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Testing and modeling dynamic properties of returns volatility

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Abstract

As a part of the essential financial theory it is assumed that asset returns are normally distributed with some mean value and some constant variance. This assumption is based on the large number law and is applicable to the significantly large number of observations. However, it has been observed that even for significantly long time periods this assumption is highly questionable. It has also been noticed that volatility of returns is varying over time in such a manner that reveals its' dependency within time. The dynamic models such as the class of ARCH/GARCH models has been developed for capturing dynamics of volatilities. In brief and generally, the ARCH/GARCH models express future volatility value based on the present and past values. The objective of this thesis is to survey the properties of returns and model volatility of returns with ARCH(1) and GARCH(1,1) models.

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Preface and acknowledgement

This is a Bachelor thesis, which will lead to a Bachelor's Degree in Mathematical Statistics at the Department of Mathematics at Stockholm University.

I would like to express my deep, sincere appreciation and gratitude to my supervisor Joanna Tyrcha, professor at the division of Mathematical Statistics, Stockholm University. Joanna's inspiration, help and guidance have been invaluable for me during the work with this thesis.

I would also like to thank my family and people close to me for their support and encouragement.

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1 Introduction

This thesis is inspired by the research work done by Robert F. Engle III over the past several decades with respect to the studying the changes of markets and economies over time.

Robert F. Engle's contribution to the economic science has been recognized by The Sveriges Riksbank Prize in Economic Science in Memory of Alfred Nobel in 2003. Robert F. Engle shared the prize with Clive W.J. Granger. Both of them have discovered different important properties of the economic time series and developed statistical methods for studying and modelling of these.

Robert F. Engle developed methods for examination of the volatility of economic time series. The ARCH model which is the shorted name for 'autoregressive conditional heteroscedasticity' was introduced by Engle in 1982, while applying it to the historical inflation in the U.K.. Engle promoted the methods which may identify and describe changes on financial markets with sequences of turbulent and calm periods. Finance is the field where the variety of ARCH models have been greatly important since the risks and returns are primary linked to each other and are available on at least daily basis.

The essential theory of the financial mathematics and econometrics, as for example Black-Scholes model and Capital Asset Pricing Model, is based on the assumption that the volatility is constant and use the historical volatility for the purposes of the financial modelling.

The theory about dynamic volatility of an asset returns appears to be very modern, interesting and important in the light of the recent financial instability and crisis. As a consequence the ARCH model has been discussed, further investigated and developed during the last thirty years.

We have become interested in the research done and also the models developed by Robert F. Engle for the financial time series and want to investigate the models of dynamic volatilities within the Swedish financial market. In this thesis we survey the properties of the returns time series and model time varying volatility for Swedish financial data. We follow up the exchange rate between the Swedish krona and another currency, its development within time and investigate its dynamic properties. This analysis is done by testing the returns distribution and dependency within time. Further we test the returns fluctuations on the basis of the ARCH(1) and GARCH(1,1) models.

2 Data

The data analysed in this thesis is the price of the U.S. dollar (USD) expressed in the Swedish krona (SEK), i.e. the exchange rate between the Swedish krona and the U.S. dollar. Since currencies are traded on a daily basis the changes in the price of one currency expressed in another are presented as time series data. As the world has become more international during the several past decades, the cross border trading of different kind of assets and also services has contributed to the additional impact on the currency changes among the economic variables such as inflation and economy's growth. Consequently the currency market becomes more linked to other markets. We want to investigate if the time series data presented by the exchange rate appears to have similar dynamic properties as such financial data as for example stock prices.

According to the definition on the Sveriges Riksbank's web site, every day the Swedish banks calculate a fixing rate according to the formula (bid+ask)/2 which are further established as a joint mid-price as the average value of the banks' fixing rates.

The data is represented by the daily value from January 2004 to December 2011. This period has been chosen in order to cover different conditions on the market both calm and turbulent and further to be sufficient for model estimation.

The data is available and has been downloaded from the Sveriges Riksbank's web site.

3 Theory

In this part of the thesis we will describe the theoretical references for the different parts of our analysis. This theory will be further used in our reasoning. The data we investigate further is the daily change in the Exchange Rate USD/SEK. Due to the fact that the data is collected sequentially over time, i.e. each trading day, it is presented as time series and shows how much the value of the Swedish krona has changed for each day.

3.1 Returns and Volatility

We start by defining the return on an asset at time t as R_t , t = 1, 2, …, n

$$R_t=\frac{I_t-I_{t-1}}{I_{t-1}}\approx\ln(1+\frac{I_t-I_{t-1}}{I_{t-1}})=\ln(\frac{I_t}{I_{t-1}}) \qquad (3.1)$$

where I_t is the value of the Exchange Rate at time t and n is the number of observations. The approximation above is valid for small values of $(I_t-I_{t-1})/I_{t-1}$.

In finance fluctuations in interest rate, exchange rate or any other return rate is called volatility. With reference to Höglund T. [11], volatility for a predefined time period of one day is estimated as a square root of the sample variance, according to the following formula

$$\hat{\sigma}=\sqrt{\frac{1}{n}\sum_{t=1}^{n}(R_t-\hat{\mu})^2} \qquad (3.2)$$

where $\hat{\mu}$ is the mean value of the returns. The definition above holds due to the assumption that volatility of returns is equal to volatility of growth as a consequence of the relation between return and growth.

3.2 References for preliminary analysis of data

According to the essential theory of the financial mathematics, as for example Black-Scholes model, the historical returns are assumed to be presented by time series with constant defined expected value and finite variance, with reference to [11].

As part of this theory, volatility is predictable and constant, i.e. does not change with time. It is assumed to be approximately normally distributed if the time period is sufficiently large, { R_t } are assumed to be a purely random process with independent increments.

In our preliminary analysis we will investigate whether (A) assumption of independent increments and (B) assumption of the normal distribution appears to be reasonable for our data.

3.2.1 Test of independence

A) Autocorrelation test

For time series data the assumption of independence is jeopardized since correlation appears to be introduced by time order. Autocorrelation function is the measure of dependency of observations separated by defined time period of length k, called the lag. We assume that { R_t } is a process defined as

$$R_t = \mu + \varepsilon_t \qquad (3.3)$$

where μ is the expected value of { R_t }. Independent random variable ε_t has expected value zero and variance σ^2.

The autocorrelation function for { R_t } is denoted r(k) and is estimated as follows with reference to Tamhane A.C., Dunlope D.D. [5]:

$$\hat{r}(k) = \frac{\sum_{t=1}^{n-k} (R_{t+k} - \hat{\mu})(R_t - \hat{\mu})}{\sum_{t=1}^{n} (R_t - \hat{\mu})^2} \qquad (3.4)$$

for k < n. Where $\hat{\mu}$ is the estimated mean value of { R_t }. The estimated sample autocorrelation function is assumed to be approximately normally distributed if the expected value of { R_t } is defined and constant, the variance is finite and also if n is significantly large. If it is further assumed that { R_t } is a sequence of serially uncorrelated random variables, then $E(\hat{r}(k))=0$ for all k and the standard deviation is equal to

$$\sigma_{\hat{r}(k)}=\frac{1}{\sqrt{n}} \quad . \tag{3.5}$$

In addition, according to Kirchgässner G., Wolters J. [12], the following test statistic, developed by Ljung G.M. and Box G.E.P., can be calculated for the autocorrelation coefficients in order to test dependence within the time series data:

$$Q=n(n-2)\sum_{i=1}^{k}\frac{\hat{r}(i)^2}{n-i} \quad . \tag{3.6}$$

This test statistic is asymptotically χ^2 distributed with k degrees of freedom.

B) Heteroscedasticity test

Referring to Andersson P., Tyrcha J. [2], in the classical linear regression model it is assumed that all error terms have the same, constant variance which is equal to σ^2. It is also assumed that the regression model is linear in the parameters independently of variables. If the error terms do not have the same variance, these are called heteroscedastic. For the linear model below

$$\boldsymbol{Y}=\boldsymbol{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon}$$

where $\boldsymbol{Y}=(Y_{1,}Y_{2,}\ldots,Y_n)^T$, $\boldsymbol{\beta}=(\beta_{1,}\beta_{2,}\ldots,\beta_m)^T$ and $\boldsymbol{X}$ is $n\times m$ design matrix, the sample errors, $\boldsymbol{\varepsilon}=(\varepsilon_{1,}\varepsilon_{2,}\ldots,\varepsilon_n)^T$, have the same variance and are uncorrelated then the ordinary least square estimator is the best linear unbiased estimator. If this is the case the error terms have equal variance of σ^2 of the following form

$$E[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{\boldsymbol{T}}|\boldsymbol{X}]=\sigma^2\mathbf{I}_{\boldsymbol{n}} \quad .$$

If the error terms are heteroscedastic the variance of these conditional on the chosen values of the explanatory variables is as follows:

$$E[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{\boldsymbol{T}}|\boldsymbol{X}]=diagonal(\sigma_{1,}^2\sigma_{2,}^2\ldots,\sigma_n^2)$$

where σ_i^2 can be estimated with $\hat{\sigma}_i^2 = \hat{\varepsilon}_i^2$.

3.2.2 Test of normal distribution

It is typically assumed that the errors are normally distributed as a part of the essential theory. This could be tested by estimation of the skewness and kurtosis of the sample distribution.

Referring to [12], kurtosis and skewness are the third and the forth moment of the distribution, which could be applied to both time series data itself and to estimated regression residuals in accordance to the following:

$E[(R_t - E(R_t))^i]$ for i= 3,4.

Kurtosis is a measure of the shape of the sample distribution while skewness is a measure of the asymmetry of the sample distribution. Skewness, S and kurtosis, K can be estimated by the following:

$$\hat{S} = \frac{1}{n} \frac{\sum_{t=0}^{n} (R_t - \hat{\mu})^3}{\sqrt{\left(\frac{1}{n}\sum_{t=0}^{n} (R_t - \hat{\mu})^2\right)^3}} \qquad \hat{K} = \frac{1}{n} \frac{\sum_{t=0}^{n} (R_t - \hat{\mu})^4}{\left(\frac{1}{n}\sum_{t=0}^{n} (R_t - \hat{\mu})^2\right)^2}$$

The theoretical value of skewness of the normal distribution is equal to zero due to the distribution's symmetrical characteristics. The theoretical value of kurtosis for the normal distribution is equal to three. Referring to Engle R.F. [6], the approximate asymptotic standard error of skewness is $\sqrt{6/n}$ and is $\sqrt{24/n}$ of kurtosis. If the sample distribution's values of skewness and kurtosis differ from the theoretical values of zero respectively three it could be an indication that the sample is not normally distributed. Especially, the kurtosis value is significantly higher than three could indicate that the observations' distribution has a 'fat tale' so called leptokurtic distribution.

Further it is a good practice to investigate the sample distribution graphically. It is helpful to review the normal plots of the data. Especially quantile plot and histogram could reveal if the sample data deviates from the assumed normal distribution.

3.3 General theory with respect to time series

The aim of the time series analysis is to find a mathematical model which could provide explanation for the sample observed.

3.3.1 White Noise Process

We start with definition of a stochastic process { X_t } which is called white noise if it has the following properties:

a) $E(X_t)=\mu$ and

b) $Var(X_t)=\sigma^2$ and,

c) $Cov(X_t, X_{t+k})=0$ for any k>0.

The above defined time series is denoted $\{X_t\}\in WN(\mu, \sigma^2)$. If such { X_t } contains independent and identically distributed random variables with expected value of zero and variance σ^2 then $\{X_t\}\in IID(0, \sigma^2)$.

3.3.2 Time series

The classical general model is defined by the properties of its first and second moments only $E[(X-E(X))^i]$ for i=1,2.

With reference to Gradell J. [9], such model could be defined at time t as follows:

$X_t=T_t+S_t+Y_t$

T_t is the trend function

S_t is the seasonal function

Y_t is a stationary time series if it's properties are as follows:

a) $E(Y_t^2)<\infty$ for each t and

b) $E(Y_t)=\mu$, constant and independent of t and,

c) $Cov(Y_t, Y_{t+k})=E[(Y_{t+k}-\mu)(Y_t-\mu)]$ is independent of t for each k.

A time series Y_t is called strictly stationary if $(Y_{1,} Y_{2,} \ldots, Y_t)$ and $(Y_{1+k}, Y_{2+k}, \ldots, Y_{t+k})$ have the same joint distribution for any integer k.

The goal of the time series analysis is to define T_t and S_t and test property of Y_t in hope of this to be a stationary time series.

Financial time series in form of returns on some asset typically do not have any seasonal components and are typically centred round zero with expected value significantly close to zero.

3.4 Dynamic modelling

Returns of any financial asset, represented as time series data is typically characterized by periods of significant fluctuations. ARCH and GARCH models are designed to exhibit the dynamics of the conditional second moment of the time series. The data generating process in accordance to these models could be defined as

$$X_t=\mu_t+\varepsilon_t$$

where μ_t is the conditional expected value of the time series data.

The conditional error term ε_t is defined as follows:

$$\varepsilon_t|\boldsymbol{X_{t-1}}\in N(0,\sigma_t) \quad (3.7)$$

where σ_t^2 is the conditional variance of ε_t given $\{\boldsymbol{X_{t-1}}\}=\{X_{t-1}, X_{t-2}, \ldots, X_1\}$ at time t. The conditional variance is type of weighted variance conditioned to the observed variance which is designed specifically for each model. However the unconditional expected value is zero and also unconditional variance σ^2 is constant.

Under the alternative notation of the conditional errors ε_t are generated as

$$\varepsilon_t = u_t \sigma_t \qquad (3.8)$$

where the innovation term is assumed to be $\{u_t\} \in IID\, N(0,1)$.

These models forecast variance of the time series in terms of actual and also past observations. The model of the conditional variance varies for different models.

3.4.1 ARCH (q) Model

The autoregressive conditional heteroscedasticity model defines the conditional variance as a linear function of lagged squared errors.

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \alpha_q \varepsilon_q^2 \qquad (3.9)$$

where $\alpha_i \geqslant 0$ for i=0, 1, ...,q.

3.4.2 GARCH (p,q) Model

The general autoregressive conditional heteroscedasticity model defines the conditional variance as a linear function of lagged squared errors and also lagged conditional variance.

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \alpha_q \varepsilon_q^2 + \beta_1 \sigma_{t-1}^2 + \ldots + \beta_p \sigma_{t-p}^2 \qquad (3.10)$$

where $\alpha_i, \beta_j \geqslant 0$ for i=0, 1, ...,q and j=1, 2, ..., p.

3.5 Dynamic modelling of the volatility with ARCH(1) and GARCH(1,1)

Returns of any financial asset, represented as time series data is typically characterized by periods of significant fluctuations. This could be explained by the uncertainty of holding an asset which is changing over time.

We decide to model our data with ARCH (1) and GARCH (1,1) models. These models are the simplest with only one lag of the squared residuals and in case of GARCH(1,1) also with one lag of the variance. These models are the ground for the entire class of the dynamic volatilities models.

We make an assumption that the sample has been generated by a stationary process according (3.3) with the constant expected value μ as follows:

$$R_t = \mu + \varepsilon_t$$

This process is characterized by the error terms $\varepsilon_t = R_t - \mu$. If we further assume that $\mu = 0$ we have the model

$$R_t = \varepsilon_t . \qquad (3.11)$$

The errors are conditionally independent provided the history of the process. It can be expressed by $\varepsilon_t | \boldsymbol{R_{t-1}} \in N(0, \sigma_t)$ and σ_t^2 is the conditional variance of $\{\boldsymbol{R_{t-1}}\} = \{R_{t-1}, R_{t-2}, \ldots, R_1\}$ at time t. However, we recall that the unconditional ε_t is distributed with expected value zero and variance σ^2. We also recall the alternative notation of the conditional errors $\varepsilon_t = u_t \sqrt{\sigma_t^2}$ with the innovation term assumed to be $\{u_t\} \in IID\, N(0,1)$.

3.5.1 ARCH (1) Model

The autoregressive conditional heteroscedasticity model defines the conditional variance for the process (3.11) above based on the formula (3.9) as

$$\sigma_t^2 = \alpha_0 + \alpha_1 R_{t-1}^2 \qquad (3.12)$$

where constants are non-negative.

3.5.2 GARCH (1,1) Model

The general autoregressive conditional heteroscedasticity model defines the conditional variance for the process above (3.11) based on the formula (3.10) as

$$\sigma_t^2 = \alpha_0 + \alpha_1 R_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \qquad (3.13)$$

where constants are non-negative.

3.5.3 Parameter estimation

There are several available methods for estimating the unknown parameters $\alpha_{0,}\alpha_{1,}\beta_1$ of the conditional variance processes above. Referring to Engle R.F. [6], the ordinary least squares method could be used, however maximum likelihood is more efficient for estimation of the parameters.

The following loglikelihood function, *l*, could be maximized in order to estimate $\alpha_{0,}\alpha_1$ for ARCH(1) model and $\alpha_{0,}\alpha_{1,}\beta_1$ for GARCH(1,1) model.

$$l=\ln L=\ln(f(\varepsilon_{1,}\dots,\varepsilon_n|\boldsymbol{R_{t-1}}))=\ln(\prod_{t=1}^{n} f(\varepsilon_t|\boldsymbol{R_{t-1}}))$$

Assumed the process (3.11) with $\mu=0$ and also assumed normal distribution of the conditional errors $\varepsilon_t|\boldsymbol{R_{t-1}}\in N(0,\sigma_t)$, the loglikelihood function is as follows:

$$l=-\frac{n}{2}\ln(2\pi)-\frac{1}{2}\sum_{t=1}^{n}\ln\sigma_t^2-\frac{1}{2}\sum_{t=1}^{n}\frac{R_t^2}{\sigma_t^2} \qquad (3.14)$$

Taking into consideration the definitions of the conditional variances it gives us

$$l_{ARCH(1)}=-\frac{n}{2}\ln(2\pi)-\frac{1}{2}\sum_{t=1}^{n}\ln(\alpha_0+\alpha_1 R_{t-1}^2)-\frac{1}{2}\sum_{t=1}^{n}\frac{R_t^2}{(\alpha_0+\alpha_1 R_{t-1}^2)}$$

$$l_{GARCH(1,1)}=-\frac{n}{2}\ln(2\pi)-\frac{1}{2}\sum_{t=1}^{n}\ln(\alpha_0+\alpha_1 R_{t-1}^2+\beta_1\sigma_{t-1}^2)-\frac{1}{2}\sum_{t=1}^{n}\frac{R_t^2}{(\alpha_0+\alpha_1 R_{t-1}^2+\beta_1\sigma_{t-1}^2)}$$

Maximization of the functions above is to be done numerically with the inequality constrains.

3.5.4 Properties of unconditional error terms

Recalling the fact that the unconditional mean value is equal to zero and the unconditional variance is not changing over time based on the law of iterated expectations

$$E[Y]=E[E[Y|X]]$$

where Y is a random variable and X is relevant known data. Assuming stationarity of the process, we can state for ARCH(1)

$$\sigma^2=Var(\varepsilon_t)=E[\varepsilon_t^2]=E[E[\varepsilon_t^2|\boldsymbol{R_{t-1}}]]=E[\alpha_0+\alpha_1\varepsilon_{t-1}^2]=\alpha_0+\alpha_1 E[\varepsilon_{t-1}^2]=\frac{\alpha_0}{1-\alpha_1}$$

and also for GARCH(1,1) the unconditional variance could be expressed similarly

$$\sigma^2=Var(\varepsilon_t)=E[\varepsilon_t^2]=E[E[\varepsilon_t^2|\boldsymbol{R_{t-1}}]]=\frac{\alpha_0}{1-\alpha_1-\beta_1}$$

4 Results

4.1 Preliminary analysis of the data

We start by plotting the daily value of the exchange rate between the Swedish krona and the U.S. dollar for the entire period in order to see what appearance the time series data has. The data is presented in Figure 1. The curves show the exchange rate development during the period of approximately 8 years, i.e. 2 016 observations. We need to understand that the increase in the exchange rate means that the value of the Swedish krona decreases in comparison to the U.S. dollar.

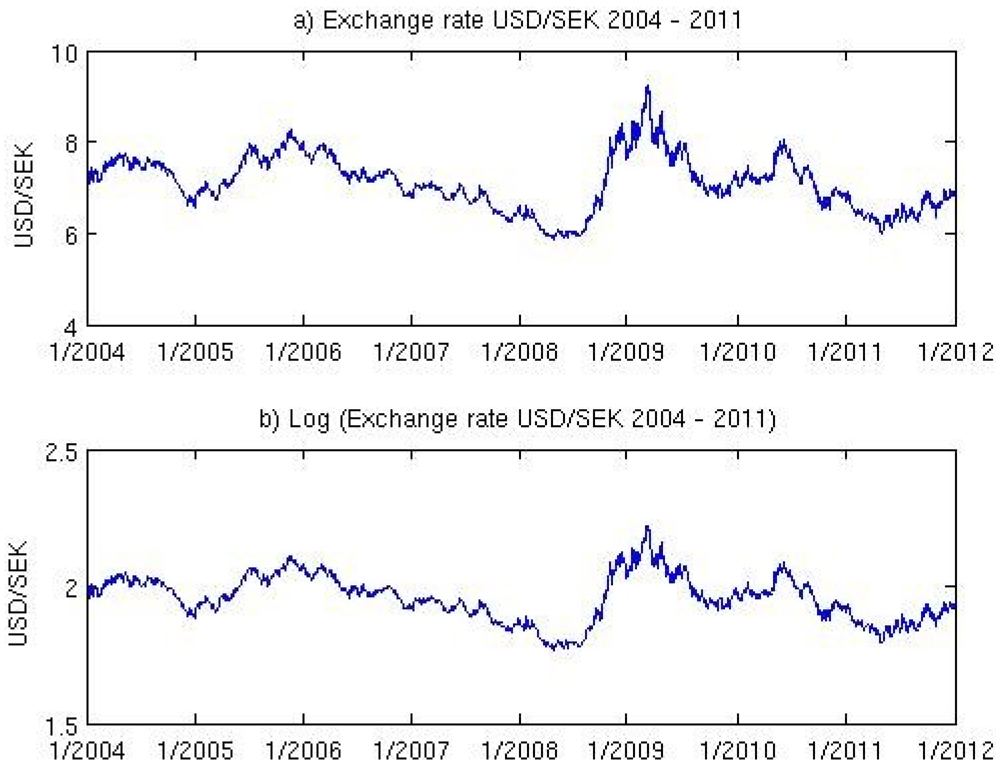

Figure 1. The Exchange Rate USD/SEK from January 2004 to December 2011.

The curve above shows the raw data which appears to have quite high amplitude of the fluctuations over certain time periods. The value of the exchange rate varies in the interval from approximately 6 to almost 10. The curve Figure 1b shows the logarithmic values of the original daily values with smaller scale which implies smaller amplitude of the fluctuations. It is seen in the picture that the exchange rate value has been moving within the band of 2 points, between the values of 6 and 8 during the first 5 years. Starting from September 2008 the value increased with more than 50% during few months time as consequence of the instability period on the financial markets around the world and particularly the impact of this on the Swedish krona. This period is characterized by the decreasing value of the Swedish krona. Since then the exchange rate has recovered and landed at even lower value than in January 2004. During the last two years, starting from January 2010 the exchange rate value has moved greatly with the upward sloping trend starting from April 2011.

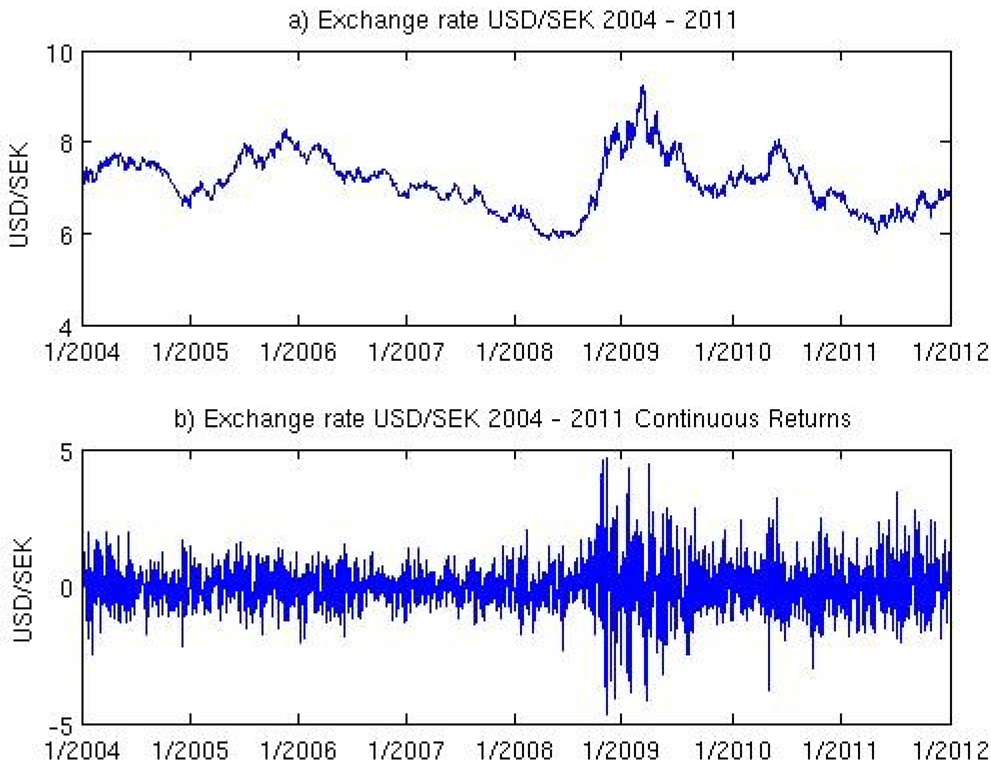

Figure 2. The Exchange Rate USD/SEK and Continuous Returns of this rate from January 2004 to December 2011.

Figure 2. presents also the daily changes of the exchange rate, as these are defined by the formula (3.1). The first observation is that the returns move around zero through the entire period of time independently of the changes in the exchange rate value. Secondly, it is easy to see that the returns change with time. Amplitude of the returns has changed within time, started as the relatively low in the beginning of the chosen

period. The amplitude increased significantly starting from the second part of 2008. Since the second part of 2009 the amplitude of the returns dropped but has remained at higher level prior to the increase. We can also see the pattern with concentration of the return movements with similar magnitude within time. This pattern reveals volatility clustering within the return of the exchange rate which represents a form of heteroscedasticity of the sample errors. The periods of volatility clustering seems to be linked to when the exchange rate increases. When the exchange rate increased considerably within quite short period which happened starting from September 2008, the magnitude of the volatility was the highest.

We continue analysis by studying the sample distribution of { R_t }. Table 1. contains several statistic features of the returns for the entire time period. Assuming the normal distribution of the estimated statistics, 95% confidence interval for these is listed below.

The daily mean value is very close to zero and gives annualized value of approx. -0.5%.

Since skewness and kurtosis give an indication on the shape of the sample distribution it is important to examine these for our time series data. The skewness of 0.2125 indicates that the observations are relatively evenly distributed round the mean value however with some displacement to the right from zero. The estimated kurtosis of 5.7884 significantly exceeds the theoretical value of the normal distribution of 3. Therefore it could be assumed that the sample distribution is symmetric and heavy tailed.

Table 1. Statistics of the Exchange Rate USD/SEK Returns.

	Value	*95% confidence interval*
Mean value ($\hat{\mu}$)	$-1.9 * 10^{-5}$	[$-3.8*10^{-4}$, $3.8*10^{-4}$]
Skewness ($\hat{S}$)	0.2125	[-0.1068, 0.1068]
Kurtosis ($\hat{K}$)	5.7884	[2.7861, 3.2138]

We can also observe the sample distribution graphically and make a similar conclusion. The quantile plot of the returns in the Figure 3. shows that the sample is not normally distributed due to the s-shape of the curve. Normally distributed observations would follow the dashed straight line. Also the histogram confirms that the sample is not normally distributed.

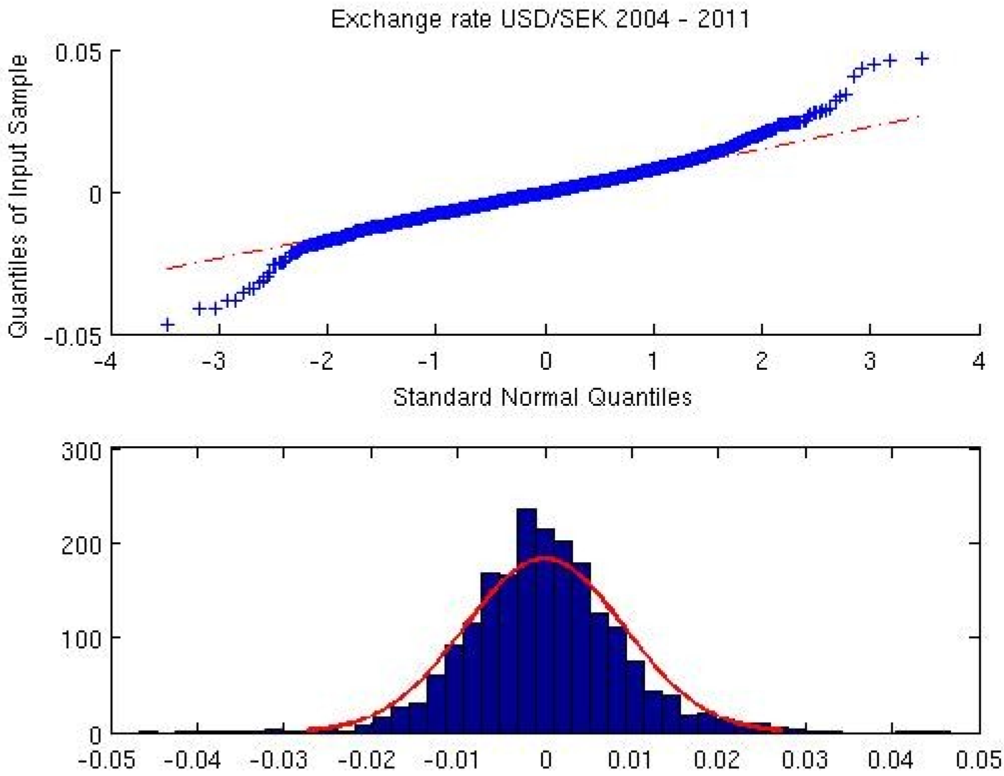

Figure 3. Quantile plot and histogram of the Exchange Rate USD/SEK Returns.

Based on these observations other distributions could be considered for the financial time series data. According to the different researches and literature, for example [12], one could assume independent and t-distributed increments for these types of time series.

Further, we assume that the sample data could be described by the constant mean model as in (3.3)

$$R_t = \mu + \varepsilon_t$$

We calculate and plot the autocorrelation function for the sample data.

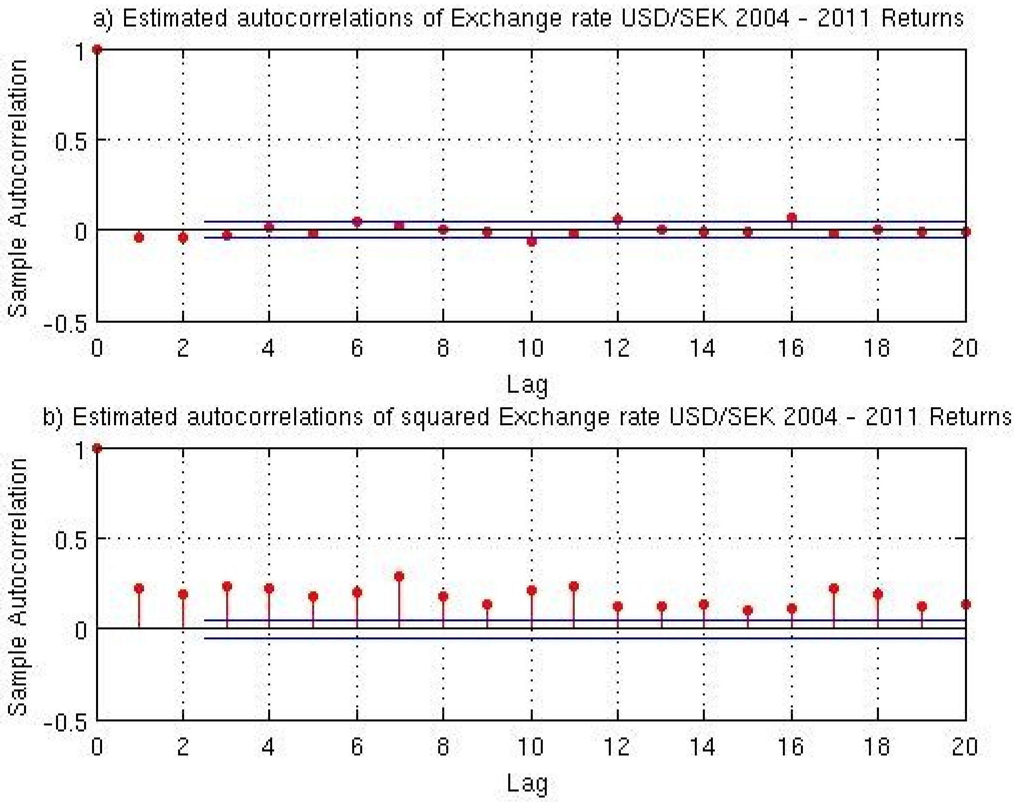

Figure 4. Estimated Autocorrelation function with lag 20 for Returns of the Exchange Rate USD/SEK.

Autocorrelation function measures dependency between the value today and the value some days ago, the lag of dependency. With reference to the literature with respect to this subject, lags up to 20 are chosen for estimation of the autocorrelation function according to the formula (3.4). The Figure 4 a) shows autocorrelation function for the exchange rate returns further Figure 4 b) shows the autocorrelation function for the squared returns. The autocorrelations of the returns in Figure 4.a) could be assumed as insignificant since just few of the values slightly exceed the confidence interval at the significance level of 5%. Referring to the theoretical background, especially the formula (3.5), estimated autocorrelations above absolute value of 0.045 to be considered as significant at the above mentioned significance level. However, if we consider the autocorrelation function for the squared returns, there is clearly dependency in second moment of the returns. The entire series of the autocorrelations are positive and strongly significant. The revealed clustering contradicts the assumption of the constant, time independent volatility.

We calculate in addition the test statistics according to the formula (3.6), which gives us Q-value of 39.3 for the returns and Q-value of $1.4*10^3$ for the squared returns. Both of these test statistics are asymptotically χ^2 distributed with k=20 degrees of freedom. This test at the confidence level of 5% confirms that the squared returns are significantly autocorrelated while the returns are also autocorrelated with the smaller p-value. However, at a different significance level the returns could be considered as not correlated.

The clustering volatility can also be detected by testing the residuals estimated by our linear model for heteroscedasticity. This test reveals whether the sample consists of some sub groups with different variabilities. If the test shows that the errors are jointly distributed with the constant volatility it demonstrates homoscedasticity. R.F. Engle developed and presented in his work "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation" in 1982 the test of no heteroscedasticity, i.e. constant volatility. This test can be calculated by applying the Lagrange multiplier principle. However, we decide to use a general test and investigate whether it is possible to detect the clustering volatility without using the specified test.

For the linear regression models with explanatory variables the general test with no requirements for the form of the errors distribution, called White test, could be used. However, in our model we have assumed no explanatory variables. Therefore we decide to test heteroscedasticity by testing the following linear autoregressive model for the squared residuals from our original linear model (3.3)

$$\varepsilon_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + u_t$$

where **ε** are the error terms from the linear model (3.3) and **u** are assumed to be white noise.

We test the null hypothesis of homoscedasticity against the alternative hypothesis of heteroscedasticity to the model above as follows:

H_0 : $\alpha_1 = 0$

H_1 : $\alpha_1 \neq 0$

As the result of linear regression, we receive estimated value of α_1 of 0.223 with the calculated test statistics of 11.28. The test statistics is t(n-2) distributed. Therefore at the significance level of 5%, the null hypothesis of homoscedasticity can be rejected. This means that the volatility of the returns is heteroscedastic, i.e. the variance varies for the time series which is a feature of the clustering volatility.

On the basis of the theory in the previous part of the thesis and taking into consideration the results of the preliminary analyses of the original time series data it could be concluded that the residuals from the assumed model (3.3) are not purely white noise process.

Consequently as the results of the preliminary analysis of the raw data we can state that regardless of the quite insignificant autocorrelation of the returns these are serially dependent. The returns are not unconditionally independent and also are not normally distributed. Two ways of analysis and modelling of the data could be considered at this point. One of the possibilities is testing and fitting different 'fat-tailed' distributions to the time series data. Secondly, the correlation between the second moment of the data could be investigated and modelled. It is also possible to combine these two analysis and allow any 'fat-tailed' distribution in the analysis of the second moment of the data.

4.2 Modelling and forecast for purpose of risk prediction

The purpose of the thesis is to consider and test dynamic models for estimation of the volatility for financial time series data. Therefore we follow the second possible analysis and investigate how ARCH and GARCH models could be fitted to our data and also how the fitted models could be used for forecast of the future development.
We decide to concentrate on the dynamic models with one lag such as ARCH(1) and GARCH (1,1) in the context of this thesis. However, it is of great value to consider further both ARCH and GARCH models of the higher orders of lags and also several other models within the class of the generalized ARCH models.

4.2.1 ARCH(1) and GARCH(1,1) model estimations

The aim is to estimate ARCH(1) and GARCH(1,1) models for the volatility of the chosen financial time series data and further to forecast volatility based on the fitted models for risk prediction.

We start by estimating the parameters for the conditional variance by using maximum likelihood functions based on the theoretical details listed in the part 3.5.3 in this thesis.

By maximizing the loglikelihood function numerically we estimate the parameters for the ARCH (1) model. We get the following equations for the Exchange Rate USD/SEK 2004 - 2011.

$$\sigma_t^2 = 5.8466 * 10^{-5} + 0.312\, R_{t-1}^2 \qquad (4.1)$$

According to the estimated ARCH(1) model we see that the first component, the constant variance which is long term average is close to zero but yet is approx. 70% of the sample variance of the returns. The second component is the latest available information, i.e. the squared error term is taken into account of approx. 31%.

For the GARCH (1.1) model we estimate the parameters by numerically maximizing the loglikelihood function and we get the following equations for the Exchange Rate USD/SEK 2004 - 2011.

$$\sigma_t^2 = 5.4116 * 10^{-7} + 0.05757\, R_{t-1}^2 + 0.93554\, \sigma_{t-1}^2 \qquad (4.2)$$

Observing the estimated GARCH(1,1) model above we see that the constant variance is even closer to zero than in the ARCH(1) model and is less than 1% of the sample variance of the returns. The latest available information, i.e. the squared error term is taken in account of approx. 6% which is far less than in the ARCH(1) model. The most of the information for the conditional variance today is coming from the forecast made yesterday, i.e. 94%.

We observe also that the value of the loglikelihood function has increased from 6,686 for ARCH(1) model to 6,860 for GARCH(1,1) model.

We plot further estimated with ARCH(1) and GARCH(1,1) models conditional variance against the squared returns. We simulate also returns as a zero mean white noise process with the constant variance estimated as the squared sample standard deviation according to the formula (3.2).

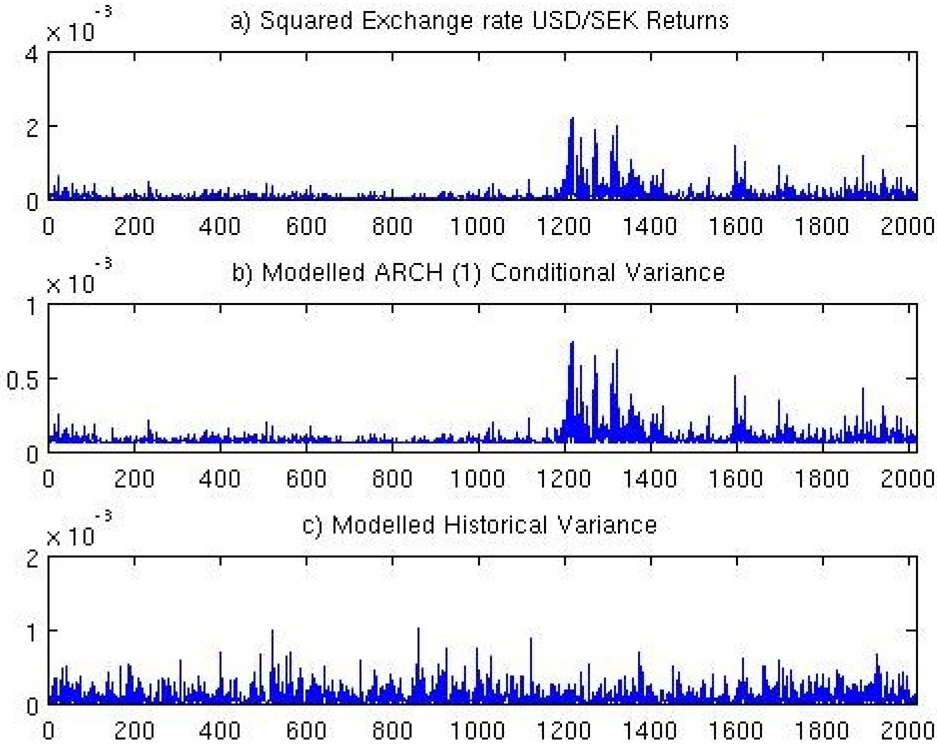

Figure 5. The Exchange Rate USD/SEK Squared Returns, Estimated Conditional Variance ARCH(1) and Variance of the Simulated Returns with White Noise.

Looking at the Figure 5, it could be seen that the estimated conditional variance ARCH(1), $\hat{\sigma}_t^2$, follows the time series data's volatility clustering closer than the volatility of the returns simulated with white noise. The historical variance above is squared simulated returns as white noise process with independent, identically and normally distributed random variables with the zero mean value and the estimated from the historical data constant volatility of 0.0091.

Further we simulate the series of the Exchange Rate returns based on the ARCH(1) model.

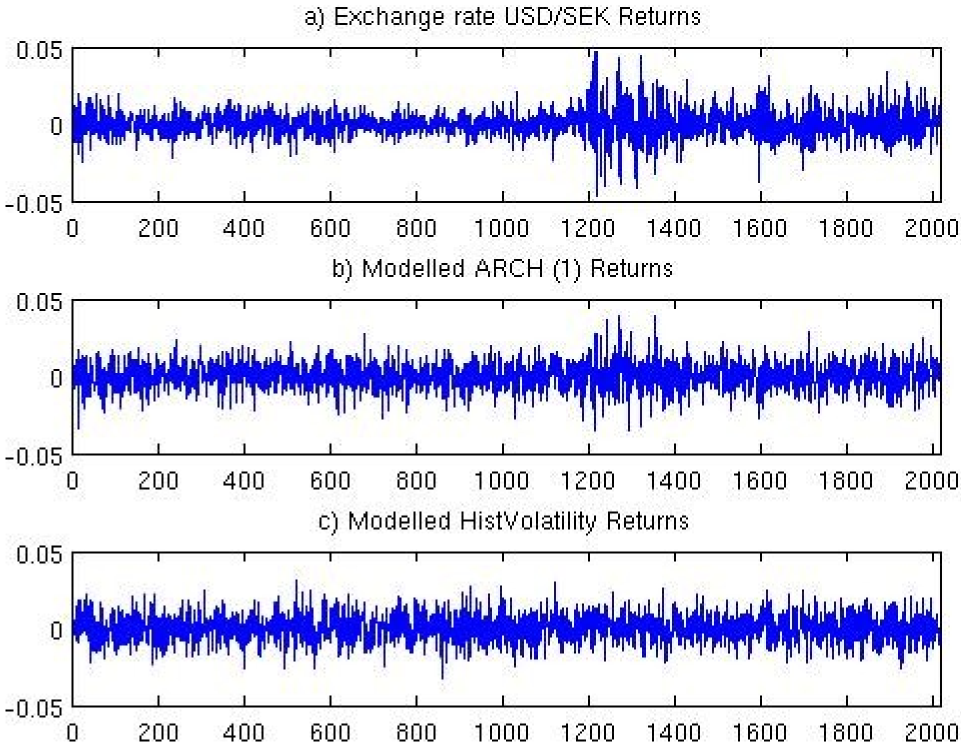

Figure 6. The Exchange Rate USD/SEK Returns, Simulated Returns with Modelled ARCH(1) process and Simulated Returns with White Noise.

We can observe in the Figure 6. that the simulated returns series based on the conditional variance modelled with ARCH(1) follows the original time series data somehow better than the simulated returns series with white noise. Due to the fact that the GARCH(1,1) model has one extra lagged term in comparison to the ARCH(1) we could consider that the GARCH(1,1) model could give us a better adjustment to the original series of returns. Therefore we produce for the GARCH(1,1) model the similar sequence of the plots as it has been done for the ARCH(1) model.

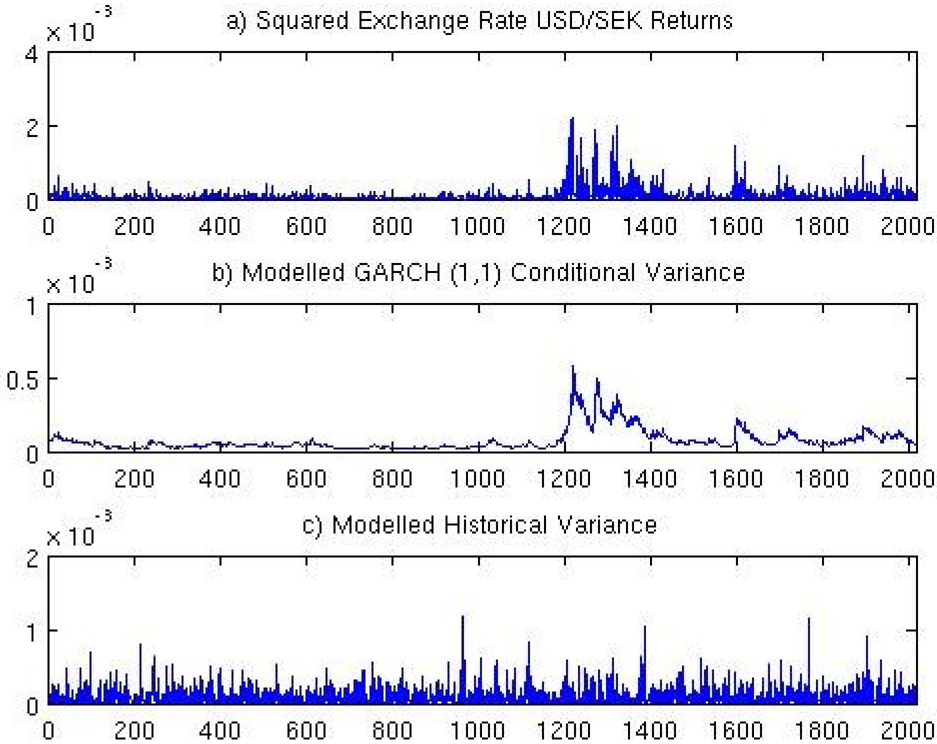

Figure 7. The Exchange Rate USD/SEK Squared Returns, Estimated Conditional Variance GARCH(1,1) and Variance of the Simulated Returns with White Noise.

We notice that the features of the conditional variance modelled with GARCH(1,1) appear to be similar to the squared original returns series in comparison to the squared simulated returns with white noise.

Further, we simulate the returns series based on the GARCH(1,1) and plot these against the original returns and also the simulated returns with white noise.

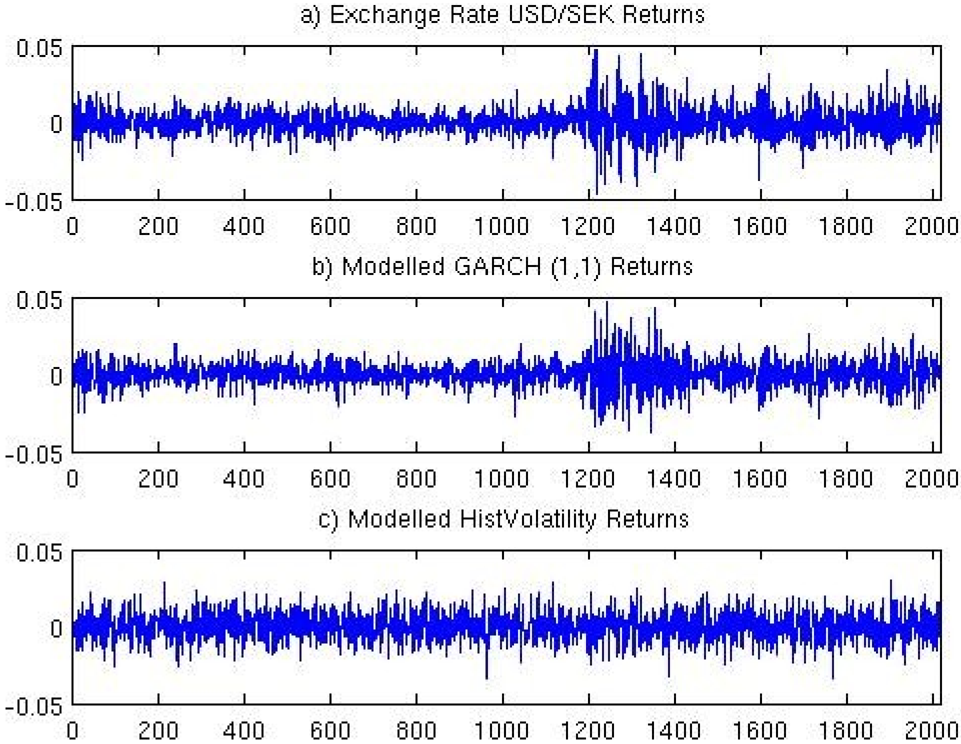

Figure 8. The Exchange Rate USD/SEK Returns, Simulated Returns with Modelled GARCH(1,1) process and Simulated Returns with White Noise.

It could be easily observed from the Figure 8. that appearance of the simulated with the GARCH(1,1) returns series is very similar to the Exchange Rate USD/SEK Returns.

4.2.2 Review of the adjusted ARCH(1) and GARCH(1,1) models

Further, we want to investigate how well the fitted ARCH(1) and GARCH(1,1) models explain the sample data. This investigation could be done by the similar approach which we used in our preliminary analysis of the returns. Since it is assumed as the part of our model that the innovation terms $u_t = \frac{R_t}{\sqrt{\sigma_t^2}}$ are the random variables such as $\{u_t\} \in IID\, N(0,1)$. These are assumed to be independent of any previous observations. We check onwards if the observed innovation terms comfort to the assumption.

We start analysis by studying the sample distribution of { u_t }. Table 2. contains several statistic features of the innovation terms for both models and also for the Exchange Rate USD/SEK returns. The mean value is very close to zero, however it has increased for each step of the modelling. The standard deviation is approximately one for both models. The skewness has decreased for the ARCH(1) model and also decreased for the GARCH(1,1), with total decrease of approx. 50%. The estimated kurtosis has decreased significantly. The kurtosis of the GARCH(1,1) innovation terms is significantly close to the value of 3 which is the theoretical kurtosis of the normal distribution. Therefore it could be assumed that the distribution of the innovation terms is more symmetric than the distribution of the returns. We could also conclude that by modelling the volatilities for the next day based on the information available today, it is possible to foresee the extreme values as these are part of the model and do not impact the innovation terms as much as the original return data. This gives the opportunity to eliminate the extremes by modelling these with the only remaining random process which is significantly closer to the white noise process than the original return data series.

Table 2. Statistics of the Exchange Rate USD/SEK Returns and ARCH(1) and GARCH (1,1) Innovations.

	Exchange Rate Returns	*ARCH(1) Innovations*	*GARCH(1,1) Innovations*
Mean value ($\hat{\mu}$)	$-1.9 * 10^{-5}$	$-4 * 10^{-3}$	-0.0102
Standard deviation ($\hat{\sigma}$)	0.0091	1.0004	1.0938
Skewness ($\hat{S}$)	0.2125	0.1414	0.1068
Kurtosis ($\hat{K}$)	5.7884	4.5851	3.3846

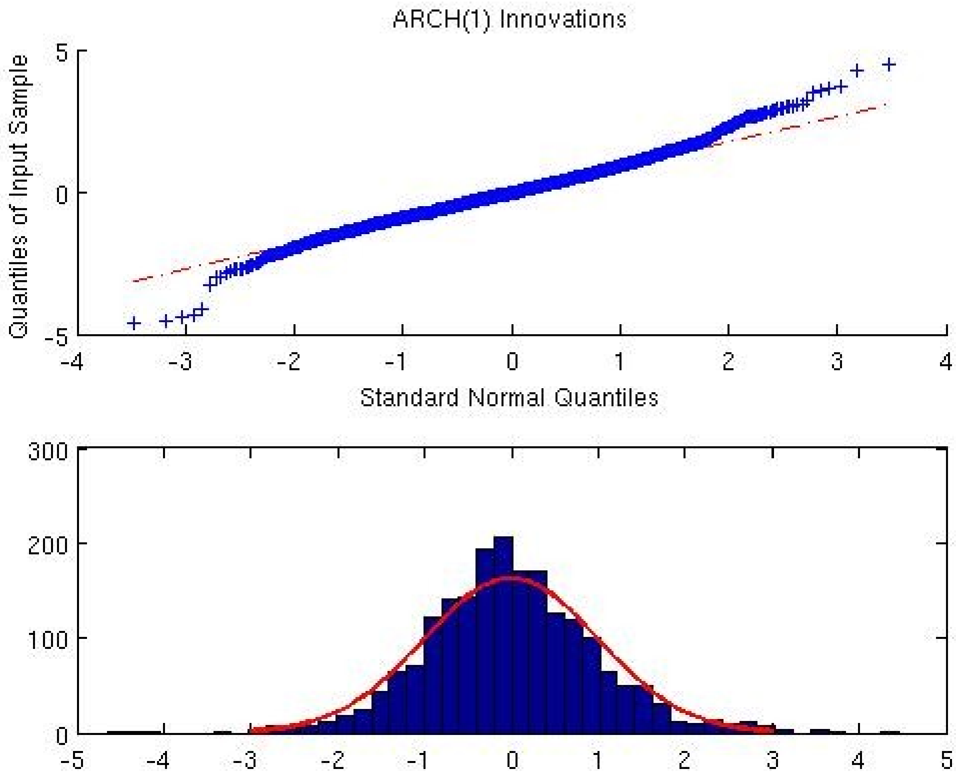

Figure9. Quantile and histogram for ARCH(1) innovation terms.

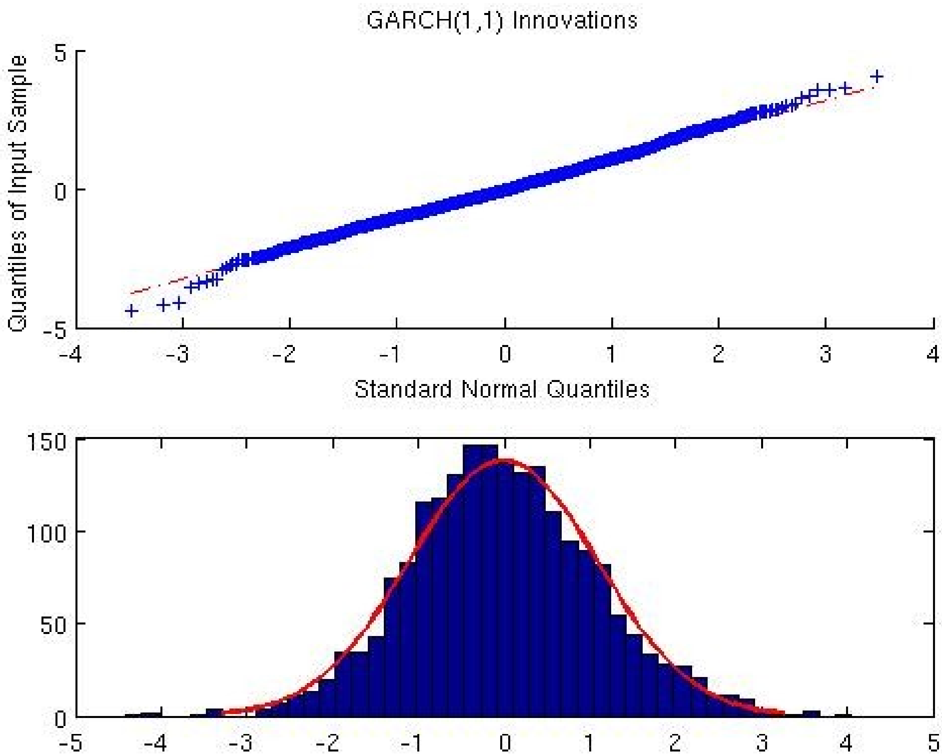

Figure 10. Quantile and histogram for GARCH(1,1) innovation terms.

We continue to observe the distribution of the innovation terms graphically. By comparing the plots in Figure 9. and Figure 10. we can note that the GARCH(1,1) innovations could probably be assumed to be normally distributed.

Assumption of the independence of the innovation terms is to be checked by estimation and test of the autocorrelation function of the innovation series itself and also for squared values.

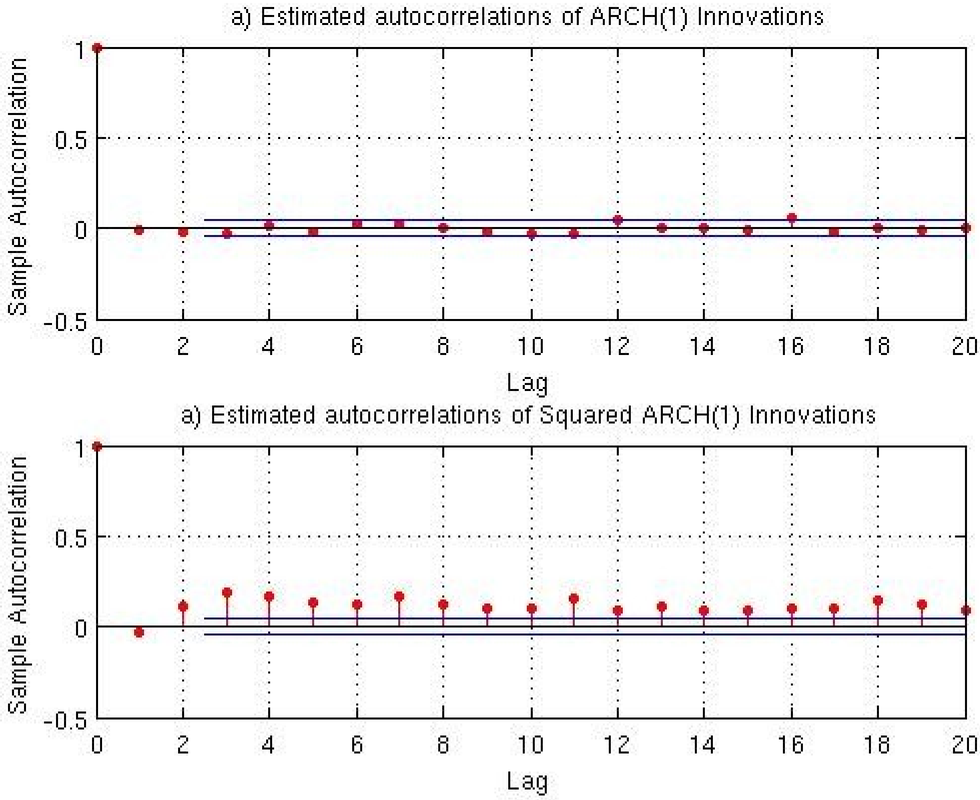

Figure 11. Autocorrelation function for ARCH(1) innovation terms.

We see in Figure 11. that the autocorrelation function of the ARCH(1) model is quite similar to the autocorrelation function of the returns, i.e. the autocorrelation of the innovation terms is almost insignificant but the autocorrelation of the squared innovations is significant. As consequence of this, the hypothesis of the independence of the innovation terms could be rejected.

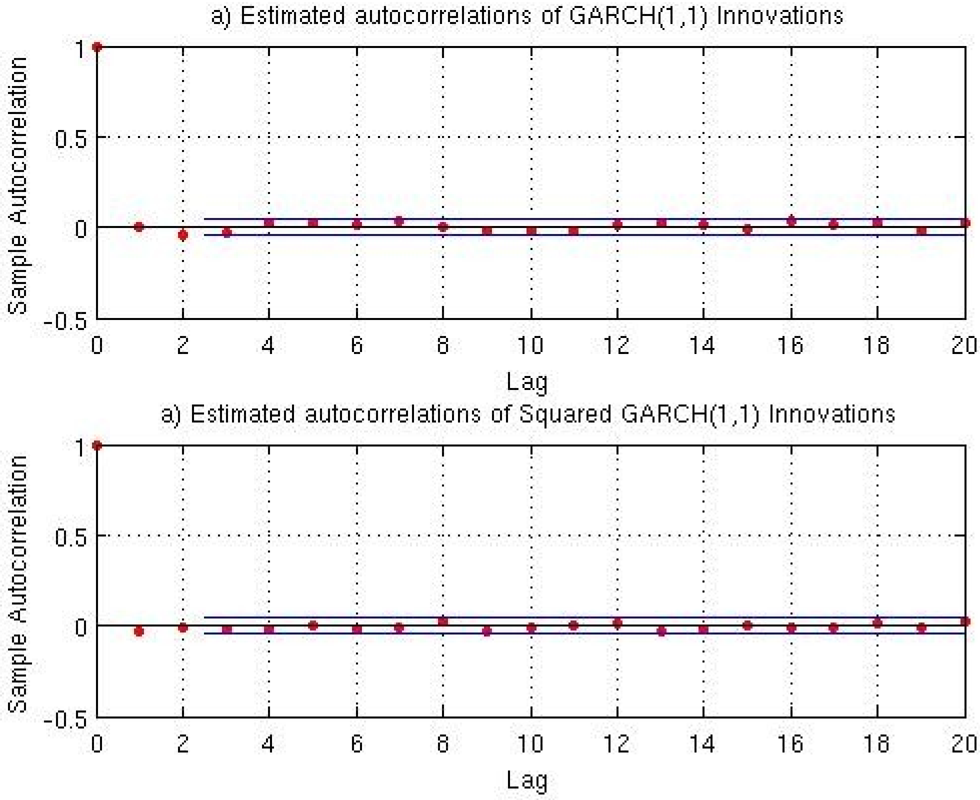

Figure 12. Autocorrelation function for GARCH(1,1) innovation terms.

The autocorrelation function of the GARCH(1,1) innovations and also of squared innovations are both almost insignificant. Therefore the innovation terms of the GARCH(1,1) model could be assumed not autocorrelated .

Consequently based on the analysis of the innovations estimated from the adapted models, we can state that the GARCH(1,1) model is the best fit to our data. This could potentially be considered as the additional confirmation of the fact that the GARCH models are widely used, according to the Englund P., Persson T., Terävirta T. [7].

4.2.3 Risk measurement and management

The behaviour of optimization rewards and risks is typical for finance.

One way of measuring the risks is to assume that these are equal to the volatility of the asset. The typical measure of the risks within some confidential interval is called Value at Risk, VaR. This measure is widely used by the different financial institutions. Several international regulations, such as rules for banks and also rules for insurance companies, refer to VaR calculations for quantification of risks and capital requirement.

The Value at Risk measures the probability of loss for some portfolio. The Value at Risk could be calculated for 1 day, 10 days or 30 days ahead. Referring to [11], mathematically the VaR for a portfolio with expected value of the return equal to zero could be expressed as follows

$$P(R<-VaR)=\varepsilon$$

for a given value $0<\varepsilon<1$ and VaR is given by

$$VaR=Z_{\varepsilon}\sigma\sqrt{length\,of\ the\ period} \tag{4.3}$$

where Z_{ε} is the given quantile of the sample distribution of the innovations.

For calculation purposes, the assumption with regards to the distribution of returns is needed. As mentioned previously, according to the essential theory returns are assumed to be normally distributed with the constant variance. As we have seen from our analysis this assumption is significantly uncertain for the observed returns. However, if we assume the conditional variance of GARCH(1,1) model the normal distribution could potentially be considered as reasonable.

There is a possibility to avoid assumption of the normal distribution by estimating the empirical distribution of the sample data and use its empirical quantiles for the calculations.

We assume that we need to estimated the Value at Risk for some portfolio. It is further assumed that the portfolio consists of one asset which implies the only risk linked to the volatility of the Exchange Rate USD/SEK, previously investigated in this thesis. In other cases, all variances and covariances for all assets in the portfolio are necessary for the calculation of VaR.

We assume that the Value at Risk is to be calculated for the 1, 10 and 30 days ahead starting from 2011-12-30, which is the last working day of the observed period. It is assumed for the calculation purposes that the portfolio is unchanged during these certain periods. This assumption is needed for the model simplification, however could possibly be considered unrealistic.

We calculate The Value at Risk both by using the historical volatility and also the dynamic volatility generated by the GACR(1,1) model. Since the actual values of the Exchange Rate USD/SEK are available for the considered dated, we can also observe the actual values of 'Loss' and also compare these to the estimated ones. The significance level assumed to be 1% which gives the normal distribution quantile of 2.33.

When the historical volatility is used for VaR calculation, we multiply the sample standard deviation with 1%-quantile of the normal distribution and also with the square root of the length of the forecast period. The value of the sample standard deviation is estimated based on the entire observed period between 2004- 2011. We understand that it is possible to argue that a shorter period could be chosen, however the purpose of this thesis is to test the dynamic models for the returns.

We have already estimated the GARCH(1,1) parameters from the data prior to the forecast period, formulas (4.1) and (4.2). Forecasts of the conditional variance are generated by the following recursion procedure for the GARCH(1,1) model for t=1, 10, 30.

$$\sigma_t^2 = 5.4116 * 10^{-7} + 0.05757\, R_{t-1}^2 + 0.93554\, \sigma_{t-1}^2$$

The date denoted as t=0, i.e. the last date in the observed period and is 2011-12-30. For n=1, i.e. as of 2012-01-02 we have

$$\sigma_t^2 = 5.4116 * 10^{-7} + 0.05757 * (-3.6 * 10^{-4})^2 + 0.93554 * 0.0075^2$$

For n=10 and n=30 we simulate the GARCH(1,1) process using normal random variables simulator 50 times and choose the worst case scenario.

Observing the estimated values of the VaR in Table 3. and compare these estimated values to the actual development of the Exchange Rate USD/SEK during January – February 2012, we see that the VaR values estimated with the historical volatility are the most extreme ones. This is probably due to the fact that the constant historical volatility accumulates over the time and as consequence the dynamic of the returns volatility is not taken into account.

Table 3. VaR at 1% significance level vs. the actual value of the possible 'Loss'.

	VaR / historical volatility (%)	VaR / dynamic volatility (%)	Actual value of 'Loss' (%)
2012-01-02	2.12	1.68	0.6
2012-01-16	6.71	2.44	-1.09
2012-02-13	11.61	1.84	4.30

By looking at the Figure 13., we can observe that the conditional volatility of the Exchange Rate USD/SEK at the end of 2011 was low and the downward sloping. The start value of the conditional variance at 2011-12-30 is 0.0075 which is lower than the historical volatility of 0.0091. However, the long run forecast, i.e. for 30 days appears to be uncertain for both approaches, this could potentially be explained by the unpredictability of the returns.

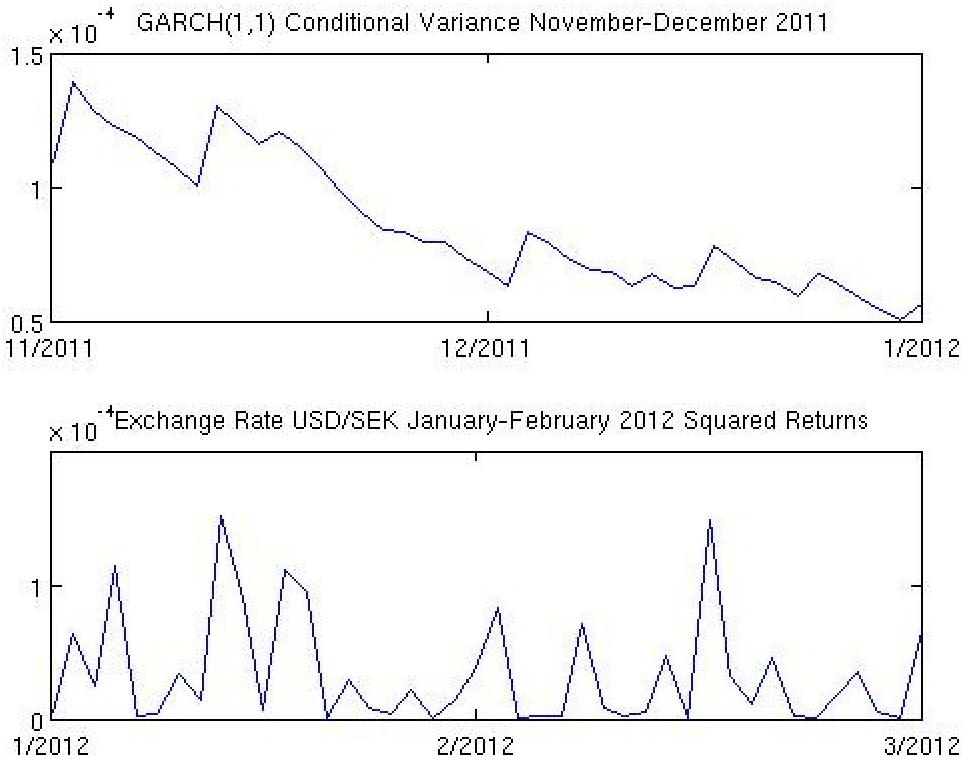

Figure 13. GARCH(1,1) Conditional Variance November – December 2011, Squared Returns of the Exchange Rate USD/SEK January – February 2012.

5 Conclusions and Discussion

In this thesis we have examined the changes in the exchange rate expressed as times series data. We have reviewed if the basic assumptions with respect to returns as a part of the essential financial theory could be considered as reasonable for the chosen sequence of the data. As a consequence of the implemented analysis for the chosen data, we can state that the assumption of the normal distribution of returns and stationarity of these, as well as the assumption of independence within the financial time series data is greatly questionable. For this issue three possible solutions could be considered. The first one is to adjust a 'fat-tailed' distribution with retained assumptions of independence within time and also assumption of the stationary. Secondly, the normal distribution could be adjusted by the conditional variance determined by the dynamic models. The third alternative could be to combine the first two, i.e. consider a 'fat-tailed' distribution on the basis of the conditional variance.

Within frame of this thesis, we have chosen to model the conditional variance with ARCH(1) and GARCH(1,1) models as the basis for the conditional normal distribution of the returns. We have observed that the GARCH(1,1) model appears to suit the selected data superior to the ARCH(1). This could be a benefit of the extra lagged term in form of the observed conditional variance. Thus the GARCH(1,1) fit could be considered sufficiently good for capturing the dynamic variation of the variance for the chosen data among the considered models. However, further investigation particularly of the GARCH model with more lags could be considered. One additional alternative, with reference to [10], could be to examine asymmetric types of the dynamic models within ARCH/GARCH family as for example EGARCH, TGARCH or GJR.

The estimated dynamic models have a significant value within the finance theory since these are used greatly for asset pricing and risk measurement. In this thesis we have forecasted the conditional variance in order to quantify risks via determination of VaR. These forecasts have been done for the periods of 1, 10 and 30 days where for the last two periods 50 simulations of the worst case scenarios have been proceeded. It appears that the short run forecasts for 1 and 10 days are closer to the real exchange rate volatility than the simulated volatility assumed to be stationary normally distributed. Further investigation of forecasts of GARCH(p,q) models or even larger number of simulations could be regarded as quality increasing approaches.

As it has been observed, the dynamic models ARCH(1) and GARCH(1,1) could potentially be considered as appropriate models for financial returns data due to the fact that these capture such properties as time variety, volatility clustering and dependency within time of the return series.

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