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An evaluation approach of embedded options and immunization strategies

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An evaluation approach of embedded options and immunization strategies

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Abstract

New accounting and Solvency regimes are soon standing at the insurance undertakings doorstep and with new rules on how to value the technical provisions. The embedded options such as surrender option and annuity conversion included in insurance policies must be valued and accounted separatly as part of the technical reserve. The usage of Black- Scholes option pricing and option valuation will give a survey over the calculations of the extra amount of reserve needed to cover future contractual obligations to the policyholders. As the liabilities in an insurance company are at market value and we constantly have fluctuations in the market interest rates it is important to be strategic in the hedging management. The most changes in the market interest rate occurs during the first ten years, which smoothen out to be almost a straight line. The appropriate management of fluctuations in the interest rate and its impact is called immunization and is an instrument to handle the problem of mismatch in present value and duration.

Keywords: Traditional life insurance mathematics. Valuation of embeded surrender and annuity conversion options, yield curve, immunization, hedging strategies, Slovency II, IFRS 4 Phase 2, Risk Capital

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Preface/Acknowledgement

This report is a 20 credit thesis in mathematical statistics, aiming at insurance mathematic. It is written at Ernst & Young in Stockholm after four years of studying at Stockholm University. The study is concentrating on an evaluation of embedded options in insurance contracts, i.e. surrender and annuity conversion options.

I want to send my best gratitude to my councellor , diplomerad aktuarie Sören Kruse and my councellor at the University of Stockholm Thomas Höglund for their great support throughout my work. I also want to thank several persons and co-workers at Ernst & Young for their help with both small and large tasks, but yet so important to me. These persons are Albert Bergendal, David Honour, Adrian Williams and David Paul.

Finally, I want to thank my family for their tremendous support to me.

Table of contents

Figures

Equations

1. Introduction

The purpose of this work is to get an insight into the embedded options that could be a part of insurance contracts. Embedded options are future cash flows such as surrender options and annuity conversion. The option to surrender a contract is often a feature of insurance products but is not as at today a part of the liabilities in that sense that insurance undertakings are generally not reserving for the expected cost if policyholders are to surrender and for that matter convert their contract. The current reserve is what you can say, “underestimated”, as it does not cover the cases of these options.

During May 2007 the IASB (International accounting standard board) have presented a discussion paper about insurance contracts. It regards the calculation of technical provisions and that insurance and reinsurance undertakings shall take into account of the value of financial guarantees and contractual options included in insurance and reinsurance policies. This is of great interest then, to explore possible estimates of these option values. So the question is how should we at time t value the surrender and conversion options? The surrender option is the option to surrender the policy at any time t during the insurance period and the conversion option is the option to convert the lump sum insured to an annuity payment at maturity.

In July 2007 CEIOPS (Committee of European Insurance and Occupational Pensions Supervisors) have presented a first draft for a future EU Solvency Directive introducing almost the same thoughts regarding valuation of financial guarantees and embedded options in insurance contracts

The second part of this paper is to find a way to match up the liabilities of a life insurance company with some given assets. The problem is to find a replicating strategy for the life insurance company to find an optimal hedge and minimizing costs (costs of capital, hedging costs, operational costs etc.). Insurance companies are making commitments to the policyholders, commitments that might last for e.g. twenty or thirty years. Assets that are available in the market to cover these liabilities do not have that long duration, and thereby the company are facing interest rate risk as well as reinvestment risk . Immunization is one possible way to hedge against changes in the interest rate curve and we are going to look into this feature and some strategies later on in this paper.

Finally, this paper is generally focusing on the balance sheet, with assets and liabilities. Hence, this means that the balance sheet is our starting point.

2. Balance sheet

The balance sheet of a life insurance company combines two different worlds. We have the financial world which refer to investing in assets and the actuarial world which is controlling liabilities. These two sides speaks different languages, comes from other backgrounds and do not share the same perception. These two worlds are then combined in the same balance sheet and this is sometimes called a paradox.

Both the asset and liability sides are split into sub groups themselves. The groups are different in the sense that their liquidity is different. It is important for the asset side to hold a certain amount of assets liquid enough in case of payment of liabilities. The secret is to have so much liberty that you are able to sell your assets at any time. A normal balance sheet can be shown as below.

Assets normally consist of bonds, stocks, properties and other assets. But due to the risk of assets an insurer is not allowed to invest all their capital in stocks. There are restrictions. The insurance companies have to be able to cover the amounts facing the technical provisions with assets that mostly consists of bond (government bonds) with a risk free interest rate and stocks at the most 25 % of technical provisions. If we have a higher consolidation fund, then we are able to invest more capital than 25 % in stocks.

Then we have the liabilities, which consists of equity and technical provisions (TP). Because of the structure of our intended insurance policy we don't include discretionary participation feature, i.e. no kind of dividend to policyholder are paid out.

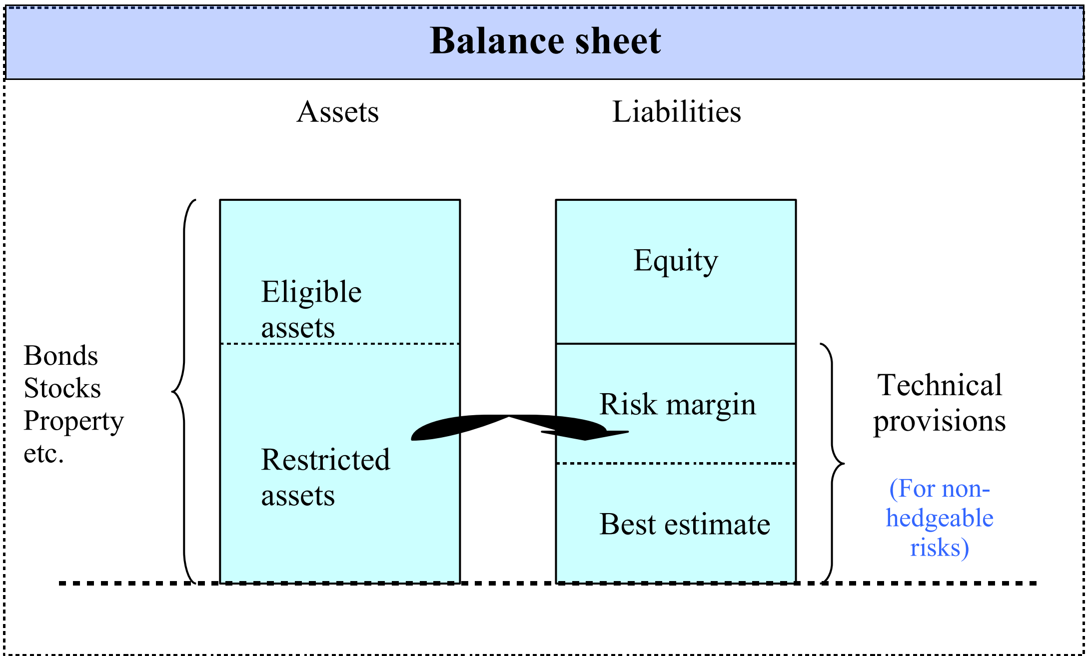


Figure 1: *Balance sheet for a typical life insurance company*

The technical provisions consist of two features. First we have a risk margin, which is the additional cost, above the best estimate, of providing capital to support the insurance obligations over the lifetime of the portfolio; or alternative, the amount an insurance undertaking would expect to have to pay today if it transfer its contractual rights and obligations immediately to another knowledgeable and willing undertaking in an arms length transaction. However we are not going to lay any more attention to this feature because of its dimension, as we could use the rest of this thesis evaluating the risk margin and that is not our intention.

Secondly, technical provisions consists of “best estimate”, i.e. the expected present value of future cash flows, taking into account all the cash in and out flows. Further information on how to value the best estimate and risk margin can be seen in part 4 – Valuation according to Solvency II.

3. Product

3.1 Description of product

The policy is an endowment assurance with a 20 years term life insurance written at the age of 40 with a lump sum payment in case of death or at maturity. This is a paid-up policy, no incoming premiums, and a run-off portfolio. The policyholder has annually the right to surrender the policy at a predefined surrender value or convert it to a 10 year annuity at age 60 at a predefined interest rate.

The predefined surrender value is the reserve of the assurance valued at the guaranteed interest rate. The predefined conversion rate at age 60 is also equal to this guaranteed interest rate.

To get an overview over the insurance policy we can look at a Markov chain consisting four states with corresponding transition probabilities. When entering the Markov chain, i.e. signing the insurance policy contract you will be in state "Alive". As time goes during the insurance period until the age of 60 there are two possible outcomes. Either you die with an intensity $\mu(x)$, related to the Makehams' formula for death probability or surrender with a probability depending on guaranteed interest rate and market interest rate. This transition intensity can be expressed in a formula depending on the assumptions of the undertaking.

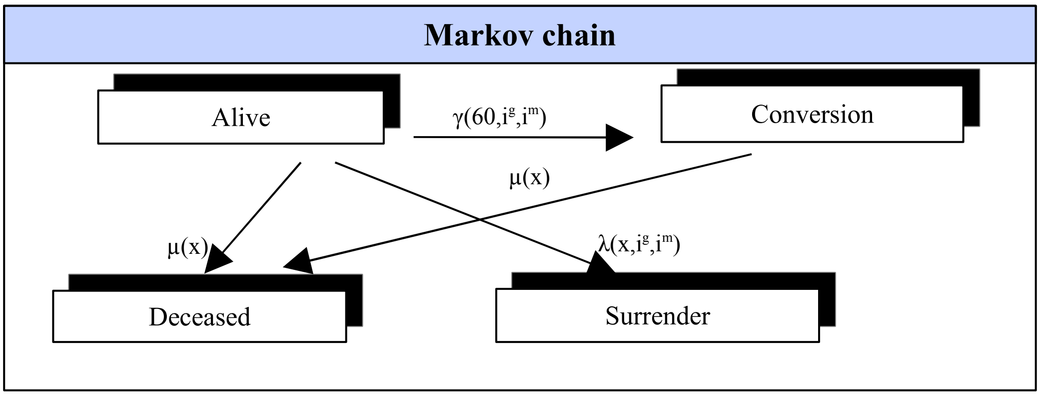


Figure 2: *Markov chain and transition intensities for intended policy*

But it is complicated in that way that policyholders are not always rational. If the market interest rate is higher than the guaranteed interest rate it will be beneficial for the policyholder to surrender, due to retrospective calculations of the reserve, and invest their money elsewhere with a higher rate of return. In this case the undertakings may not have enough assets to cover these payments because the surrender value of the policy is higher than the current market value of the policy. . But not every policyholder thinks that way or maybe not thinking about it at all. However to simplify this it is convenient to assume that all policyholders are rationale. That is to when the market value of the policy is lower than surrender value of the policy everyone will surrender and when the market value is higher no one will surrender.
We will do our valuation at market value using a yield curve. However simplified we will say that the policyholders will surrender if the market interest rate is higher

than the guaranteed interest rate because they will then be able to invest their money at a higher interest rate outside the life insurance company.

The last state is possible to enter when the policyholders are 60 years old. If policyholders convert or not is due to the market interest rate and the guaranteed interest rate just like surrender option but the other way around. If the guaranteed interest rate is higher than the market rate it is beneficial to convert the policy and for the same reason as for the surrender option the insurance company have to reserve an amount larger for the annuity than the amount they would have had to pay out for the lump sum insured. The difference between the reserve for the annuity at market value and the lump sum would be the option value.

Simplified we may again say that the policyholders will convert the lump sum to an annuity if they may be able to invest their money to a higher interest rate within the life insurance company than outside. This is what you can call "arbitrage" possibilities for the policyholders.

3.1.1 Product restrictions

Before we go any further it will be good to summarize the restrictions in this portfolio. Some of them were mentioned just above. All the restrictions have been necessary to reduce the complexity of the analysis of this portfolio.

As we are analyzing these liabilities with policyholders from age 40 to 69, it is possible to think of this as a policy that is written only by 40 years old individuals. That means that the policyholder that are older than 40 years old have once written their policy at age 40. We have also constrained our self to a run-off portfolio, no more policies are written. This means that we do not have to consider whether how many individuals that are willing to sign and enter this policy, which could be complex depending on the guaranteed interest rate and the market interest rate at a future point in time.

Also the duration of the endowment insurance is discretionary. It could be more or it could be less. It would also be possible to use a lifelong annuity instead of a term annuity.

Surrender frequency is restricted in the way that we think of the policyholder as rational individuals. Individuals who want to receive as much rate of return as possible if they got the chance. That is not always the case but it simplifies the analysis. You may say that this thesis is about valuation and hedging not about modeling policyholders' surrender behavior.

3.2 Embedded options

Currently embedded option is generally not attributed in accounting. This value of embedded option is only considered if the value have an "intrinsic value" (i.e. the value of an asset based on an underlying perception of the value), because they are currently out of or in the money. As at today the IASB (International Accounting Standard Board) have presented a discussion paper about insurance contracts. And one feature is that embedded option should be a part of the valuation models for insurance contracts. That is why these options are important to value.

The conversion and surrender feature in this insurance portfolio can be seen as an embedded options. A valuable option to end or continue the contract at fixed or constrained prices even if the risk has changed. The fixed prices could be a fixed interest rate, as in this product. If we look at a conversion option we will find that the options have a "time value", because they could be in the money at expiry. The time value is defined as "the value of an option arising from the time left to maturity (equals an option's price minus its intrinsic value), Hull (2006).

3.2.1 Conversion option

There are probably several ways to calculate the value of this option, but which is the most accurate or reliable one? One way to solve this could be, perhaps a more simplified solution, that when the insurance contract are at maturity at the age of 60 and the policyholder decides to convert the policy to an annuity, the insurer uses the lump sum payment to buy an immediately starting temporary annuity by a one time premium. Below is a figure of the reserve development of an immediately starting temporary annuity, where we can that see that the reserve is decreasing as no premiums are being paid.

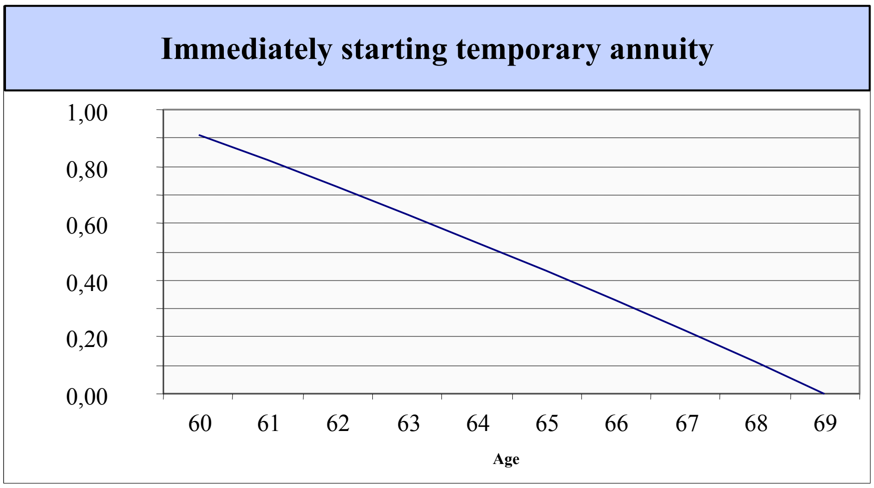


Figure 3: *Development of remaining reserve; an annuity payment of the sum insured*

The approach we are going to look into in this portfolio is through option pricing via bond futures option formula. Intuitive, we look at this as the option of the policyholder to buy an annuity, where the price of the bond (underlying asset) is the discounted cash flows from the guaranteed rate discounted at the market rate, and the exercise price is the lump sum in the endowment insurance. Every policyholder possesses a call option with the rights to convert their insurance to an annuity for ten years (10 years is in this matter arbitrary). The value of this option at the age of 60 is then:

$$\text{Call option value} = \mathit{Annuity} \times \max\left(a^{m}_{60:10} - a^{g}_{60:10};\, 0\right) \qquad (1)$$

Annuity is the annual payment for ten years:

$$Annuity = \frac{Face_{amount}}{\frac{N(60)-N(70)}{D(60)}} = \frac{D(60)}{N(60)-N(70)} \times Face_{amount}$$, and

$$a^{m}_{60:10} = \sum_{s=0}^{9} \frac{l_{60+s}}{l_{60}} \times v_{m}(s)$$

, annual payment of 1 unit (e.g. 1 €) for ten years discounted with market rates

$$a^{g}_{60:10} = \sum_{s=0}^{9} \frac{l_{60+s}}{l_{60}} \times v_{g}$$

, annual payment of 1 unit (e.g. 1 €) for ten years discounted with guaranteed rate, independent of s

Here l_x denotes the survival function for an x year old individual and v_m and v_g denotes the market interest rate and the guaranteed interest rate. Face amount is the sum insured.

This conversion option is also to be considered throughout the insurance period and have to be calculated. It is done for every accounting year and cohort and the future annuity payments discounted back to the accounting year for both the guaranteed interest rate and the market interest rate. The option value is the difference between an annuity payment of 1 unit (e.g. 1 €) from the age of 60 years for ten years discounted with a market rate and the corresponding with guaranteed interest rate times the annuity payment of the insured amount;

$$Annuity \times \max({}_{60-x}a^{m}_{60:10} - {}_{60-x}a^{g}_{60:10} ; 0) \quad , \quad 40 \le x \le 59 \qquad (2)$$

A positive option value means that the undertakings have to reserve more for the annuity than the lump sum if the policyholders chose to convert their policy to an annuity. This is because the annuity, i.e. the discounted cash flows from the guaranteed rate, discounted at current rate is higher than the insured lump sum. And this is possibility something the undertakings have to take into concern when reserving.

Simplified we say that if:

Guaranteed interest rate (g_r) > Market interest rate (m_r) ,(see picture 4)

$\Rightarrow$ Call (conversion) option value > 0

3.2.2 Surrender option

The surrender option can be valued in a similar way. But we are going to use a put option instead of a call option. We are now facing a more complex situation as the policyholder have the right to surrender at several times during the insurance period. We also have to consider whether the policyholder's is alive or not at these points in time.

The price of the bond is the value of the reserve discounted with the market interest rate, V_m. We also have the strike price which is the "surrender value", i.e. the reserve discounted with the guaranteed interest rate, V_g.

The surrender put option value has the following formula:

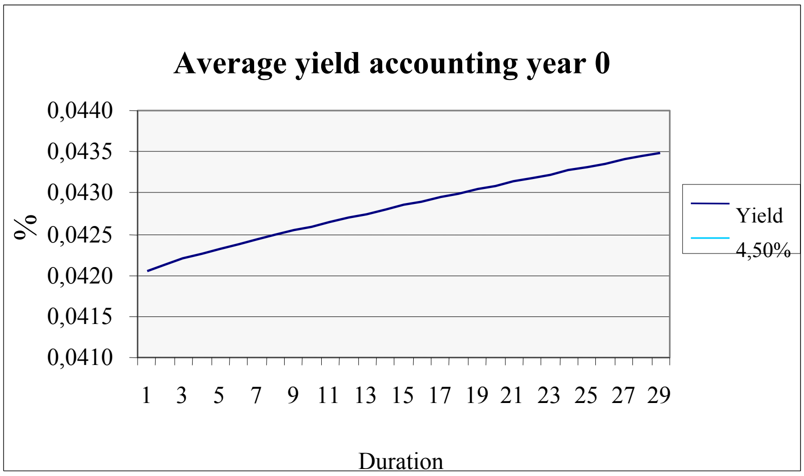
$$\sum_{t=1}^{\min(60-x;\inf(t)\{V_g(x,t)>V_m(x,t)\})} \frac{l_{x+t}}{l_x} \times v_m(t) \times \max(V_g(x,t)-V_m(x,t);0) \ , \quad 40 \le x \le 59 \qquad (3)$$

Simplified we say that if:

Guaranteed interest rate (g_r) < Market interest rate ($m_{r(t)}$)

⇒ Put (surrender) option value(t) > 0

As we can see the formula only summarizes the first positive value from the projection. For example, to receive the option value for that accounting year and the corresponding cohort a projection of future cash flows are made and the first time the value of ($V_{gr} – V_{mr)}$) is positive value and we have our option value. At this time we stop locking forward in the projection because the policyholder is logically only going to surrender one time.



Picture 4: *Yield curve vs. Guaranteed interest rate*

Once again the value of the option to surrender is only positive when market interest rate is higher than the guaranteed interest rate. This is because the present value of the reserve is lower when discounted at a higher interest rate. And the conversion option value is positive when the guaranteed interest rate is lower than the market interest rate. Hence, undertakings have to increase their reserve with the option value to be able to answer for their obligations.

3.2.2.1 Intensity-based surrender option valuation

The next two valuations of the surrender options is a survey of what Thomas Moller and Mogens Steffensen is describing in their book “Market-Valuation methods in life and pension insurance” (2007). This first intensity-based approach is based on a three-state Markov chain model. It is the same Markov chain described earlier except the conversion state. At any time between the ages 40 to 59 the policyholder can reach the state surrender by an intensity $\lambda(x, i^g, i^m)$. This intensity is dependent on age of the policyholder, the guaranteed interest rate and the market rate.

Moller and Steffensen are using Thiele’s differential equation for a three-state model to approach the value of the reserve. An explicit formula can be written as follows:

$$V^{sur}(t) - V(t) = \int_t^n \exp\left[-\int_t^s (r+\mu+\lambda)\right]\lambda(s)(G(s)-V(s))\,ds$$

Were $V^{sur}(t)$ is total reserve including surrender value, and subtracting the reserve without surrender value V(t) will get us the estimate of the value of the surrender option. Further, $\lambda(t)$ is the surrender intensity at time t, interest rate r and mortality intensity μ. This expression involves the future second order basis through future value of G(t) (surrender value) and V(t) (reserve). Locking at different cases when the reserve excluding surrender is larger or lower than the technical reserve V*, we get a more simplified expression of the estimate of the surrender option. Calculation of this result can be seen by M. Steffensen & T. Moeller (2007). The result is hence:

$$V^{sur}(t) - V(t) \text{ ¿ } \max(p(t)\times(G(t)-V(t));0) \qquad (4)$$

$$p(t) = \int_t^n \exp\left[-\int_t^s \lambda\right]\lambda(s)\,ds$$

The interpretation of p(t) is that it represent a kind of surrender intensity. It seems to summarize the future intensities “valued” to present time. And the option value is the difference between the surrender value and the current reserve.

This is another way of estimating the surrender option value and also a similar result to our approach with the Black-Scholes method. But the approach is different. Moller and Steffensen are starting from Thiele’s equation and approach by looking at different scenarios where the reserve is higher or lower than the technical reserve and ends with an inequality.

3.2.2.2 Intervention-based surrender option valuation

This intervention-based approach is saying that if it is optimal for a policyholder to surrender they will do so immediately. One problematic circumstance that Steffensen (2007) is pointing out is that there is very little historical evidence from were surrender is an optimal choice. If it is profitable to surrender the regulatory authority would probably set up some kind of protection to avoid arbitrage possibilities, costs that will make the surrender option less attractive. Either way this is an alternative approach and thus similar to the approach we have discussed earlier with the assumption of rational policyholders.

As a rational policyholder they may think that it is a beneficial strategy to surrender if the market interest rate is higher than the guarantee or more precisely.

Therefore if:

Vg(x,t) - Vm(x,t) > 0 => surrender intensity $\lambda = 1$
Vg(x,t) – Vm(x,t) < 0 => surrender intensity $\lambda = 0$

4. Valuation of technical provision

In July 2007 a Solvency II draft directive was establish since the present Solvency I rules are seen as outdated. Thus, the new solvency II regime is not expected to come into force until 2012. The Solvency II framework will form the basis of valuation of liabilities in this thesis. According to the new directive liabilities shall be valued at the amount for which they could be transferred, or settled, between knowledgeable willing parties in an arm's length transaction, i.e. market value. And there shall not be any adjustments to take into account of the own credit standing of the insurance undertaking.

Generally, an insurance undertaking shall base the technical provisions on their current exit value, i.e. the amount an insurance or reinsurance undertaking would expect to have to pay if it transferred its contractual rights and obligations immediately to another undertaking.

Therefore, the value of technical provisions shall be equal to the sum of a best estimate and a risk margin, as discussed earlier in chapter two. Best estimate shall be equal to the probability-weighted average of future cash-flows, taking account the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure. Furthermore, the calculations shall be based on creditable and adequate information while using actuarial methods and statistical techniques. The cash-flow projections used in the calculation of best estimate shall take account for all cash in –and out-flows required to settle the insurance and reinsurance obligations over the lifetime thereof.

Calculation of risk margin shall be such as to ensure that the value of the technical provisions is equivalent to the amount insurance and reinsurance undertakings would be expected to require in order to take over and meet the insurance and reinsurance obligations.

Best estimate and risk margin is to be calculated separately. Because of the inherent risk in the obligations that is transferred during a potential take over, the risk margin shall be calculated by determining the cost of providing an amount of eligible own funds equal to the solvency capital requirement necessary to support the insurance and reinsurance obligations over the lifetime thereof.

The rate used in the determination of the cost of providing that amount of eligible own funds is called "Cost-of-Capital" rate and shall be the same for all insurance and reinsurance undertakings. The Cost-of-Capital rate used shall be equal to the additional rate, above the relevant risk-free interest rate, that an insurance and reinsurance undertaking holding an amount of eligible own funds equal to the solvency capital requirement would incur to hold those funds.

However, where the future cash flows associated with insurance or reinsurance obligations can be replicated using financial instruments for which a market value is directly observable, the value of technical provisions shall be determined on the basis of the market value of those financial instruments. This mean that a separate calculation of the best estimate and the risk margin is not required.

5. Life insurance mathematics

5.1 Introduction to life insurance

What is an insurance contract? The definition of an insurance contract from the current international accounting standard on insurance contracts IFRS 4 is: “an insurance contract as a contract under which one party (the insurer) accepts significant insurance risk from another party (the policyholder) by agreeing to compensate the policyholder if a specified uncertain future event (the insured event) adversely affects the policyholder”. This means that we need to know how much to compensate the policyholder, or rather know what the expected payments would be. As the compensation is dependent on an uncertain future event it is impossible on before hand to know the exact amount and we can only make an estimate of the future payments and set up a reserve accordingly.

To do that insurance companies use well developed mathematic techniques. In this portfolio an important factor is mortality and it is central to know whether payments are made at either death or at maturity. We need to know what the probability is that a policyholder will die during a certain year. Thus, a probability-weighted estimate is needed .

5.2 Mortality

The mortality assumptions applied by insurers around Europe are different from country to country. For example in Scandinavia, most of the undertakers are using the Makeham formula. Particular in Sweden the Makeham formula is used for equalization of observed mortality. Generally there is two ways of using mortality. First we have Makeham, using smoothed mortality data on population from previous periods. Second is to use experience data for different classes of business and applying a well-throughout method of graduation.

The model for length of life is defined through the following assumption of the mortality rate, Makeham formula:

$$\mu(x) = \alpha + \beta * \exp(\delta * x), \quad x \geq 0 \tag{5}$$

where $\mu(x)$ is the intensity for dying and x is the age of the insured.

In order to estimate the parameters in the Makeham formula taking account for future development in mortality it is possible to use a model called Lee-Carter model with Poisson assumptions. The Lee-Carter model is a popular and an easy model to apply when to project future patterns of mortality.

$$\mu(x,t) = \exp[a(x) + \kappa(t) b(x)]$$

where t is a future point in time

The financial supervisory authority in Sweden (Finansinspektionen) has recently made these calculations to change the parameters due to changes in the actual mortalities. Historical data is needed in order to make these projections and the maximum likelihood-method to estimate the parameters. Number of deaths at age x and during calendar year t, D(x,t), is assumed to be Poisson distributed:

$$D(x,t) \propto Poisson(E(x,t) \times \mu(x,t))$$

E(x,t) is the exposure, which is the average persons at age x during calendar year t. κ(t) is describing how mortality changes with time.

The survival function l(x), for an x year old policyholder, is defined as follows:

$$-\ln(l_x) = \begin{cases} ax+(\frac{b}{c})(\exp(cx)-1) & , x \leq w \\ -\ln(l_w)+\mu_w(x-w)+(\frac{k}{2})(x-w)^2 & , x > w \end{cases} \tag{6}$$

were a, b and c are given parameter in the Makeham formula and w is the age at which you are adjusting the survival function due to change in expected life length. The financial supervisory authority is expecting it to be 97.

To get more appropriate assumptions of mortality regarding the policy that lies within the portfolio in this thesis we will use the one-year-mortality q_x instead of μ_x as presented in the Markov chain, were x represent the age of a policyholder. This is because we want to see how many of the policyholder who is expecting to die every year during the insurance period. To calculate q_x we will look at a relation between q_x and μ_x.

We have that

$$q_x = P(Tx \leq 1) = 1 - P(Tx > 1)$$

were T_x is remaining length of life. The probability that you will live longer than one year can be expressed as:

$$P(Tx > 1) = Exp[-\int_x^{x+1} \mu(s)ds]$$

This will give us

$$q_x = 1 - Exp[-\int_x^{x+1} \mu(s)ds]$$

In the search of the relation between q_x and μ_x we do some rearrangement and take the logarithm of both sides in this equation and furthermore gets

$$-\ln(1-q_x) = -\int_x^{x+1} \mu(s)ds$$

To get an approximation of μ it is suitable to use a straight line due to the short intervals that we are looking at, one-year intervals (x, x+1), and choose the value in the middle.

$-\ln(1-q_x) \approx \mu_{x+½}$

$$-\ln(1-q_x) \approx \frac{q}{(1+\frac{q}{2})}$$, for small non-negative values of q

$$\rightarrow \quad \mu_{x+½} \approx \frac{q(x)}{(1+\frac{q(x)}{2})}$$

$$\rightarrow \quad q_x \approx \frac{\mu(x+½)}{(1+\frac{\mu(x+½)}{2})} \tag{7}$$

5.3 Commutation functions

Pricing insurance contract it is normally about valuation of future payments, such as premiums and future benefits. Future payments are also dependent if an insurance event occurs or not, which itself is due to the life of the policyholders. To be able to value the future payments we have to calculate the capital value, i.e. the present value of future payments discounted due to interest rate and mortality.

The commutation functions that we will introduce are used to calculate the capital value can be expressed in discrete and continuous forms. As we are building this insurance model in discrete time it is convenient to use discrete functions. But commutation functions are only valid for a fixed interest rate. Our intention is to evaluate the possibility to either surrender or convert their policy and for that we have to calculate the reserves with both the guaranteed interest rate and a possible market interest rate and also the value of these options. Hence, we will have to use a probability-weighted average of future cash-flows, taking into account the time value of money (expected present value of future cash-flows). So we are only going to use commutation functions to calculate the annuity payments based on the guaranteed rate of interest.

5.3.1 Definition of discrete commutation function with fix interest rate

x represent the age of the policyholder and i the interest rate.

- q_x = probability of death
- $p_x = 1 - q_x$ = probability of survival
- l_x = number of survivals
- $d_x = q_x*l_x$ = number of dead
- $v_x = 1/(1+i)$ = discounting factor
- $D_x = l_x*v_x$ = discounted number of survival at a certain point of time
- N_x = sum of D_x = sum of discounted number of survival
- $C_x = v_x*d_x$= discounted number of dead
- M_x = sum of C_x = sum of discounted number of dead

- R_x = sum of M_x

5.3.2 Calculation using a yield curve

To calculate the capital value, i.e. the reserve, we will have to use a probability-weighted average of future cash-flows, taking into account the time value of money (expected present value of future cash-flows). This is because we can not use commutation functions while using a yield curve. The commutation functions requires a fixed interest rate. The reasons are as follows:

We have that

$$D(x)=v(x)\times l(x) \text{ ,}$$
$$N(x)=\sum_{t=0}^{\infty} D(x+t) \text{ ,}$$
$$a(x,n)=\sum_{t=x}^{x+n-1} v(t-x)\times\frac{l(t)}{l(x)}$$
$$a(x,n)=\frac{N(x)-N(x+n)}{D(x)}=\frac{v(x)\times l(x)+v(x+1)\times l(x+1)+\ldots+v(x+n-1)\times l(x+n-1)}{v(x)\times l(x)}=$$
$$=1+\frac{v(1)\times l(x+1)}{l(x)}+\ldots+\frac{v(n-1)\times l(x+n-1)}{l(x)}$$

An it is not possible to divide with v(x) unless v(x) is fixed.

6. Option pricing

If we again concentrate on the embedded options we have discussed earlier we could see that the option have a financial value, or a time value. The options can be either in-the-money, at-the-money or out of the money. A call options is referred to as in-the-money if the price of the underlying asset is larger than the exercise price and at-the-money we have equality between underlying asset and exercise price. Then, of course, this will lead to that out-of-the-money is when underlying asset price is lower than exercise price. And the opposite applies for put options.

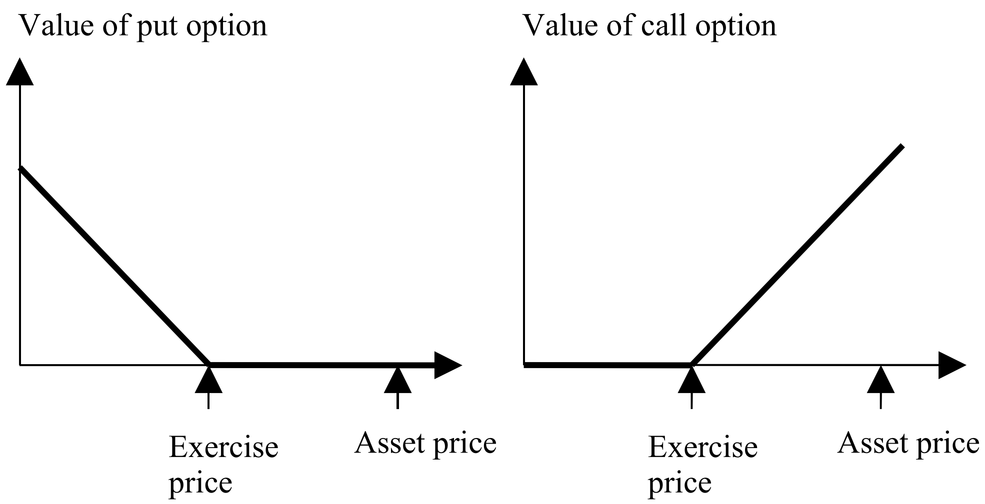


Figure 5: *Value of option at expiration.*

As told before, the underlying assets for the embedded option is not really an asset. For the put option, which is referring to the surrender option, the price is the value of the reserve discounted with the market interest rate and the strike price is the ”surrender value”, i.e. the reserve discounted with the guaranteed interest rate. The price of the underlying conversion option is the discounted cash flows from the guaranteed rate discounted at the market rate and the strike price is the insured amount.

This means that we are facing a different kind of option management, when you have to discover the new features when valuating the options. To be able to use this kind of approach we have to obtain volatility for bonds rates. And there might not be a certain way or a correct way to obtain this but it can be derived from volatility of market interest rates. This is essential, because the decision of the policyholder to convert or surrender is highly dependent on the market interest rate and its relationship whit the guaranteed interest rate, the volatility of the market interest rate has then an influence on the option value.

These options that we are offering to the policyholder must also have a price. One way forward could therefore be to see the problem as if the policyholder was charged with the corresponding price of the option. The price of the conversion option is much easier to compute and to settle because there is only one option per policyholder and they can only exercise the option when they turn 60 years old. However, the surrender option can be exercised every year up to age 59 and the price will be different during the insurance period. So, should the insurance company charge the policyholder when they buy the insurance or charge it every year. We can look at it as the policyholder buys the right to surrender at the beginning of every year and therefore the policyholder will be charged for a one-year option.

The formula to value a European option on a non-dividend stock was derived by Black and Scholes 1973 and was extended by Merton 1973 to allow for a continuous dividend yield. Fisher Black was the first to show that the futures price have the same lognormal property that are assumed for stock prices, this was 1976. The formula is therefore called Black-76 and is probably the most widely used model to price interest rate options. Thus, the formula is:
.

Call: $c = \exp(-rT)[FN(d_1) - XN(d_2)]$ (8)

Put: $p = \exp(-rT)[XN(-d_2) - FN(-d_1)]$ (9)

Where

$$d_1 = \frac{\ln\left(\frac{F_0}{X}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{F_0}{X}\right) - \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

F_0	=	Futures price at time zero
X	=	Exercise price
σ	=	Volatility of the relative price change of the underlying asset (vol. of market interest rate)
b	=	Cost of carry rate (i.e. the cost of interest plus any additional cost) = 0 on a future contract. B = r-q, q = dividend
r	=	volatility of market interest rates
N()	=	The cumulative normal distribution function

As described in the table, the parameter b is the cost of carry and equals to zero when pricing a futures option. Black proves that the drift of the futures price in a risk-neutral world is zero. So from the fact that a risk-free process for the stock price is given by

$$dS = (r - q)Sdt + \sigma Sdz$$

This gives that the futures price behaves in a similar way to a stock providing a dividend yield q equal to r.

$dF = \sigma Fdz$, where σ is a constant

The formula that Black derived in 1976 does not require that the length of the futures contract is the same as for the option contract; they do not have to mature at the same time.

Going back to the pricing and financial part, the insurance undertakings have to hold additional assets to cover the embedded options. An idea of how to set the additional price is to calculate an average price based on the simulations. In other words, in the 10 000 simulation we are calculating the option value at the time we expect the policyholder to surrender and discount it back to the accounting year, if we then calculate the price for that specific surrender value (taking into account the probability of surviving) and discounting back to the accounting year for every scenario and then set an average price.

The interpretation of how to calculate the surrender option price at time t and age x will be similar to the surrender option value:

Surrender option price (t) =

$$= \sum_{t=1}^{\min(60-x;\inf(t)\{V_g(x,t)>V_m(x,t)\})} \frac{l_{x+t}}{l_x} \times v_m(t) \times [X \times N(-d_2) - F_o ¿ N(-d_1)] \ , \quad 40 \le x \le 59 \quad (10)$$

Where d_1 and d_2 is the same as before, except that F_0 equals $V_m(x,t)$ and X equals $V_g(x,t)$. v_m (t) is the t year discount factor.

The corresponding conversion option price for an x year old policyholder:

$$\text{Conversion option price} = v_m(60-x)[F \times N(d_1) - X \times N(d_2)] \ , \quad 40 \le x \le 59 \quad (11)$$

Where F_0 equals $Annuity \times a^{m}_{60:10}$ and X equals $Annuity \times a^{g}_{60:10}$ = Face amount

When it comes to managing risk you might consider looking at the sensibility of the option. There are ways to measure the sensibility when there is a small change in a parameter of the pricing formula. The option sensitivity is called the "Greeks", where each Greek letter measures a different dimension to the risk of an option position. If we can manage the Greeks, then we are in a great position to see to that all risks are acceptable. We will not focus our attention on the Greek management in this thesis but is worthwhile to mention and what kind of interest it brings to this topic.

The Greeks are:

Delta	the option's sensitivity to small changes in the underlying asset price
Gamma	the Delta's sensitivity to small changes in the underlying

	asset price
Vega	the option's sensitivity to a small change in the volatility of the underlying asset
Theta	the option's sensitivity to a small change in time to maturity
Rho	the option's sensitivity to small changes in the risk-free interest rate

7. Valuation of reserve

Valuation of the reserve is made through a probability-weighted estimation value as described in chapter four. We are using the probabilities of mortality and the discounting factors to do the projections. We are calculating the surrender option and the conversion option separately and adding it to the reserve that corresponds to the endowment insurance and we get the total reserve.

To understand the complexity of this reserve we have to see it in several dimensions.

- Firstly we have our triangle which contains the reserve for every accounting year until the expiration of the policy.
- Secondly, for every accounting year we are calculating the reserve for every cohort of policyholders, i.e. 19 cohorts for 30 accounting years ahead (depending on age of policyholder), thus it includes reserve for both the endowment and annuity payment.
- Thirdly, for every accounting year and cohort we do projections of future cash flows and discounting them back to today.

Accounting year	**Age**	40	41	42	…	60	…	68	69
	0	-	-	-	-				
	1		-	-	-	-			
	2			-	-	-	-		
	…				-	-	-	-	
	…					-	-	-	-
	…						-	-	-
	28							-	-
	29								-

Figure 6: *Example of reserving triangle*

At the start of accounting year 0 we have set the number of policyholder's to be 100 for every age group. The predefined annuity that the policyholder may buy at age 60 years is the following with different guaranteed interest rates with the insured amount of 1 €:

Interest rate	**3,25 %**	**4,50 %**
Annuity	0,13202	0,13830

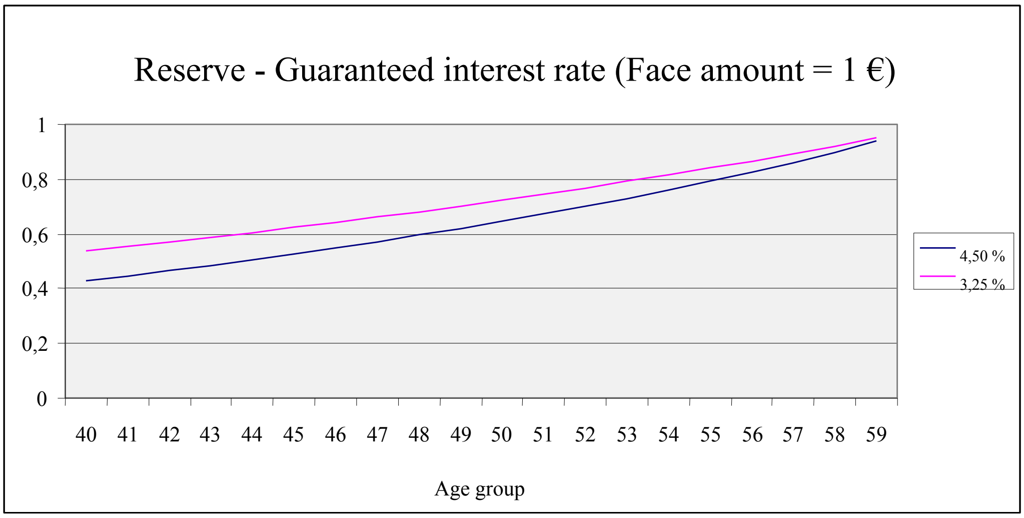


Figure 7: *Reserve calculated with 3,25 % and 4,50 % guaranteed interest rate*

As we can see from the picture above lower guaranteed interest rate will generate a higher guaranteed value for the policyholder. (Paid-up policy)

To be able to do these projections I have used data from an Economic Scenario Generator (ESG). This generator has given me prices of zero coupon bonds, discount rates and spot rates for thirty years ahead for 10 000 scenarios.

This data was supplied by Ernst & Young LLP using Barrie & Hibbert's Economic Scenario Generator (ESG). Barrie & Hibbert is the leading supplier of stochastic asset modelling software to the global ALM (Asset Liability Management) community to assess the rate of return on investment, yield curve, bond prices up to 30 year etc.

ESG simulation (10 000 simulations) is based on the following:

- Interest rate model - Monthly Libor Market Model - calibrated to government's yields
- Equity model - Deterministic time varying volatility - calibrated to EUROSTOXX
- Property model - Lognormal constant volatility model - calibrated to EURO property
- Calibration Date - 30/06/07
- Time Horizon - 30yrs
- Antithetic variables used

The probability-weighted reserve is as described before the present value of expected payments in the future taking into account the probability of being alive at certain point in time $l_x(t)$, mortality intensity $\mu_x(t)$ and the discount factor $v_m(t)$. This leads us to en explicit formula for the traditional reserve for the endowment policy, V(m);

$$\boxed{\left(\sum_{t=0}^{59-x} \frac{l_x(t)}{l_x(0)} \times \mu_x(t) \times v_m(t) + \frac{l_x(60-x)}{l_x(0)} \times v_m(60-x)\right) \times face_{amount}} \quad , 40 \le x \le 59 \quad (12)$$

$v_m(t)$ is considering discounting at market interest rate. If we ought to use this reserve as our final we could end up reserving a to small amount, if the market interest rate is lower than the guaranteed interest rate, and that is not acceptable. Because of the possible differences between the guaranteed payment to the policyholder and the reserve in our portfolio developed with a market interest rate we have to consider both the surrender option value and the conversion option value. This leads us to the total reserve:

Total reserve = $V(m)+surr.opt.value+conv.opt.value$

Furthermore, with the assumptions of rational individuals, the criterion for the choice of surrender or conversion would be:

Surrender if $V(g)>V(m)+surr.opt.value+conv.opt.value$ (13)

We talked earlier about rational policyholders that will surrender if it is beneficial to them. But when is it beneficial to surrender? If a policyholder would consider it advantageous to surrender they should take into account all the future benefits. That would of course be very difficult for policyholders to know, they might only have expectations of the future of their own. Nevertheless, it is of great importance for the insurance company to know the values of the embedded options. Therefore the criterion for surrender should be when the surrender value is higher than the sum of mortality reserve plus both option values.

The argument is that in this case the policyholders should be able to buy the same future payments in the market for a price less that then guaranteed surrender value.

When we value the liabilities in our portfolio it is interesting to see what the spread would be in different scenarios. The estimated reserve for the life of the policy is then an average of the 10 000 simulations of the reserve. But as we can see from the figures down below there are a wide spread in the beginning of the lifetime, which diminishing later on. Also worth to mention is that the higher interest rate we guarantee the more policyholder are likely not to surrender and a larger spread in the total reserve.

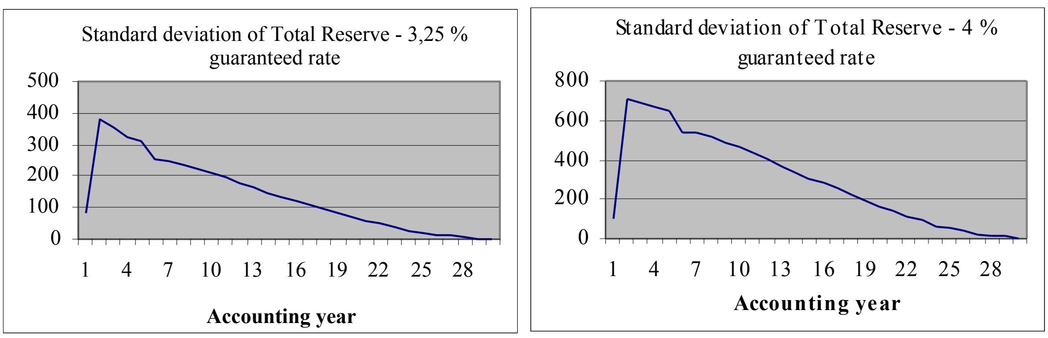


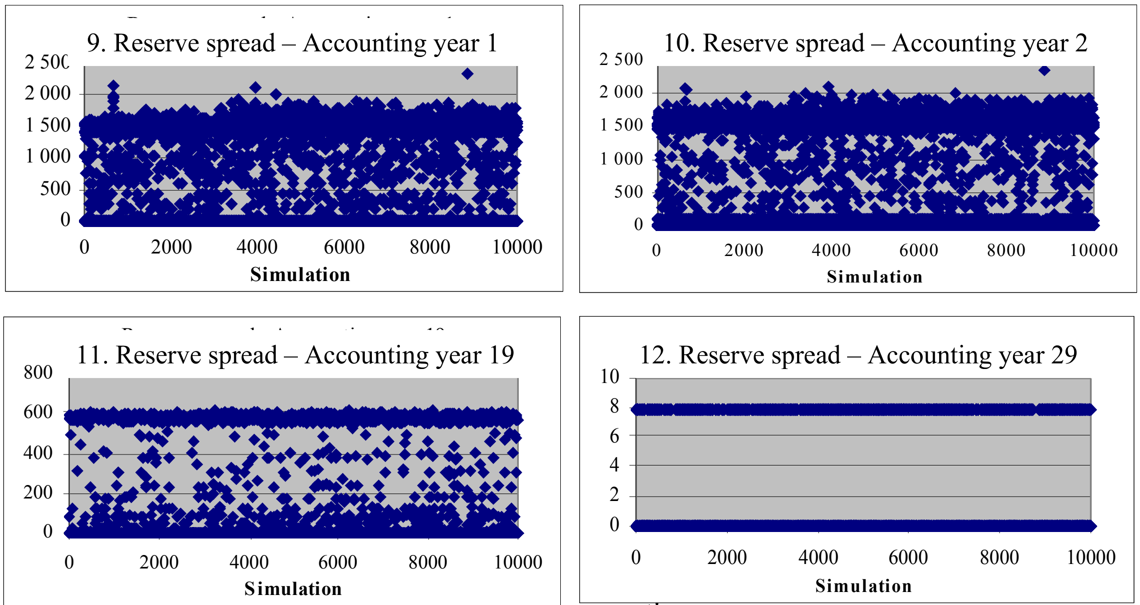
Figure 8: *Standard deviation of total reserve with 3,25 % and 4 % guaranteed interest rate*

The reason for such low spread with a guaranteed interest rate such as 3,25 % is because that the guaranteed rate is somewhat lower than the average yield and the most of the policyholders will surrender. When the guaranteed rate is close to average yield we will see more spread in the reserve and when the guaranteed rate is much higher than the average yield we will again see a low spread in the reserve.

The widely spread can be explained by the fact that we see our policyholder as rational and therefore will surrender as soon as it is beneficial to them. After a choice of conversion or not, policyholders will stay in the portfolio until the last annuity payment or if they decease and therefore a low spread in the tail. If we instead had made a model based on historical patterns of surrender, insurance companies will have more control over the active policyholders and how many that is expected to surrender and we would see a smaller spread in the reserve. This will also make the undertakings more certain about their estimated reserve and future cash flows.

The simulation has been done with different guaranteed interest rates. The result shows us that the lower the guaranteed interest rate is the fewer active policyholders we have left in our portfolio. And that is also what we would have expected with our criterion of being rational. We will also see cases where all the policyholders will surrender after the first year and thus the policy will run-off and it will also lower the estimated average reserve.

The pictures down below present for us the spread that we could expect throughout the simulation with a guaranteed interest rate of 4 % and with assumption of rational policyholder's.



Picture 9-12: *Reserve spread of four different accounting year*

The pictures above also present that the interest rate (yield curve) have a great impact on the technical provision, valued at market value. These pictures can be seen in a larger scale in the appendix.

We can conclude from the simulations and the pictures that reserve is very sensitive to fluctuations in the interest rate curve. This is a problem for insurance companies and have to be tackled in the right way by choosing a relevant hedging strategy. That would be a strategy that will reduce the impact of changes in the interest rate, such as immunization. If an insurance company has not set up a hedging strategy

against fluctuations in the interest rate they could end up underestimating the reserve.
From the specific simulation above, with a guaranteed interest rate of 4 %, it would be interesting to present the number of simulations where the surrender options may be applied and respectively the conversion option may be applied.

Accounting year	Number of surrender applied									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Number	6742	1137	566	374	1000	124	142	116	95	70
Percentage	67,43%	11,37%	5,66%	3,74%	10,00%	1,24%	1,42%	1,16%	0,95%	0,70%
Average	12,80	1,63	0,76	0,44	0,90	0,07	0,11	0,10	0,08	0,06
	11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	
Number	73	61	58	48	43	33	32	17	21	
Percentage	0,73%	0,61%	0,58%	0,48%	0,43%	0,33%	0,32%	0,17%	0,21%	
Average	0,05	0,04	0,04	0,03	0,02	0,01	0,01	0,01	0,00	

Accounting year	Number of conversion applied									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Number	3785	3017	2579	2281	2043	1331	1213	1292	1294	1272
Percentage	37,85%	30,17%	25,79%	22,81%	20,43%	13,31%	12,13%	12,92%	12,94%	12,72%
	11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Number	1223	1169	1113	1049	1018	977	939	913	873	890
Percentage	12,23%	11,69%	11,13%	10,49%	10,18%	9,77%	9,39%	9,13%	8,73%	8,90%

Table: *Numbers from simulation, 4 % guaranteed interest rate*

With 4 % guaranteed interest rate we have a high percentage of cases when the policyholders surrender their policy and that means that all the policyholder in the portfolio will surrender. The third row shows the average number of cases when the surrender option have been applied. From accounting year two the percentage is diminishing fast except for the accounting year five where it goes up to 10 %.

The number of conversion applied are dependent on how many that are surrendering, but it does not mean that the number have to summarize to 10 000. It is not useful to set an average on number of conversion because either everyone left in the portfolio converts at a certain accounting year or no one is converting their policy.

8. Portfolio

8.1 Assets

The investment management is today split into two categories: a) “assets covering technical provisions”, and are subject to a number of quantitative restrictions (e.g. asset eligibility criteria and quantitative limits); and b) “free assets”, i.e. other assets which are not subject to quantitative restrictions.

Investment management is going through changes when the new Solvency II regime, will start in a couple of years. It will only be one category and instead equity investments along with all other assets will be subject to a capital requirement proportional with the level of the risk of the asset. This is due to experience in the past that shows that inappropriate investments strategies can threaten the soundness of an insurer and its ability to meet its obligations. It is then possible to invest in any assets the insurers wish but it will also mean that the higher risk or the more volatile the asset tends to be, the higher requirement for holding extra risk capital. But the new rules will also incite insurers to invest in more high risk investment, which will provide more in capital return.

8.2 Hedging portfolio

Typical hedging instruments used to secure insurers ability to meet their obligations are; government bonds with risk-free interest rate, zero coupon bonds for any of the fixed cash flows, interest rate swaption, swaps for interest rate risk and equity options.

A zero coupon bond has the advantage of being free of reinvestment risk. Zero coupon bonds are conducive to being sensitive to fluctuations in interest rates, because there are no coupon payments to reduce the impact of interest rate changes. In addition, markets for zero coupon bonds are relatively illiquid.

Swaption is an option to enter a swap contract. The contractual owner of the swap option is not obliged but have the right to enter the specified swap contract at a predefined time in the future if it is beneficial to the option holder.

An interest rate swap is a deal between banks or companies where borrowers switch floating-rate loans for fixed rate loans. It is called an interest rate swap if both cash flow streams are in the same currency and are defined as cash flow streams that might be associated with some fixed income obligations. Generally speaking, swaps are wanted by firms that want a type of interest rate structure that another firm can provide less expensively.

The most popular interest rate swaps are fixed-for-floating swaps under which cash flows of a fixed rate loan are exchanged for those of a floating rate loan. Among these, most common use a 3-month or 6-month Libor rate (or Euribor, if the currency is the Euro) as their floating rate. These are called vanilla interest rate swaps.

Equity options is a derivative that give the option holder the right (but not the obligation) to buy or sell a specified number of shares of stock, at a specified price for a certain time period that is limited. Usually one option equals 100 shares of stock.

Another derivative that concerns equity is an equity default swap. An equity default swap is an instrument for one party to provide another party protection against some possible event relating to some asset. With an equity default swap, the reference asset is some company's stock and protection is provided against a dramatic decline in the price of that stock. For example, the equity default swap might provide protection against a 70% decline in the stock price from its value when the equity default swap was initiated. The event being protected against is called the trigger event or knock-in event.
There are two parties to the agreement. Maturities are for several years, with five years being typical. The party buying protection pays the other a fixed periodic coupon for the life of the agreement. The other party makes no payments unless the trigger event occurs. If it does occur, the equity default swap terminates, and the protection seller makes a specified payment to the protection buyer. This is calculated as

notional amount (1 – recovery rate)

where the notional amount is simply a sum. The recovery rate for an equity default swap is fixed—typically at 50%.

When looking at the balance sheet, we will see two different sides. They are different in many ways but especially they occur in different durations. For example in Sweden, life insurer has typically liabilities of 20 – 30 years to policyholders and the assets which are set to cover the liability have only a duration of 10 to 15 years. The European market can instead provide assets like bonds that will last for up to 30 years and possible even longer in the future. The risks insurance companies are facing are the interest rate and reinvestment risk.

By hedging this portfolio it means that we are trying to ensure that the average duration of their assets equals the average duration of their liabilities. How often the undertakings are rebalancing their portfolio is different from company to company. It depends on the expected changes in the market and varies from 1 year to several years. When the company rebalance their portfolio is not the most important in this study, it is that the present value of the old portfolio when they rebalance discounted with the interest curve at the rebalancing year do not differ to much from the present value of the portfolio they want to buy.

This is known as portfolio immunization. What is does is that it ensures that a small parallel shift in interest rate will have a small effect on the value of the portfolio of assets and liabilities.

8.3 Immunization of portfolio

A way to immunize is to provide a portfolio of bonds, **y.** This portfolio might contain a small number of bonds and have different durations then the cash flow, **x**, we want to immunize against, but they have the same present value and will react in a same manner on shifts in interest rates curves which is very important. In this case **x** is the estimated reserve set up by the insurance company.

The spot rate curve is not constant but varies with time and there are several shift to the spot interest rate curve that we could immunize against; parallel shift, steepening (i.e. rates with time to maturity shorter than approximately 2 years moves in one direction and other rates in the other direction) and bending (i.e. short and long yields moves in one direction and the other rates in the other direction).

The most common shifts in spot rate curves are parallel shifts. Immunization against parallel shifts will have an effect of explaining around 83 % of the total variance in the spot rate. Even though a lot of the variance is explained it is not enough. We may want hedge our self against steepening as well. It will increase the rate of explanation from 83 % to 93 %, and if we want to add bending in our strategy it will increase it further to 96 %.

Going back to our hedging portfolio **y,** we will only consider to hedge against parallel shifts and steepening. We then want to immunize **y-x** against these shifts in the spot rate and we have to look upon some conditions that will arrive (Luenberger 1998). D is the duration of respective portfolio.

$$D_Y = D_X \quad \text{, parallel shift condition}$$
$$D_Y^{(1)} = D_X^{(1)} \quad \text{, steepening condition}$$

We have that **X** = ($-x_1$, $-x_2$, …, $-x_{28}$, $-x_{29}$) is our future cash flow and d = (d_1, d_2, …, d_{28}, d_{29}) are the discount factors. We can then express the duration:

$$D_X = \sum_{k=1}^{29} \frac{k \times x_k \times d_k}{P_x} \quad D_X^{(1)} = \sum_{k=1}^{29} \frac{r_k \times k \times x_k \times d_k}{P_x} \quad \text{, k = 1, …, 29}$$

The identities say that we want the duration of our future cash flow to be the same for our bond portfolio.
Furthermore, we assumes that our portfolio **y** is compiled of bonds $\mathbf{b}_1$,…, **bn** and that a_k is the number of $\mathbf{b}_k$ bonds in the portfolio. k=1,…, n.

The composition of portfolio **y** is therefore: **y** = $a_1b_1 + \ldots + a_nb_n$

If we let P_k and D_k represent prices and duration for bond b_k, the above equation can be expressed as follows:

$$a_1 P_1 + \ldots + a_n P_n = P_x$$
$$a_1 P_1 D_1 + \ldots + a_n P_n D_n = P_x D_x$$
$$a_1 P_1 D_1^{(1)} + \ldots + a_n P_n D_n^{(1)} = P_x D_x^{(1)}$$

or

$$v_1 + \ldots + v_n = 1$$
$$v_1 D_1 + \ldots + v_n D_n = D_x$$

$$v_1 D_1^{(1)} + \ldots + v_n D_n^{(1)} = D_x^{(1)} \quad , v_k = \frac{a_k P_k}{P_x}$$ = weight of bond $\mathbf{b}_k$ in the portfolio

If we only want to secure our portfolio against parallel shift in spot rate curve we only need two bonds and we have to solve the two first equations above. This leads us to the weights of bond A and B respectively:

$$v_A = \frac{D - D_B}{D_A - D_B} \quad \text{and} \quad v_B = \frac{D_A - D}{D_A - D_B}$$, for two bonds A and B.

But we want to immunize against steepening as well, so the weights for the required zero coupon bonds can be extracted from the three equations above. Thus, three equations require three bonds. The weights are:

$$v_A = \frac{(D_B - D_C) \times (D_x^{(1)} - D_C^{(1)}) - (D_x - D_C) \times (D_B^{(1)} - D_C^{(1)})}{(D_A^{(1)} - D_C^{(1)}) \times (D_B - D_C) - (D_A - D_C) \times (D_B^{(1)} - D_C^{(1)})}$$

$$v_A = \frac{(D_A - D_C) \times (D_x^{(1)} - D_C^{(1)}) - (D_x - D_C) \times (D_A^{(1)} - D_C^{(1)})}{(D_B^{(1)} - D_C^{(1)}) \times (D_A - D_C) - (D_B - D_C) \times (D_A^{(1)} - D_C^{(1)})}$$

$$v_C = 1 - v_A - v_B$$

What could be interesting is to see what strategy to prefer. Since it is possible to buy bonds up to 30 years of maturity in Europe it would be convenient to bye a certain amount of a one-year zero coupon bond, two-year zero coupon bond and so on, to make a perfect hedge. But it would not be interested to analyze and for that matter there might not be available bonds that mature after 1, 2, … up to 29 years. There might be some of them that are available in the market at that time. So we have chosen a couple of bonds that are at our disposal in the market, bonds with maturity of 1, 5, 9, 10, 15, 16, 18, 19, 20, 23, 24, 29 years.

To get en understanding of the rebalancing of the portfolio is done we consider an example of one bond portfolio, for example the first group which contains duration years 1 to 9. At time 0, as at today, we buy our portfolio with bonds with maturity of 1, 5, and 9 year. We are selling our portfolio after one year and buying our new portfolio of bonds with maturities of 1, 4 and 8. The amount we receive when selling our current portfolio is the present value discounted with the interest rate curve from the following year. Since the zero coupon bond with maturity of 9 year at time point 0 has lapsed for one year and is now considered as an 8 year old zero coupon bond and so on.

So we want to split our obligated cash flows into two or three groups so that we can see how much our immunization differ from each other and the perfect hedge. We can also compare the analysis without immunize against steepening to see how much steepening is bringing to explain the variance.

The suggested groupings of our cash flow are groups of two and three. The groups are (year 1 to 9, year 10 to 18, year 19 to 29) compared to (year 1 to 15, year 16 to

29), (year 1 to 9, year 10 to 29). This is our starting point and we will have to regroup and change the composition of the portfolio as time going.

If we for example start with three portfolios in the beginning our intention is to proceed with three portfolios as long as we can. Our intention is also to compare differences between immunization against a parallel shift in the interest rate curve and immunization against both parallel shifts and steepening. The idea behind this choice of groups is based on a statistical study made in 1997 by J. Frye where he showed that the most changes in the interest rate curve will happen within the first ten years. This is shown very well in the picture below. So it is interesting to see what strategy that is most appropriate to apply and furthermore provide a solid portfolio with low margin of error.

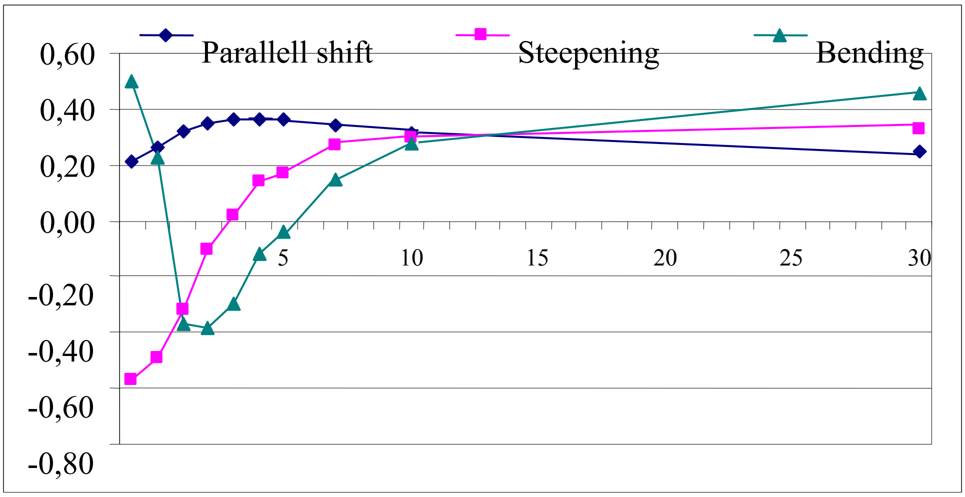


Figure 13: *The three most important components that will explain shifts in the interest rate curve.*

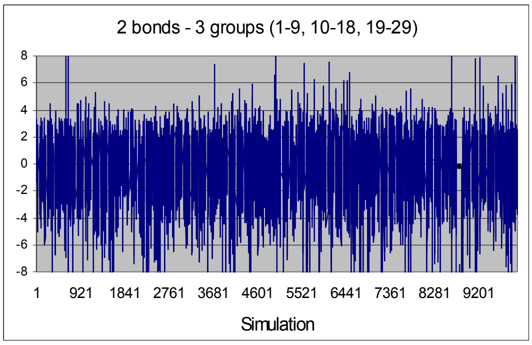
To analyze this we first decide to rebalance our portfolio once a year. The measure that is of our interest is the fault that arrives when we are selling our current portfolio and for that money byes our new portfolio. We then rebalance our portfolio every year in a lifetime period of thirty years and the result gives us the total fault of that single simulation. This is done for 10 000 simulations with 10 000 triangles of interest rate curves, i.e. 300 000 interest rate curves. These interest rates are a result from the ESG simulation given by Barry & Hibbert.

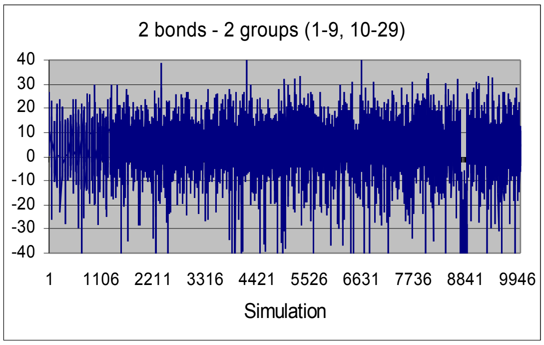
What we can see from picture 6 is that the interest rate fluctuates a lot the first ten years and then smoothens out to a straight line. Even though the curve do not vary too much we have a relative large difference in the rate in year 30 compared to year 10. We also know from real market the interest rate is not very certain in the future, many things can happen that would have an influence on the interest rate. From out of that it seems that the long period from about ten years up to thirty years is not to be seen as one period but several periods. How many? It is hard to say. We have chosen at least to divide the latter period into two groups to see if there are improvements in the immunization.

8.4 Matching

The result from the immunization simulation shows that it is better to immunize against both parallel shifts and steepening in the interest rate curve. That is also what we have had expected and should be the case.

If we only restrict us to the scenario of immunization against parallel shifts it would be better to split the future cash flow into three groups, i.e. three portfolios. The suitable composition of portfolio is to compile the first portfolio with bonds that have maturities of 1 and 9 years, second with bonds that correspond to duration 10 and 18, and finally a group of bonds that mature in 19 and 29 years.

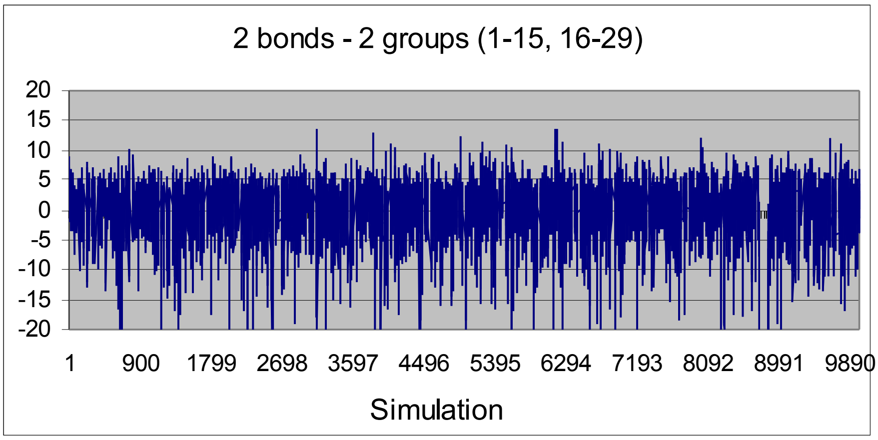




Picture 14 and 15: *Result from simulation of immunization against parallel shift in the interest rate curve*

The total fault of the immunization is shown in the pictures above and we can see more fluctuations in the right picture with only two groups. With only two bonds covering about 19 years and the fact that zero coupon bonds are very sensitive to changes in the interest rate the result is not unexpectedly. So to split up the latter period into two would be a great idea when we only immunize against parallel shifts.

If we use a set of two periods that have about the same duration we will get a better result, but not better than the first case with three groups. So if we only consider to immunize against parallel shifts we ought to compile a portfolio with three sub portfolios with similar duration length.

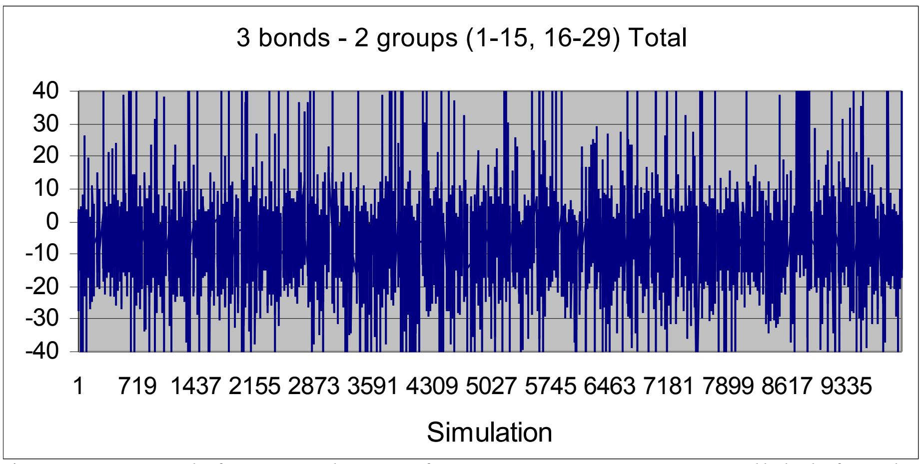


Picture 16: *Result from simulation of immunization against parallel shift in the interest rate curve*

The fault that we can see in the pictures are either money left after rebalancing the portfolio or money that are missing to be able to buy your desired portfolio. Negative values represent money left after the purchase. So to finance the purchase when there are not enough assets from selling the current portfolio would be money coming from a buffer. Hopefully it is enough, but we don't want the possible outcomes from the immunization to change like it does in the figures above.

Going forward in our analysis of strategies, we will consider the steepening condition as well. This time we will use three bonds to build our portfolio and try to match the present value and duration as best as we can. We have run three different simulations that correspond to three kinds of grouping.

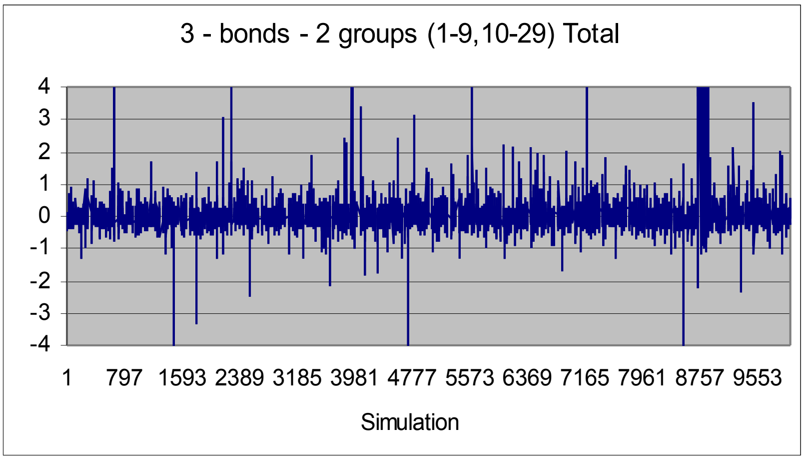
We start with the portfolio which contains the two periods: (1-15, 16-29). The first segment is chosen to stretch over the period of time that the interest rate is fluctuating at the most. This is to see whether it is best to focus on the first ten years or if that is not necessary.



Picture 17: *Result from simulation of immunization against parallel shift and steepening in the interest rate curve*

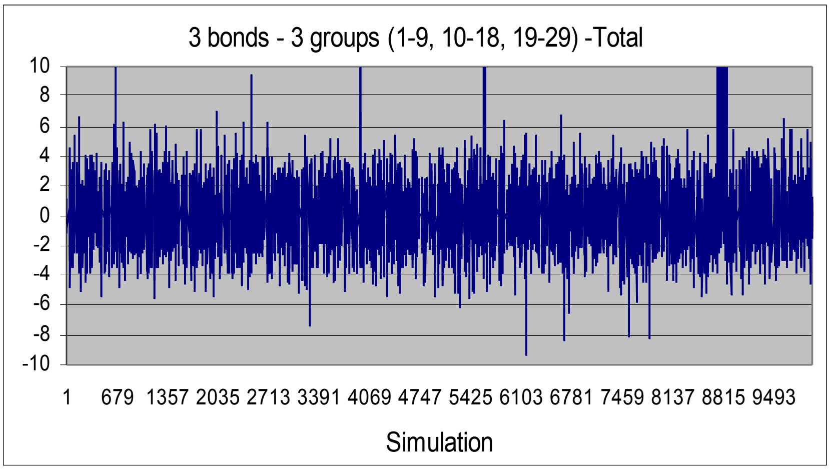
There seems not to be a good idea to compile the portfolio in this way. It is even worse than the situation when we only have two bonds and immunize against parallel shifts with the same grouping situation. This might depend on the need of regrouping your portfolio and there will be a mismatch.

If we organize this portfolio in a different way and use a set of bonds to handle the first ten years and another set to handle the rest we ought to get a better result. And we do! We will see that this composition is much better. This is really a desirable portfolio that captures the fluctuations in the market interest rate at the time where the most change would appear. But there are also very high peaks to be aware of and might result in large losses.



Picture 18: *Result from simulation of immunization against parallel shift and steepening in the interest rate curve*

The question is whether the long period after ten years ought to be divided to capture more fluctuations in the interest rate curve. But looking at the result from the simulation of immunization with three bonds and three groups does not make it better. This is not really what we had anticipated. The hypothesis was that dividing the stream of cash flows with duration from 10 up to 30 years would give us better result than this and maybe the best strategy as well.



Picture 19: *Result from simulation of immunization against parallel shift and steepening in the interest rate curve, 3 groups*

We can’t see any great differences between this last strategy and the corresponding strategy with two bonds. Possible reasons for that might be that the time period after ten years do not contribute to large changes in the interest rate, which we know from the study from 1997 we discussed earlier. So in this case it doesn’t matter if we add steepening to the immunization.

Thus, the result presents the strategy of hedging against both parallel shifts and steepening and divides the portfolio into groups that concerns the first nine years and the rest to be the best one.

More graphs are found in the appendix. The presented graphs are those pictures presented above in a larger scale and also, for the same strategies, the simulation result from different periods in time. The first six years, up to sixteen year and above. This is to see how the matching is proceeding during the thirty years of rebalancing.

8.5 Risk capital

In the table we have calculated the relative error at time 1. This is an expression of the need for risk capital at time 1 as a percentage of the assets at time 0. The need for risk capital may be positive as well as negative.

Following standards set up for the current Swedish traffic light system and the draft Solvency II directives a probability of 0,5 % of not meeting the obligations within 1 year is acceptable. The 99.5 % percentile of the distribution of the relative error at time 1 is therefore an expression for the necessary risk capital at time 0.

Hedging strategy	**Average error**	**99.5 % percentile**
1. Parallel shift (1-9, 10-18, 19-29)	-0,11 %	0,025 %
2. Parallel shift and steepening (1-9, 10-18, 19-29)	0,021 %	0,028 %
3. Parallel shift (1-9, 10-29)	-7,8 %	0,045 %
4. Parallel shift (1-15, 16-29)	-0,62 %	0,076 %
5. Parallel shift and steepening (1-15, 16-29)	0,180 %	0,107 %
6. Parallel shift and steepening (1-9, 10-29)	2,62 %	0,439 %

As it may be seen from the table the percentiles are dependent on the actual strategy but in all cases very low. This is because we have managed almost fully to hedge our portfolio.

The strategies are ranked as the strategy with the smallest risk capital amount necessary at time 0 in first following by strategies with a higher necessary risk capital. Compared to the pictures above, the total fault during a time of 30 years of rebalancing, we have a slight difference in the order. We have to remember that the result in the table is only referring to the rebalancing of the portfolio between time 0 and 1.

What seem to be the best strategy as a whole was in this sense the strategy which demands a higher risk capital in a one-year period. The strategy and the corresponding picture 18 also showed some high peaks which resulted in this relatively high risk capital. It is high compared to the other strategies. The first two strategies in the table is thus the strategies which require the smallest amount of risk capital and it is a chocking result that the strategy with only immunization against parallel shift is in the top. But the differences are very small and looking at the average error we get a better result when we immunize against both parallel shift and steepening.

9. Conclusions and discussion

The most important task for insurance undertakings is to meet their obligations to the policyholders. The company will go bankrupt even before they will get that far due to capital requirement; the companies have to stay solvent to be able to serve as an insurance undertaking. To set the reserve by calculating a best estimate insurance companies have to make sure that they are not underestimating the reserve and are taking into account the value of financial guarantees and contractual options included in insurance and reinsurance policies.

Embedded options such as the option to surrender a contract at a predefined interest rate or the option to convert a capital insurance to an annuity are both contractual option. This thesis describes a possible approach to value these options and the result it is quite similar to the approach with Thiele's differential equation that is presented by Thomas Moller and Mogens Steffensen.

Our assumption of individuals being rational, that they will use their right to surrender if it is beneficial to them, is not the main core of the valuation approach. That will only make it less complex in the calculation of estimates and the result of the option value stays the same. In reality, arbitrage possibilities are hard to find. If it is profitable to surrender and the insurance undertakings may not have the capital to meet these surrenders, the regulatory authority would probably set up some kind of protection to avoid arbitrage, costs that will make the surrender option less attractive.

One purpose of this thesis was the handling of interest rate risk. Either when insurance companies are valuing their reserve at market value or facing fluctuations in the yield curve when managing hedging strategies. The analysis of possible immunization strategies showed us that the most suitable strategy, in the sense of minimizing the margin of error when rebalancing the portfolio, depends on number of bonds and how we divide the portfolio into sub portfolios to handle the fluctuations in interest rates.

The best strategies among the chosen was to immunize against both parallel shifts and steepening, and also split your portfolio into two and consider the first nine years, where the most changes in the interest rate would appear, and then a portfolio to handle the last nineteen years of duration. Zero coupon bonds are conducive to being sensitive to fluctuations in interest rates, because there are no coupon payments to reduce the impact of interest rate changes. But there are no mayor fluctuations in the past nineteen year and that might be the explanation of this result.

When we use immunization to match assets and liabilities we are considering the restricted assets, as we would call it today. As the immunization is an instrument to reduce the impact from fluctuations in the interest rate structure, that will generate a mismatch in the present value and duration, it doesn't solve the mismatch in liquidity. It was mentioned earlier in this thesis that markets for zero coupon bonds are relatively illiquid. For this matter insurance or reinsurance undertakings have to hold other assets that are more liquid and even a buffer. As mentioned before, the secret is to have so much liberty that you are able to sell your assets at any time. The risk capital analysis resulted in a different way. Immunization against only parallel shift resulted in a smaller amount of risk capital but not by much compared to immunization against both parallel shift and steepening. Both strategies was divided into 3 cash flows and the latter resulted in a smaller average error when rebalancing. So in a risk capital point of view the hypothesis that the strategy of immunization against both parallel shifts and steepening ought to be the best one is true.

Finally, we want to mention that the key findings and conclusion of this thesis is concluded on the basis of this particular insurance policy and assumptions and might be different for other types of insurance policies.

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11. Appendix

11.1 Greek formulas

Delta:

$$\Delta_{call} = \exp[(b-r)T]N(d_1) \qquad , > 0$$
$$\Delta_{put} = \exp[(b-r)T][N(d_1)-1] \qquad , < 0$$

Gamma:

$$\Gamma_{call,\,put} = \frac{n(d_1)\exp[(b-r)T]}{F\sigma\sqrt{T}}$$

Vega:

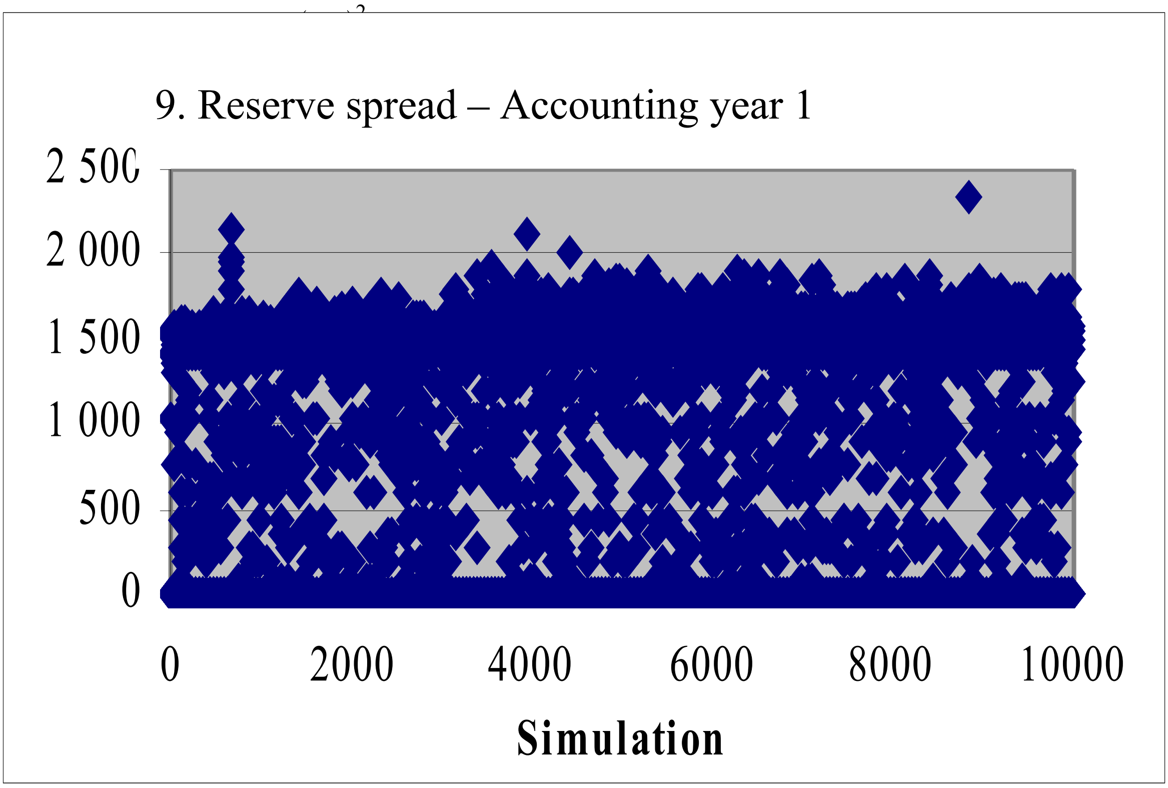
$$Vega_{call,\,put} = F\exp[(b-r)T]n(d_1)\sqrt{T} \qquad , > 0$$

Theta:

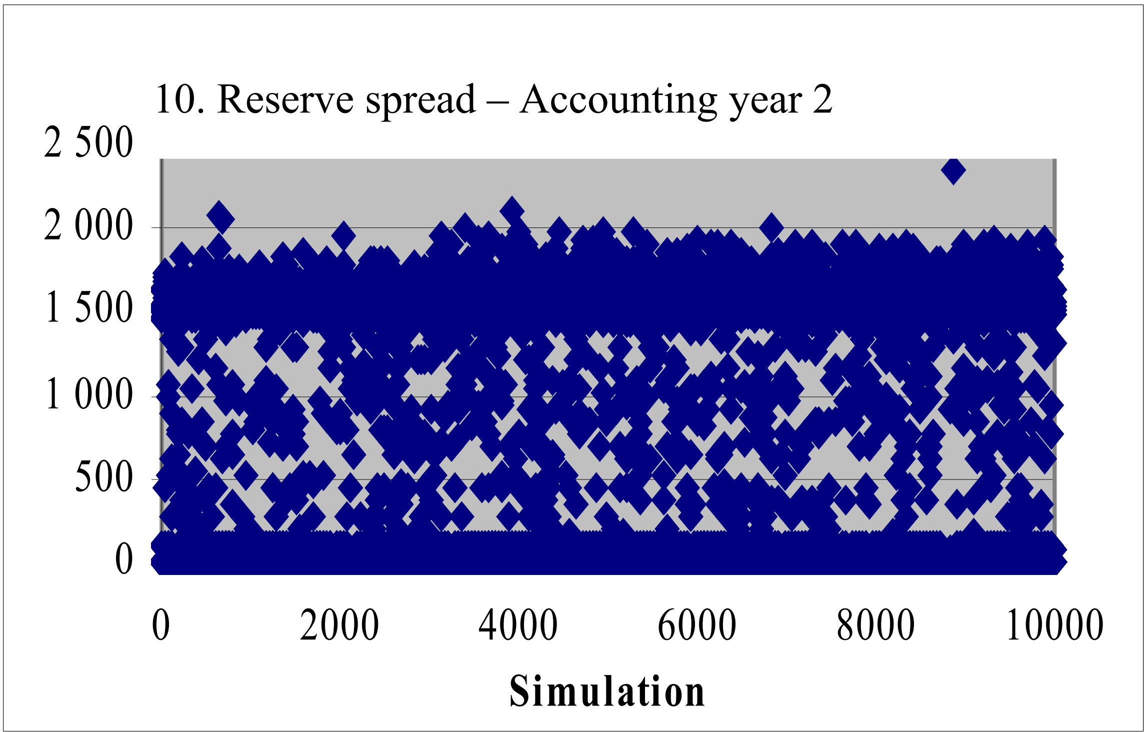
$$\Theta_{call} = -\frac{F\exp[(b-r)T]n(d_1)\sigma}{2\sqrt{T}} - (b-r)F\exp[(b-r)T]N(d_1) - rX\exp[-rT]N(d_2)$$
$$\Theta_{put} = -\frac{F\exp[(b-r)T]n(d_1)\sigma}{2\sqrt{T}} + (b-r)F\exp[(b-r)T]N(-d_1) + rX\exp[-rT]N(-d_2)$$

Rho:

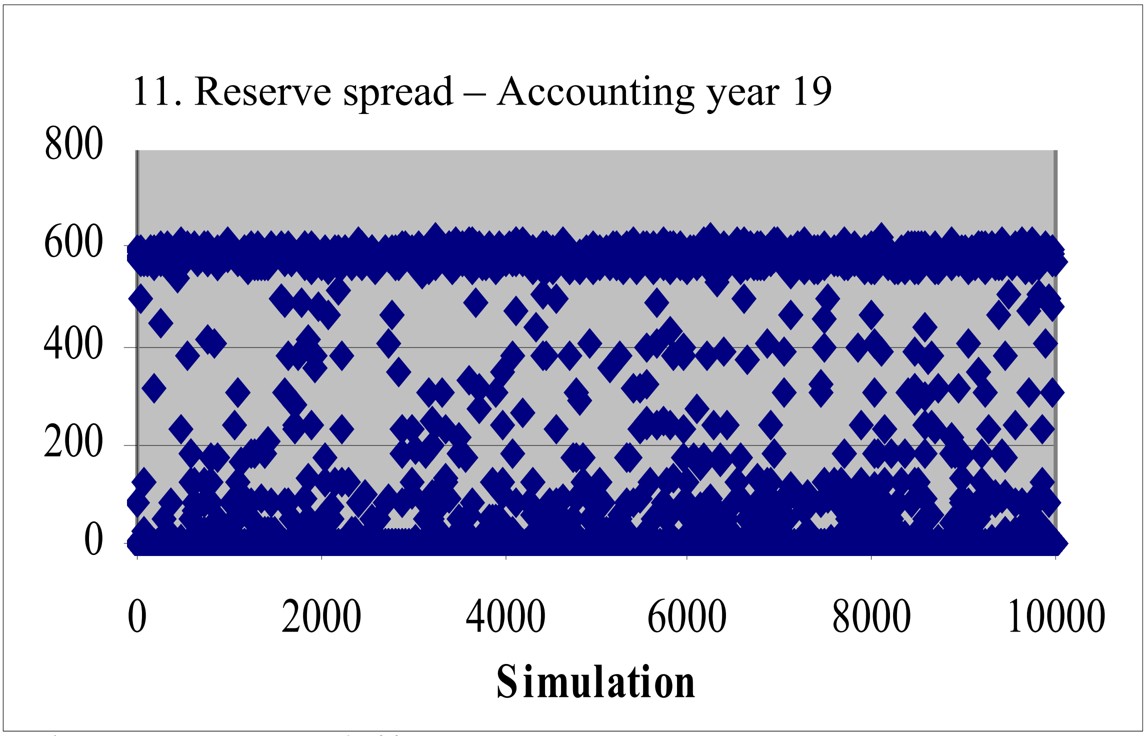
$$\rho_{call} = TX\exp[-rT]N(d_2) \qquad , > 0$$
$$\rho_{put} = -TX\exp[-rT]N(-d_2) \qquad , < 0$$



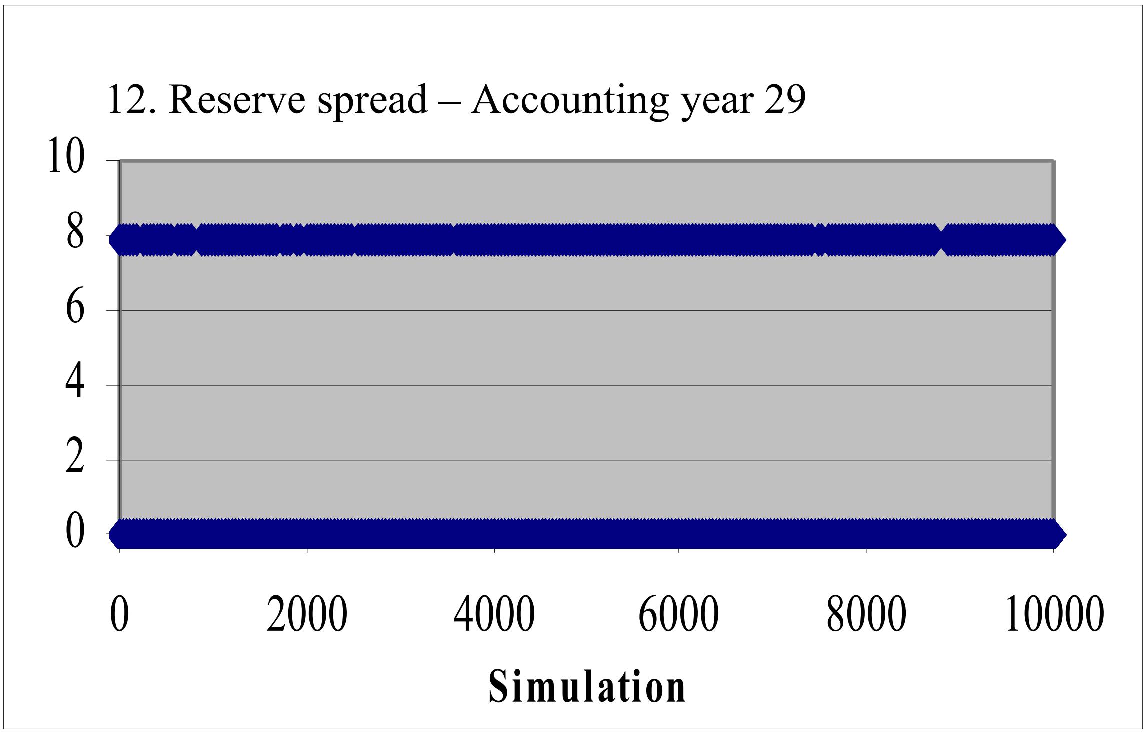
Picture 9: *Reserve spread of four accounting year 1*



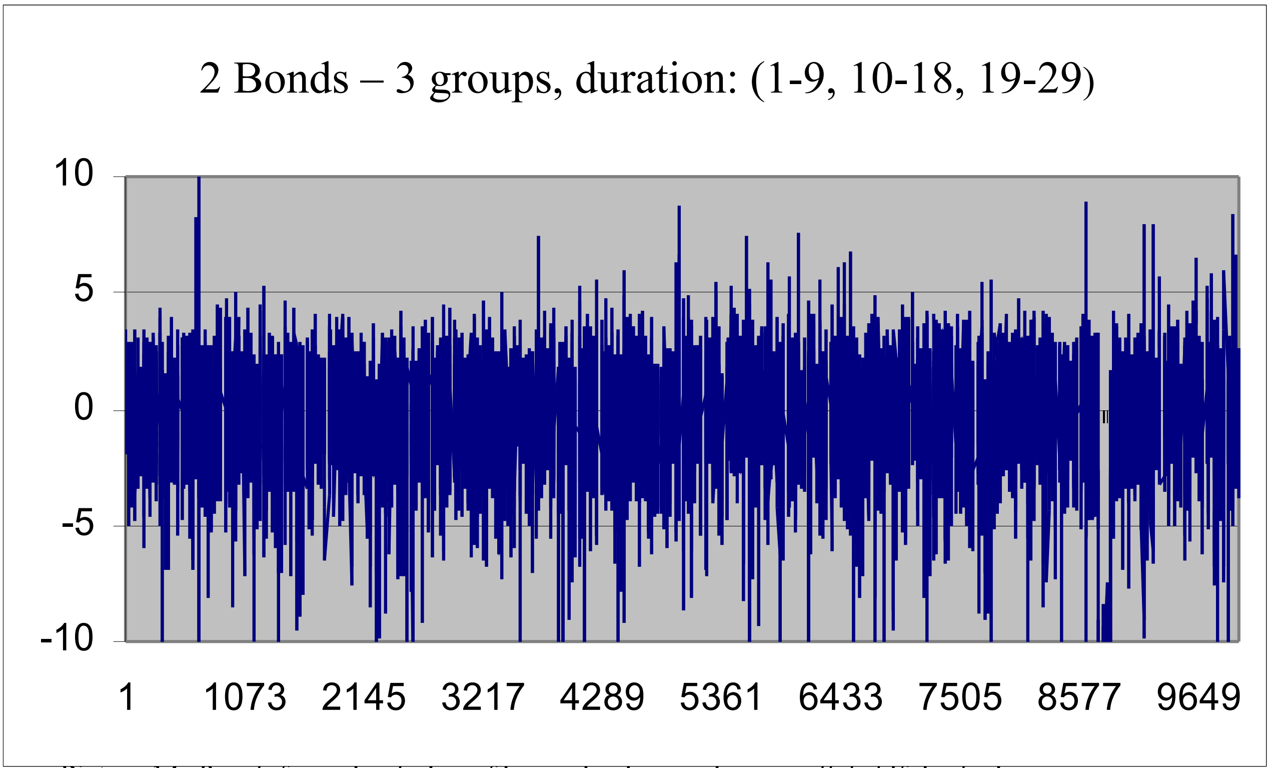
Picture 10: *Reserve spread of four accounting year 2*



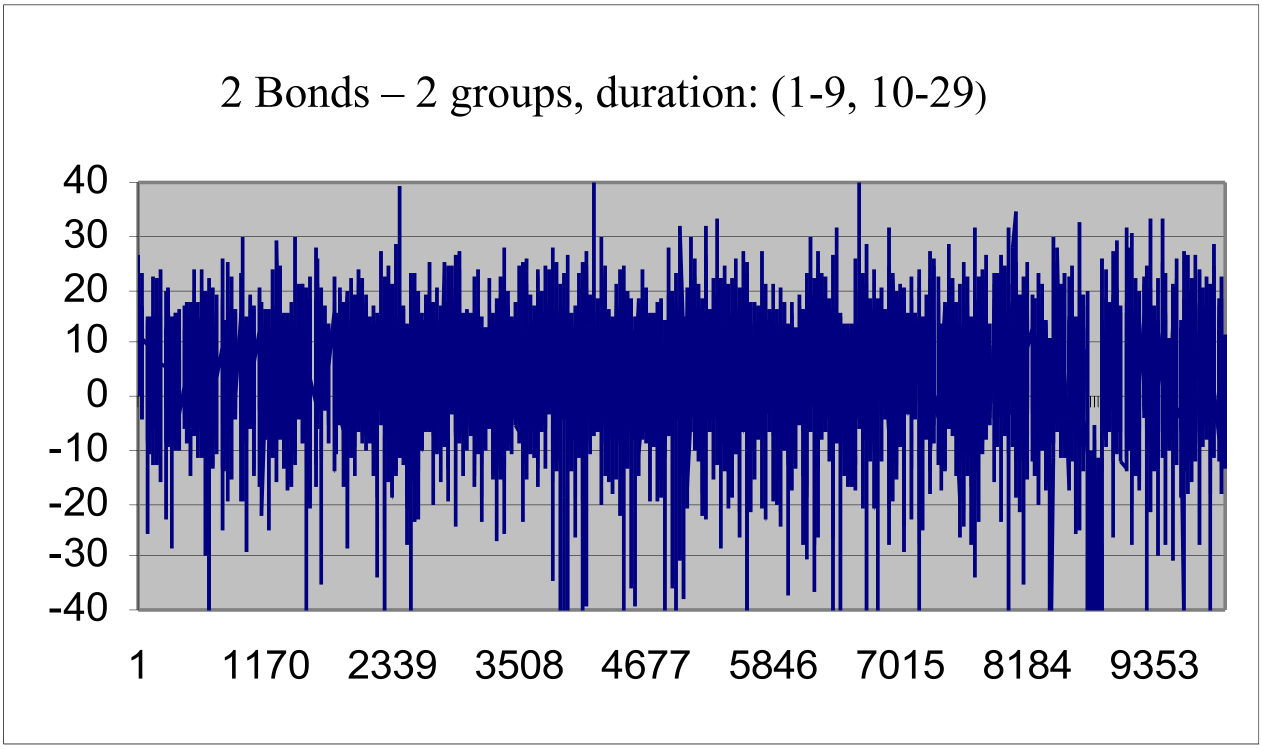
Picture 11: *Reserve spread of four accounting year 19*



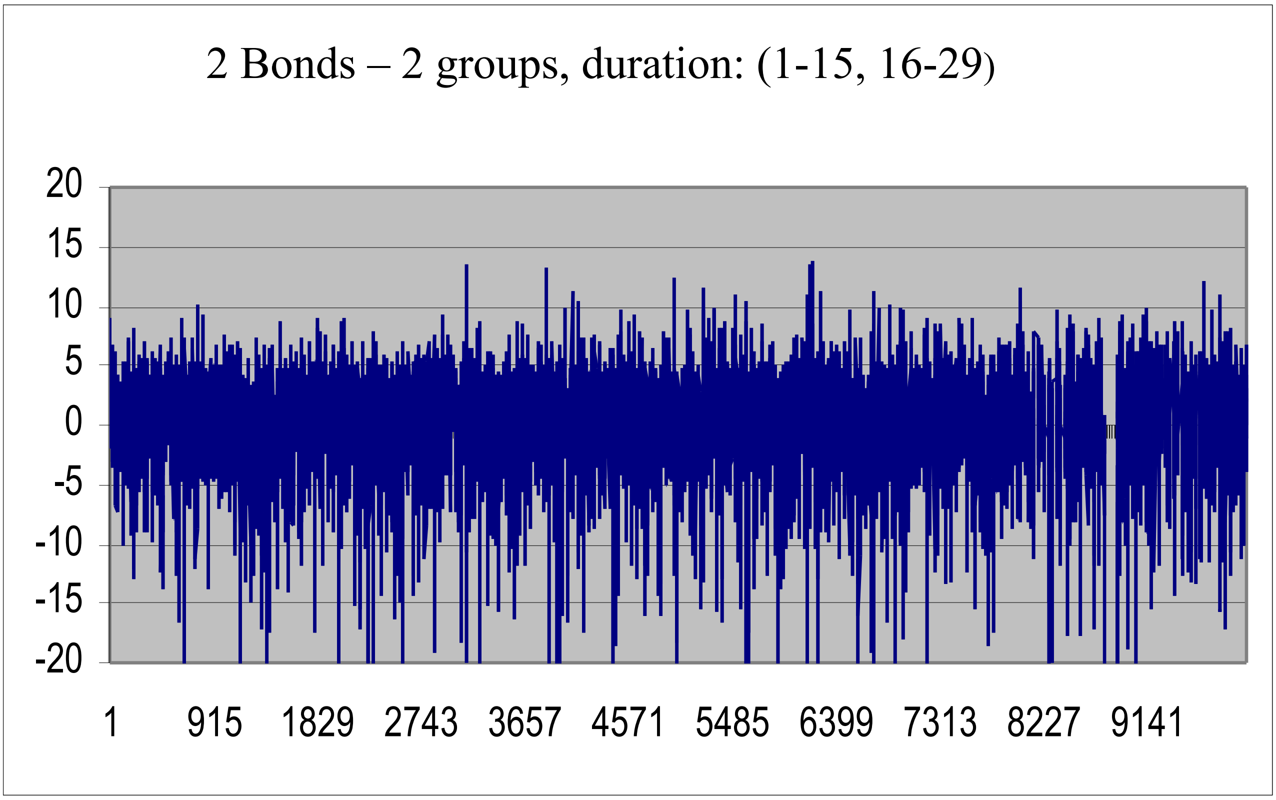
Picture 12: *Reserve spread of four accounting year 29*



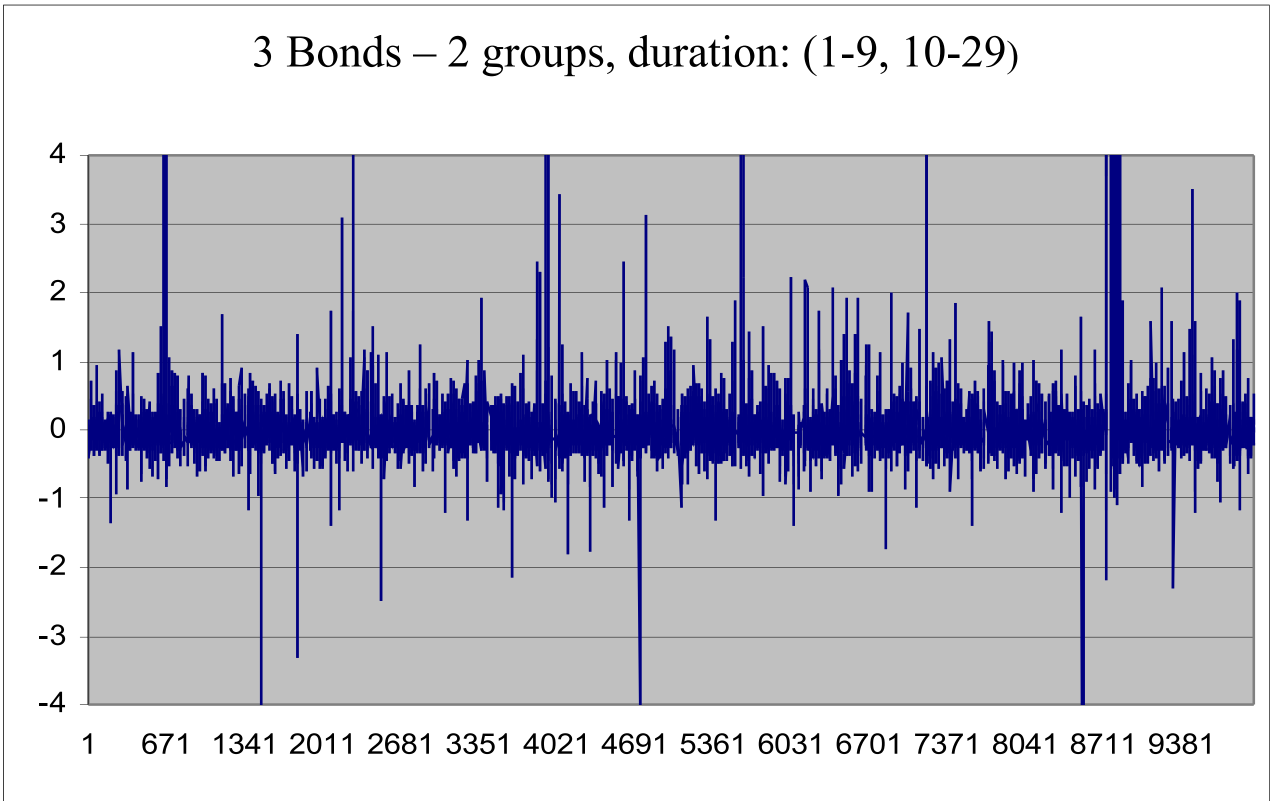
Picture 14: *Result from simulation of immunization against parallel shift in the interest rate curve*



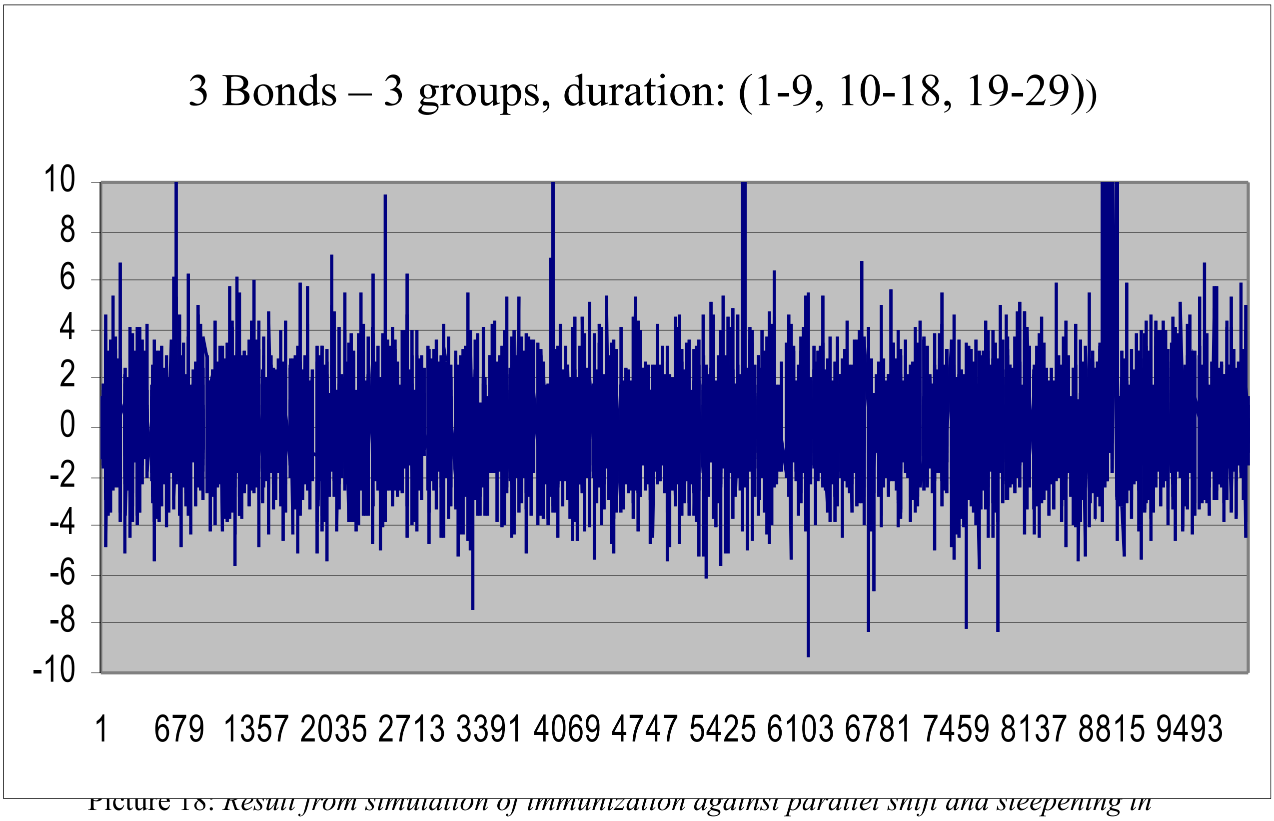
Picture 15: *Result from simulation of immunization against parallel shift in the interest rate curve*



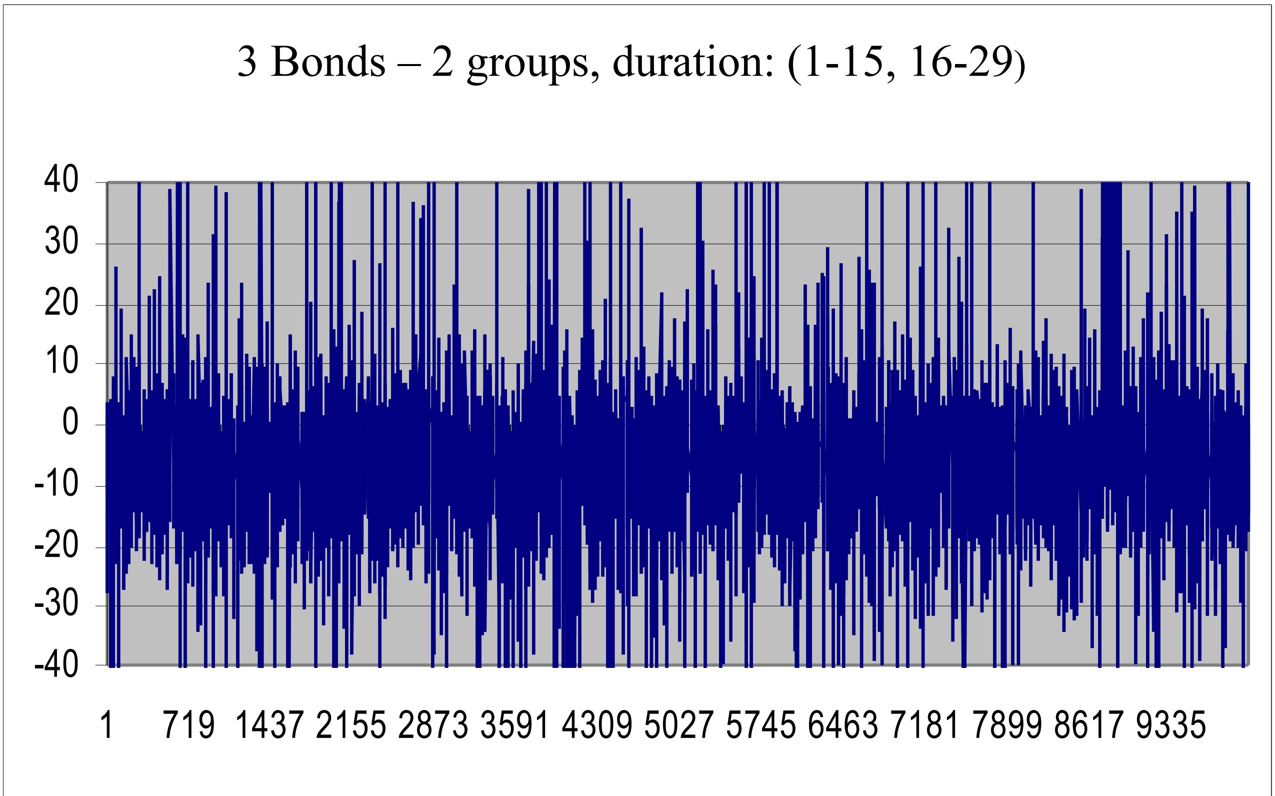
Picture 16: *Result from simulation of immunization against parallel shift in the interest rate curve*



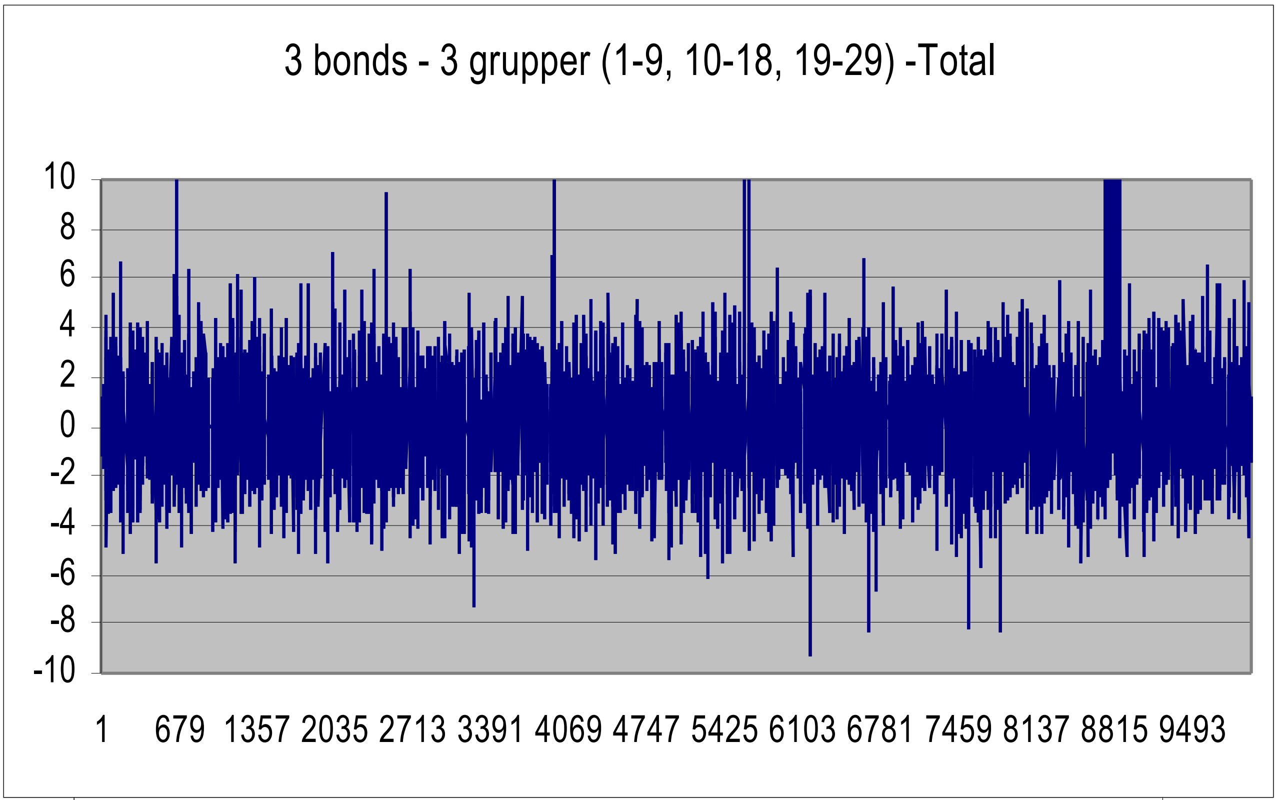
Picture 17: *Result from simulation of immunization against parallel shift and steepening in the interest rate curve*



Picture 18: *Result from simulation of immunization against parallel shift and steepening in the interest rate curve*



Picture 19: *Result from simulation of immunization against parallel shift and steepening in the interest rate curve, 3 groups*



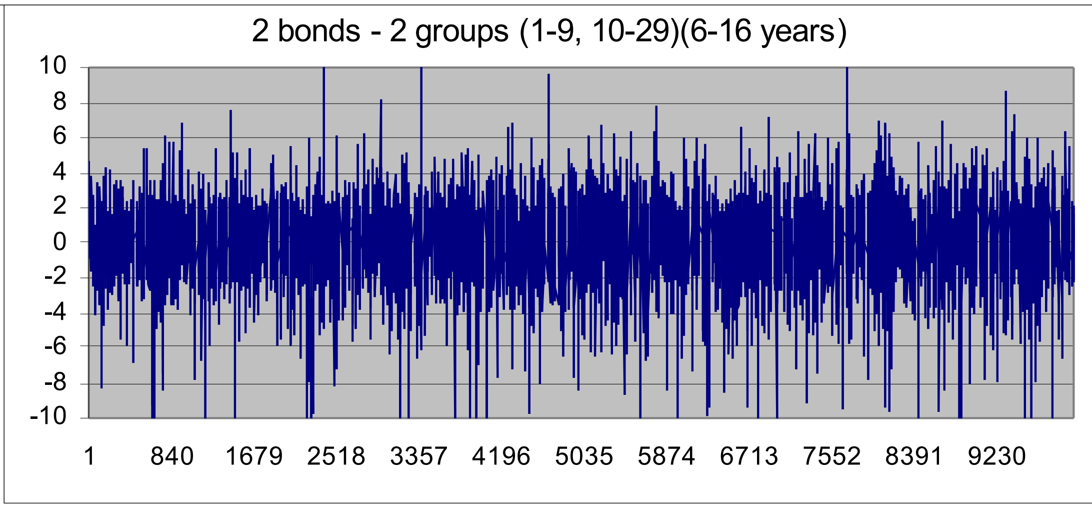


Figure 21: *Result from immunization against parallel shift between year 6 and 16*

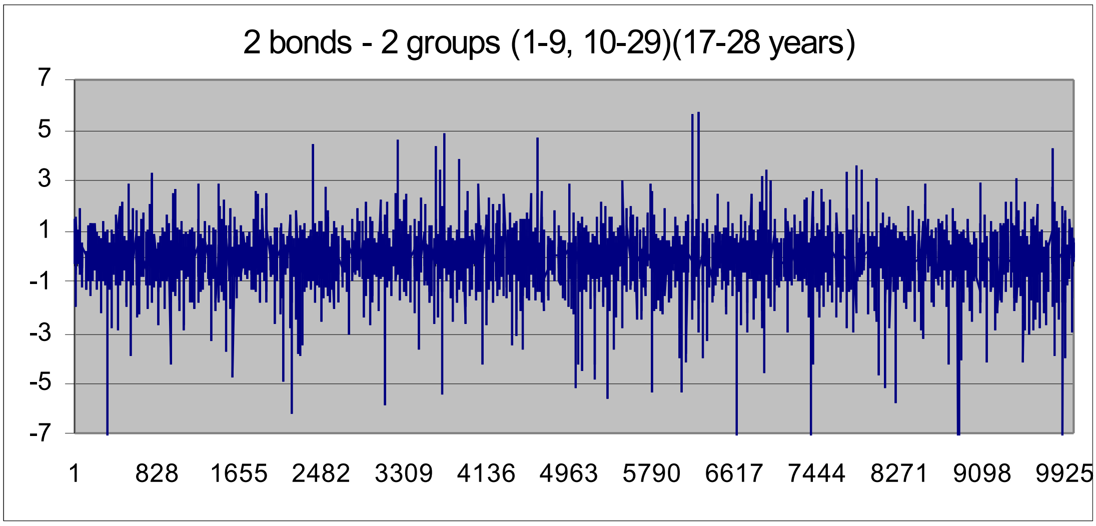


Figure 22: *Result from immunization against parallel shift between year 17 and 28*

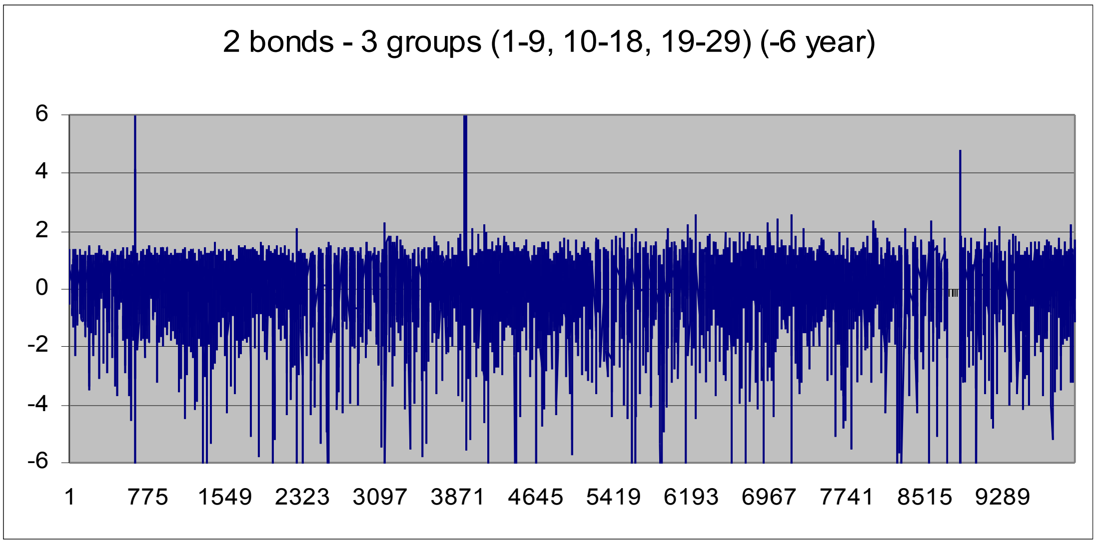


Figure 23: *Result from immunization against parallel shift up to balancing year 6*

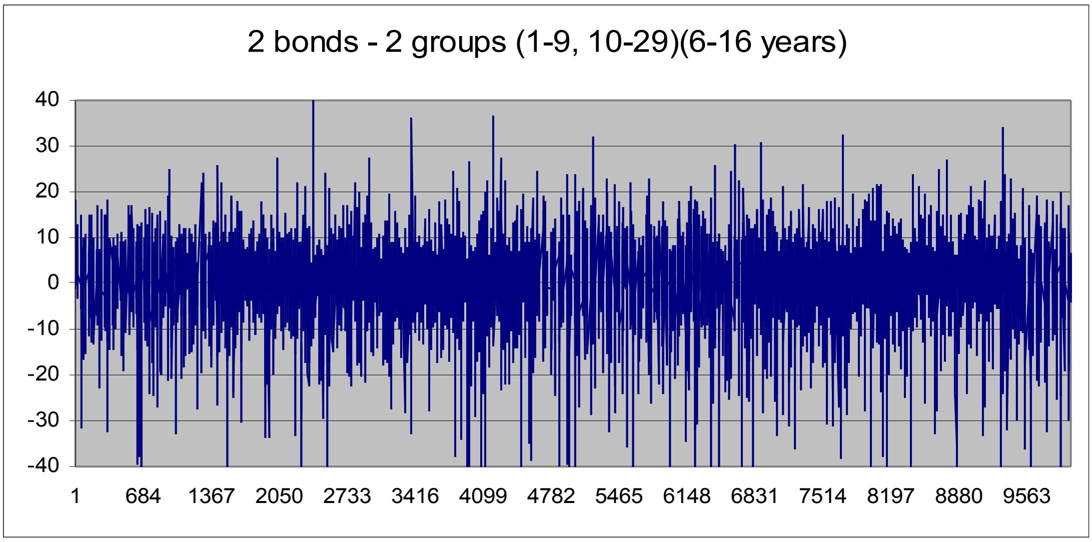


Figure 24: *Result from immunization against parallel shift between year 6 and 16*

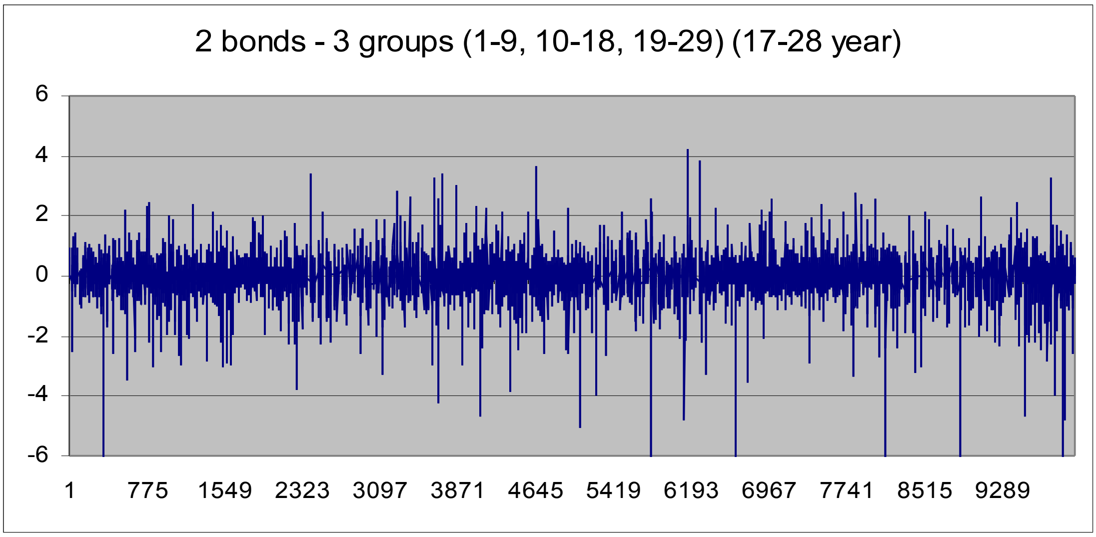


Figure 25: *Result from immunization against parallel shift between year 17 and 28*

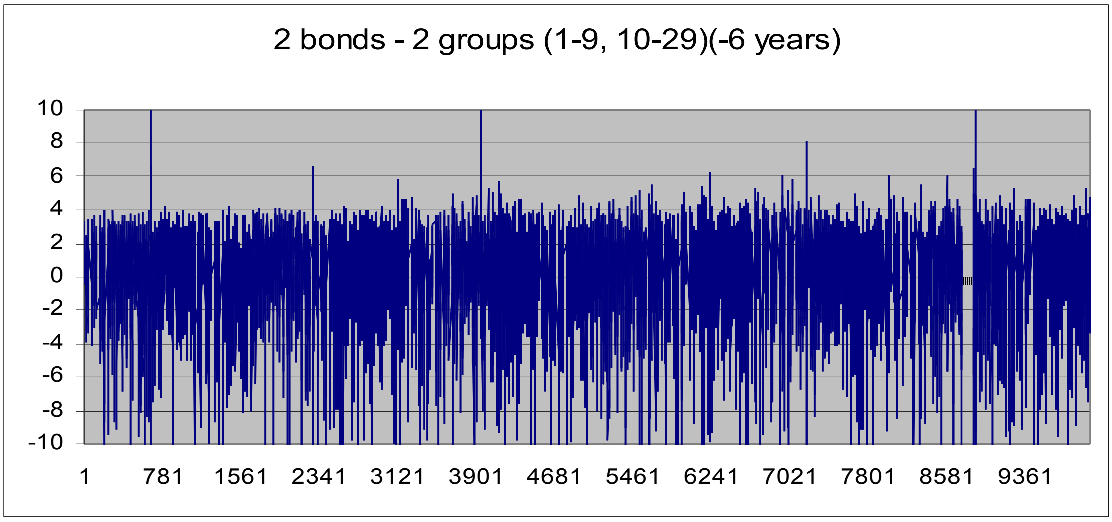


Figure 26: *Result from immunization against parallel shift up to balancing year 6*

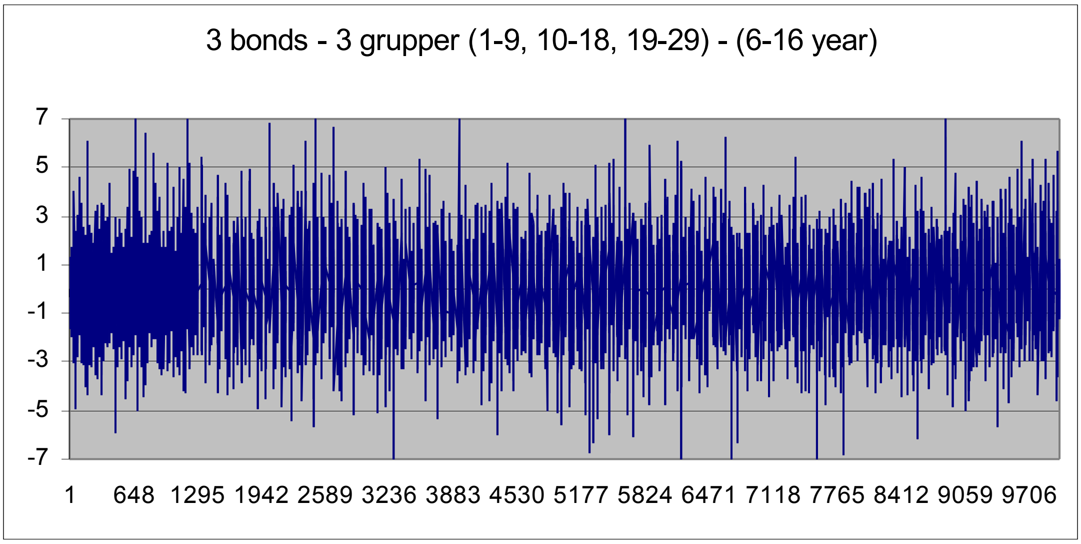


Figure 27: *Result from immunization against parallel shift between year 6 and 16*

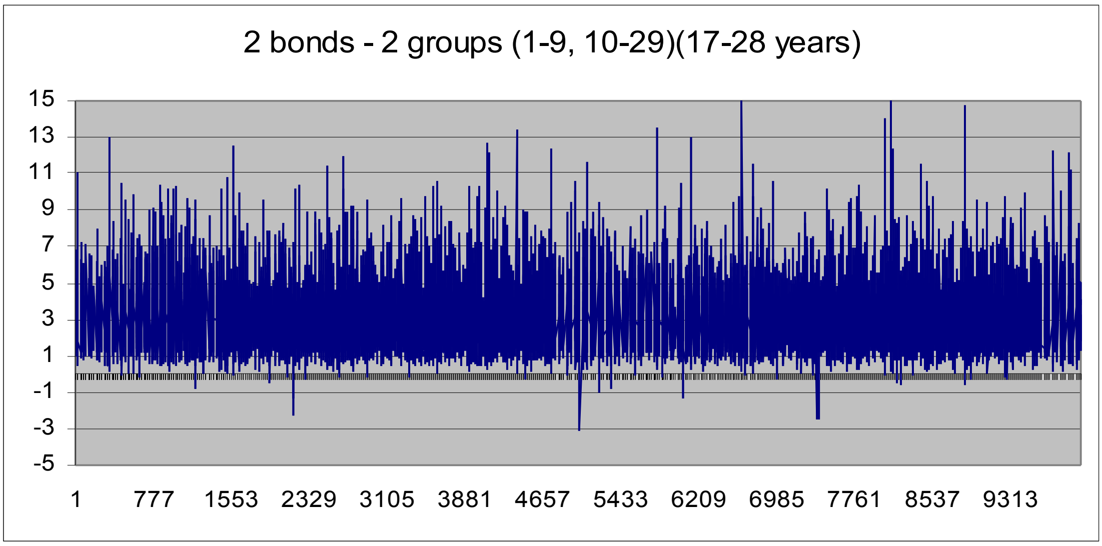


Figure 28: *Result from immunization against parallel shift between year 17 and 28*

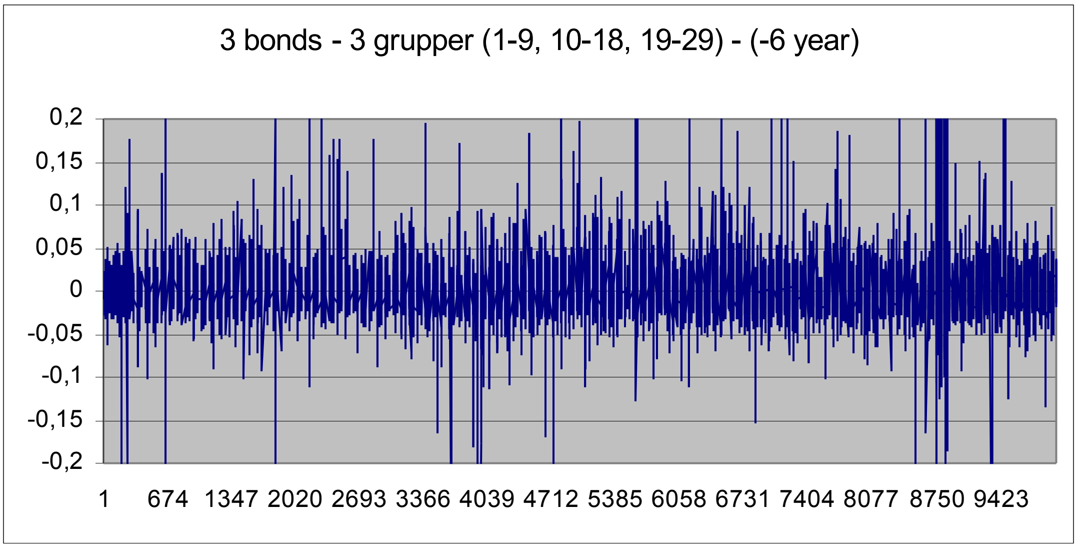


Figure 29: *Result from immunization against parallel shift and steepening up to balancing year 6*

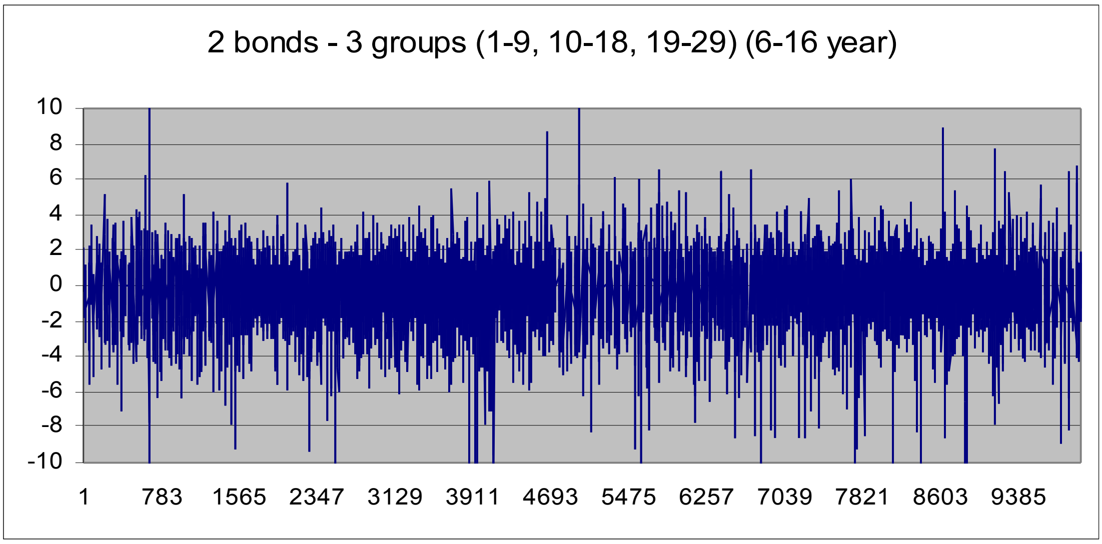


Figure 30: *Result from immunization against parallel shift and steepening between year 6 and 16*

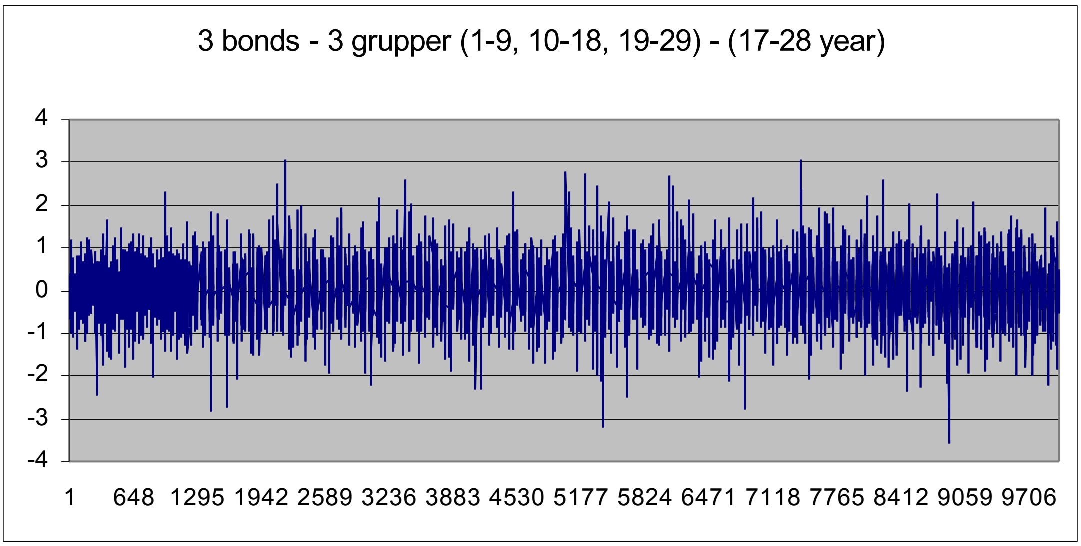


Figure 31: *Result from immunization against parallel shift and steepening between year 17 and 28*

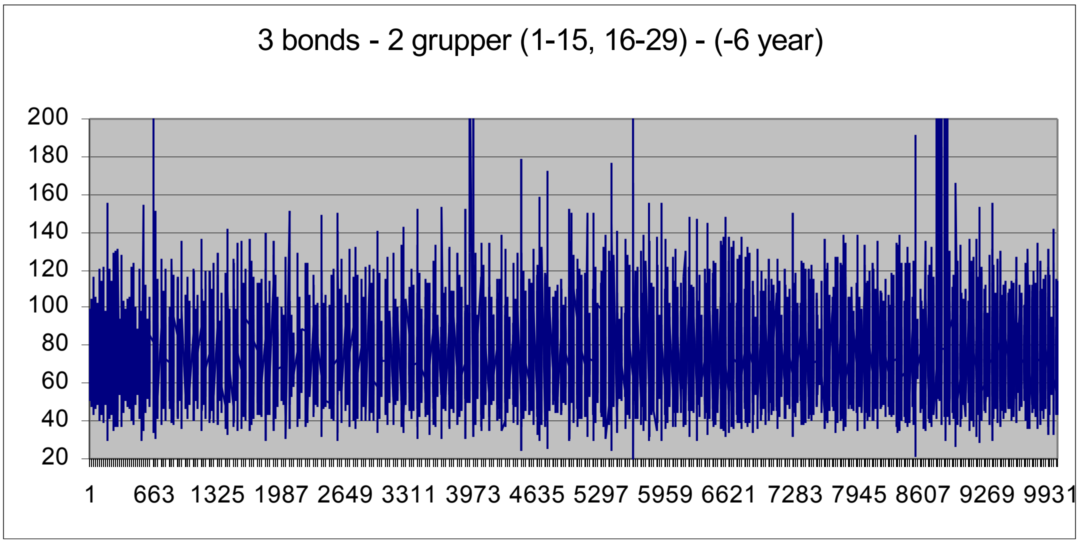


Figure 32: *Result from immunization against parallel shift and steepening up to balancing year 6*

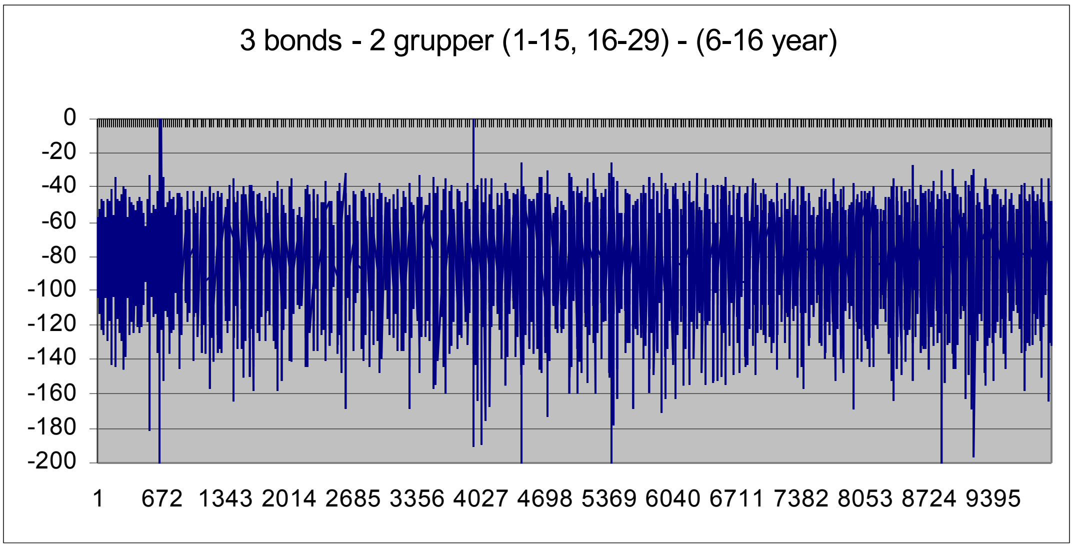


Figure 33: *Result from immunization against parallel shift and steepening between year 6 and 16*

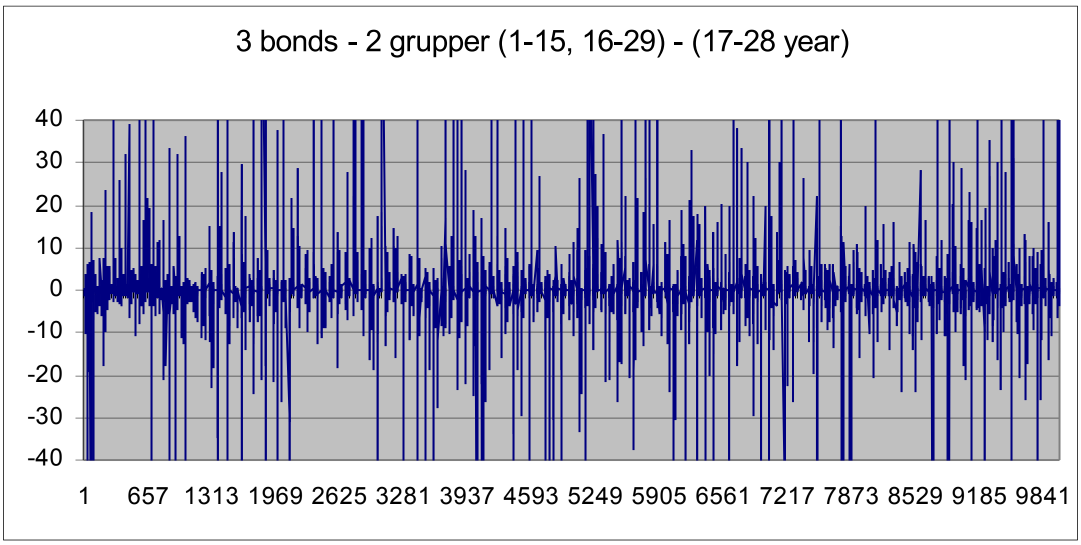


Figure 34: *Result from immunization against parallel shift and steepening between year 17 and 28*

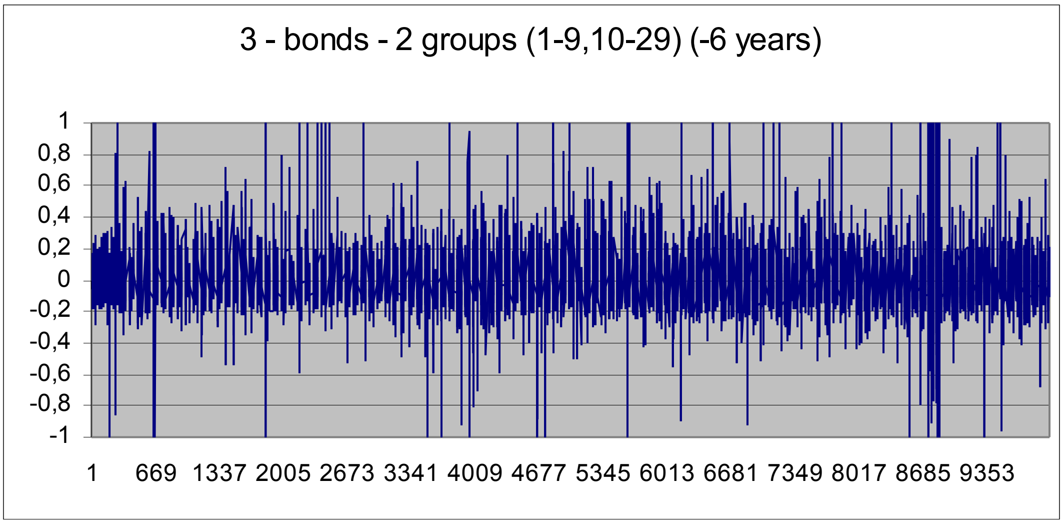


Figure 35: *Result from immunization against parallel shift and steepening up to balancing year 6*

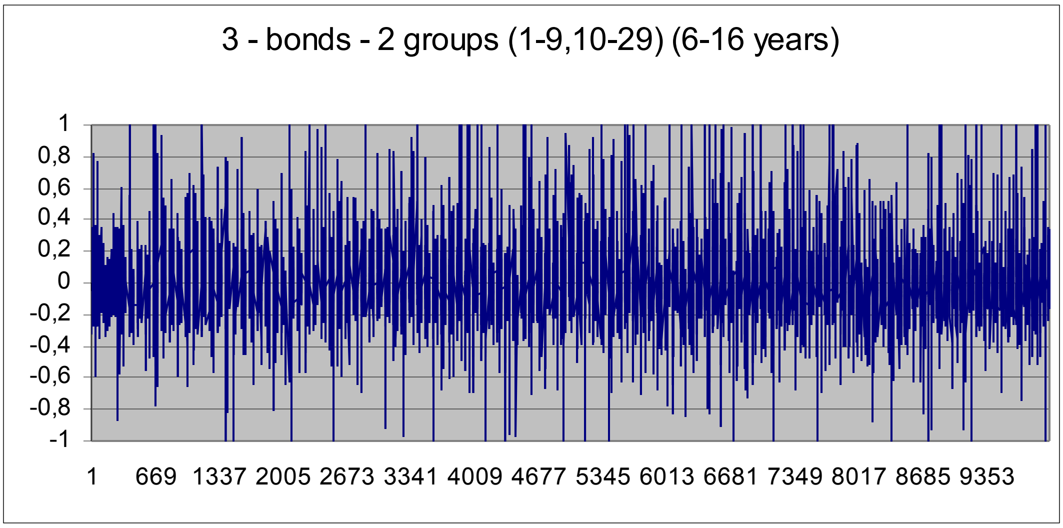


Figure 36: *Result from immunization against parallel shift and steepening between year 6 and 16*

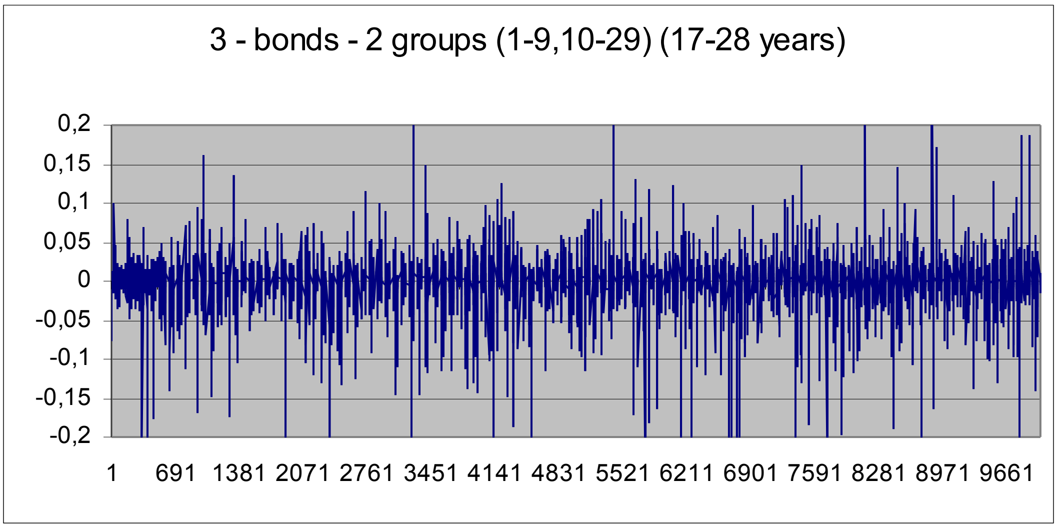


Figure 37: *Result from immunization against parallel shift and steepening between year 17 and 28*