deduces $g_\phi(\alpha+\beta) = 0$, whence $g_\phi = 0$ almost everywhere. This contradicts the non-triviality of ϕ.

We have found that g_ϕ is a measurable homomorphism of $\mathbb{T}$ into $\mathbb{C}\setminus\{0\}$. It is well known - see for example [K] - that this forces $g_\phi(\theta) = e^{in\theta}$ for some $n \in \mathbb{Z}$, i.e., ϕ corresponds to a character of the circle group. The claimed result is precisely this statement.

We introduce the translation operator on ${L^1_{b0}}^*$ by declaring ϕ_τ of $\phi \in {L^1_{b0}}^*$ to be determined by $\phi_\tau \sim \sum \phi_n e^{-in\tau} e^{in\theta}$, that is $(\phi_\tau)_n = e^{-in\tau}\phi_n$. The result is that the representative for ϕ_τ is $[g_\phi]_\tau$ and the expected duality $\langle f, \phi_\tau\rangle = \langle f_{-\tau}, \phi\rangle$ obtains.

Observation 6.4. *${L^1_{b0}}^*$ is translation invariant and $\|\phi_\tau\|_* = \|\phi\|_*$. In addition, $\phi_\tau \xrightarrow{w} \phi$ as $\tau \to 0$.*

Since $\|f_{-\tau}\| = \|f\|$ and $\langle f, \phi_\tau\rangle = \langle f_{-\tau}, \phi\rangle$, we deduce $\|s_N\phi_\tau\|_* = \|s_N\phi\|_*$. By Proposition 6.1 $\phi_\tau \in {L^1_{b0}}^*$ and $\|\phi_\tau\|_* = \|\phi\|_*$. Furthermore,

$$\bigl|\langle f, \phi - \phi_\tau\rangle\bigr| = \bigl|\langle f - f_{-\tau}, \phi\rangle\bigr| \leqslant \|\phi\|_*\, \|f - f_{-\tau}\|.$$

The continuity in L^1_{b0} for the translation operator demonstrates that ϕ_τ converges weak-$*$ to ϕ.

Remark. This observation shows that ${L^1_{b0}}^*$ is a homogeneous space in its weak topology. Whether the same is true in the norm topology is unclear; it remains to confirm or refute $\|\phi - \phi_\tau\|_* \to 0$.

Proposition 6.5. *${L^1_{b0}}^* \subsetneq L^1_{b0} \subsetneq L^1$ as sets and $\|\phi\|_1 \leqslant \|\phi\| \leqslant \|\phi\|_*$ for all $\phi \in {L^1_{b0}}^*$, given the identification of ϕ with g_ϕ in their identical action on L^1_{b0}.*

Proof. Proposition 6.2 shows $\|s_N g_\phi\|_1 \leqslant \|s_N\phi\|_* \leqslant \|\phi\|_*$ for all N, whence $\|g_\phi\| \leqslant \|\phi\|_*$ and ${L^1_{b0}}^* \subseteq L^1_{b0}$ via $\phi \leftrightarrow g_\phi$, recalling $g_\phi \in L^2 \subseteq L^1_{b0}$. On the other hand, $\sum_{n\geqslant 2}(\log n\, \log\log n)^{-1} e^{in\theta}$ will later in Example 6.13 (a result independent of the present argument) be seen to belong to $L^1_{b0} \setminus {L^1_{b0}}^*$.

It will be clear from later examples that there exist many functionals in ${L^1_{b0}}^*$ which arise from unbounded functions.

Theorem 6.6. *Letting $h_\phi(\theta) = g_\phi(-\theta)$, the duality pairing may be expressed*

i) $\langle f, \phi\rangle = \lim_{N\to\infty} s_N f * h_\phi(0)$,

ii) $s_N f * h_\phi(\tau) = \langle f, s_N \phi_\tau\rangle = \langle f_{-\tau}, s_N\phi\rangle$.

For $f \in L^1_{b0}$ and $\phi \in {L^1_{b0}}^*$, the expression $\lim_{N\to\infty} f * s_N\phi(\theta)$ defines a continuous function. Likewise $\tau \mapsto \langle f, \phi_\tau\rangle$ introduces a continuous function denoted $f \star \phi$. In fact, they are identical functions. We have also $\|f \star \phi\|_{C(\mathbb{T})} \leqslant \|f\|\,\|\phi\|_*$.

This last result has to await its proof until the already pending proof has been completed! The second representation is instrumental in demonstrating ${L^1_{b0}}^*$ to be a convolution subalgebra of L^1.

Proof of Theorem 6.2. Consider a finite $p > 1$. We know from the theory of Riesz' that $\|\ \| \leqslant C_p \|\ \|_p$, whence for polynomial f

$$\big|\langle f, \phi\rangle\big| \leqslant \|\phi\|_*\,\|f\| \leqslant C_p\,\|\phi\|_*\,\|f\|_p.$$

In consequence, the restriction of ϕ to L^p belongs to L^{p*} with $\|\phi\big|_{L^p}\|_{L^{p*}} \leqslant C_p\,\|\phi\|_*$; this being based on $L^p \subseteq L^1_{b0}$. We conclude the existence of a function $g_p \in L^{p'}$ such that $g_p = \phi\big|_{L^p}$ as elements in L^{p*}.

Each polynomial h gives $\langle h, \phi\rangle = \langle h, \phi\big|_{L^p}\rangle = \langle h, g_p\rangle = \int h g_p\,dm$, whence

$$\langle f, \phi\rangle = \lim_{N\to\infty} \langle s_N f, \phi\rangle = \lim_{N\to\infty} \int s_N f\, g_p\,dm, \quad \text{all } f \in L^1_{b0}.$$

From the inequality

$$\big|\langle f, s_N g_p\rangle\big| = \big|\langle s_N f, \phi\big|_{L^p}\rangle\big| = \big|\langle s_N f, \phi\rangle\big| \leqslant \|\phi\|_*\,\|s_N f\|,$$

the membership $g_p \in {L^1_{b0}}^*$ obtains, with $\|g_p\|_* \leqslant \|\phi\|_*$.

In case $p, q > 1$ are considered, $\|g_p\|_1 \leqslant \|g_p\|_{p'} = \|\phi\big|_{L^p}\|_{L^{p*}} \leqslant C_p\,\|\phi\|_*$ as well as $\|g_q\|_1 \leqslant C_q\,\|\phi\|_*$, so $g_p - g_q$ is integrable. Since polynomial h gives

$$\int h(g_p - g_q)\,dm = \langle h, \phi\rangle - \langle h, \phi\rangle = 0,$$

we deduce that $g_p = g_q$ almost everywhere. This means that for $p > 1$

$$\|g_2\|_{p'} = \|g_p\|_{p'} \leqslant C_p\,\|\phi\|_* = C_{p'}\,\|\phi\|_*,$$

since $C_p = C_{p'}$. From $C_2 = 1$ every $1 \leqslant p \leqslant 2$ gives $\|g_2\|_p \leqslant \|g_2\|_2 \leqslant \|\phi\|_*$, so the general property $C_p \leqslant C \cdot p$ for $p \geqslant 2$ demonstrates that the choice $g_\phi = g_2$ has all properties claimed for it in the statement.

Proof of Theorem 6.6. Since $\widehat{h_\phi}(n) = \phi_{-n}$ it is clear that all three expressions in ii) evaluate to the common value $\sum_{-N}^N \widehat{f}(n)e^{in\tau}\phi_{-n}$; hence the claim. Taking the particular case $\tau = 0$ and letting $N \to \infty$ certifies i) by definition of the pairing.

As $f \in L^1_{b0}$ and $\phi \in {L^1_{b0}}^*$, every $(s_N f) * h_\phi$ is a continuous function. Furthermore, for all $\tau \in \mathbb{T}$

$$\big|s_N f * h_\phi(\tau) - s_M f * h_\phi(\tau)\big| = \big|\langle s_N f - s_M f, \phi_\tau\rangle\big| \leqslant \|s_N f - s_M f\|\,\|\phi\|_*.$$

It follows that

$$\|s_N f * h_\phi - s_M f * h_\phi\|_{C(\mathbb{T})} \leqslant \|s_N f - s_M f\| \, \|\phi\|_* \to 0, \quad \text{as } N, M \to \infty,$$

which shows that $\{s_N f * h_\phi\}_{N=0}^{\infty}$ is a Cauchy sequence in $C(\mathbb{T})$. Consequently, $\lim_{N\to\infty} s_N f * h_\phi$ is a continuous function.

The identity $\langle s_N f, \phi_\tau\rangle = s_N f * h_\phi(\tau)$ and the claim in the last paragraph show that also $\langle f, \phi_\tau\rangle$ is a continuous function of τ. Simultaneously, the argument above, taking this last relation into consideration, establishes that the two functions in the claim coincide point-wise. By construction the inequality

$$\|f \star \phi\|_{C(\mathbb{T})} \leqslant \sup_{N\geqslant 0} \|\tau \mapsto \langle s_N f, \phi_\tau\rangle\|_\infty \leqslant \|f\| \, \|\phi\|_*$$

clarifies the last statement.

Now the preparation needed to show ${L^1_{b0}}^*$ to be an L^1-subalgebra has been completed.

Lemma 6.7. *For all $f \in L^1_{b0}$ and $\phi \in {L^1_{b0}}^*$, the membership $f * h_\phi \in L^\infty \subset L^1_{b0}$ obtains, and furthermore, $f * h_\phi = f \star \phi$ almost everywhere.*

Proof. Since $\langle s_N f, \phi_\tau\rangle = \sum_{|j|\leqslant N} \widehat{f}(n)\widehat{h_\phi}(n)e^{in\tau}$ it is clear from the construction above that $f\star\phi$ and $f * h_\phi$ have identical Fourier coefficients; hence they are equal almost everywhere and $f * h_\phi \in L^\infty$. The algebra L^1_{b0} being an ideal in L^1, it is obvious that $f * h_\phi \in L^1_{b0}$.

Remark. Remember the inclusion $L^1_{b0} \supseteq L^\infty$ from $\|g\| \leqslant \|g\|_2 \leqslant \|g\|_\infty$.

Proposition 6.8. *Define $\phi \star \psi$ in ${L^1_{b0}}^*$ for $\phi, \psi \in {L^1_{b0}}^*$ by $\langle f, \phi\star\psi\rangle = \langle f\star\phi, \psi\rangle$. Then ${L^1_{b0}}^*$ is a commutative algebra with the multiplication $g_{\phi\star\psi} = g_\phi * g_\psi$.*

Proof. Lemma 6.7 shows that $\langle f\star\phi, \psi\rangle$ makes sense and

$$\big|\langle f\star\phi, \psi\rangle\big| \leqslant \|f * h_\phi\| \, \|\psi\|_* \leqslant \|f\| \, \|h_\phi\|_1 \, \|\psi\|_* \leqslant \|f\| \, \|\phi\|_* \, \|\psi\|_*$$

by the norm inequality in Theorem 6.6. This means that $\phi\star\psi \in {L^1_{b0}}^*$ with $\|\phi\star\psi\|_* \leqslant \|\phi\|_* \, \|\psi\|_*$.

The formal series for $\phi\star\psi$ has coefficients

$$(\phi\star\psi)_n = \langle e^{-in\theta}, \phi\star\psi\rangle = \langle e^{-in\theta} * h_\phi, \psi\rangle = \widehat{h_\phi}(-n)\, \langle e^{-in\theta}, \psi\rangle = \phi_n\, \psi_n,$$

so $\phi\star\psi$ and $\psi\star\phi$ have the same expansion $\sum \phi_n\, \psi_n\, e^{in\theta}$. It follows that $g_{\phi\star\psi} = g_\phi * g_\psi$ as well as $h_{\phi\star\psi} = h_\phi * h_\psi$. All the claims have been verified.

Observe that we have obtained the identity

$$\langle f, \phi\star\psi\rangle = f * h_{\phi\star\psi}(0) = f * h_\phi * h_\psi(0) = \langle f, g_\phi * g_\psi\rangle$$

for all polynomials f and all $\phi, \psi \in {L^1_{b0}}^*$.

In case $\phi \in L^1 \cap {L^1_{b0}}^*$ we have $g_\phi = \phi$ almost everywhere and $h_\phi = \widetilde{g_\phi} = \widetilde{\phi}$, where $\widetilde{f}(\theta) = f(-\theta)$. In consequence, each $\psi \in {L^1_{b0}}^*$ gives $g_{\phi\star\psi} = \phi * g_\psi$ and $h_{\phi\star\psi} = \widetilde{\phi} * h_\psi$ viewed as L^1-functions.

Definition. For $\phi \in L^1$ and $\psi \in {L^1_{b0}}^*$ the object $\phi\star\psi$ is determined by $g_{\phi\star\psi} = \phi * g_\psi$ and $h_{\phi\star\psi} = \widetilde{\phi} * h_\psi$.

This notion coincides with the previous notation for the cases where ϕ belongs either to L^1_{b0} or to $L^1 \cap {L^1_{b0}}^*$; the latter class appears as long as ϕ and g_ϕ are not identified. In contrast to $L^1_{b0} \star {L^1_{b0}}^* \subseteq L^\infty$, by Lemma 6.7, the space $L^1 \star {L^1_{b0}}^*$ does contain unbounded functions; Example 6.13 can be used to construct some instances.

Proposition 6.9. i) *For $\phi \in L^1$, $\psi \in {L^1_{b0}}^*$ the membership $\phi \star \psi \in {L^1_{b0}}^*$ obtains. In addition, $\|\phi \star \psi\|_* \leqslant \|\phi\|_1 \, \|\psi\|_*$ and $\langle f, \phi \star \psi\rangle = \langle f * \widetilde{\phi}, \psi\rangle$.*

ii) *$\big({L^1_{b0}}^*, \star\big)$ is a commutative algebra and an ideal of L^1 under the convolution $\star$, containing the subspace of all polynomials. In other words, ${L^1_{b0}}^*$ is a Segal algebra.*

Remark. By definition of $\star$ it becomes the ordinary L^1-convolution as soon as the functional $\phi \in {L^1_{b0}}^*$ is identified with either g_ϕ or h_ϕ.

Proof. $g_{\phi\star\psi}$ and hence also $\phi\star\psi$ have a formal expansion $\sum \widehat{\phi}(n)\psi_n \, e^{in\theta}$. It follows that for $f \in L^1_{b0}$

$$\langle f, s_N(\phi \star \psi)\rangle = \sum_{|n|\leqslant N} \widehat{f}(n)\widehat{\phi}(-n)\psi_{-n} = \sum_{|n|\leqslant N} \widehat{f * \widetilde{\phi}}(n)\psi_{-n} = \langle f * \widetilde{\phi}, s_N\psi\rangle.$$

In particular,

$$\big|\langle f, s_N(\phi \star \psi)\rangle\big| \leqslant \|f * \widetilde{\phi}\| \, \|\psi\|_* \leqslant \|f\| \, \|\widetilde{\phi}\|_1 \, \|\psi\|_* = \|f\| \, \|\phi\|_1 \, \|\psi\|_*.$$

Letting $N \to \infty$ we deduce the three properties in (i).

Furthermore, $\phi * g_\psi$ is the g-representative for $\phi \star \psi$, so $\|\phi * g_\psi\|_1 \leqslant \|\phi \star \psi\|_* \leqslant \|\phi\|_1 \, \|\psi\|_*$, thereby proving ${L^1_{b0}}^*$ to be a convolution ideal of L^1 using the identification $\phi \mapsto g_\phi$.

At this stage a supply of non-trivial functionals in ${L^1_{b0}}^*$ would be most useful; this we do next.

Proposition 6.10. *Assume that $\phi \sim \sum_{-\infty}^{\infty} \phi_n e^{in\theta}$ has the properties*

$$\frac{|n|+1}{\log(|n|+2)} \, |\phi_n| \leqslant A \quad \textit{and} \quad \sum\nolimits_{n=-\infty}^{\infty} \frac{|n|+1}{\log(|n|+2)} \, \big|\, |\phi_n| - |\phi_{n+1}| \,\big| \leqslant A.$$

Then there is an absolute constant C, independent of A and ϕ, such that $\phi \in {L^1_{b0}}^*$ and $\|\phi\|_* \leqslant C\,A$.

Proof. One needs to prove $|\langle f, s_N\phi\rangle| \leqslant C\,A\,\|f\|$ for all N and f. Let for this purpose $f \sim \sum c_n e^{in\theta} \in L^1_{b0}$. According to Salem–Zygmund's theorem both $d_N = \sum_{n=0}^{N} |\phi_n|$ and $d'_N = \sum_{n=-N}^{-1}$ are majorised by $C\,\|f\|\,N/\log(N+1)$ for $N \geqslant 1$. A calculation completes the proof:

$$|\langle f, s_N\phi\rangle| = \big|\sum\nolimits_{n=-N}^{N} c_n\phi_{-n}\big| \leqslant \sum\nolimits_{n=-N}^{N} |c_n| \, |\phi_{-n}|$$

$$
\begin{aligned}
&= |c_0| + d_N\,|\phi_{-N}| + d'_N\,|\phi_N| + \sum_{n=1}^{N-1} d_N\,\big(|\phi_{-n}| - |\phi_{-n-1}|\big)\\
&\qquad + \sum_{n=1}^{N-1} d'_N\,\big(|\phi_n| - |\phi_{n+1}|\big)\\
&\leqslant |c_0| + \frac{CN\|f\|}{\log(N+1)}\Big\{|\phi_{-N}| + |\phi_N|\Big\}\\
&\qquad + C\,\|f\| \sum_{n=1}^{N-1} \frac{n}{\log(n+1)}\Big\{\big|\,|\phi_{-n}| - |\phi_{-n-1}|\,\big| + \big|\,|\phi_n| - |\phi_{n+1}|\,\big|\Big\}\\
&\leqslant C\,\|f\|\Big\{\frac{N+1}{\log(N+2)}\big(|\phi_{-N}| + |\phi_N|\big)\\
&\qquad\qquad + \sum_{n=-N}^{N} \frac{|n|+1}{\log(|n|+2)}\,\big|\,|\phi_n| - |\phi_{n+1}|\,\big|\Big\}\\
&\leqslant C\,A\,\|f\|.
\end{aligned}
$$

Corollary 6.11. *Let $\{\phi_n\}_{-\infty}^{\infty}$ have $|\phi_n|$ decreasing in each direction as $n \to \pm\infty$ and $|n| \geqslant M$. There is a universal constant such that $\phi \sim \sum_{n=-\infty}^{\infty} \phi_n e^{in\theta}$ satisfies*

$$
\|\phi\|_* \leqslant C\Big\{\,\|s_{M-1}\phi\|_* + (M+1)\,\|\,\{\phi_n/\log(|n|+2)\}\,\|_{\ell^1}\,\Big\}.
$$

Proof. Write $\psi = \phi - s_M\phi$; clearly $\|\phi\|_* \leqslant \|s_M\phi\|_* + \|\psi\|_*$.

The decrease of coefficients shows

$$
|\psi_{\pm n}| \leqslant \frac{\log(|n|+2)}{n-M+1} \sum\nolimits_{|k|\leqslant n} \frac{|\phi_k|}{\log(|k|+2)},
$$

which obviously provides for all n, since $\psi_n = 0$ if $|n| < M$,

$$
\frac{n+1}{\log(n+2)}\,|\psi_{\pm n}| \leqslant (M+1)\,\left\|\frac{\phi_k}{\log(|k|+2)}\right\|_{\ell^1}.
$$

Furthermore, the variation inequality

$$
\begin{aligned}
\sum_{n=M+1}^{N} \frac{n}{\log(n+1)}\,\big|\,|\psi_n| - |\psi_{n+1}|\,\big| &= \sum_{n=M+1}^{N} \frac{n}{\log(n+1)}\,\big(\,|\psi_n| - |\psi_{n+1}|\,\big)\\
&= \frac{(M+1)|\phi_{M+1}|}{\log(M+2)} - \frac{N\,|\phi_{N+1}|}{\log(N+1)} + \sum_{n=M+2}^{N} |\phi_n|\,\left\{\frac{n}{\log(n+1)} - \frac{n-1}{\log n}\right\}\\
&\leqslant \frac{(M+1)|\phi_{M+1}|}{\log(M+2)} - \frac{N\,|\phi_{N+1}|}{\log(N+1)} + C\sum \frac{|\phi_n|}{\log(n+1)}\\
&\leqslant C\,(M+1)\,\left\|\frac{\phi_n}{\log(|n|+2)}\right\|_{\ell^1}
\end{aligned}
$$

is independent of the upper limit N. A perfectly similar calculation shows also

$$
\sum_{n=M+1}^{N} \frac{n}{\log(n+1)}\,\big|\,|\psi_n| - \psi_{n+1}|\,\big| \leqslant C\,(M+1)\,\left\|\frac{\phi_n}{\log(|n|+2)}\right\|_{\ell^1}.
$$

The previous proposition displays that $\|\psi\|_*$ is dominated by a multiple of the ℓ^1-norm in the present statement, so the full claim for $\|\phi\|_*$ follows.

Corollary 6.12. *Consider $\phi \sim \sum \phi_n e^{in\theta}$ such that $\phi_n \geqslant 0$ and ϕ_n, $\phi_{-n} \searrow 0$ for $n \in [M, \infty[$. Then $\phi \in {L^1_{b0}}^*$ if and only if $\sum_{|n|\geqslant 2} \phi_n / \log |n|$ converges.*

Proof. Should the series converge, the previous corollary proves ϕ to be an element in ${L^1_{b0}}^*$.

Assume for the other implication $\phi \in {L^1_{b0}}^*$ and consider the by now well known $\ell(\theta) = \sum_{|n|\geqslant 2} (\log |n|)^{-1} e^{in\theta}$. One recalls that $\ell \in L^1_b$ and $\|\ell - s_N \ell\| \to 4/\pi^2$, so the collection of polynomials $s_N \ell$ is uniformly bounded in L^1_{b0}. This implies $|\langle s_N \ell, \phi\rangle| \leqslant \|s_N \ell\| \, \|\phi\|_* \leqslant C \, \|\phi\|_*$ independently of N. Observing $\langle s_N \ell, \phi\rangle = \sum_{2 \leqslant |n| \leqslant N} \phi_{-n} / \log |n|$ it is clear that the claimed convergence of the series is a consequence.

Remark. The slightly more general case $\zeta \xi^n \phi_n \geqslant 0$, for some ζ, $\xi \in \mathbb{T}$, may be reduced to the above case simply by considering $\zeta \phi_\tau$ with $\xi = e^{i\tau}$. It is functionals with more irregular oscillation of $\arg \phi_n$ that present difficulties.

Example 6.13. By Corollary 6.12 the function $\sum_{n=2}^{\infty} \frac{e^{in\theta}}{n \log^a n}$ belongs to ${L^1_{b0}}^*$ for every $a > 0$. These functions are point-wise unbounded as soon as $0 < a \leqslant 1$; in particular, ${L^1_{b0}}^* \supsetneq L^\infty$.

Likewise $\sum_{n=3}^{\infty} \frac{\cos n\theta}{n \log^a \log n}$ represents an element in ${L^1_{b0}}^*$ if and only if $a > 1$. Observe that no restrictions on a are needed to make these to be elements in L^1_{b0}; this follows already from Lemma 1.5.

7. Relations to other norms.

We have already seen in Theorem 6.2 that L^p-norms of g_ϕ are controlled by the norm of $\phi \in {L^1_{b0}}^*$, but that the L^∞-norm is not approachable of natural reasons. In view of the construction of ϕ from all $s_N \phi$ in ${L^1_{b0}}^*$, it seems natural to consider a particular norm put on each $s_N \phi$ and then try to recover information on $\|\phi\|_*$.

The first result, which is essentially sharp by Corollary 7.3, explains why Theorem 6.2 could not say anything on L^∞-norms.

Proposition 7.1. *Let $\phi \in {L^1_{b0}}^*$. Then for all N*

$$\|s_N \phi\|_\infty \leqslant C \, \{1 + \log(N+1)\} \, \|\phi\|_*.$$

Proof. The translated Dirichlet polynomial $D_N(\theta - t)$ has

$$\langle D_N(\cdot - t), \phi\rangle = \sum_{n=-N}^{N} \phi_{-n} e^{-int} = s_N \phi(t).$$

In particular,

$$|s_N \phi(t)| = \left|\langle (D_N)_t, \phi\rangle\right| \leqslant \|D_N\| \, \|\phi\|_* \leqslant C \, \{1 + \log(N+1)\} \, \|\phi\|_*,$$

which is the claim.

It is convenient to reinterpret Salem–Zygmund's theorem:

Proposition 7.2. *Let p_N be a trigonometric polynomial of degree N. If a linear form $\langle \cdot, p_N\rangle$ is defined on M_b using $\langle \mu, p_N\rangle = \sum \widehat{\mu}(n) \, \widehat{p_N}(-n)$, then $p_N \in M_b^*$ with $\|p_N\|_* \leqslant C \frac{N+1}{\log(N+2)} \|\widehat{p_N}\|_\infty$.*

Proof. Clearly $\langle \mu, p_N \rangle$ is linear in $\mu \in M_b$. For every $N \geqslant 2$, by Salem–Zygmund's theorem,

$$|\langle \mu, p_N \rangle| \leqslant \sum_{|n| \leqslant N} |\widehat{\mu}(n)|\, |\widehat{p_N}(-n)| \leqslant C\, \tfrac{N}{\log N}\, \|\widehat{p_N}\|_\infty\, \|\mu\|.$$

Clearly $|\langle \mu, p_1 \rangle| \leqslant 3\, \|\widehat{p_N}\|_\infty\, \|\widehat{\mu}\|_\infty \leqslant 3\, \|\widehat{p_N}\|_\infty\, \|\mu\|$, which establishes the claim.

Proposition 7.3. *In ${L^1_{b0}}^*$ and M_b^* the value $\|D_N\|_* = \frac{N}{\log N} \cdot \mathcal{O}(1)$ obtains.*

Proof. That the norm cannot exceed the stated value follows from Proposition 7.2, whereas $\|D_N\|_{M_b^*} \geqslant \|D_N\|_{{L^1_{b0}}^*}$ and Proposition 7.1, observing $\|D_N\|_\infty = 2N+1$, clearly give the lower estimate needed for the claim.

Since every $\phi \in L^\infty$ has $\|\phi\|_* \leqslant \|\phi\|_\infty$, it is clear that the inequality in Proposition 7.1 cannot be reversed. It is possible to achieve some limited understanding of $\|\phi\|_*$ if all $\|s_N \phi\|_\infty$ are taken into account, as the last two results in this section show.

Proposition 7.4. *Assume $\phi \in {L^1_{b0}}^*$ has $\phi_n \geqslant 0$. There is a constant independent of ϕ such that for all $N \geqslant 0$*

$$\sum_{n=0}^{\infty} \frac{\|s_n \phi\|_\infty}{(n+2) \log^2(n+2)} \leqslant C\, \|\phi\|_*.$$

Remark. In particular, the increase exhibited in Proposition 7.1 is never attained for this specialized kind of functionals.

Proof. Positivity yields $\|s_N \phi\|_\infty = \sum_{-N}^{N} \phi_n$. Therefore

$$\begin{aligned}
\sum_{-N}^{N} \frac{\phi_n}{\log(|n|+2)} &= \frac{\|s_0\phi\|_\infty}{\log 2} + \sum_{1}^{N} \frac{\|s_n\phi\|_\infty - \|s_{n-1}\phi\|_\infty}{\log(n+2)} \\
&= \frac{\|s_0\phi\|_\infty}{\log 2} + \frac{\|s_N\phi\|_\infty}{\log(N+2)} + \sum_{n=1}^{N} \|s_n\phi\|_\infty \left([\log(n+2)]^{-1} - [\log(n+3)]^{-1} \right) \\
&= \frac{\|s_0\phi\|_\infty}{\log 2} + \frac{\|s_N\phi\|_\infty}{\log(N+2)} + \sum_{n=1}^{N} \frac{\|s_n\phi\|_\infty \log\left(1 + \frac{1}{n+2}\right)}{\log(n+2)\log(n+3)}.
\end{aligned}$$

The first member can conveniently be seen as the action of ϕ, so this is equal to

$$\left\langle \sum_{n=-N}^{N} \frac{e^{in\theta}}{\log(|n|+2)}, \phi \right\rangle \leqslant C\, \|\phi\|_*,$$

since $\| \sum_{-N}^{N} [\log(|n|+2)]^{-1} e^{in\theta} \| \leqslant C$ independently of N. The claim follows from this inequality, simply by estimating from below the quotient of the three logarithms in the form of the quantity $C\, [(n+2)\log^2(n+2)]^{-1}$.

Corollary 7.5. *Consider $\phi \sim \sum \phi_n e^{in\theta}$ such that ϕ_{-n} and ϕ_n are decreasing to zero for $n \geqslant 0$. In ${L^1_{b0}}^*$ the two quantities $\sum_{n=0}^{\infty} \frac{\|s_n\phi\|_\infty}{(n+2)\log^2(n+2)} + \sup_{n \geqslant 0} \frac{\|s_n\phi\|_\infty}{\log(n+2)}$ and $\|\phi\|_*$ are equivalent as norms of this particular kind of functional.*

Proof. According to Corollary 6.11, the assumptions on ϕ implies

$$\|s_N\phi\| \leqslant C \sum_{n=-N}^{N} \frac{\phi_n}{\log(|n|+2)} \leqslant C\Big\{\frac{\|s_N\phi\|_\infty}{\log(N+2)} + \sum_{n=0}^{N-1} \frac{\|s_n\phi\|_\infty}{(n+2)\log^2(n+2)}\Big\},$$

where the last inequality is extracted from the proof of the previous proposition.

On the other hand, the reverse inequality for bounding the complicated expression by $C\,\|\phi\|_*$, follows without any assumption of decrease, but keeping the positivity, from Propositions 7.1 and 7.4. The proof is complete.

8. The remaining dual spaces.

This section focuses on some features of the dual spaces $L^1_b{}^*$ and $H^1_{b0}{}^*$. The first of these displays awkward properties in comparison with $L^1_{b0}{}^*$, such as the failure of continuity under translation as well as the non-denseness of partial sums even in the weak-$*$ topology. The analytic space on the other hand shows better behaviour than its general counterpart, as of course is to be expected.

To display the new behaviour of $L^1_b{}^*$ we start with a construction, which will lead to an element Ψ in $L^1_b{}^*$ with the necessary properties. In essence the shortcoming of $L^1_b{}^*$ is that its elements are not approximable in norm by polynomials, so that, as will be demonstrated, there are functionals $\psi \in L^1_b{}^*$ whose partial sums $s_N\psi$ yield no information on ψ at all.

Lemma 8.1. *The family of functionals ψ_N on M_b, for $N \geqslant 2$, defined by*

$$\langle \mu, \psi_N\rangle = \frac{\log N}{N} \sum_{n=N}^{2N} \widehat{\mu}(n)$$

is a bounded set in M_b^*.

Proof. Trivially every ψ_N is a member of M_b. By Salem–Zygmund's theorem

$$\Big|\tfrac{\log N}{N} \sum_{n=N}^{2N} \widehat{\mu}(n)\Big| \leqslant \tfrac{2\log N}{\log 2N} \cdot \tfrac{\log 2N}{2N} \sum_{|n|\leqslant 2N} |\widehat{\mu}(n)| \leqslant C\,\|\mu\|$$

for all $\mu \in M_b$ and uniformly in $N \geqslant 2$. This is the claim.

Recall from Proposition 5.8 that the linear space V in the following statement is a closed subalgebra of L^1_b.

Lemma 8.2. *The formula $\langle g, \psi\rangle = \lim_{N\to\infty}\langle g, \psi_N\rangle$ defines a bounded functional on $V = \{\mu * \ell + f\,;\, \mu \in M_{\mathrm{d}}(\mathbb{T}),\, f \in L^1_{b0}\}$. More specifically, the action of ψ on V is described by $\langle \mu * \ell + f, \psi\rangle = \mu\big(\{0\}\big)$.*

Proof. We need to make sense out of the notation

$$\langle g, \psi\rangle = \lim_{N\to\infty} \frac{\log N}{N} \textstyle\sum_{n=N}^{2N} \widehat{g}(n)$$

as a functional on V. Clearly the linearity is automatic, and by Lemma 8.1 the complex numbers on the right, as depending on N for fixed $g \in V$, are contained in the closed set $\{z \in \mathbb{C}\,;\, |z| \leqslant C\,\|g\|\}$ for an absolute constant C independent of N and $g \in V$. Consequently the claim follows once it is shown that the limit in fact has value $\mu(\{0\})$ for the element $g = \mu * \ell + f$ in V.

Let μ be a discrete measure and $f \in L^1_{b0}$. For such functions Salem–Zygmund's theorem says that (see [Z], page 289)

$$\frac{\log 2N}{2N} \sum\nolimits_{|n| \leqslant 2N} |\widehat{f}(n)| = o(1),$$

so $\langle f, \psi \rangle = 0$ exists.

On the other hand, one finds that

$$\frac{\log N}{N} \sum_{n=N}^{2N} \widehat{\mu * \ell}(n) = \int_{\mathbb{T}} \left(\sum_{n=N}^{2N} \frac{e^{-in\theta}}{1 + \frac{\log(n/N)}{\log N}} \frac{1}{N} \right) d\mu(\theta).$$

The integrand has modulus not exceeding $\frac{N+1}{N}$, and as $N \to \infty$ it is point-wise convergent to zero, except for $\theta = 0$ where the convergence is to 1. By dominated convergence it follows that the relevant expression $\langle \mu * \ell, \psi_N \rangle$ tends to $\mu(\{0\})$ with increasing N. This completes the proof of the statement.

Theorem 8.3. *There is a functional $\Psi \in L^{1*}_b$ such that*

i) $\Psi\Big|_{L^1_{b0}} = 0$, *i.e., $s_N \Psi = 0$ for all N, and*

ii) $\langle \ell, \Psi_\tau \rangle = \begin{cases} 1, \ \tau = 0, \\ 0, \ \tau \neq 0, \end{cases}$ *for the translates of Ψ.*

Corollary 8.4. (1) $L^{1*}_b \supsetneq {L^1_{b0}}^*$ *in the sense that not all functionals on L^1_b are determined by their action on L^1_{b0}.*

(2) *It is not true that $s_N\phi$ tends to ϕ in the weak-$*$ topology of L^{1*}_b, much less in its norm topology.*

(3) *The mapping $\tau \mapsto \phi_\tau$ is not continuous in L^{1*}_b.*

(4) *L^{1*}_b is not separable.*

Proof. The properties appearing in the four claims are verified by the functional $\Psi \in L^{1*}_b$ from Theorem 8.3. For the last claim just observe that $\tau \neq \sigma$ in $\mathbb{T}$ yields

$$1 = \langle \ell_\tau, \Psi_\tau - \Psi_\sigma \rangle \leqslant \|\ell_\tau\| \, \|\Psi_\tau - \Psi_\sigma\|_*,$$

so $\|\Psi_\tau - \Psi_\sigma\|_* \geqslant 1/\|\ell\|$. This prevents separability.

Proof of Theorem 8.3. Let Ψ be any Hahn–Banach extension to L^1_b from V of the functional $\psi \in V^*$ constructed in Lemma 8.2. Since the restrictions of Ψ and ψ to ${L^1_{b0}}^*$ agree and the latter is zero there, the first claim follows. The second claim is simply the obvious calculation

$$\langle \ell, \Psi_\tau \rangle = \langle \ell_{-\tau}, \Psi \rangle = \langle \delta_{-\tau} * \ell, \psi \rangle = \begin{cases} 1, \ \tau = 0, \\ 0, \ \tau \neq 0. \end{cases}$$

The desired properties of Ψ are thus evident.

These few results give a first indication that the structure of L^{1*}_b is essentially different from ${L^1_{b0}}^*$, which of course reflects the more complicated geometry of L^1_b in comparison to that of L^1_{b0}. The failure for polynomials to approximate the elements in L^1_b is at the heart of the matter. When proceeding to the analytic space ${H^1_{b0}}^*$, the added structure in H^1_{b0} ensures stronger results than those holding for ${L^1_{b0}}^*$.

In dealing with ${H^1_{b0}}^*$ we keep the notation $\| \, \|_*$ for the dual norm and trust that the circumstances make clear that it is with the analytic spaces in mind statements are being made.

The development in section 6 leading up to and including Theorem 6.2 can obviously be repeated *mutatis mutandis* for ${H^1_{b0}}^*$ with one alteration: the representative g_p of $\phi\big|_{H^p}$ can be taken in $\overline{H^{p'}}$, where the bar indicates complex conjugation. We formalize the outcome for completeness. Observe that the dual pairing intentionally is taken to be sesquilinear when dealing with the analytic spaces, and so the conventions now differs from the handling of ${L^1_{b0}}^*$.

Proposition 8.5. *${H^1_{b0}}^*$ constitutes a subclass of expansions $\phi \sim \sum_{-\infty}^{\infty} \phi_n e^{in\theta}$ with a dual pairing $\langle f, \phi\rangle$ described by $s_N\phi(\theta) = \sum_{-N}^{N} \phi_n e^{in\theta}$, $\phi_n = \overline{\langle e^{in\theta}, \phi\rangle}$, and*

1. $\langle f, s_N\phi\rangle = \displaystyle\int f\, \overline{s_N\phi}\, dm = \sum_{n=0}^{N} \widehat{f}(n)\, \overline{\phi_n}$.
2. *For some constant C and all $f \in H^1_{b0}$, $N \geqslant 0$*

$$\big|\langle f, s_N\phi\rangle\big| \leqslant C\, \|s_N f\|, \qquad \langle f, s_N f\rangle = \langle s_N f, \phi\rangle.$$

The minimal constant C coincides with $\|\phi\|_*$. In particular, $\|s_N\phi\|_* \leqslant \|\phi\|_*$.

For each $\phi \in {H^1_{b0}}^*$ there exists an analytic function $h_\phi \in \bigcap_{p\geqslant 1} H^p$ such that the boundary value function of h_ϕ has Fourier series $\sum_{n=0}^{\infty} \phi_n e^{in\theta}$. The function $g_\phi(\theta) = h_\phi(e^{i\theta})$ verifies $\langle f, \phi\rangle = \langle f, g_\phi\rangle$ for all $f \in H^1_{b0}$. In addition, $\|h_\phi\|_p \leqslant \|\phi\|_*$ for all $1 \leqslant p \leqslant 2$, and in general $\|h_\phi\|_p \leqslant C\cdot p\, \|\phi\|_*$.

Two methods for constructing members of ${H^1_{b0}}^*$ make a suitable starting point. The first is a simple application of lacunarity.

Proposition 8.6. *Let $0 = k_0 < k_1 < k_2 < \dots$ be a sequence of integers with $k_{n+1} \geqslant \rho\, k_n$ for $n \geqslant 1$ and a fixed $\rho > 1$. To every $b = \{b_n\}_{n=0}^{\infty} \in \ell^2$ there is an element $\psi_b \sim \sum_{n=0}^{\infty} b_n\, e^{ik_n\theta}$ in ${H^1_{b0}}^*$ with the property $\|\psi_b\| \leqslant C_\rho\, \|b\|_2$. Here the constant C_ρ depends only on ρ.*

Conversely, assume $\psi \in {H^1_{b0}}^*$ to have support $\{n \geqslant 0\,;\, \phi_n \neq 0\}$ being a lacunary sequence in $\mathbb{N}$ with lacunarity parameter ρ. Then $\|h_\psi\|_2 \leqslant \|\psi\|_* \leqslant C_\rho\, \|h_\psi\|_2$.

Proof. Paley's inequality generates a constant C_ρ such that for every $f \in H^1$

$$\Big[\sum_{n=0}^{\infty} |\widehat{f}(k_n)|^2\Big]^{1/2} \leqslant C_\rho\, \|f\|_1.$$

In particular, each $f \in H^1_{b0}$ satisfies

$$\begin{aligned}\big|\langle f, \psi_b\rangle\big| = \Big|\sum_{n=0}^{\infty} \widehat{f}(k_n)\overline{b_n}\Big| &\leqslant \big\|\{\widehat{f}(k_n)\}_{n=0}^{\infty}\big\|_2\, \|b\|_2 \leqslant C_\rho\, \|f\|_1\, \|b\|_2 \\ &\leqslant C_\rho\, \|b\|_2\, \|f\|,\end{aligned}$$

from which $\psi_b \in {H^1_{b0}}^*$ with $\|\psi_b\|_* \leqslant C_\rho\, \|b\|_2$ follows.

For the second part, Proposition 8.5 gives the left inequality even without any lacunarity, whereas the part of the statement that already has been treated provides the calculation

$$\|\psi\|_* \leqslant C_\rho\, \|\{\psi_n\}_0^{\infty}\|_2 = C_\rho\, \|h_\psi\|_2.$$

The equivalence of $\|\psi\|_*$ and $\|h_\psi\|_2$ has therefore been established.

Corollary 8.7. *Let $M : \mathbb{N} \to \mathbb{R}_+$ be unboundedly increasing, but otherwise arbitrary. There is an element $\phi \in {H^1_{b0}}^*$ such that* $\limsup_{n\to\infty} M(n)\,\phi_{-n} = +\infty$.

Proof. A suitable choice of $k_0 < k_1 < k_2 < \dots$ for $\phi \sim \sum_{n=0}^{\infty} \frac{1}{n+1} e^{ik_n\theta}$ will resolve the problem. Take $k_0 = 0$ and $k_1 = 1$. Recursively we desire $k_{n+1} \geqslant 2k_n$ and $M(k_{n+1})\, b_{n+1} \geqslant 2M(k_n)\, b_n$, where $b_n = (n+1)^{-1}$. Since M is increasing, it is to each k_n possible to determine k_{n+1} satisfying both requirements. Thus $\{k_n\}$ is lacunary of quotient at least 2, so the previous proposition yields $\|\phi\|_* \leqslant \sqrt{2}\,\|b\|_2 = \pi/\sqrt{3}$, using the determination $C_2 = \sqrt{2}$ due to Fournier.

Finally, by construction, $M(k_n)\, b_n \geqslant 2^{n-1} M(k_1)\, b_1$ for all $n \geqslant 1$. The intended limes superior follows from this.

Remark. The result says that the decay to zero can be arbitrarily slow. Soon we will see that the such decay can only take place on a small set of indices.

To describe the second method of construction we use the expression $\|\phi\|_m = \sup_{n\geqslant 0} (n+1)|\phi_n|$ applied to $\phi \sim \sum_{-\infty}^{\infty} \phi_n e^{in\theta}$. Obviously $\|\ \|_m$ is a semi-norm.

Lemma 8.8. *Seen in ${H^1_{b0}}^*$ the inequality $\|\phi\|_* \leqslant \pi\,\|\phi\|_m$ obtains.*

Proof. Consider $f \in H^1_{b0}$ and $N \geqslant 0$. Then

$$\begin{aligned}\big|\langle f, s_N\phi\rangle\big| &\leqslant \sum_{0\leqslant n\leqslant N} |\widehat{f}(n)|\,|\phi_n| \leqslant \|\phi\|_m \sum_{0\leqslant n\leqslant N} \frac{|\widehat{f}(n)|}{n+1}\\ &\leqslant \pi\,\|\phi\|_m\,\|f\|_1 \leqslant \pi\,\|\phi\|_m\,\|f\|,\end{aligned}$$

where the second to last transition is due to Hardy's inequality. The inequalities do not depend on N, so $\phi \in {H^1_{b0}}^*$ with $\|\phi\|_* \leqslant \pi\,\|\phi\|_m$ follows.

Corollary 8.9. *The class of formal series $c_0 + \sum_{n\neq 0} \frac{c_n}{n} e^{in\theta}$, for $c = \{c_n\}_{-\infty}^{\infty} \in \ell^\infty$ and $\|\{c_n\}_0^\infty\|_\infty \leqslant A$, constitutes a uniformly bounded subspace of ${H^1_{b0}}^*$, but which is not contained in ${L^1_{b0}}^*$.*

Proof. The norm for the formal series is in ${H^1_{b0}}^*$ at most $\pi\left(|c_0| + \|\{\frac{n+1}{n}\, c_n\}_1^\infty\|_\infty\right) \leqslant 2\pi A$. The boundary function of $\log(1-z) = \sum_{n\geqslant 1} n^{-1} z^n$ represents an element in ${H^1_{b0}}^* \setminus {L^1_{b0}}^*$, according to Corollary 6.12.

For one kind of functionals the above decay of rate $\mathcal{O}(n^{-1})$ is optimal, as we demonstrate presently.

Proposition 8.10. *Let $\phi \in {H^1_{b0}}^*$ with $\phi_n \geqslant 0$. There is a constant C independent of ϕ such that*

$$\sum_{n=0}^{N} \frac{\phi_n}{\log(n+2)} \leqslant C\,\log\log(N+2)\,\|\phi\|_*.$$

This rate of growth is attained by the member $\sum_1^\infty n^{-1} e^{in\theta}$ of ${H^1_{b0}}^*$.

Proof. By the Lemma 5.10

$$\sum_{n=0}^{N} \frac{\phi_n}{\log(n+2)} \leqslant \Big\| \sum_{n=0}^{N} \frac{z^n}{\log(n+2)} \Big\|\,\|\phi\|_* \leqslant C\,\log\log(N+2)\,\|\phi\|_*.$$

It is clear that this rate of growth is achieved by $\sum_1^\infty n^{-1} e^{-in\theta}$, which represents an element in ${H^1_{b0}}^*$ by Corollary 8.9.

Corollary 8.11. *There is $C > 0$ such that every $\phi \in {H^1_{b0}}^*$ with all $\phi_n \geqslant 0$ has the property* $\liminf_{n\to\infty} n\phi_n \leqslant C\,\|\phi\|_*$.

Remark. As for ${L^1_{b0}}^*$ the same result is valid for the case that $\xi^n\phi_n \geqslant 0$ by considering ϕ_τ and $\xi = e^{i\tau}$.

Proof. Choose for $\delta < \liminf n\phi_n$ an $N_\delta > 0$ such that $n \geqslant N_\delta$ implies $\delta \leqslant n\phi_n$. The previous proposition now shows

$$C\,\log\log N\,\|\phi\|_* \geqslant \sum_{n=N_\delta}^{N} \frac{\phi_n}{\log(n+2)} \geqslant \delta \sum_{n=N_\delta}^{N} \frac{1}{n\log(n+2)}$$
$$\geqslant C_1\,\delta\big(\log\log N - \log\log N_\delta\big).$$

It follows that
$$\delta\Big(1 - \frac{\log\log N_\delta}{\log\log N}\Big) \leqslant C\,\|\phi\|_*.$$

As $N \to \infty$ we deduce $\delta \leqslant C\,\|\phi\|_*$ independently of N_δ, so taking supremum over all possible δ the result has been established.

Remarks. i) Corollary 6.12 and Proposition 8.10 illustrate that elements of ${L^1_{b0}}^*$ and ${H^1_{b0}}^*$, respectively, show different behaviour for their Fourier coefficients.
ii) A similar method of proof as in Proposition 8.10 using $1 + \cdots + e^{iN\theta}$ as test function demonstrates that with the same assumption on positivity $\|s_N\phi\|_\infty = \sum_{n=0}^N \phi_n \leqslant C\,\log(N+2)\,\|\phi\|_*$, an inequality which has the same appearance for ${L^1_{b0}}^*$.

Corollary 8.12. *Let $\phi \sim \sum_{-\infty}^{\infty} \phi_n e^{in\theta}$ with $\phi_n \geqslant 0$ as $n \geqslant 0$ and with $\{n\phi_n\}_{n=0}^\infty$ monotone. Then $\phi \in {H^1_{b0}}^*$ if and only if $\{n\phi_n\}_{n=0}^\infty \in \ell^\infty$.*

Proof. From $\{n\phi_n\}_{n=0}^\infty \in \ell^\infty$ Lemma 8.8 produces $\phi \in {H^1_{b0}}^*$. Conversely, Corollary 8.11 yields from $\phi \in {H^1_{b0}}^*$ and $\phi_n \geqslant 0$ the existence of $\liminf_{n\to\infty} n\phi_n$ as a finite quantity. By monotonicity the membership in ℓ^∞ obtains.

Remark. The example $\sum_{k=1}^\infty k^{-3/4} z^{2^k} \in {H^1_{b0}}^*$ shows that monotonicity is decisive.

Similarly to the observations regarding ${L^1_{b0}}^*$ we quote for ${H^1_{b0}}^*$ results how L^∞-norms of $s_N h_\phi$ relate to the norm of ϕ. For the next two results, the optimality of the quantities $\log\log N$ and $\log N$, respectively, is once more demonstrated by the particular example mentioned in Proposition 8.10.

Proposition 8.13. *Let $\phi \in {H^1_{b0}}^*$ with $\phi_n \geqslant 0$. There is a universal constant independent of ϕ such that*

$$\frac{\|s_N h_\phi\|_\infty}{\log(N+2)} + \sum_{n=0}^{N-1} \frac{\|s_n h_\phi\|_\infty}{(n+2)\log^2(n+2)} \leqslant C\,\big\{1 + \log_+\log_+ N\big\}\,\|\phi\|_*.$$

Proof. The technique of proof applied in Proposition 7.4, now incorporating Lemma 5.10, demonstrates that the left-hand side in the statement is less that

$$C\sum_{n=0}^{N} \frac{\phi_n}{\log(n+2)} \leqslant C\,\Big\langle \sum\nolimits_{n=0}^{N} \frac{z^n}{\log(n+2)}, \phi\Big\rangle \leqslant C\,\big\{1 + \log_+\log_+ N\big\}\,\|\phi\|_*.$$

In each step the constants are independent of N and ϕ.

Corollary 8.14. *There is a constant C such that each $\phi \in {H^1_{b0}}^*$, with $\phi_n \geqslant 0$ for $n \geqslant 0$, satisfies*

$$\liminf_{N\to\infty} \frac{\|s_N h_\phi\|_\infty}{\log N} \leqslant C\,\|\phi\|_*.$$

Proof. Given ϕ of the kind described above, consider any $\rho \geqslant 0$ such that for $N \geqslant M = M_\rho$ the growth $\|s_N h_\phi\|_\infty \geqslant \rho \log N$ obtains. As a result of the previous proposition

$$\log\log N\,\|\phi\|_* \geqslant C\rho \sum_{n=M}^{N-1} \tfrac{1}{n\log n} \geqslant C\rho\,\{\log\log N - \log\log M\},$$

as soon as $N > M$, with the positive constant C independent of N, M, ρ, and ϕ. Hence

$$\rho\left\{1 - \tfrac{\log\log M}{\log\log N}\right\} \leqslant C\,\|\phi\|_*.$$

Letting $N \to \infty$ one deduces $\rho \leqslant C\,\|\phi\|_*$, from which the claim is an obvious consequence.

References

[An] Andersson M.E., *Some Banach spaces with uniformly norm-bounded partial sums*, Research reports in mathematics **5** (1999), Dept. of Math., University of Stockholm.

[A] Armitage, D., *The convergence of polynomial expansions of positive harmonic functions*, Math. Proc. Camb. Philos. Soc. **117** (1995), no. 1, 165–174.

[B] Burnham, *The relative completion of an A-Segal algebra is closed*, Proc. of AMS **49** (1975), no. 1, 116–122.

[D] Doss, R., *Harmonic analysis of oscillations*, J. London Math. Soc. (2) **7** (1973), 41–47.

[F] Fournier, J.J.F., *On the coefficients in Fourier-Stieltjes series with norm-bounded partial sums*, Colloquium Math. **XLIX** (1985), 257–265.

[H] Helson, H., *Proof of a conjecture by Steinhaus*, Proc. Nat. Acad. Sci. USA **40** (1954), 205–206.

[Ho] Hoffman, K., *Banach spaces of analytic functions*, Prentice-Hall Inc., New Jersey, 1962.

[K] Katznelson, Y., *An introduction to harmonic analysis*, 2nd ed., Dover Publ., New York, 1976.

[K2] ———, *Trigonometric series with positive partial sums*, Bull. Amer. Math. Soc. **71** (1965), 718–719.

[PS] Pigno, L. and B. Smith, *Quantitative bahaviour of transforms with norm-bounded partial sums*, Amer. J. Math. **102** (1980), 321–326.

[S] Sidon, S., *Ein Satz über positive harmonische Polynome*, Jahresbericht d. DMV **XXXV** (1926), 97–99.

[Wa] Wang, H-C., *Homogeneous Banach algebras*, Lecture notes in pure and applied mathematics vol. 29, Marcel Dekker Inc., New York, 1977.

[We] Weiss, M., *On a problem of Littlewood*, J. London Math. Soc. **XXXIV** (1959), no. 133, 217–221.

[Wo] Wojtaszczyk, P., *Some remarks about the space of measures with uniformly bounded partial sums and Banach-Mazur distances between some spaces of polynomials.*, Lecture Notes in Mathematics, LNM 1470, Proc. U. of Texas, 1987–1989, Springer, Heidelberg, 1991, pp. 60–67.

[Z] Zygmund, A., *Trigonometric series I*, 2nd ed., Cambridge University Press, Cambridge, 1959.

Mats Erik Andersson, Matematiska institutionen, Stockholms universitet, 106 91 Stockholm

E-mail address: matsa@matematik.su.se