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Variance Reduction
Three Approaches to Control Variates

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Abstract

In option price simulations, simulation-time is of great importance. Control variates is a variance reduction technique that can reduce simulation-time. Three approaches to the use of control variates in Monte Carlo option pricing are presented and evaluated. Employed methods include ordinary control variate implementation, a replicating delta hedge and re-simulation. Ordinary control variates uses a highly correlated random variable with known mean to reduce variance. The delta hedge tries to replicate the option and is constructed with an approximative delta formula, which is new to stock markets. The third method evaluated, called re-simulation, is a new method which use an earlier simulated option price as control variate. Applying an earlier option price as control variate results in a more generic method, since earlier simulated prices often exists. The three models are evaluated on Asian and Cliquet options, either in the standard Black and Scholes model or in Merton's jump diffusion model. Presented results show that the re-simulation method almost always yield a more efficient simulation procedure compared with the other methods. For some Cliquet options the simulation speed up over crude Monte Carlo is remarkable.

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Contents

1 Introduction

When pricing options and other financial derivatives, time and accuracy are of greatest essence. In real time trading, accurate prices and option greeks are needed updated according to real time prices. For some simple derivatives there exists closed analytical pricing formulas and these questions are not an object. Some payoff structures can, however, not be evaluated using closed analytical formulas. This due to the too complex payoffs structures of many exotic options or that the underlying is a basket of assets. In these cases prices are determined, for example through the widely used Monte Carlo method. Though the Monte Carlo method is easy to use, it is time consuming to simulate a price within reasonable bounds of accuracy for certain contracts. To reduce computational time, several variance reduction techniques have been proposed, among others antithetic variates, control variates and importance sampling. If the variance reduction achieved is not offset by the additional computational time, a more efficient simulation is achieved.

This thesis will focus on the method of control variates, one of the most popular and effective methods used. Three different approaches regarding control variates are presented. First, ordinary control variates are constructed for Asian and Cliquet options. Second, a delta hedge is applied as control variate for an Asian option in a jump diffusion model. For this purpose, a delta approximation formula for the Asian option in a Merton's jump diffusion model is derived. Approximation formulas are new to control variates in stock markets, however, it has been tested in libor markets by Jensen and Svenstrup [8]. Implementing these two methods is often a question of finding a reasonable set of control variates, which can be a problem. Moreover, there is even harder, if all possible, to find a general set of control variates that can contribute with significant variance reduction. This means that constructing efficient control variates for every contract clearly can be a burden, especially under real market conditions.

The third method presented, solves this problem. This new method makes it possible to use the same control variate scheme regardless which contract being priced. Since the control variate is constructed with an earlier simulated price, it is well suited in applications were the price is frequently updated. Results from this new technique turns out to be very good and for some contracts the decreased simulation-time is remarkable. A Cliquet option, for example, converges up to 500 times faster with re-simulation than for crude Monte Carlo.

This thesis will proceed as follows, the fundamental simulation details are first presented followed by the control variate theory. Implementation of the methods to some specific contracts are then described, highlighting some possible problems with different methods. The thesis sums up with an efficiency ratio comparison between the methods.

2 Preliminaries

In this section a quick recap of the Monte Carlo method is followed by how simulation of multiple correlated assets is carried out. A jump diffusion model, which include discontinuous jump's to the simulation process, is also presented.

2.1 The Monte Carlo Method

The Monte Carlo method is a numerical method used in various applications, for example used to determine the expected value of a random variable. The method builds upon the law of large numbers and thereby large samples of random numbers. The law of large numbers assure that a sample of independent, identically distributed (i.i.d.) random variables, converges to the sample mean as the sample size, n increases. This can be written

$$\bar{Y} = \frac{Y_1 + \cdots + Y_n}{n} \xrightarrow{p} \mathrm{E}[Y] \quad as \quad n \to \infty,$$

where Y_i are i.i.d. with finite mean. The proof and conditions can be found in Gut [6]. The standard deviation of this estimator is $n^{-1/2}\sigma^2$ and the convergence rate can thus be expressed $O(n^{-1/2}\sigma^2)$.

In option pricing, the price is expressed by the discounted expected value of the option payoff at maturity, under the risk neutral measure. The Monte Carlo method was introduced to option pricing to evaluate this expected value, which is a function of random variables. The procedure now is to evaluate a function of random variables $f(\boldsymbol{Y}_i)$ until the sample mean converges. Further the vector's of random variables $\boldsymbol{Y}_i$ are i.i.d. but the random variables within each vector can be correlated.

As a simple example, let $S(T)$, the asset price at time T, be a random variable with known distribution. Now let $\Phi(S(T)) = \max(S(T) - K, 0)$ denote the payoff of an European call option with strike price K. $\mathrm{E}^Q[\Phi(S(T))]$, the expectation under the risk-neutral measure Q, can now be determined by drawing independent trials of $S(T)$ and calculating $\Phi(S(T))$ until the sample mean converges,

$$\frac{1}{n}\sum_{i=1}^{n}\Phi\Big(S_i(T)\Big) \xrightarrow{p} \mathrm{E}^Q\Big[\max(S(T)-K,0)\Big] \quad as \quad n\to\infty.$$

A more comprehensive study on the Monte Carlo method can be found in, for example Glasserman [5].

2.2 Multiple Asset Dynamics

Contracts in this thesis will mainly include multiple underlying assets. Multiple assets are modeled with multidimensional Geometric Brownian Motion (GBM). GBM are defined by the following systems of Stochastic Differential Equations (SDE's)

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dX_i(t), \quad i=1,\dots,d, \tag{2.1}$$

where X_i denotes correlated standard Brownian Motion. The system $\boldsymbol{S} = (S_1,\dots,S_d)$ can now be defined as $\mathrm{GBM}(\boldsymbol{\mu},\boldsymbol{\Sigma})$ where $\boldsymbol{\mu} = (\mu_1,\dots,\mu_d)$ is the vector of drift parameters and $\boldsymbol{\Sigma}$ is the matrix of covariance parameters. Each element in $\boldsymbol{\Sigma}$ is given by $\Sigma_{ij} = \sigma_i\sigma_j\rho_{ij}$ where σ_i is the volatility parameter of S_i and ρ_{ij} is the correlation between $X_i(t)$ and $X_j(t)$. Further, since a Brownian Motion $\mathrm{BM}(0,\boldsymbol{\Sigma})$ can be expressed by $\boldsymbol{AW}(t)$ where $\boldsymbol{W}$ is a standard $\mathrm{BM}(0,\boldsymbol{I})$ and $\boldsymbol{A}$, preferably the Cholesky factorization of $\boldsymbol{\Sigma}$, satisfies $\boldsymbol{AA'} = \boldsymbol{\Sigma}$, equation (2.1) is equal to

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sum_{j=1}^{d} A_{ij} dW_j(t), \quad i=1,\dots,d.$$

From this representation the solution to the SDE is given by

$$S_i(t) = S_i(t)\exp\Big\{(\mu_i - \frac{1}{2}\sigma_i^2)dt + \sum_{j=1}^{d} A_{ij}W_j(t)\Big\}. \tag{2.2}$$

To simulate correlated GBM's now evolves to be a fairly simple task. At times $t_0 < t_1 < \cdots < t_n$, the vaule of asset i, $i = 1, \ldots, d$, is given by

$$S_i(t_{k+1}) = S_i(t_k) \exp\Big\{(\mu_i - \frac{1}{2}\sigma_i^2)(t_{k+1} - t_k) + \sqrt{t_{k+1} - t_k} \sum_{j=1}^{d} A_{ij} Z_{k+1,j}\Big\},$$

where $Z_{k+1,j}$ are i.i.d. $N(0,1)$ for $k = 0, 1 \ldots, n-1$ and $j = 1, \ldots, d$.

2.3 Jump Diffusion Dynamics

Asset prices do not usually posses such smooth price paths that results from simulating in the framework in section 2.2. A more realistic model would capture the abrupt jumps that occur in price processes upon arrival of news. Such a model is based on the model above and a counting process delivering jumps with a certain intensity. The SDE can be specified by

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dW(t) + dJ(t), \tag{2.3}$$

where $S(t-)$ is the asset value just before a potential jump at time t and J is a process independent of W. The process J has piecewise constant sample paths and is given by

$$J(t) = \sum_{j=1}^{N(t)} (Y_j - 1),$$

where $N(t)$ is a counting process and $Y_1, Y_2, \ldots$ are random variables. In Merton's [14] framework $N(t)$ is a Poisson process with intensity λ and the jumps, Y_j independent of $N(t)$, are log-normally distributed $Y_j \sim LN(a, b^2)$. J is in this case called a *compound Poisson process*. The solution to (2.3) is now given by

$$S(t) = S(0) \exp\Big\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\Big\} \prod_{j=1}^{N(t)} Y_j.$$

For $e^{-rt}S(t)$ to now fulfill the martingale property under the risk-neutral measure Q, where r denotes the risk free interest rate, the drift has to be adjusted. The drift is adjustment by the expected value of the jump diffusion process. For Z i.i.d. $N(a, b^2)$,

$$\begin{aligned} \mathrm{E}\Big[e^{\sum_{j=1}^{N(t)} Z_j}\Big] &= \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \mathrm{E}\Big[e^{\sum_{j=1}^{N(t)} Z_j} \Big| N(t) = i\ \Big] \\ &= \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \phi(1)^i = e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(\lambda t \phi(1))^i}{i!} = e^{\lambda t(\phi(1)-1)}, \end{aligned} \tag{2.4}$$

where

$$\phi(\theta) = \mathrm{E}\Big[e^{\theta Z}\Big] = e^{a\theta + \frac{1}{2}b^2\theta^2}, \tag{2.5}$$

denotes the *moment generating function* for Z.

The adjusted drift that makes $e^{-rt}S(t)$ a martingale, under the risk-neutral measure Q, can now be expressed as $\mu = r - \lambda(\phi(1) - 1)$. A closed analytical formula exists for European options in this model based on the properties above. An analytical solution is possible because the distribution of $S(t)$ conditional of the number of jumps is known and log-normal. This is called the Merton option pricing formula and has the solution

$$e^{-rT}\mathrm{E}^Q\Big[\max(S(T) - K, 0)\Big] = \sum_{n=0}^{\infty} e^{-\lambda' T} \frac{(\lambda' T)^n}{n!} BLS(S(0), K, r_n, T, \sigma_n), \tag{2.6}$$

where $\lambda' = \lambda\phi(1)$, $\sigma_n^2 = \sigma^2 + b^2 n/T$, $r_n = r - \lambda(\phi(1) - 1) + n \ln \phi(1)$ and BLS denotes the standard Black and Scholes formula; see Glasserman [5].

3 Variance Reduction Techniques

General for all variance reduction techniques are to increase accuracy in the estimated variable by a decreased sample standard deviation, instead of larger samples. This is far more effective, since a standard deviation reduction by 10 times is equal to increase the number of simulations by 100 times.

3.1 Control Variates

The control variate technique is popular because of a effective variance reduction in a simple theoretical framework. The method, first introduced to option pricing by Boyle [1], takes advantage of random variables with known expected value and positively correlated with with the variable under consideration. Let Y be a random variable whose mean is to be determined through simulation and X a random variable with known mean μ_X. Now, for each trial the outcome of X_i is calculated along with the output of Y_i. Further suppose, that the pairs $(X_i, Y_i),\ i = 1, \dots, n$ are i.i.d., the definition of the control variate estimator $\bar{Y}_{CV}$ of $\mathrm{E}[Y]$ is then

$$\bar{Y}_{CV} = \bar{Y} - \bar{X} + \mu_X = \frac{1}{n}\sum_{i=1}^{n}(Y_i - X_i + \mu_X). \tag{3.1}$$

Lemma 3.1. *The control variate estimator* (3.1)*, is unbiased and consistent.*

Proof. The expected value of (3.1) yields the unbiasedness

$$\mathrm{E}[\bar{Y}_{CV}] = \mathrm{E}[\bar{Y}] - \mathrm{E}[\bar{X}] + \mu_X = \mathrm{E}\Big[\frac{1}{n}\sum_{i=1}^{n} Y_i\Big] = \mathrm{E}[Y]$$

and the following limit guarantees consistency

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} Y_{CV}(i) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(Y_i - X_i + \mu_X) = \mathrm{E}[Y - X + \mu_X] \stackrel{p}{=} \mathrm{E}[Y],$$

see Glasserman [5]. □

The resulting variance for the control variate estimator is

$$\mathrm{Var}(\bar{Y}_{CV}) = \frac{1}{n}\mathrm{Var}(Y) = \frac{1}{n}\mathrm{Var}(Y - X + \mu_X) = \frac{1}{n}(\sigma_Y^2 + \sigma_X^2 - 2\rho_{XY}\sigma_X\sigma_Y),$$

indicating that the control variate estimator $\bar{Y}_{CV}$, will have lower variance than $\bar{Y}$, if $\sigma_X^2 < 2\rho_{XY}\sigma_X\sigma_Y$.

To fully take advantage of the control variate, a parameter β is introduced and optimized to minimize the variance of $\bar{Y}_{CV}$. The parameterized control variate estimator is defined by

$$\bar{Y}_{CV}(\beta) = \bar{Y} - \beta(\bar{X} - \mu_X),$$

with resulting variance

$$\mathrm{Var}\Big(\bar{Y}_{CV}(\beta)\Big) = \frac{1}{n}(\sigma_Y^2 + \beta^2\sigma_X^2 - 2\beta\rho_{XY}\sigma_X\sigma_Y). \tag{3.2}$$

Minimizing the variance with respect to β yields

$$\beta^* = \frac{\sigma_Y}{\sigma_X}\rho_{XY} = \frac{Cov(X, Y)}{Var(X)}. \tag{3.3}$$

The variance reduction is as mentioned highly dependent of the correlation between the estimated variable and the control variate. Inserting (3.3) in (3.2) yields the minimum variance as a expression of ρ_{XY}, more precisely

$$\mathrm{Var}\Big(\bar{Y}_{CV}(\beta^*)\Big) = (1 - \rho_{XY}^2)\frac{\sigma_Y^2}{n}. \tag{3.4}$$

The importance of high correlation with the control variate for effective variance reduction can in (3.4) be seen with clarity. Another interesting property to notice when the β parameter is introduced, is that the correlation equally well can be negative. This is an effect of using the parameterized version, without the optimal β^* the influence of a negatively correlated control variate would increase the variance.

A problem when minimizing the variance with respect to β is that in practice $Cov(X, Y)$ never is known and has to estimated. The sample estimators

$$
\begin{aligned}
S_{XX} &= \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2, \\
S_{XY} &= \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y}),
\end{aligned}
$$

gives the β^* estimator as

$$
\hat{\beta}^* = S_{XY}S_{XX}^{-1}.
$$

The strong law of large numbers assure that $\hat{\beta}^*$ converges to β^* with probability 1. Estimating β^* can be done by running the simulation procedure a couple of pilot runs, a small fraction of the total number of simulations, then estimate β from those. Another alternative is to make use of all of the simulation runs to estimate β. From the latter method a bias in the estimation μ_Y arises since $\hat{\beta}^*$, X and Y are dependent. The bias results in some implications, first the variance can not be calculated directly by (3.4) due to the dependence. Second, a confidence intervals based on the t-distribution can not directly be used since

$$
\frac{\bar{Y}_{CV}(\hat{\beta}^*)-\mu_Y}{\sigma(\bar{Y}_{CV}(\hat{\beta}^*))},
$$

is not t-distributed. Lavenberger [11] proves under an assumption that (X, Y) have a multivariate normal distribution that $\bar{Y}(\hat{\beta}^*)$ is an unbiased estimator of $\mathrm{E}[Y]$. Under the normal assumption the variance of the control variate estimator is

$$
\mathrm{Var}\Big(\bar{Y}_{CV}(\hat{\beta}^*)\Big) = \frac{n-2}{n-3}(1-\rho_{XY}^2)\frac{\sigma_Y^2}{n},
$$

which can be used to derive a valid confidence interval. As the sample size get large though, an asymptotically valid variance and confidence interval for μ_Y is obtained since $\hat{\beta}^*$ converges to the true value. In option pricing the control variate technique relies on the asymptotic results since option payoff functions rarely can be assumed to follow a normal distribution.

To put the ideas into action, a simple control variate is introduced to the simulation of an European call option. A simple control variate will be the underlying asset itself. This is a generic control variate, available to all contracts, but not always that efficient. The payoff function of a European call option is $\Phi(S(T)) = \max(S(T) - K, 0)$ resulting in the option price $C(S(0)) = e^{-rT}\mathrm{E}^{Q}[\Phi(S(T))]$, where $S(0)$ is the asset price at time 0. Further the underlying asset has known expected value $\mathrm{E}^{Q}[S(T)] = S(0)e^{rT}$, which yields the estimation setup

$$\bar{C}(S(0)) = \frac{1}{n}\sum_{i=1}^{n}\left(\Phi(S_i(T)) - \hat{\beta}^{*}(S_i(T) - S(0)e^{rT})\right).$$

$\bar{C}(S(0))$ will now be a unbiased an consistent estimator of $C(S(0))$ with lower variance than if $C(S(0))$ was determined by crude Monte Carlo.

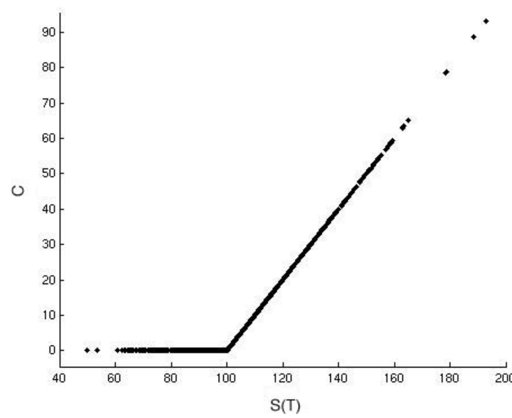

Figure 1: Correlation between underlying asset and option price.

Figure 1 illustrates the correlation between $C(S(0))$ and $S(T)$ for an at the money call option. The correlation in this example can be seen to be rather high, resulting in a $\beta = 0.67$ and decreases standard deviation almost 3 times. Correlation this high can, however, not be expected between more complex payoff functions and the underlying asset.

3.2 Multiple Control Variates

Often several possible control variates can be put together into one control variate, however, that is not always possible. The existents of several different control variates makes the question of multiple controls interesting. The previous results are easily extended to the case of multiple control variates. Letting $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ be a vector of control variates with known expected values corresponding to $\boldsymbol{\mu}_Z = (\mu_1, \dots, \mu_d)$ then

$$\bar{Y}_{CV}(\boldsymbol{\beta}) = \bar{Y} - \boldsymbol{\beta}'(\boldsymbol{Z} - \boldsymbol{\mu}_Z),$$

is an unbiased and consistent estimator of μ_Y. The proof is just straight forward from the one dimensional case; see lemma 3.1. The optimal parameter values in the vector $\boldsymbol{\beta}$ are now given by

$$\boldsymbol{\beta^*} = \boldsymbol{\Sigma}_{ZZ}^{-1}\boldsymbol{\Sigma}_{YZ},$$

where $\boldsymbol{\Sigma}_{ZZ}$ and $\boldsymbol{\Sigma}_{YZ}$ corresponds to the covariance matrix of $\boldsymbol{Z}$ and the vector of covariances between $(Y, \boldsymbol{Z})$ respectively. Likewise the estimation of $\boldsymbol{\beta^*}$ is given by

$$\hat{\boldsymbol{\beta}}^* = \boldsymbol{S}_{ZZ}^{-1}\boldsymbol{S}_{YZ},$$

where

$$\begin{aligned} S_{Z^{(j)}Z^{(k)}} &= \frac{1}{n-1}\sum_{i=1}^{n}\Big(Z_i^{(j)} - \bar{Z}^{(j)}\Big)\Big(Z_i^{(k)} - \bar{Z}^{(k)}\Big), \qquad j,k = 1,\dots,d \\ S_{YZ^{(j)}} &= \frac{1}{n-1}\sum_{i=1}^{n}\Big(Y_i - \bar{Y}\Big)\Big(Z_i^{(j)} - \bar{Z}^{(j)}\Big), \qquad j = 1,\dots,d, \end{aligned}$$

are the sample counterpart to $\boldsymbol{\Sigma}_{ZZ}$ and $\boldsymbol{\Sigma}_{YZ}$. Further, with the assumption that $(Y, \boldsymbol{Z})$ is multivariate normal, Lavenberger [11] derives the variance with $\boldsymbol{\beta}$ estimated from the samples to be

$$\text{Var}\Big(\bar{Y}_{cv}(\hat{\boldsymbol{\beta}}^*)\Big) = \frac{n-2}{n-d-2}(1 - R^2)\text{Var}(\bar{Y}),$$

where

$$R^2 = \frac{\boldsymbol{\Sigma}'_{YZ}\boldsymbol{\Sigma}^{-1}_{ZZ}\boldsymbol{\Sigma}_{YZ}}{\sigma^2_Y}.$$

R^2 is the squared multiple correlation coefficient, a measure of the variance in $\bar{Y}$ explained by $\boldsymbol{Z}$ as in regression analysis.

3.3 A Hedge as Control Variate

If a lack of suitable control variates occur, one idea is to replicate the contract with a hedge. Clewlow and Carverhill [3] derives a technique for control variates with hedges based on the various option greeks. The most significant variance reduction is obtained by using the delta hedge, which can be complemented by the gamma hedge for highly nonlinear option prices. The delta hedge control variate is formulated in [3] as follows

$$\Delta_{CV} = \sum_{i=1}^{N-1} \frac{\partial C(S(t_i))}{\partial S(t_i)} \Big(\Delta S(t_{i+1}) - \mathrm{E}^Q\Big[\Delta S(t_{i+1})\Big]\Big) e^{r(t_N - t_{i+1})}, \qquad (3.5)$$

where

$$\begin{aligned} \Delta S(t_{i+1}) &= S(t_{i+1}) - S(t_i), \\ \mathrm{E}^Q\Big[\Delta S(t_{i+1})\Big] &= S(t_i)(e^{r(t_{i+1}-t_i)} - 1). \end{aligned}$$

The gamma hedge can correspondingly be represented as, adjusted from some misprint in [3],

$$\Gamma_{CV} = \sum_{i=1}^{N-1} \frac{\partial^2 C(S(t_i))}{\partial S(t_i)^2} \Big((\Delta S(t_{i+1}))^2 - \mathrm{E}^Q\Big[(\Delta S(t_{i+1}))^2\Big]\Big) e^{r(t_N - t_{i+1})}, \quad (3.6)$$

where

$$\mathrm{E}^Q\Big[(\Delta S(t_{i+1}))^2\Big] = S^2(t_i)(e^{(2r+\sigma^2)(t_{i+1}-t_i)} - 2e^{r(t_{i+1}-t_i)} + 1).$$

In the gamma hedge (3.6) the squared asset price difference must be used since the underlying asset itself is gamma neutral. In the same manner the rho and vega hedges could be constructed for models possessing non constant risk free rate and volatility. The problem with the greeks is that they are not usually known if the pricing function itself is unknown.

Obtaining hedge parameters can be handled as in the ordinary control variate case by applying the greeks from some similar option with known analytical formula. For some options, however, approximation formulas are at hand which can be used to get a good approximation of the option greeks.[1] Approximation can also be applied to simplify the model, if for example the simulation is carried out in a jump diffusion framework, the greeks could be calculated as in the standard framework.

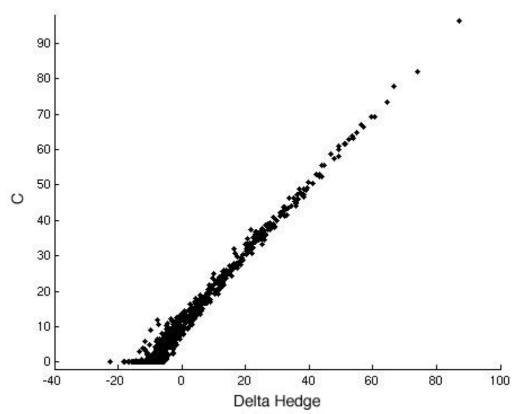

Figure 2: Correlation between delta hedge and option price.

An European call option will again be considered as an example. For a European call option the delta is known to be $N(d_1)$, where N is the normal distribution function and d_1 determined by Black and Scholes formula. A control variate hedge is thus easily constructed, in this example the option

[1]This idea proposed by Ola Hammarlid to apply in the stock market. However, later it was found that a similar idea has been applied to libor market models by Jensen and Svenstrup [8].

has 1 year to maturity and the hedge is rebalanced every month. In figure 2 the correlation can be seen to be high, again because of the simple contract and the fact that delta is known, resulting in a $\beta = 0.9$ and 7 times decreased standard deviation. The negative outcomes of the delta hedge seen in figure 2 arise when the underlying asset decrease much in the period.

The resulting variance reduction will of course be dependent of the number of rebalancing times for the hedge. In the case when options only depend upon the terminal value of the underlying asset, the hedge will, most likely, not serve well as a control variate. There may also be a trade off between the variance reduction achieved and the simulation-time. This can be used to analyze the number of rebalancing times to be used. The most natural is however to use the same number of rebalancing times as time steps simulated for the underlying asset.

3.4 A New Approach Introduced

In practice market actors will always have access to relevant prices. Therefore, when a new price is to be simulated, an old price can be recycled as a control variate. To take advantage of this an idea is to use the latest simulated option price as a control variate when updating the price.[2] This idea will be presented and examined in comparison with the other methods presented.

The idea builds upon re-simulation of a past price by replicating trajectories from the current simulation and simulate new trajectory parts between the two simulation points. In other words; when a simulation run have been made from the current time and market conditions, an additional control variate simulation is made. This simulates trajectories from the past time and up to current time under the past prevailing market conditions. From the current time and forward each control variate trajectory will be mirrored by a trajectory from the current spot in time, adjusted to the right price level.

This control variate simulation will have expected value equal to the last price and are likely to be highly correlated with the current price simulation, since it is the same option beeing priced. The performance of the method will depend on the time elapsed between the past and current simulation as well as the changed market conditions and the options sensitivity to the market. If the time elapsed between simulations is rather short the correlation are likely to be high.

[2]This idea was provided by Ola Hammarlid.

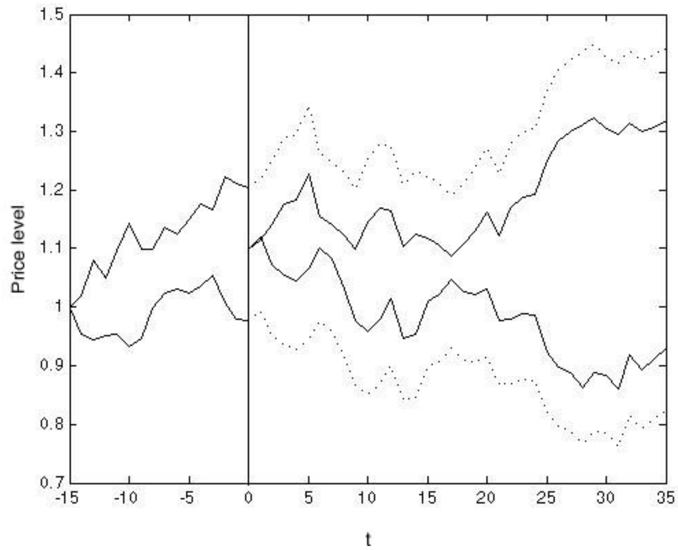

Figure 3: Trajectories simulated from current time and trajectories simulated up to that time, the dashed lines is the mirrored trajectories from time t = 0.

Figure 3 will help explain the idea. Two trajectories are simulated from a current spot in time, $t = 0$, where the asset price have increased 10% from a past spot in time, $t = -15$. From this past spot in time and the asset price 1, two additional trajectories have been simulated up to current time, $t = 0$. From time, $t = 0$, and forward the new trajectories will be mirrored to follow the same pattern as the two first trajectories simulated, as the dashed line demonstrate.

The value used as expected value in this method is not obtained by a closed analytical formula which so far been the case. The expected value is itself obtained through Monte Carlo simulation and may thereby include some uncertainty. However, the law of large numbers assure that the value will converge to the true expected value with probability 1. Mathematically this can be expressed as

$$\bar{Y}_{CV}(t_i, \beta) = \bar{Y}(t_i) - \beta \left(\bar{Y}(t_{i-1}) - \mathrm{E}\Big[\bar{Y}(t_{i-1})\Big] \right),$$

where

$$\frac{\sum_{i=1}^{n} Y_i(t_{i-1})}{n} \overset{p}{=} \mathrm{E}\Big[\bar{Y}(t_{i-1})\Big] \quad as \quad n \to \infty.$$

Advantages with this method is clearly that it demands little extra knowledge about the contract beeing priced. No similar contract with known expected value nor any approximation formula must be known. The only details need to be known are the earlier simulated price and market conditions. With this knowledge, control variates to any option can be constructed if a simulation procedure to price the option exists. This results in the opportunity to construct a general control variate technique for all kinds of derivatives.

3.4.1 Adjustment for Parameter Changes

The control variate trajectories are constructed with the same trajectories as used in the simulation, adjusted to the right price level. If, however, any of the underlying asset parameters change between simulations, additional adjustments must be done. This must be done so that the price available as control variate, is simulated under the right circumstances. If, for example, the volatility parameter change between simulations, an old simulated price is simulated under an other volatility. Therefore, the new trajectories must be adjusted with respect to the volatility difference. Most of the parameter changes can be adjusted in the trajectories with exception for the jump intensity parameter.

Therefore, to keep full flexibility in the model, an requirement is that two additional processes $W(t)$ and $N(t)$ as well as the sequence of jumps Y has to be recorded. Easier implementation can be achieved, by omitting the following results and simulate without control variates whenever any simulation parameter changes.

Lemma 3.2. *Let parameters denoted by t_0 be those used when the control variate price was simulated and parameters denoted t_1 be those used when the new price is simulated.*

Changes in the interest rate parameter for asset S_i is adjusted by the following factor

$$\alpha_r(t) = \exp\Big\{\Big(r(t_0) - r(t_1)\Big)t\Big\}.$$

Volatility changes are adjusted by the following factor

$$\alpha_\sigma(t) = \exp\Big\{\Big(\sigma(t_0) - \sigma(t_1)\Big)W(t) - \frac{1}{2}\Big(\sigma^2(t_0) - \sigma^2(t_1)\Big)t.\Big\},$$

which in the multiple asset case becomes

$$\alpha^i_\sigma(t) = \exp\Big\{\sum_{i=1}^{d}\Big(A_{ij}(t_0) - A_{ij}(t_1)\Big)W(t) - \frac{1}{2}\Big(\sigma_i^2(t_0) - \sigma_i^2(t_1)\Big)t.\Big\},$$

for $i = 1, \ldots, d$*, where* A *is defined in section 2.2.*

Proof. The results will be proved for volatility changes, but are analogous for interest changes. The volatility adjustment are obtained by solving the equation

$$\begin{aligned} &S(t_1)\exp\Big\{\Big(r - \frac{1}{2}\sigma(t_0)\Big)t + \sigma(t_0)W(t)\Big\} \\ &= \alpha_\sigma(t)S(t_1)\exp\Big\{\Big(r - \frac{1}{2}\sigma(t_1)\Big)t + \sigma(t_1)W(t)\Big\}, \end{aligned}$$

which gives the result.

□

Lemma 3.3. *Changes in the jump parameter* Y *for asset* S *can also be handled. If* $Y \sim N(a, b^2)$ *is rewritten to* $a + bZ$*, where* $Z \sim N(0,1)$*, is the standardization of* Y*, the adjustment is given by*

$$\alpha_{\{a,b\}}(t) = \exp\Big\{\lambda t\Big(\phi_{t_0}(1) - \phi_{t_1}(1)\Big) + \sum_{j=1}^{N(t)} a(t_0) - a(t_1) + \Big(b(t_0) - b(t_1)\Big)Z_j\Big\},$$

where $\phi_{t_k}(1)$ *is defined in* (2.5) *with the* a *and* b *parameters at time* t_k*.*

Proof. The proof is analogous the proof to lemma 3.2.

□

Now, a change in the volatility parameter between simulations can be handled by adjust the trajectories by $\alpha_\sigma(t)$, before using them for the control variate, that is,

$$S_{CV}(t) = S(t)\alpha_\sigma(t).$$

4 Control Variate Construction

In this chapter different options of interest are examined for suitable control variates which then are implemented, if possible, to suit the required needs. Two options to be taken under consideration is the Asian option and the Cliquet option, exotic options which are a common component in *structured products*, but also used in a range of other situations. Method implementations described are ordinary control variates and the hedging technique, the re-simulation method needs no further description.

4.1 The Arithmetic Asian Option

The arithmetic asian option is an exotic option because of that it is path dependent of the underlying asset during some part or the total lifetime of the option. More precisely the payoff is determined by the arithmetic average value of the underlying asset during the period. The average value can either be sampled continuously or as almost always in reality, at predetermined time steps. The option is as mentioned common in *structured products*, often with a basket of underling assets. An arithmetic Asian call option with strike price K can accordingly be described by the payoff function

$$\Phi\Big(S(t_1),\ldots,S(t_N)\Big) = \max\left(\frac{1}{N}\sum_{i=1}^{N} S(t_i) - K, 0\right), \tag{4.1}$$

where $S(t_i)$ denotes the asset price at time t_i, $i = 1,\ldots,N$, satisfying $t_1 < \cdots < t_N$. Correspondingly the option price, with time T to maturity, could be expressed in terms of risk-neutral valuation as

$$C(S(t_0)) = e^{-rT}\mathrm{E}^Q\Big[\Phi\Big(S(t_1),\ldots,S(t_N)\Big)\Big], \tag{4.2}$$

where the expectation is evaluated under the risk-neutral measure Q and r denotes the risk free interest rate.

In the pricing formula (4.2) a problem arises since the sum of the log-normally distributed asset prices are not log-normally distributed. This is the reason why no analytical formulas for this option can be found, because standard frameworks relies on log-normal assets. The same problem arises when the option has a basket of underlying assets, defined by $V(t) = \sum_{j=1}^{d} q_j S_j(t)$, where q_j denotes the quantity of asset S_j. Although a number of approximation formulas exist, the most accurate prices can be obtained by simulation.

4.1.1 Control Variate with Geometric Asian

A geometric average option can in contrast to the arithmetic average option be priced within a closed formula. This can be done, since the geometric average of log-normally distributed variables

$$\bar{S}_G = \prod_{i=1}^{N} S(t_i)^{\frac{1}{N}}, \tag{4.3}$$

is itself log-normally distributed and the standard Black and Scholes framework can be applied. In an article presented by Kemna and Vorst [9] the geometric Asian option is first used as a control variate when simulating the arithmetic Asian. They use a continuous sample of the average value. Here a discrete sample will be used and the model will be extended to handle multiple underlying assets as well as discrete dividends.

Analytical formulas for the geometric Asian option can be derived by calculating the mean and variance of (4.3) and applying the standard Black and Scholes formula; see Glasserman [5]. This analytical formula can now be implemented as an control variate for Asian options written on a single underlying asset. To handle the case of multiple underlying assets the multiple control variate technique can be applied with one control variate corresponding to each underlying asset. Another option is to form a weighted geometric portfolio of the underlying assets defined by

$$V_G(t) = \prod_{j=1}^{d} S_j(t)^{q_j},$$

where q_i is the quantity of asset S_j; see appendix A.1 for details. Now the geometric time average $\bar{S}_G$, for the geometric portfolio can be determined. The expected value for this option can then be calculated using the same formula as for the single underlying case after first forming the input parameters for a geometric portfolio; see appendix A.1. Depending on the option setup, this method will show to be somewhat more efficient as the number of underlying assets increases. An one dimensional control variate can also preferred, for example correlations comparisons.

4.1.2 Hedging in the Jump Diffusion Model

When simulating the Asian option in the jump diffusion model, defined in section 2.3, good variance reduction may be of even greater value. This is beacuse of a more time consuming simulation process in itself, but also a higher variance due to the effect of the jumps. In this case the geometric Asian option, used in the standard model, can not be used as a control variate. This is because no analytical pricing formula exist for the option under Merton's [14] asset dynamics. Instead an approximative delta hedge can be constructed and implemented as a control variate as in section 3.3.

To obtain a delta approximation formula, first an approximation formula for the arithmetic Asian option is derived, from which then delta is calculated. A good existing approximation formula for the Asian option in a standard framework is proposed by Levy [12]. Levy's approximation is derived by moment matching the sum in (4.1) to a log-normal distribution. These results are extended to include the jump diffusion part, referring to appendix A.3 for details. The approximation formula with the moment matching parameters becomes

$$C\big(S(t_0)\big) = e^{-rT}\Big(M_1\mathrm{N}(d_1) - K\mathrm{N}(d_2)\Big), \tag{4.4}$$

where N denotes the normal distribution function and M_1, is given by

$$M_1 = \frac{S(t_0)}{N}\sum_{i=1}^{N} e^{rt_i}$$

and d_1 and d_2, are given by

$$d_1 = \frac{\mu_M - \ln K + \sigma_M^2}{\sigma_M},$$
$$d_2 = d_1 - \sigma_M = \frac{\mu_M - \ln K}{\sigma_M},$$

where μ_M and σ_M denotes moment matched parameters, derived in appendix A.3. The approximation formula in itself turns out not to be particularly good, despite the good approximation in the standard model. The delta will, however, do as a control variate.

Lemma 4.1. *An approximative delta formula is given by*

$$\frac{\partial C(S(t_0))}{\partial S(t_0)} = e^{-rT}\left(\frac{\partial M_1}{\partial S}\mathrm{N}(d_1) + K\mathrm{N}'(d_2)\frac{\partial \sigma_M}{\partial S}\right), \tag{4.5}$$

where

$$\frac{\partial \sigma_M}{\partial S} = \frac{\frac{1}{M_2}\frac{\partial M_2}{\partial S} - \frac{2}{M_1}\frac{\partial M_1}{\partial S}}{2\sigma_M},$$

with the moments, M_1 and M_2, derived in appendix A.3.

Proof. Differentiating formulation (4.4) with respect to S, where the time variable is suppressed, yields

$$\frac{\partial C}{\partial S} = e^{-rT}\left(\frac{\partial M_1}{\partial S}\mathrm{N}(d_1) + M_1\mathrm{N}'(d_1)\frac{\partial d_1}{\partial S} - K\mathrm{N}'(d_2)\frac{\partial d_2}{\partial S}\right), \tag{4.6}$$

further differentiating the relation $d_2 = d_1 - \sigma_M$, yields

$$\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} - \frac{\partial \sigma_M}{\partial S}. \tag{4.7}$$

With the relation (4.7), equation (4.6) can be rewritten to

$$\frac{\partial C}{\partial S} = e^{-rT}\left(\frac{\partial M_1}{\partial S}\mathrm{N}(d_1) + K\mathrm{N}'(d_2)\frac{\partial \sigma_M}{\partial S} + \left(M_1\mathrm{N}'(d_1) - K\mathrm{N}'(d_2)\right)\frac{\partial d_1}{\partial S}\right).$$

By the properties of the normal probability density function, $\mathrm{N}'(d_2)$ can be rewritten to

$$\begin{aligned}\mathrm{N}'(d_2) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d_1^2 - 2\sigma_M d_1 + \sigma_M^2)} \\ &= \mathrm{N}'(d_1)\frac{1}{K}e^{\mu_M + \frac{1}{2}\sigma_M^2} = \mathrm{N}'(d_1)\frac{M_1}{K},\end{aligned}$$

which reduces (4.6) to (4.5).

□

A special case of the formula needs to be derived so it can handle in-period valuation, which is necessary when hedging at different time-steps.

Lemma 4.2. *The in-period delta approximation formula, where asset values $S(t_0), \ldots, S(t_n)$, $0 \leq n < N$, are known, is given by*

$$\frac{\partial C(S(t))}{\partial S(t)} = \frac{N-n}{N}e^{-rT}\left(\frac{\partial M_1}{\partial S}\mathrm{N}(d_1) + K_{in-period}\mathrm{N}'(d_2)\frac{\partial \sigma_M}{\partial S}\right),$$

for $t_n \leq t < t_{n+1}$, where

$$K_{in-period} = \frac{N}{N-n}\left(K - \frac{1}{N}\sum_{i=1}^{n} S(t_i)\right),$$

with all summations in the moment calculations in appendix A.3 derived from $n+1$ to N and averaged by $N-n$.

Proof. By rewriting the payoff function with the already known asset values $S(t_0), \ldots, S(t_n)$, $0 \leq n < N$, as

$$
\begin{aligned}
&\max\left(\frac{1}{N}\left[\sum_{i=1}^{n} S(t_i) + \sum_{i=n+1}^{N} S(t_i)\right] - K, 0\right) \\
&= \max\left(\frac{1}{N}\sum_{i=n+1}^{N} S(t_i) - \left[K - \frac{1}{N}\sum_{i=1}^{n} S(t_i)\right], 0\right) \\
&= \frac{N-n}{N}\max\left(\frac{1}{N-n}\sum_{i=n+1}^{N} S(t_i) - \frac{N}{N-n}\left[K - \frac{1}{N}\sum_{i=1}^{n} S(t_i)\right], 0\right),
\end{aligned}
$$

results in the price function, for $t_n \leq t < t_{n+1}$,

$$
C(S(t)) = \frac{N-n}{N} e^{-rT}\Big(M_1 \mathrm{N}(d_1) - K_{in-period}\mathrm{N}(d_2)\Big). \tag{4.8}
$$

Applying the derivation to lemma 4.1 to (4.8) and consider $S(t_0), \ldots, S(t_n)$, in the strike price as constants gives the result. The proof can also be found in Dennemark [4].

□

The delta approximation for a put option can be obtained from the Asian put-call parity, which can be found in Dennemark [4].

Implementation now proceeds according to formulation (3.5). The resulting variance reduction turns out to be quite good, but the additional computational time take away some of the glory. For example, a standard deviation reduction by 6 times, for a single asset, increases the computational time to almost the double. Further, the gamma hedge is also implemented, but the additional variance reduction is minimal to the cost of many extra calculations why these results are omitted.

Underlying baskets are handled in the same manner as in the geometric Asian case, by a geometric portfolio. In the jump diffusion model, the expected value of $V_G(t)$ is derived in appendix A.2. Further, the drift and moment matching formula have to be adjusted, which is described in appendix A.3, resulting in that the jump parameters must be specified for each asset.

4.2 The Cliquet Option

In the last decade, the so called Cliquet option has contributed to somewhat safer investment opportunities in the derivatives market. A global floor level makes the losses bounded at a predetermined level, while at the same time gains in intervals of the options lifetime is capitalized. These properties are usually desirable for options included in *structurded products*. The payoff in each interval, called reset period, is determined by the rate of return

$$R_n = \frac{S(t_n)}{S(t_{n-1})} - 1,$$

which then are truncated with a cap, C, and a floor, F, according to

$$R_n^* = \max(\min(R_n, C), F) \qquad F < C. \tag{4.9}$$

Most generally the Cliquet option is now determined by summing the truncated returns over the reset periods and truncate with a global cap, C_g, and global floor, F_g. Accordingly the price function is given by

$$C(S(0)) = e^{-rT}\mathrm{E}^Q\left[B \cdot \max\left(\min\left(\sum_{i=1}^{N} R_i^*, F_g\right), C_g\right)\right], \tag{4.10}$$

where B is a nominal amount. Further, the following limit relation is to be satisfied, $NF < F_g < C_g < NC$, where N is the number of reset times. Each payoff in (4.9) can be determined analytically and is actually a *bull call spread*. The problem for a closed formula arises in the outer option structure in (4.10). Since each return level can be determined analytically, these will first be tried as control variates. Another alternative is to use the similar *start forward option* which also is a *spread option*.

4.2.1 Control Variates with Bull Call Spreads

The analytical solution to the truncated rate of return can be determined by rewriting the expression (4.9), easiest by drawing a figure, to

$$
\begin{aligned}
R_n^* &= \max(\min(R_n, C), F) = F + \max(R_n - F, 0) - \max(R_n - C, 0) \\
&= F + \max\left(\frac{S(t_n)}{S(t_{n-1})} - (1+F), 0\right) - \max\left(\frac{S(t_n)}{S(t_{n-1})} - (1+C), 0\right).
\end{aligned}
$$

This is just a sum of constant and two call options payoffs. The spread $S(t_n)/S(t_{n-1})$ is given by

$$
\frac{S(t_n)}{S(t_{n-1})} = \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(t_n - t_{n-1}) + \sigma\Big(W(t_n) - W(t_{n-1})\Big)\right\},
$$

indicating that the spread option can be calculated with standard dynamics and initial price 1. If Black and Scholes formula is applied the price has to be adjusted by the discount factor, that is $\max(R_n - F, 0) = e^{r(t_n - t_{n-1})}C(1)$, where C denotes the Black and Scholes call option price. R_n^* are then calculated at every reset point and implemented as multiple control variates or summed together to one control variate.

The additional calculation needed to implement this control variate are, for a single asset, few. Since the truncated returns, $\max(\min(R_n, C), F))$, already have been determined in the ordinary simulation procedure, they do not have to be recalculated. For a basket of underlying assets the geometric portfolio, derived in appendix A.1, is again employed and these calculations have to be performed. Since the control variate is constructed with ordinary European options, no complications except switching to the Merton option pricing formula (2.6) will appear when introducing the jump diffusion model 2.3. When baskets appear, though, the geometric portfolio will not be used, due to some parameter estimation problems. Instead multiple control variates is applied with the *bull call spreads* of each underlying asset.

4.3 Discrete Dividends

In the simulation process, underlying assets are adjusted for discrete dividends after the raw trajectories have been simulated. Discrete dividends yields a much more accurate model of reality than using a continuous dividend yield. Unfortunately, discrete dividends are not easy implemented to analytical formulas, if at all possible.

To handle discrete dividends in the control variate setup they are adjusted in the strike price for, for example an European option. This is done, by adding the sum of dividends paid out during the lifetime of the option, inflated by the discount factor, to the strike price. Then the raw trajectories (without dividend adjustment) can be used for the control variates. For the Asian option this procedure becomes, adding the mean of the cumulative sum of dividends inflated by the discount factor to the strike price. Dividend adjustment for the Asian option can accordingly be expressed by the new strike price

$$K_{dividend} = K + \frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{i} e^{r(T-t_k)}D(t_k),$$

where $D(t_k)$ is the dividend paid between time t_k and t_{k-1}. In the case of the Cliquet option these operations has to be made to adjust the floor and cap for the dividend effect.

5 Results

In this section some results about simulations efficiency improvement will be presented. The results will not be presented in terms of absolute variance reduction nor simulation-time. Instead the results are presented as the simulation-time ratio of crude Monte Carlo in terms of the variance reduced Monte Carlo, that is,

$$\textit{Efficiency ratio} = \frac{\textit{time crude Monte Carlo}}{\textit{time variance reduced Monte Carlo}}$$

where *time* denotes the simulation-time until a specified convergence limit is reached. Presenting the results in this way makes them more independent from the simulation procedure itself, as well as computer speed and other simulation specific settings. The test will be rather general and not carried out in a range of different setups, so the results will mainly indicate which ratios that can be achieved. Since the ratios tend to differ between simulations for some setups, a mean of 3 ratios will be presented.

Simulation parameters are specified in table 1 for the purpose of the re-simulation model, where it is of interest to know the difference of the variables. The re-simulation will be performed with values one day before and one week before the current simulation. For simplicity the interset-rate,

Simulation parameters

Asset No	Interest rate	Volatility	Asset Price Current	-1 Day	-1 Week
1	0.05	0.1	100	103	110
2	0.05	0.2	100	95	92
3	0.05	0.3	100	102	103
4	0.05	0.4	100	103	95
5	0.05	0.5	100	97	103
6	0.05	0.5	100	96	94
7	0.05	0.4	100	98	97
8	0.05	0.3	100	100	95
9	0.05	0.2	100	101	105
10	0.05	0.1	100	101	92

Table 1: Asset parameters used in the simulation process with current prices, 1 day before and 1 week before.

volatility and correlation will be assumed to be constant, leaving only the asset price to change over time.

Further the assets are for simplicity assumed to be independent and have the same jump diffusion parameters, namely intensity $\lambda = 10$ and jumps $Y \sim LN(-0.03, 0.1^2)$. The tests will be performed for options written on asset 1, a basket of asset 1-5 and a basket of asset 1-10. The new re-simulation method will be tested against the other method described for the Asian and Cliquet option, in the standard as well as jump diffusion model. The convergence limit tend to have a rather large impact on the efficiency ratio. In these tests, simulation is terminated when the standard error is below 0.001.

The Asian option is a call option with 3 years lifetime and averaging points every second month, the strike price is finally set to be at the money.

In table 2 it is noted that the geometric Asian control variate is outstanding for a contract with a single underlying asset but the efficiency improvement decrease as the number of underlying assets increase. Efficiency ratios achieved with the delta hedge control variate are poor, all increasing the simulation-time. Although the absolute variance reduction is good, it is offset by the additional calculations associated with the hedging technique. The re-simulation method can be seen to perform better on the two basket options and the efficiency ratio seems more independent of the basket size.

Asian call - Black and Scholes model

	Control Variate			
Basket Size	Geometric Asian	Delta Hedge	Re-simulation -1 Day	Re-simulation -1 Week
1	17.83	0.49	5.11	2.50
5	2.33	0.52	6.09	2.97
10	2.56	0.63	6.34	2.59

Table 2: Increased simulation efficiency for the Asian option.

As mentioned before, the delta hedge control variate contribute with fairly good variance reduction but the additional calculation time reduces the efficiency ratio. The results are not very good as table 3 indicates. However, the results are better than in the standard framework. This is because of that crude simulation take much longer time in the jump model and make variance reduction more valuable. The re-simulation on the other hand has an even greater impact on the efficiency ratio in the jump diffusion model. The reason for this is probably, again, that crude simulation is more time consuming in Merton's model.

Asian call - Merton model

	Control Variate		
Basket Size	Delta Hedge	Re-simulation -1 Day	Re-simulation -1 Week
1	2.75	10.81	5.18
5	1.88	8.94	4.47
10	1.83	8.39	4.22

Table 3: Increased simulation efficiency for the Asian option in the jump diffusion model.

The Cliquet option tested has $F = -0.05$, $C = 0.05$, $F_g = 0$ and $C_g = 0.5$, the life time is 3 years and the reset time is every second month.

For the Cliquet option the re-simulation was expected to perform better than for the Asian option. Since the price change, most likely, only will have effect in one return period, only that period will be differ the simulations appart. The results for single asset options in table 4 is yet a nice surprise. Re-simulation results in a good efficiency ratio for all contracts and again seems rather independent of the basket size. Somewhat surprisingly, the call

Cliquet - Black and Scholes model

Basket Size	Control Variate Bull Call Spreads	Re-simulation -1 Day	Re-simulation -1 Week
1	1.78	364.76	195.40
5	1.00	12.28	7.31
10	1.18	12.18	8.41

Table 4: Increased simulation efficiency for the Cliquet option.

spreads results in a poor efficiency ratio, which not had been expected.

In the jump diffusion model, the re-simulation again shows some remarkable results for the Cliquet option on a single underlying asset. An interesting result is that the one week re-simulation performs better that the one day re-simulation for the single asset option. The efficiency ratios are in this model even poorer with the call spread control variate. The poor control variate is because the call spreads is much more time consuming to compute in the Merton model. In this setup, simulation-time even increases. Call spreads achieve a slight variance reduction, but the reduction is totally offset by the additional computational time. The poor results from the call spread method clearly indicates that it is not appropriate as control variate.

Cliquet - Merton model

Basket Size	Control Variate Bull Call Spreads	Re-simulation -1 Day	Re-simulation -1 Week
1	1.38	354.49	474.07
5	0.21	15.10	7.34
10	0.15	15.58	7.43

Table 5: Increased simulation efficiency for the Cliquet option in the jump diffusion model.

Results are as mentioned only presented for one option setup for each option so further testing of the re-simulation model is of value. However, tests performed with different option setup and underlying asset yields efficiency ratios at the same levels for the re-simulation method. The geometric Asian control variate have a tendency to perform less good as the option become more out of the money. This effect is not at all unexpected since the geometric average always is lower than the arithmetic average.

6 Conclusions

In this thesis, three different approaches to control variates (ordinary control variates, hedging and re-simulation) have been presented and tested. The new re-simulation method presented has, as seen, impressive impact with nearly 500 times faster simulation-time for some Cliquet options. For the remainder of option setups tested, the re-simulation method mostly outerperform the other methods, except for the single asset Asian option where the geometric Asian option serves as an excellent control variate.

The hedging technique is a very interesting approach to control variates and the technique can contribute with significant variance reduction. Though, the high computational cost for calculating the hedging parameters limit the simulation speed up. A more efficient estimation procedure of the hedging parameters would increase the efficiency ratio for this method. Simpler approximation formulas may be able to produce good variance reduction in less time, which may yield a higher efficiency ratio. The importance with the delta approximation, is that the hedge is driven in the right direction. The magnitude of the replicating hedge can then be adjusted with the β coefficient.

Mostly the speed up for re-simulation over crude Monte Carlo is above 10 times for re-simulation within one day. In reality prices are most likely updated several times each day so the efficiency ratios are likely to further be improved in these cases. Ordinary and hedging control variates tested, deliver different results depending on the number of underlying assets. Re-simulation, however, seems more independent regarding the underlying basket size.

Further investigation of the re-simulation model could include applying some variance reduction technique to the additional trajectory part that is simulated. Important sampling, one of the most powerful variance reduction techniques, is easy implemented when just raw trajectories are simulated. The use of important sampling, to change to a variance minimizing drift in the control variate simulation, could further decrease the variance, when the performance is less good. Implementing a this simple form of importance sampling should not increase computational time remarkable.

A Additional Calculations

A.1 Weighted Geometric Portfolio

By equation (2.2), the weighted geometric average can be expressed as

$$V_G(t) = \prod_{i=1}^{d} S_i(t)^{q_i} = V_G(0) \exp\Big\{ \sum_{i=1}^{d} q_i(\mu_i - \frac{1}{2}\sigma_i^2)t + \sum_{i=1}^{d}\sum_{j=1}^{d} q_i A_{ij} W_j(t) \Big\}, \tag{A.1}$$

where $V_G(0) = \prod_{i=1}^{d} S_i^{q_i}(0)$ is the weighted geometric mean of the asset values at time = 0. The variance parameter in (A.1) can be calculated as follows

$$\sigma_G^2 = \boldsymbol{q}\boldsymbol{A}\boldsymbol{A}'\boldsymbol{q}' = \boldsymbol{q}\boldsymbol{\Sigma}\boldsymbol{q}'.$$

The geometric drift parameter $\mu_G = \sum_{i=1}^{d} q_i \mu_i$. The expected value of $V_G(t)$ is now, however, not $V_G(0)e^{\mu_G t}$, but

$$\mathrm{E}\Big[V_G(t)\Big] = V(0) \exp\Big\{ \Big(\mu_G - \frac{1}{2}\sum_{i=1}^{d} q_i \sigma_i^2 + \frac{1}{2}\sigma_G^2\Big)t \Big\},$$

due to the new variance parameter in (A.1). Interpreting the difference

$$\delta = \frac{1}{2}\Big(\sum_{i=1}^{d} q_i \sigma_i^2 - \sigma_G^2\Big), \tag{A.2}$$

as a continuous dividend yield, the standard Black and Scholes can be applied to evaluate the option price.

A.2 Weighted Geometric Portfolio with Jump Diffusions

In a geometric portfolio in the jump diffusion model, the same results as in appendix A.1 hold for the GBM part, but the jump diffusion part yields some extra calculations. For the jump diffusion part the following should hold

$$\mathrm{E}\Big[e^{-\lambda(\phi(1)-1)+\sum_{j=1}^{N(t)} Y_j}\Big] = 1, \tag{A.3}$$

that is, it is a martingale. In a geometric portfolio with weights q_i, relation (A.3) do not hold anymore. Instead the exectation can be calculated as in (2.4) to be

$$\mathrm{E}\Big[e^{-q_i\lambda(\phi_i(1)-1)+q_i\sum_{j=1}^{N_i(t)} Y_j^i}\Big] = e^{-q_i\lambda(\phi_i(1)-1)+\lambda(\phi_i(q_i)-1)},$$

which results in that the expected value of the weighted geometric portfolio $V(t)$ is

$$\mathrm{E}\Big[V_G(t)\Big] = V_G(0)e^{(\mu_G-\delta-\gamma)t},$$

where $\gamma = \sum_{i=1}^{d} -q_i\lambda_i(\phi_i(1)-1)+\lambda(\phi_i(q_i)-1)$ and δ and μ_G is given in A.1.

A.3 Moment Matching

In this section the calculations leading to the approximative delta function for an Asian option in the jump diffusion model are presented. The moment matching approximation assumes that

$$\bar{S}_A = \frac{1}{N}\sum_{i=1}^{N} S(t_i) \sim LN(\mu_M, \sigma_M^2). \tag{A.4}$$

The first two moments of A should now be matched by the first two moments of (A.4). The moments of A is given by

$$\begin{aligned} M_1 &= \mathrm{E}^Q[\bar{S}_A] = e^{\mu_M+\frac{1}{2}\sigma_M^2}, \\ M_2 &= \mathrm{E}^Q[\bar{S}_A^2] = e^{2(\mu_M+\sigma_M^2)}. \end{aligned} \tag{A.5}$$

From (A.5) the mean and volatility parameters are solved to be

$$\begin{aligned}\mu_M &= 2\ln M_1 - \frac{1}{2}\ln M_2,\\ \sigma_M &= \ln M_2 - 2\ln M_1.\end{aligned} \tag{A.6}$$

Further the first moment of (A.4) is given by

$$M_1 = \frac{S(t_0)}{N}\sum_{i=1}^{N} e^{rt_i},$$

and the second moment can, for clarity, be calculated in two parts one for the GBM part and one for the jump process. Beginning expectation of the GBM part for $u < t$

$$\begin{aligned}&\mathrm{E}^Q\Big[S(t_0)\exp\Big\{(\mu - \frac{1}{2}\sigma^2)u + \sigma W(u)\Big\}S(t_0)\exp\Big\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\Big\}\Big]\\ &= S(t_0)^2\exp\Big\{(\mu - \frac{1}{2}\sigma^2)(u+t)\Big\}\mathrm{E}^Q\Big[\exp\Big\{\sigma\Big(W(t) - W(u)\Big) + 2\sigma W(u)\Big\}\Big]\\ &= S(t_0)^2\exp\Big\{(\mu - \frac{1}{2}\sigma^2)(u+t)\Big\}\exp\Big\{\frac{1}{2}\Big(\sigma^2(t-u) + 4\sigma^2 u\Big)\Big\}\\ &= S(t_0)^2\exp\Big\{\mu(t+u) + \sigma^2 u\Big\}.\end{aligned}$$

The expectation for the jump diffusion part for $u < t$, using the expectations derived in (2.4), becomes

$$\begin{aligned}\mathrm{E}^Q\left[\exp\left\{\sum_{j=1}^{N(u)} Y_j + \sum_{j=1}^{N(t)} Y_j\right\}\right] &= \mathrm{E}^Q\left[\exp\left\{2\sum_{j=1}^{N(u)} Y_j + \sum_{j=N(u)+1}^{N(t)} Y_j\right\}\right]\\ &= \exp\left\{\lambda(\phi(2)-1)u + \lambda(\phi(1)-1)(t-u)\right\}.\end{aligned}$$

A general formula for the second moment can now be expressed

$$
\begin{aligned}
M_2 &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{E}^Q\Big[S(t_i)S(t_j)\Big] \\
&= \frac{S(t_0)^2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\exp\Big\{\mu(t_i+t_j)+\sigma^2\min(t_i,t_j)\Big\} \\
&\times\ \exp\Big\{\lambda(\phi(2)-1)\min(t_i,t_j)+\lambda(\phi(1)-1)|t_i-t_j|\Big\}.
\end{aligned}
$$

Now the mean and variance parameters in (A.6) can be calculated by the derived moments, M_1 and M_2. The Black and Scholes formula can then be used by adjusting the drift and volatility to be at annual rate.

If the underlying object is a geometric basket of assets, as the case frequently is in this thesis, a problem arises when the basket parameters are derived. This is due to that the jump parameters not directly can be converted into geometric basket parameters. This means that the relation

$$
\sum_{j=1}^{N(t)} X_j = q_1\sum_{j=1}^{N_1(t)} Y_j^1 + \cdots + q_d\sum_{j=1}^{N_d(t)} Y_j^d, \tag{A.7}
$$

do not hold if X_j normally distributed. Thereby the process (A.7) has to evaluated for the drift adjustment and jump part in the moment matching. The new drift adjustment is calculated as follows

$$
\begin{aligned}
&\mathrm{E}\left[\exp\Big\{q_1\sum_{j=1}^{N_1(t)} Y_j^1+\cdots+q_d\sum_{j=1}^{N_d(t)} Y_j^d\Big\}\right] \\
&= \exp\Big\{\lambda_1 t(\phi_1(q_1)-1)+\cdots+\lambda_d t(\phi_d(q_d)-1)\Big\} \\
&= \exp\Big\{\sum_{k=1}^{d}\lambda_k t(\phi_k(q_k)-1)\Big\},
\end{aligned}
$$

indicating that the drift should be adjusted by $\sum_{k=1}^{d}\lambda_k(\phi_k(q_k)-1)$. In the same manner the moment matched jump part can be determined to, for $u < t$,

$$
\begin{aligned}
&\mathrm{E}^Q\left[\exp\left\{q_1\sum_{j=1}^{N_1(u)}Y_j^1+q_1\sum_{j=1}^{N_1(t)}Y_j^1+\cdots+q_d\sum_{j=1}^{N_d(u)}Y_j^d+q_d\sum_{j=1}^{N_d(t)}Y_j^d\right\}\right]\\
&=\mathrm{E}^Q\left[\exp\left\{\sum_{k=1}^{d}2q_k\sum_{j=1}^{N_k(u)}Y_j^k+q_k\sum_{j=N_k(u)+1}^{N_k(t)}Y_j^k\right\}\right]\\
&=\exp\Big\{\sum_{k=1}^{d}\lambda_k(\phi_k(2q_k)-1)u+\lambda_k(\phi_k(q_k)-1)(t-u)\Big\}.
\end{aligned}
$$

The jump diffusion part in moment M_2 should be replaced by

$$
\sum_{k=1}^{d}\lambda_k(\phi_k(2q_k)-1)\min(t_i,t_j)+\lambda_k(\phi_k(q_k)-1)|t_i-t_j|,
$$

when a geometric basket is used. For this purpose, the jump parameters λ, a and b must be specified for each asset, while remaining parameters are to be specified as geometric basket parameters.

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