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Impact of Interest Rate Risks on Life Insurance Assets and Liabilities

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Abstract

This paper examines the impact of interest rate risk that a life insurer is subject to, especially the effect of interest rate risk on the restricted assets and the technical provisions. The main work has been estimating the yield curves and scenario analyzing. Stress test has been applied for this purpose, aiming to get a suitable match between the assets and liabilities in such a way that changes in interest rates by shifts do not affect the financing of liabilities.

Keywords : Extended Nelson-Siegel model, yield curve, technical provisions, restricted assets, and traffic light system.

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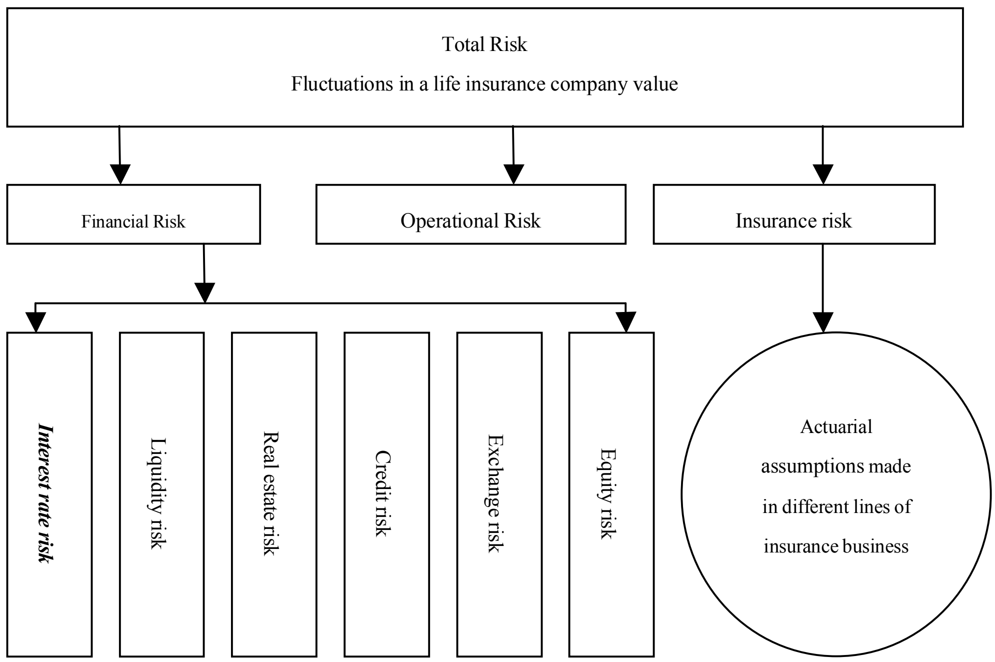
CONTENTS

1 INTRODUCTION

On January 1, 2006 the *Directive on Institutions for Occupational Retirement Provisions* (IORP Directive) came into force in Sweden. This is the beginning of a series of regulatory changes in the insurance area over the next few years. The *Traffic Light System*, as a new supervisory tool is designed by The Swedish Financial Supervisory Authority (Finansinspektionen) aiming to measure financial risks (see figure 1) that life insurance companies and occupational pension funds may be exposed to. Finansinspektionen believes the earlier identification of insurers with high financial risk, the better protection can be achieved for the policyholders. The main risk that undertakings are subject to is interest rate risk, with consideration to the fact that insurance liabilities often have long durations. In our study, the insurance business consists of occupational life long pensions on a defined benefit basis, occupational temporary pensions on a defined contribution basis and private pensions, temporary and life long. Such a composition obviously generates a long duration of our liabilities that are very sensitive to the changes in interest rates. So in this paper, the risk of interest rates stands in the focus.

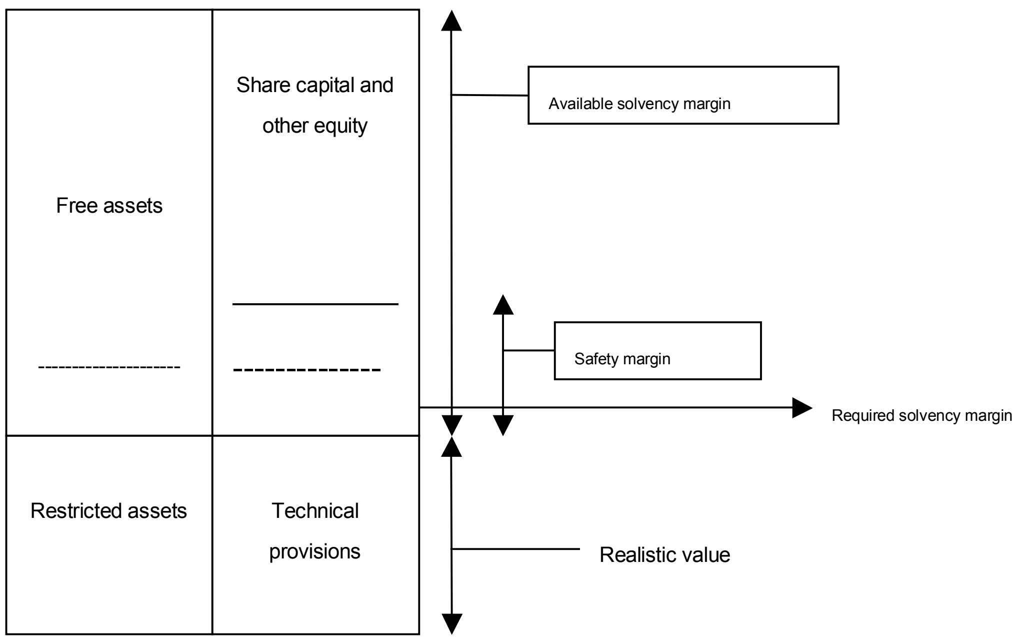
Figure 1: Overview of financial risks



According to the *Traffic Light System*, life insurers are required to follow the *Prudent Person Principle* in the case of valuing their technical provisions. Finansinspektionen emphasizes that each transaction in an insurance contract must essentially be discounted individually using the risk free rate of interest that corresponds to the duration of the liabilities, provided that an institution can categorize its liabilities in this manner. For an institution that cannot apply cash flow categorization, the problem can be solved by other approximate measures, which we will not take into account in this work.

As we mentioned above, the cornerstone of the *Prudent Person Principle* is valuing the technical provisions at their realistic value. It gives rise to a series of changes in the annual report. First of all, as an opposite financial position in a balance sheet, the restricted assets have to respond to the changes in the technical provisions. In other words, this principle requires insurers to increase their restricted assets in both quantities and qualities, so the assets at least will satisfy the solvency ratio requirement. Furthermore, from the view of risk taking a proper balance between the assets and liabilities is expected for a life insurer. In the following figure, we try to summarize the structure of balance sheet affected by the *Prudent Person Principle*.

Figure 2: Overview of existing balance sheet in a life insurance company after introducing the Traffic Light System



2 FIXED-INCOME SECURITIES

2.1 TERM STRUCTURE OF INTEREST RATE

Term structure theory puts aside the notion of yield, and instead focuses on the relationship between financial securities of different terms. The term structures of interest rates, also known as the yield curves therein interest rates are determined by their terms. The curves usually slope gradually upward as maturities increase. Such typical shape of yield curves reflects the expectation hypothesis in which market's expectation for the future interest rates is explained on the basis of the current market conditions.

Government treasuries are considered risk free. Their yields are often observed as benchmarks for the fixed-income securities with the same maturities. Here we give some examples for the most popular financial securities, zero-coupon bonds and coupon bonds.

- Zero coupon bonds

A zero coupon bond is a debt security that does not pay coupons during its life, but it is traded at a deep discount to its face value, which will be worth when the bond matures or comes due. Zero-coupon bonds have an important advantage of being free of reinvestment risk, though such bonds cannot enjoy the effects of an interest rate rise. Zero coupon bonds tend to be very sensitive to the changes in interest rates. Their prices fluctuate more than other types of bonds in the secondary market in the since that there are no coupons during their life to reduce the effect of the changes in interest rates.

- Coupon bonds

A coupon bond is a debt obligation with coupons affixed to the bond itself, and each coupon represents a single interest payment.

The current price of a bond should be the same as the present value of the stream of future cash flows, which is the nominal amount of money to change hands at some future date, discounted to account for the time value of money, e.g. interest rate. (http://en.wikipedia.org/wiki/Present_value). Present value of a stream of cash flows can be obtained by adding discounted magnitudes of the individual cash flows due to the present value is additive. It should be noted that sooner the money is received, more value it is worth if it is compared with the same amount of money that is received later since interest can be earned by loaning money out.

Suppose there is a stream consisting of several payments at the end of each period for total of n periods $(X_1, X_2, ..., X_n)$ and m times per year. Interest rate r is the nominal annual compounded interest rate. In the formula below, PV is used to denote the present value of the stream of payments (consisting of the coupon payments and the final face-value redemption payment).

Compound interest rate $$PV = \sum_{k=0}^{n} \frac{X_k}{[1 + r/m]^k}$$

Continuous compounding $$PV = \sum_{k=0}^{n} X_{t_k} \cdot e^{-r \cdot t_k} \qquad \text{where } t_k = \frac{k}{m}$$

In a context of finance, discounting is referred to a process of calculating the present value of future monetary amounts. With the help of discount factor, a future cash flow can be valuated at a given time that we are interested in. Let d_t indicate the discount factor for its term t. The value of d_t can be obtained by the formulas below with the consideration of different types of interest rates. Suppose that the fixed annual interest rate r compounded m times per year for total t periods, and then the appropriate discount factor is

$$d_k = \frac{1}{\left[1 + r/m\right]^k}$$

In the case of continuous compounding, we gave the following formula

$$d_k = e^{-r \cdot t_k} \qquad \text{where } t_k = \frac{k}{m}$$

2.2 RISK MEASURES FOR THE TERM STRUCTURES

We start this section with introducing a conception of yield. Yield for a bond is the effective rate of interest paid on the bond, at which the present value for a stream of payments is exactly equal to the current price of the bond. The stream here consists of all coupon payments and the final face-value redemption payment. This effective rate of interest is always quoted on an annual basis, and termed more properly as yield to maturity (*YTM*).

Formulas below are given for the bond prices therein coupon payments made at the end of each period of total n periods, and m times per year. Time to maturity is T years, which equates with n/m. *YTM* is constant and denoted by λ, the payment that made in period k is indicated to C_k, and its present value is denoted by PV_k.

$$\text{With compound interest rate: Bond price } P = \sum_{k=1}^{n} PV_k, \quad \text{where } PV_k = \frac{C_k}{[1 + \frac{\lambda}{m}]^k}$$

$$\text{With continuous compounding: Bond price } P = \sum_{k=1}^{n} PV_k, \quad \text{where } PV_k = C_{t_k} \cdot e^{-\lambda \cdot t_k} \text{ and } t_k = \frac{k}{m}$$

Duration is a measure of time until a bond gives a profit. It is useful in the sense that it directly measures the sensitivity of prices to the effects of changes in *YTM*. For a zero-coupon bond, duration is the same as its time to maturity. For a coupon bond, duration is strictly less than their life period. We start our discussion about this measure with introducing two kinds of durations. Suppose we have coupon payment made m times per year in T years, YTM is λ and C_k is the payment that made in period k. Macaulay duration of such a bond can be found according to the following formula.

$$D_{macaulay} = \frac{1}{P}\sum_{k=1}^{n}\frac{k}{m}\cdot\frac{C_k}{\left[1+\left(\lambda/m\right)\right]^{k}}$$

It is easy to realize that Macaulay duration is the average of the stream of payments over the life of a bond, each coupon payment is discounted on the basis of a common yield curve. The other duration we want to mention here is Modified duration. The relationship between these two durations can be described by the following.

$$D_{\text{modified}} = \frac{D_{macaulay}}{1+\frac{\lambda}{m}}$$

It should be noted that for large values of m - the number of payments made per year or small values of YTM, we have $D_{macaulay} \cong D_{modified}$. With the help of modified duration, derivation of bond price with respect to λ can be reduced to the following expression.

$$\frac{\partial P}{\partial\lambda} = \sum_{k=1}^{n}\frac{\partial PV_k}{\partial\lambda} = -\frac{1}{1+\frac{\lambda}{m}}\times D_{\text{macaulay}}\times P \equiv -D_{\text{modified}}\times P$$

That can be even rewritten as following

$$(*) \qquad \frac{\partial P}{P} = -D\text{modified}\times\partial\lambda + o(\partial\lambda^2)$$

With the help of the formula above, we readily realized that modified duration reveals the relative slope of the price-yield curve at a given point. A certain percentage change of bond price leads to a corresponding change of the yield curve. That gives a straight-line approximation to the price-yield curve.

Another risk measure for the sensitivity of price-yield curve is convexity, which measures a certain percentage change in the modified duration if the yield increases with one basis point. Convexity is defined as below

$$\text{Convexity} = \frac{1}{P}\frac{\partial^2 P}{\partial\lambda^2}$$

If the bond price P is described in terms of cash flows, the formula above will have a different appearance. In the following formula, coupon payments are paid m times per year for total n periods; C_k is the payment that made at period k, and its present value is denoted by PV_k .

$$\text{Convexity} = \frac{1}{P}\sum_{k=1}^{n}\frac{\partial^2 PV_k}{\partial\lambda^2} = \frac{1}{P\left[1+\left(\lambda/m\right)\right]^{2}}\sum_{k=1}^{n}\frac{C_k}{\left[1+\left(\lambda/m\right)\right]^{k}}\cdot\frac{k(k+1)}{m^2}$$

With the help of convexity, the formula (*) can be derived even further. A better approximation for the price-yield curve can be achieved by taking in the second-order derivative.

$$\frac{\partial P}{P} = -D_{\text{modified}} \times \partial\lambda + \frac{1}{P\left[1+(\lambda/m)\right]^2} \sum_{k=1}^{n} \frac{C_k}{\left[1+(\lambda/m)\right]^k} \cdot \frac{k(k+1)}{m^2} \times \partial\lambda^2 + o(\partial\lambda^3)$$

As we mentioned earlier in this section, the current price of a bond is exactly equal to the present value for the stream of its coupon payments. The way to calculate Macaulay duration and Modified duration for a single bond can be utilized to a portfolio as well. Suppose there is a portfolio, in which several bonds (say m) with different durations are assembled. Let P_i and D_i denote the price and duration for these bonds respectively, and $i= 1, 2 \ldots m$. The value of such portfolio can be obtained by the formula below therein P and D are used to denote the value of this portfolio and its duration, respectively.

$$P = P_1 + P_2 + \ldots + P_m$$

$$D = w_1 D_1 + w_2 D_2 + \ldots + w_m D_m \qquad \text{where } w_i = \frac{P_i}{P} \quad \text{and } i = 1,2,\ldots m$$

3 PARAMETRIC MODEL FOR THE YIELD CURVES

Estimates of the yield curves are required to have enough flexibility in order to represent the shape associated with the curves, which means the estimates should provide a maximal approximation to the observed data. God precision of the estimates is another requirement with the consideration of the analytical demands in the context of monetary policy.

In 1987, Nelson, C.R. and Siegel, A.F. published their parsimonious modeling of yield curves (so-called Nelson-Siegel Model), which successfully seized a trade-off between the smoothness of the estimated curve and the flexibility. In 1994, Svensson, L.E.O extended Nelson-Siegel Model by his work 'Estimating and interpreting forward interest rates' (so-called Extended Nelson-Siegel Model), in which he demonstrated the use of forward interest rates as a monetary policy indicator. He also pointed out 'the forward rate curve more easily allows a separation of expectations for the short, medium and long term than does the yield curve'.

Litterman and Scheinkman in 1991 claimed that most of the observed variation in bond returns could be explained by three factors baptized to the name – level, slope and curvature. Their findings provided another interpretation of short-, medium- and long-term components for the estimates of the yield curves, which were used in the paper written by Nelson and Siegel.

According to BIS (Bank for International Settlements), most central banks have adopted either Nelson-Siegel or Extended Nelson-Siegel Model, except those countries Canada, Japan, the U.K and the U.S. In the case of Sweden, Riksbank - Sweden's central bank adopted the 'smoothing splines' method in 2001, but it still reports Extended Nelson-Siegel estimates to the BIS Data Bank.

3.1 NELSON-SIEGEL MODEL

This model is motivated by, but not dependent on, the expectation theory of the term structure. It offers a parsimonious representation of the range of shapes associated with the term structure of interest rate: monotonic, humped and *S* shaped. The model has advantages of estimating lesser number of parameters and therefore ensuring a smooth forward curve.

In order to understand the ideas of Nelson-Siegel model, we start our discussion about this model with recalling the definition of instantaneous forward rate and finite-maturity forward rate. Suppose that $s_{t,m}$ is the spot interest rate for a zero coupon bond traded at a given point of time *t* and matures at time *m*; $f(t,i,m)$ is the forward rate with trade date *t*, settlement date *i* and maturity date *m*. Below we gave the formulas that describe the relationship amongst spot interest rate, instantaneous forward rate (the forward rate for a forward contract with an infinitesimal investment period after the settlement date) and finite maturity forward rate. It should be noted that the possible value of *m* must be higher or equal to the value of *i*.

$$f(t,i,m) = \left[\frac{(1+s_{t,m})^m}{(1+s_{t,i})^i}\right]^{1/(m-i)} - 1 \qquad f(t,i,m) = \frac{\int_{u=i}^{m} f(t,u)du}{m-i}$$

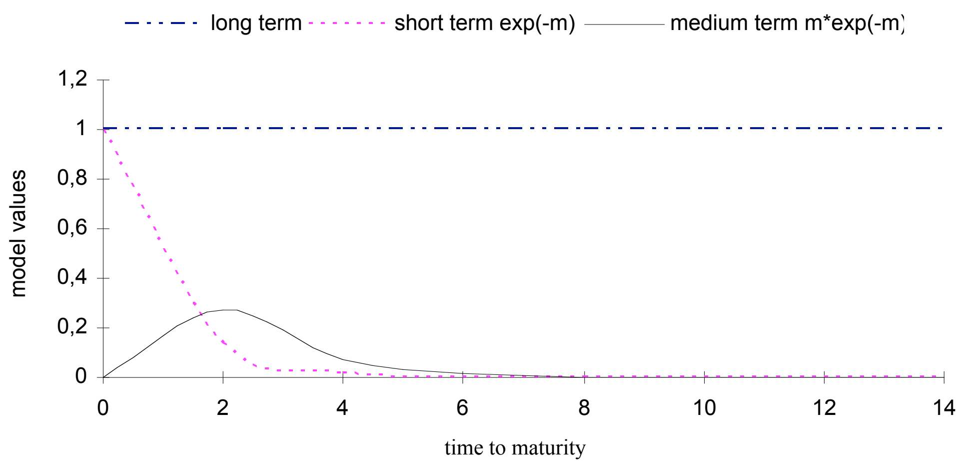
$$f(t,i) = \lim_{m \to i} f(t,i,m) \qquad s_{t,m} = \frac{\int_{u=t}^{m} f(t,u)du}{m-t}$$

Nelson-Siegel Model derives the instantaneous forward rate $f(m,\Theta,t)$ at maturity *m* in a functional form given below where Θ denotes the parameters β_0, β_1, β_2, τ_1 that need to be estimated from the observed data, and *t* indexes to the point of time at which estimation is carried out.

$$f(m,\Theta,t) = \beta_{0,t} + \beta_{1,t} \cdot \exp(-m/\tau_{1,t}) + \beta_{2,t} \cdot \left[\frac{m}{\tau_{1,t}} \cdot \exp(-m/\tau_{1,t})\right]$$

This forward rate model generates a family of forward rate curves that take on monotonic, humped, or *S* shapes depending on the values of beta-parameters. Nelson and Siegel interpreted the coefficients of each component as indicators that measure the strengths of the long-, short-, and medium-term components for the forward rate curve. The long-term component is constant that does not decay to zero in the limit. The short-term has the fastest decay of all functions that decay monotonically to zero. The medium-term starts out at zero and decays to zero. Figure 3.1 shows the characteristics of these components for the forward rate curve. There the contribution of the long-term component β_0 is equal to 1.

Figure 3.1: Components of forward rate curve estimated by Nelson-Siegel Model



Based on the forward rate model, the yield for zero-coupon bonds with different maturities is denoted as $R(m,\Theta,t)$, which can be obtained by integrating the forward rate function from zero to *m* and then divided by *m*. $R(m,\Theta,t)$ is actually the average of forward rates over time.

$$R(m,\Theta,t)=\beta_{0,t}+\beta_{1,t}\cdot\left\{\frac{1-\exp(-m/\tau_{1,t})}{m/\tau_{1,t}}\right\}+\beta_{2,t}\cdot\left\{\frac{1-\exp(-m/\tau_{1,t})}{m/\tau_{1,t}}-\exp(-m/\tau_{1,t})\right\}$$

We realize that the value of $R(m,\Theta,t)$ converges to $\beta_{0,t}$ as maturity goes to infinity. That is the reason that $\beta_{0,t}$ is interpreted as contribution of the long-term component of the curve. It indicates the level of the term structure of interest rates. The estimated value of $\beta_{0,t}$ should be obviously positive.

$R(m,\Theta,t)$ converges to $\beta_{0,t}+\beta_{1,t}$ as soon as m decreases to zero. To understand this, it is better to go back to the forward rate model therein $\beta_{1,t}$ is the coefficient of a term decaying monotonically and fast to zero. $\beta_{1,t}$ determines the starting value of the curve in terms of deviation from the asymptote $\beta_{0,t}$, and this is why Nelson and Siegel considered this parameter as contribution for the short-term component. $\beta_{1,t}$ indicates the slope of the yield curve.

In the yield function $R(m,\Theta,t)$, the coefficient of the third component - $\beta_{2,t}$ specifies curvature of the yield curves. As long as the sign of $\beta_{2,t}$ is determined, hump (positive) or *U* shape (negative) is made. The absolute value of $\beta_{2,t}$ indicates the magnitude of the curvature.

In the case of parameter $\tau_{1,t}$, it should be positive and determines the position of the curvature for the estimated yield curve.

3.2 EXTENDED NELSON-SIEGEL MODEL

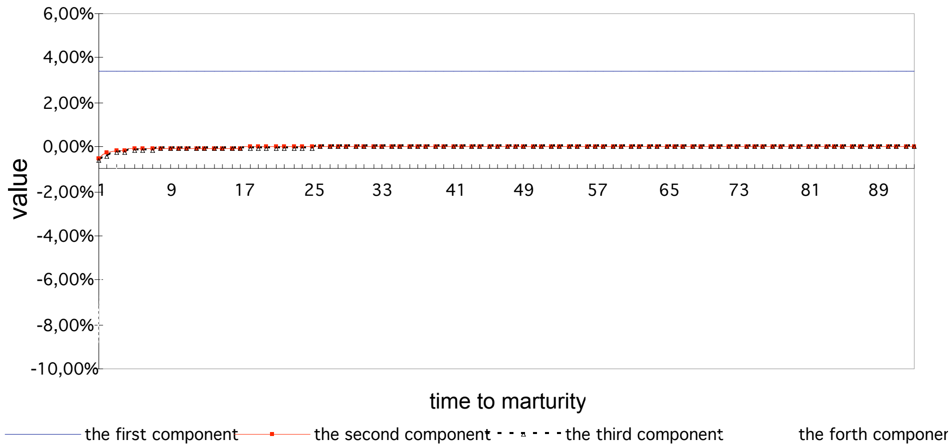
Svensson L.E.O. extended the Nelson-Siegel model by an additional component in which two parameters $\beta_{3,t}$ and $\tau_{2,t}$ were involved. In other word, Extended Nelson-Siegel model consists of total six parameters $\beta_{0,t}, \beta_{1,t}, \beta_{2,t}, \beta_{3,t}, \tau_{1,t}$ *and* $\tau_{2,t}$. The forth component creates an additional turning point in the estimated curve. In the context of parsimonious modeling of the yield curves, Nelson-Siegel model is usually considered as a restrictive application for the Extended Nelson-Siegel model. The yield function is given below

$$R(m,\Theta,t)=\beta_{0,t}+\beta_{1,t}\cdot\left\{\frac{1-\exp(-m/\tau_{1,t})}{m/\tau_{1,t}}\right\}+\beta_{2,t}\cdot\left\{\frac{1-\exp(-m/\tau_{1,t})}{m/\tau_{1,t}}-\exp(-m/\tau_{1,t})\right\}+\beta_{3,t}\cdot\left\{\frac{1-\exp(-m/\tau_{2,t})}{m/\tau_{2,t}}-\exp(-m/\tau_{2,t})\right\}$$

The additional parameter $\tau_{2,t}$ should be greater than zero, and indicates the position of the second hump or the U-shape on the curve. Parameter $\beta_{3,t}$ has the same function as parameter $\beta_{2,t}$, its sign determines the shape of the estimated curve and its absolute value indicates the magnitudes of the second curvature.

Figure 3.2 below shows the characteristics of these four components in Extended N-S model. Total sex parameters are estimated from the data set selected on 28, April 2006, and their estimated value are $\beta_{0,t}$ = 3,40%, $\beta_{1,t}$ = -1,37%, $\beta_{2,t}$ = -2,03%, $\tau_{1,t}$ = 0,43, $\tau_{2,t}$ = 5,57 and $\beta_{3,t}$ = 1,97%.

Figure 3.2: Components of the yield curve estimated by Extended N-S model.



4 ESTIMATION FOR THE YIELD CURVES

4.1 CRITERION FOR THE ESTIMATION

Parameters in Nelson-Siegel or Extended Nelson-Siegel model can be estimated by minimizing either the sum of squared bond-price errors or the sum of squared yield errors. Decision of whether the first or the later should be applied depends on the purpose of estimation.

However, as pointed out by Svensson (1994) minimizing price errors sometimes results in fairly large yield errors for bonds with short maturities. This is because prices are very insensitive to yields for short maturities. BIS (Bank for International Settlements) also noted that using bond prices in the estimation irrespective to their durations would lead to over-fitting of the long-term bond prices at the expense of the short-term prices. Facing this problem, several approaches have been introduced amongst which the most popular remedy is non-linear least squares algorithm e.g. the price error of each bond is weighted by the inverse of its duration, so-called interest rate sensitivity factors of price.

In the formula below, y indicates the yield, P_j is the market price for bond j and P_j^e is the theoretical price for bond j. ϕ_j is denoted for the weight given to *j:th* bond (interest rate sensitivity factor for *j:th* bond). In our case the parameters $\beta_0, \beta_1, \beta_2, \beta_3, \tau_1$ and τ_2 are estimated by minimizing the sum of squared bond price errors weighted by $1/\Phi$:

$$\min_{\beta_0,\beta_1,\beta_2,\beta_3,\tau_1,\tau_2} \sum_{j=1}^{n} \left\{ \frac{P_j - P_j^e(\beta_0,\beta_1,\beta_2,\beta_3,\tau_1,\tau_2)}{\phi_j} \right\}^2$$

$$\text{where} \qquad \phi_j = \frac{D_j(macauly) * P_j}{1+y}$$

4.2 DATA SELECTION

Theoretically, a term structure of interest rates with continuous time is related to a full set of zero-coupon bond with default risk. Unfortunately, most bonds are coupon bonds with time to maturity beyond 12 months, which means the yield of such bonds cannot be used as *YTM* directly. In Sweden, government bonds (nominal) are medium- and long-term coupon bonds, but the longest one is much shorter than duration that most life insurance companies have on their debt. Therefore we have to estimate a yield curve with enough long maturities.

For the purpose of yield curve estimation we selected data sets from ECOWIN at three different points of time (Mars 31, April 28 and August 1, 2006). Table 4.2-1 below provided the information about spot interest rates for zero coupon bonds (with maturities less or equal to 12 months), Swedish government bonds without SO-1035 and SO-1038 due to their relatively short maturities. It should be noted that Swedish government bonds have convention 30/360, e.g. every calendar month has exactly 30 days.

Table 4.2-1: Data stets selected on different points of time

	Mars 31,2006			April 28, 2006			August 1, 2006		
Name	Time to maturity	Yield	Coupon	Time to maturity	Yield	Coupon	Time to maturity	Yield	Coupon
1 Month	2006-04-30	2,00%	0,00%	2006-05-28	2,00%	0,00%	2006-08-31	2,24%	0,00%
2 Month	2006-05-30	2,00%	0,00%	2006-06-27	2,00%	0,00%	2006-09-30	2,27%	0,00%
3 Month	2006-06-29	2,02%	0,00%	2006-07-27	2,07%	0,00%	2006-10-30	2,39%	0,00%
6 Month	2006-09-27	2,15%	0,00%	2006-10-25	2,16%	0,00%	2007-01-28	2,62%	0,00%
9 Month	2006-12-26	2,25%	0,00%	2007-01-23	2,28%	0,00%	2007-04-28	2,80%	0,00%
12 month	2007-03-26	2,37%	0,00%	2007-04-23	2,42%	0,00%	2007-07-27	2,96%	0,00%
SO-1037	2007-08-15	2,64%	8,00%	2007-08-15	2,69%	8,00%	2007-08-15	2,93%	8,00%
SO-1040	2008-05-05	2,95%	6,50%	2008-05-05	3,03%	6,50%	2008-05-05	3,20%	6,50%
SO-1043	2009-01-28	3,16%	5,00%	2009-01-28	3,25%	5,00%	2009-01-28	3,40%	5,00%
SO-1034	2009-04-20	3,26%	9,00%	2009-04-20	3,35%	9,00%	2009-04-20	3,50%	9,00%
SO-1048	2009-12-01	3,31%	4,00%	2009-12-01	3,43%	4,00%	2009-12-01	3,54%	4,00%
SO-1045	2011-03-15	3,46%	5,25%	2011-03-15	3,61%	5,25%	2011-03-15	3,67%	5,25%
SO-1046	2012-10-08	3,56%	5,50%	2012-10-08	3,73%	5,50%	2012-10-08	3,76%	5,50%
SO-1041	2014-05-05	3,61%	6,75%	2014-05-05	3,81%	6,75%	2014-05-05	3,80%	6,75%
SO-1049	2015-08-12	3,66%	4,50%	2015-08-12	3,88%	4,50%	2015-08-12	3,84%	4,50%
SO-1050	2016-07-12	3,69%	3,00%	2016-07-12	3,92%	3,00%	2016-07-12	3,87%	3,00%
SO-1047	2020-12-01	3,70%	5,00%	2020-12-01	3,96%	5,00%	2020-12-01	3,91%	5,00%

4.3 ESTIMATES OF PARAMETERS

The purpose of comparing yield curves estimated at different points of time is to answer the question if the estimates of parameters by Extended Nelson-Siegel model reflect the changes in time horizon, in other words if the estimated yield curves are time dependent. In table 4.3-1 below, we listed the estimated values of the parameters and the yield with maturity 10 years corresponding to the yield curves estimated on the different points of time. Price errors after optimization are given in the last raw of the table, they are obtained by the criterion that we mentioned in the previous section:

$$\min_{\beta_0,\beta_1,\beta_2,\beta_3,\tau_1,\tau_2} \sum_{j=1}^{n} \left\{ \frac{P_j - P_j^e(\beta_0,\beta_1,\beta_2,\beta_3,\tau_1,\tau_2)}{\phi_j} \right\}^2$$

The results confirmed that the estimated value of parameters varied from time point to time point. In the cases of beta-parameters, the magnitude of their variations is relatively higher than do the parameters $\_{1}$ and $\_{2}$. That partly reflects the changes of time horizon, and partly depends on the specific properties of these parameters. Therefore we will investigate more about these parameters in the next section.

Table 4.3-1: Estimated value of the parameters in Extended Nelson-Siegel Model based on data selected on the different points of time.

Parameter	Mars 31, 2006	April 28, 2006	August 1, 2006
$\beta_0(t)$	3,63%	3,40%	3,99%
$\beta_1(t)$	-1,75%	-1,37%	-1,30%
$\beta_2(t)$	-2,22%	-2,03%	-1,74%
$\tau_1(t)$	0,43	0,43	0,512
$\tau_2(t)$	5,57	5,57	6,747
$\beta_3(t)$	0,56%	1,97%	-0,07%
$R(10,\Theta,t)$	3,6310%	3,8421%	3,8130%
Price Error after optimization	3,9E-07	1,3E-06	5,5E-07

Table 4.3-1 can be rewrite in the form of the function of yield.

$$R(m,\Theta,t) =$$

$$t = March\ 31, 2006$$

$$0{,}0363 - 0{,}0175 * \left[\frac{1-\exp\left(\frac{-m}{0{,}43}\right)}{\frac{m}{0{,}43}}\right] - 0{,}0222 * \left[\frac{1-\exp(\frac{-m}{0{,}43})}{\frac{m}{0{,}43}} - \exp\left(\frac{-m}{0{,}43}\right)\right] + 0{,}0056 * \left[\frac{1-\exp\left(\frac{-m}{5{,}57}\right)}{\frac{m}{5{,}57}} - \exp\left(\frac{-m}{5{,}57}\right)\right]$$

$$t = April\ 28, 2006$$

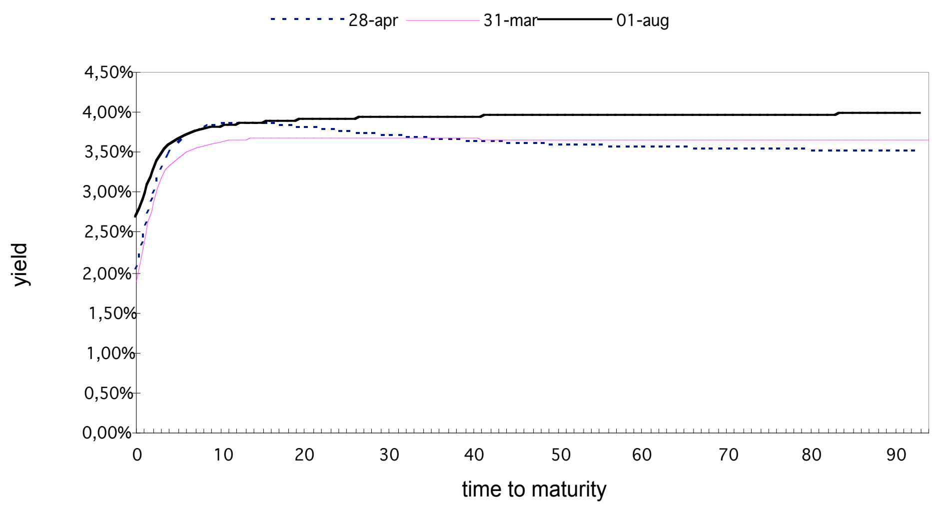
$$0{,}034 - 0{,}0137 * \left[\frac{1-\exp\left(\frac{-m}{0{,}43}\right)}{\frac{m}{0{,}43}}\right] - 0{,}0203 * \left[\frac{1-\exp(\frac{-m}{0{,}43})}{\frac{m}{0{,}43}} - \exp\left(\frac{-m}{0{,}43}\right)\right] + 0{,}0197 * \left[\frac{1-\exp\left(\frac{-m}{5{,}57}\right)}{\frac{m}{5{,}57}} - \exp\left(\frac{-m}{5{,}57}\right)\right]$$

$$t = August\ 1, 2006$$

$$0{,}0399 - 0{,}0130 * \left[\frac{1-\exp\left(\frac{-m}{0{,}512}\right)}{\frac{m}{0{,}512}}\right] - 0{,}0174 * \left[\frac{1-\exp(\frac{-m}{0{,}512})}{\frac{m}{0{,}512}} - \exp\left(\frac{-m}{0{,}512}\right)\right] - 0{,}0007 * \left[\frac{1-\exp\left(\frac{-m}{6{,}75}\right)}{\frac{m}{6{,}75}} - \exp\left(\frac{-m}{6{,}75}\right)\right]$$

Furthermore, with the consideration of making easier to compare these estimates at a certain point (time-to-maturity) we plotted all these estimated yield curves together in figure 4.3-1, in which we could see clearly that the discrepancy between estimates for April 28 and August 1 is more legible when maturity is going longer. But in the case of comparison between the yield curves estimated on April 28 and Mars 31, we did not find the same tendency. The discrepancy between estimates April 28 and Mars 31 is smaller for the maturities first from 0 up to 6 years and then from 30 up to 55 years.

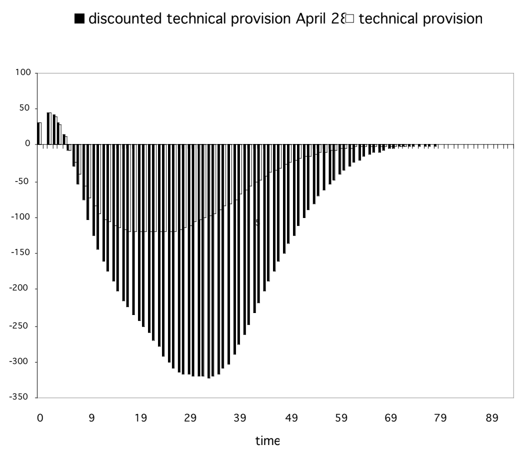
Figure 4.3-1: Yield curves estimated at the different points of time

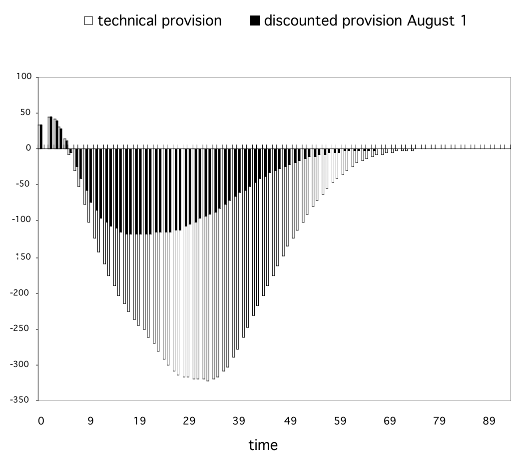


4.4 IMPACT OF THE YIELD CURVES

Valuing the technical provisions into the value of market (realistic value) is the cornerstone of the Solvency System aiming to protect the policyholders from the financial risks. It is a crucial issue for any life insurance company to estimate a reasonable yield curve, which will be used as risk free rate of interest in present value computation for its technical provisions. In our case, we suppose that a life insurance company can apply cash flow categorization following the regular that Finansinspektionen requires. In the figures below, we plotted the technical provisions together with their realistic values. The realistic values of the technical provisions are obtained on the basis of yield curves estimated on the different points of time April 28 and August 1, 2006.

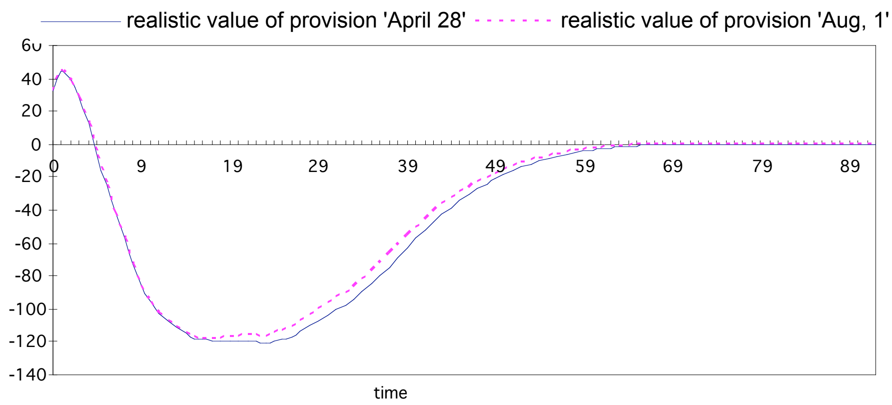
Figure 4.4-1.2: Realistic value of the technical provisions corresponding to the yield curves estimated on April 28 and August 1, 2006. Unit value is million Swedish kronor.





With the help of figures above, we can see clearly the impact of the yield curves on the realistic values of the technical provisions. But unfortunately, we could not see a legible difference between these two realistic values here. So we made a graph instead of using histograms where the realistic values are plotted against each other.

Figure 4.4-3: Realistic value of the technical provisions corresponding to the yield curves estimated at April 28 and August 1, 2006. Unit value is million Swedish kronor.



4.5 ALTERNATIVE YIELD CURVES

Before we finish this section, let us talk, in short about the alternative yield curves that are acceptable from the view of Finansinspektionen to be used in the present value calculation. According to the decision maid by Finansinspektionen, alternative yield curves cannot be utilized in the assert-liability valuation if they are not derived under the following conditions:

- Derive from interest rates with low credit risk.
- Exclude any credit risk premiums
- Are not higher than the interest rate that could reasonably be expected on a risk-fee fixed-income instrument with good liquidity and with corresponding duration
- Follow the prudent person principle in such a manner that the assumption that provides the lowest interest rate is used in cases where there is uncertainty regarding the choice of assumption
- Are calculated on the basis of recognized methodology.

In practice, several life insurance companies tried to derive their alternative yield curves based on the interest rate swaps, which are often used by companies to alter their exposure to interest rate fluctuations. However, interest rate swaps are not considered as risk-free due to the involved credit risk. This kind of risk comes into play in the case where one of the parties is in the money, then that party faces credit risk of possible default by another party. The purpose for using such alternative yield curve is that interest rate swaps have much longer maturity, e.g. we have more observed data. Secondly, the assets for any life insurance companies normally have higher return then government bonds do, so it seems to be more motivated to utilize alternative yield curves in valuing their insurance liabilities. But using interest rate swaps to derive the yield curves brings us a spiny trouble that is how to identify the size of credit risk premiums on the swaps.

To identify the premium of the credit risks is a problem that needs to be discussed more in details. But in this paper, we will not go further than doing a comparison between the yield curves derived from government bonds and interest rate swaps. This comparison was carried out at the same point of time - Mars 31, 2006 and showed in figure 4.5-1. Moreover, in order to seize the impact of these two estimates of the yield curves on the realistic valuation, we plotted together the realistic values calculated on the basis of these yield curves.

Figure 4.5-1: Comparison between the yield curves estimated on the basis of different term structures – government bonds and interest rate swaps

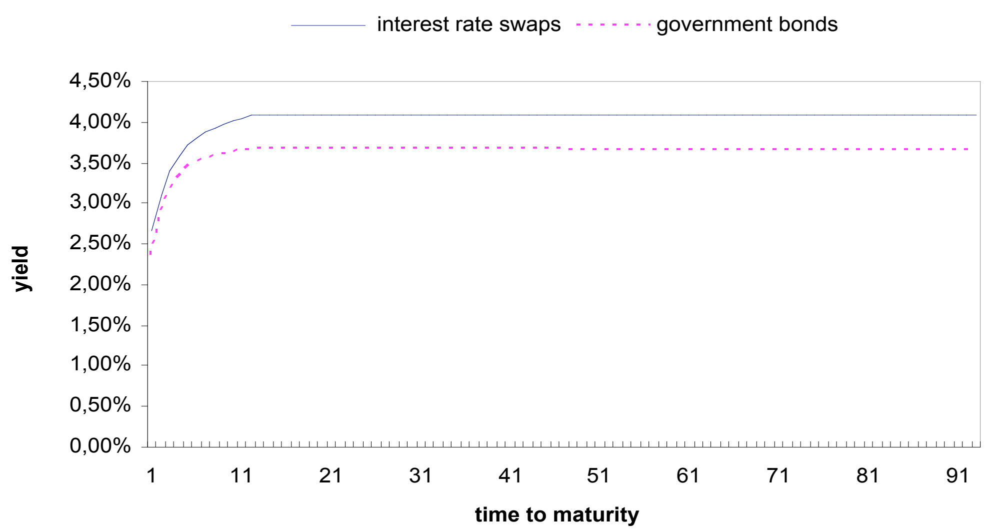
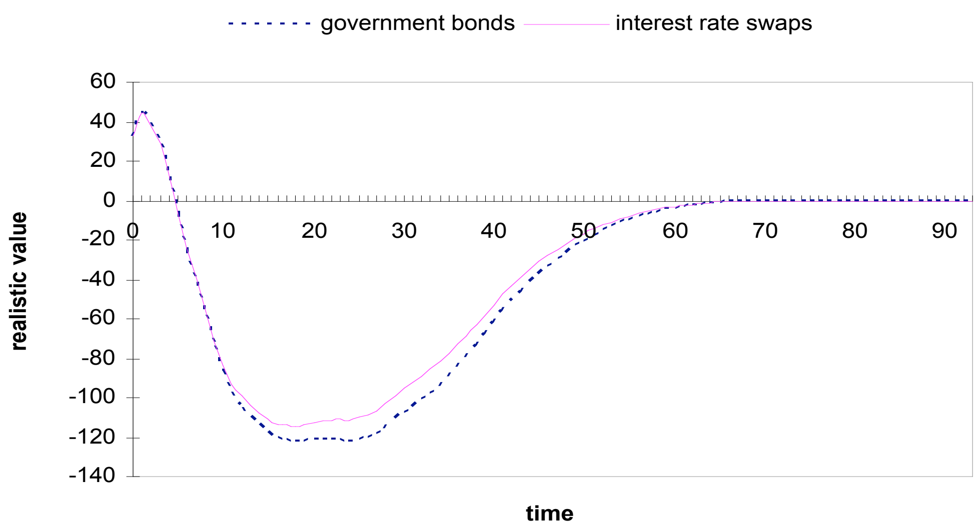


Figure 4.5-2: Realistic values of the technical provisions discounted on the basis of different yield curves



5 PARAMETRIC EFFECT ON THE YIELD CURVES

In section '*Parametric model for the yield curves*', we theoretically briefed on the functions of the parameters in the estimates of the yield curves. But in this section, we want shift our focus onto the quantification analysis for the effect of these parameters. We want to elucidate to which extent a parameter can contribute the changes in the estimates. It is useful for prognosticating the development of the assets and liabilities, especially to answer the question what will happen if interest rate increases by a certain percentage.

We start our analysis by taking one parameter into account at a time, which means we let β_0 very within a reasonable interval and keep other parameters unchanged. We want to seize the effects of these parameters individually and elucidate their influences on the level, slope and curvature of the yield curves.

The yield curves used for our analysis are estimated from the data set selected on August 1, 2006. The main results are summarized in the form of tables in which the point in brackets indicates the analyzed parameter, *YTM(.; 10)* is used to denote the yield with maturity 10 years; *d(.;10)* is discount factor corresponding to 10 years; $N_{i=10}(.)$ is the present value of the technical provision categorized in the 10*th* year; *F(.)* is obtained by adding each N_i, in our case $i= 0, 1, 2,\ldots 94$.

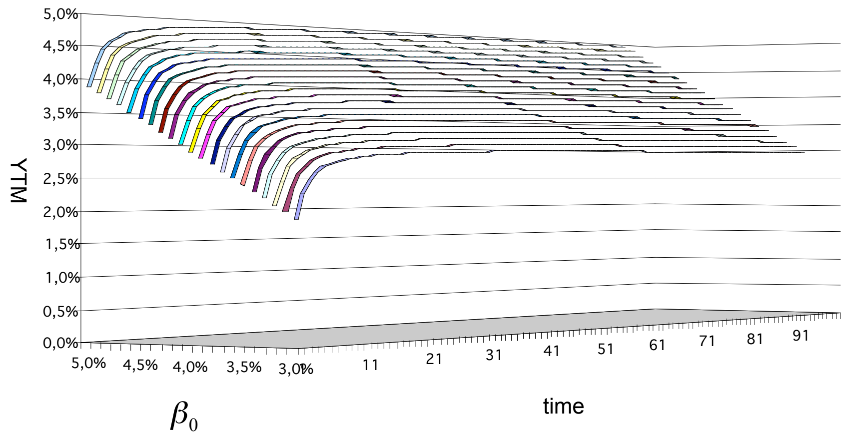
5.1 INFLUENCE OF PARAMETER β_0

We already knew the value of *YTM* will converge to β_0 as maturity goes to infinity. β_0 is considered as a contribution for the long-term component. We did a series of variations for β_0 from 2.99% up to 4.99% with unit step 0.1%. We verified that the variation caused by β_0 leads to a parallel shift on the yield curve, which means the value of *YTM* goes up or down with exact the same percentage corresponding to the percentage changes in β_0.

Table 5.1: Impact of β_0 on *YTM(10)*, $N_{i=10}$ *and F(.)*

Parameter	Estimate	*YTM(10)*	*d(.;10)*	$N_{i=10}(.)$	*F(.)* *(million)*	Diff Estimate	Diff *YTM*	Diff $N_{i=10}(.)$ (million)	Diff *F (.)*
Parameter β_0	3.99%	3.81%	0.6878	-85 305 963	-3572	-	-	-	-
Lower boundary	2.99%	2.81%	0.7577	-93 975 896	-4649	-1.0%	-1.0%	-9	-1.1E+09
Upper boundary	4.99%	4.81%	0.6250	-77 507 781	-2780	1.0%	1.0%	8	7.92E+08

Figure 5.1: The impact of β_0 on the yield curves



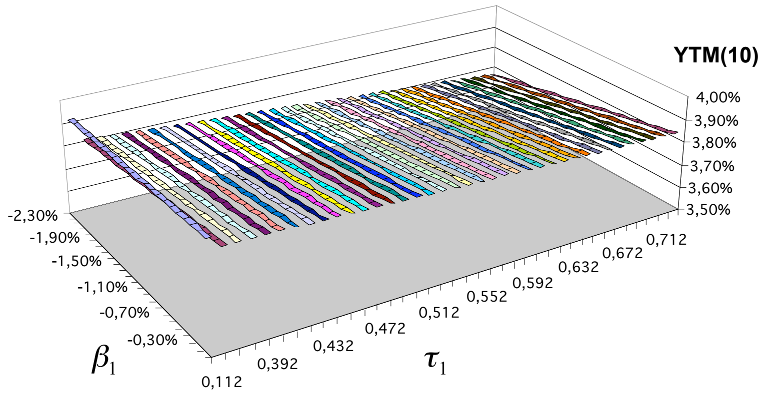
5.2 INFLUENCE OF PARAMETER β_1 AND τ_1

An increment and reduction of β_1 leads to a proportional increment and reduction of *YTM,* respectively, but the magnitude of changes on *YTM* is smaller than what β_0 did. An increment and reduction of τ_1 makes a proportional reduction and increment on *YTM,* respectively, but the corresponding variation on *YTM* is much smaller than what beta parameters generated. But unfortunately, the impact of a combination of these two parameters is hard to be quantified due to their adverse characteristics.

Table 5.2: Impact of β_1 and τ_1 on *YTM(10)* and $N_{i=10}$

Parameter (.)	Estimate	*YTM(.;10)*	*d(.;10)*	$N_{i=10}$(.)	Estimate Diff	YTM(.;10) Diff	$N_{i=10}$(.) Diff
Parameter β_1	-1.30%	3.81%	0.6878	-85305963	-	-	-
Lower boundary	-2.30%	3.76%	0.6912	-85727807	-1.00%	-0.05%	-421844
Upper boundary	-0.10%	3.87%	0.6838	-84802762	1.20%	0.06%	503201
Parameter τ_1	0.5120	3.81%	0.6878	-85305963	-	-	-
Lower boundary	0.1120	3.93%	0.6798	-84313751	-0.4000	0.12%	992212
Upper boundary	0.7220	3.75%	0.6921	-85831995	0.2100	-0.06%	-526032
Combination of β_1 and τ_1	(-1,30 ; 0,5120)	3.81%	0.6878	-85305963	-	-	-
Lower boundary	(-2,30% ; 0,1120)	3.92%	0.6806	-84404638	(-1% ;-0.4)	0.11%	901325
Upper boundary	(-0,10% ; 0,7220)	3.84%	0.6863	-85118531	(1.2% ; 0.21)	0.03%	187432

Figure 5.2: The impact of β_1 and τ_1 on *YTM(10).*



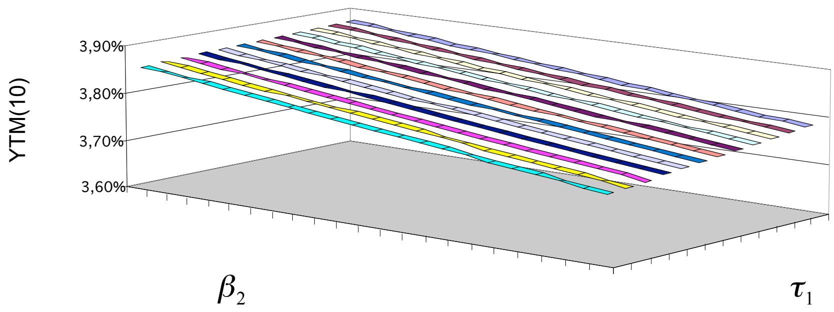
5.3 INFLUENCE OF PARAMETERS β_2 AND τ_1

An increment and reduction of β_2 leads to a proportional increment and reduction of *YTM,* respectively*,* the magnitude of changes of *YTM* caused by this parameter is very small compared with what β_0 did on the *YTM.* However, the impact of a combination of parameters β_2 and τ_1 is hard to be quantified due to their adverse characteristics.

Table 5.3: Impact of β_2 and τ_1 on *YTM(10)* and $N_{i=10}$

Parameter	Estimate	*YTM (.;10)*	*d(.;10)*	$N_{i=10}$ *(.)*	Estimate Diff	*YTM* Diff	$N_{i=10}$*(.)* Diff
Parameter β_2	-1.74%	3.81%	0.6878	-85305963	-	-	-
Lower boundary	-2.74%	3.76%	0.6912	-85727807	-1.00%	-0.05%	-421844
Upper boundary	-0.64%	3.87%	0.6841	-84844571	1.10%	0.06%	461392
Parameter τ_1	0.5120	3.81%	0.6878	-85305963	-	-	-
Lower boundary	0.1120	3.93%	0.6798	-84313751	-0.4000	0.12%	992212
Upper boundary	0.7220	3.75%	0.6921	-85831995	0.2100	-0.06%	-526032
Combination of β_2 and τ_1	(-1,74% ;0,5120)	3.81%	0.6878	-85305963	-	-	-
Lower boundary	(-2,74 ; 0,1120)	3.78%	0.6899	-85561170	(-1% ; -0.4)	-0.03%	-255207
Upper boundary	(-0,64% ;0,7220)	3.86%	0.6849	-84939618	(1.1% ; 0.21)	0.05%	366345

Figure 5.3: The impact of β_2 and τ_1 on *YTM(10)*



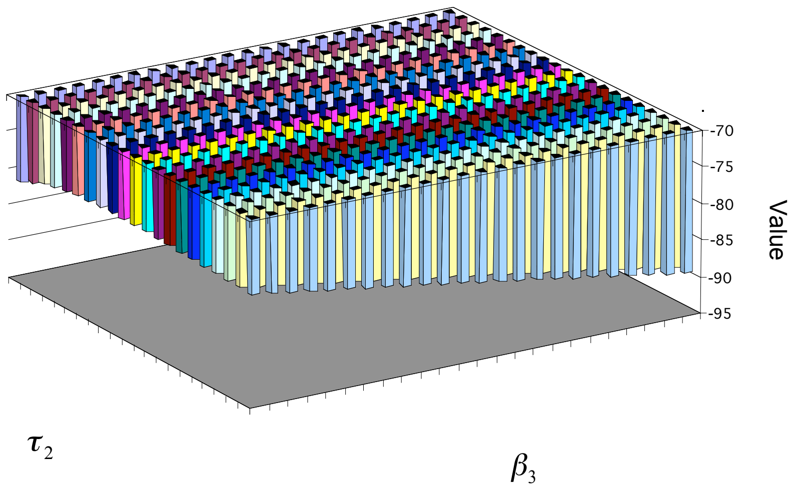
5.4 INFLUENCE OF PARAMETERS β_3 AND τ_2

A series of variations has been maid on the parameters β_3 and τ_2. An increment and reduction of β_3 leads to a proportional increment and reduction of *YTM,* respectively. But the magnitude of changes in *YTM(10)* corresponding to the effect of β_3 is considerably small. In the case of τ_2, an increment and reduction of it does not make a corresponding linear variation of *YTM*, and moreover the magnitude of changes of *YTM(10)* effected by the variation of τ_2 is extremely small. Figure 5.4 showed us what kind of impact that τ_2 has on $N_{i=10}$. To quantify the impact of these parameters' combination is a delicate matter in the sense that parameter τ_2 has a complicated property.

Table 5.4: Impact of β_3 and τ_2 on *YTM(10)* and $N_{i=10}$

Parameter	Estimate	YTM(.;10)	d(.;10)	N_i=10 (.)	Estimate Diff	YTM(.;10) Diff	N_i=10 (.) Diff
Parameter β_3	-0.07%	3.81%	0.6878	-85305963	-	-	-
Lower boundary	-2.07%	3.22%	0.7281	-90296746	-2.00%	-0.59%	-4990783
Upper boundary	2.53%	4.58%	0.6391	-79266073	2.60%	0.77%	6039890
Parameter τ_2	6.75	3.810%	0.6878	-85305963	-	-	-
Lower boundary	1.75	3.822%	0.685	-85231830	-5.00	0.012%	74133
Upper boundary	11.75	3.817%	0.6843	-85277177	5.00	0.007%	28786
Combination of β_3 and τ_2	(-0,07% ; 6,75)	3.81%	0.6878	-85305963	-	-	-
Lower boundary	(-2,07% ; 1,75)	3.48%	0.7103	-88089233	(-2.0% ; -5)	-0.33%	-2783270
Upper boundary	(2,53% ; 11,75)	4.46%	0.6466	-80189361	(2.6% ; 5)	0.65%	5116602

Figure 5.4: The impact of β_3 and τ_2 on $N_{i=10}$. Unit value is million Swedish kronor.

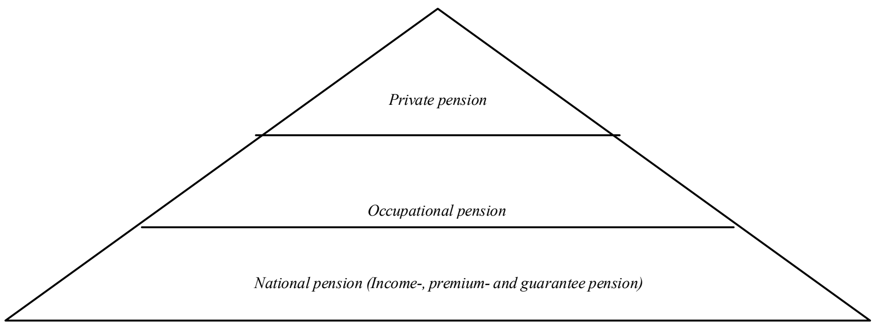


6 LIFE INSURANCE MATHEMATICS

6.1 PENSION SYSTEM IN SWEDEN

A pyramid with three layers can generalize the Swedish pension system (see the figure 6.1 below). The basis of the pyramid is *national pension*, interlayer is *occupational pension* and the top-level is *private pension.* The national pension is a statutory pension, which is paid out by the National Insurance Office. Individuals earn money for their national pension during their entire life. The old system (basic pension and national supplementary pension) remains for people born in 1937 or earlier. The new pension system covers people born in 1938 or later and started to pay out pension in 2003. The national pension comprises three components: income pension, premium pension and guarantee pension. Occupational pension is built up on an agreement between trade unions and employers for the benefit of the employees, and paid out by different pension institutions, depending on whether you work within private industry, local government/county council etc. Generally the occupational pension approximately amounts to 10 per cent of salary at the year before he/she retires on his/her pension. The total period of employment, moreover, affects the outcome of occupational pension. Individuals create private pension savings as supplement for their pension. There are several life insurance products that people can choose to fit their own life situation, such as traditional pension insurance, unit-link fund insurance and ordinary savings.

Figure6.1: Swedish pension system shaped in a form of pyramid.



6.2 LIFE INSURANCE RISKS

As we mentioned before in section one '*Interest rate risk in the focus'*, Finansinspektionen emphasizes that each transaction in an insurance contract must essentially be discounted individually using the risk-free rate of interest that correspond to the duration of the liabilities.

In reality, it's not easy for a life insurance company to evaluate these future transactions exactly, because the magnitude of in- and out-cash flows are not determined. Quantifying the changes that could be caused by the decisions made from insurers and policyholders requires a more complicated calculating approach in which distribution of profit, requirement of repayment cover etc. can be taken into account.

In this paper we don't keep our focus on risk factor identification, but how to quantify these uncertainties. Furthermore, we are not going to discus all factors that possibly have statistical effects on valuing cash flows, it will be impossible with the consideration of paper space. We decided to discuss risks that have a great deal of impact on life assurance mathematics.

Transfer risk: A policyholder signs up for a certain insurance contract period but changes his mind after a while and wants his accumulated capital transferred to another insurance company before the maturity date of the policy. This is defined as transfer risk. It should be noted that surrender is a term most often for life policies but transfer might often have the same economic impact on the insurance business. In this paper we have not taken transfer risk into account.

$$P^S(t) = \prod_{k=1}^{t-1}(1 - P_A^S(k)) * P_A^S(t)$$

Unconditional probability of a transfer at year t

$$P_A^S(t) = P(\text{Transfer at year } t \mid \text{policy is Active at the end of year } (t\text{-}1))$$

Conditional probability of a transfer at year t

Paid-up risk: Another possible opportunity for those who wish to discontinue premium payment is the paid-up policy. Such a policy remains in-force but no further premiums will be paid in. In practice, using a standard paid-up assumption seems to be a solution to deal with such problems, e.g. each year a certain percentage of policies with regular premiums will be converted to paid-up.

Mortality risk: Discussion around mortality risk is a hot topic, in the sense that mortality risk is the most important uncertainty in the context of life-assurance mathematics, and especially as longevity is continually rising. For the purpose of focusing on mortality risk, we assume an insurance product where only mortality risk has impact on valuing technical provision. Based on this assumption, liability does neither lose nor profit from other risk factors than mortality risk from now on.

In Sweden, most life insurance companies apply Gompertz-Makeham model to describe the development in mortality. We cannot deduce this approach completely because of restrictions in space, but it will be helpful for us to run through Gompertz-Makeham model. We assume that the mortality intensity μ_x for a person at age x ($x \geq 0$) is continuously differentiable as a function of age. It is the same if we regard μ_x as probability for that, a person at age x will die in a very short time interval $(x, x + dx)$. Let us neglect the gender aspect at the moment; we have a formula of μ_x according to the law of mortality in Gompertz-Makeham model. In practice, we even work more often with another variable l_x, so-called survival function. It describes the probability that an x-year old person can survive t years longer. If we introduce T_x for the remainder of life for an x-year old person, then l_x can be expressed with the help of T_x in the form $P(T_x > t) = l_x(t)$.

$$\mu_x = \alpha + \beta \cdot e^{\gamma \cdot x} = \frac{f(x)}{1-F(x)} \quad \text{where } x \geq 0 \text{ and } f(x) = F'(x)$$

$$F_x(t) = P(T_x \leq t) \qquad \text{where } T_x = \text{remainder of life for an } x \text{ year old person}$$

$$l_x(t) = \frac{l_0(x+t)}{l_0(x)} = e^{-\int_x^{x+t} \mu(s)ds} \qquad \text{where } t \geq 0$$

With the specification in the current official Swedish mortality table M90, we have the explicit formula for mortality intensity, which is utilized also in this paper.

$$\mu_x = \alpha + \beta \cdot e^{\ln(10)\cdot\gamma\cdot(x-f)}$$

where, $x \geq 0,\ \alpha = 0.001,\ \beta = 0.000012$ and $\gamma = 0.044$
$f = 0$ or 6, for men and women respectively

In practice, commutation functions *D* (x) and *N* (x) are introduced into valuing provision for the purpose of reducing complexities. *D* (x) and *N* (x) are defined as below

$$D(x) = l(x) \cdot e^{-\delta \cdot x}, \ N(x) = \int_x^{\infty} D(u)du$$

where, δ is discount factor, determined by $\ln(1 + r \cdot \text{tax}) - (\text{premium charge})$
r is discount rate and tax effect of tax deduction

In this paper, out-cash flow in a life insurance company is approximately equal to the expected value of benefit on the collective level, which will be paid out from the company successively in the future. We also assume that in-cash flow is equal to the expected value of premium on the same level as benefit, which will be paid in from policyholders successively during the signed contract period. Net cash flow is simply equal to the difference between the out- and in-cash flows under the condition that policyholders are alive during this period.

6.3 CASH FLOW VALUATION

We start with a simple insurance product for the needs of occupational pension, wherein benefit *S* will be paid out successively to a policyholder at the year from which he has reached 65, as long as he is alive but no longer than 5 years. During the signed contract period, the policyholder will pay his premium until he is 65. Our policyholder here is 60 years old at the point of time *t* when the calculation is carried out.

At the point of time *t*, out- and in-cash flows for a policyholder are calculated by the formulas below.

$$\text{out – cash flow} \quad \text{b(t)} = \frac{N(65) - N(70)}{D_t(60)} * S \quad \Leftrightarrow \quad \text{b(t)} = S * \frac{\int_{65}^{70} D(u)du}{D_t(60)}$$

$$\text{in – cash flow} \quad \text{a(t)} = \frac{N_t(60) - N(65)}{D_t(60)} * P \quad \Leftrightarrow \quad \text{a(t)} = P * \frac{\int_{60}^{65} D(u)du}{D_t(60)}$$

It should be noted that functions a(t) and b(t) are not directly dependent on *t,* but the age of our policyholder at the time point *t*. At the point of time *t,* the calculation of out- and in-cash flow on the collective level can be carried out in a similar way. In the formula below, *w(60)* is denoted for a total number of individuals who are 60 years old in the year of calculation and they have exactly the same insurance contract.

$$\text{out – cash flow} \quad \text{B(t)} = \frac{N(65) - N(70)}{D_t(60)} * S * w(60)$$

$$\text{in – cash flow} \quad \text{A(t)} = \frac{N_t(60) - N(65)}{D_t(60)} * P * w(60)$$

Formulas above give us a comprehensive view of development of benefit and premium from statistic perspective, but now we want to shift our focus into technical detail, which means the formula will be deduced in a discrete context.

$$\frac{N(65) - N(70)}{D_t(60)} = \frac{N(65) - N(66)}{D_t(60)} + \frac{N(66) - N(67)}{D_t(60)} + \cdots + \frac{N(69) - N(70)}{D_t(60)}$$

We can even more develop the function above under a condition that pension will be paid out from the insurance company to the policyholder in the beginning of each year.

$$\begin{aligned}\frac{N(65) - N(70)}{D_t(60)} &= \frac{N(65) - N(66)}{D_t(60)} + \frac{N(66) - N(67)}{D_t(60)} + \cdots + \frac{N(69) - N(70)}{D_t(60)} \\ &= \frac{l(65)}{l_t(60)} \cdot e^{-\delta \cdot 5} + \frac{l(66)}{l_t(60)} \cdot e^{-\delta \cdot 6} + \cdots + \frac{l(69)}{l_t(60)} \cdot e^{-\delta \cdot 9}\end{aligned}$$

Furthermore, let *V* (*t*) denote the present value of net cash flows in a life insurance company at point of time *t*, e.g. *V* (*t*) =*B (t) - A (t).*

7 PROBLEM OF MISMATCH

7.1 MISMATCH BETWEEN THE ASSET AND THE LIABILITY

Mismatching between the assets and the liabilities is a very problematic issue in the financial area because it has a multiple property. Mismatching in a life insurance company is sculptured out of life insurance mathematics due to the actuarial assumptions made in insurance subsidiaries. A traditional way to scrutinize this problem is to partition it into segments, in which the assets and liabilities are mismatched. It should be noted that in this paper the mismatching problem has been localized on the level where we are supposed to get a suitable balance between the restricted assets and the technical provisions.

Let us for the moment leave the provisions aside and concentrate upon the restricted assets. For an asset portfolio consists of interest rate instruments, there are at least three mismatched segments that a life actuary must to take into account. These segments are mismatched present values, durations and liquidities. As a well known financial solution, immunization has been used to reduce the impact of interest rate fluctuations that generate the mismatch in the present value and duration. By the help of this approach, these two mismatched segments can successfully bridged over, we can even get a better match between the assets and liabilities if their convexities could be equal. But unfortunately, the immunization cannot contribute anything for solving the mismatch in the liquidity, which means at a certain moment the value of asset portfolio is not enough for the liabilities or adversely much more than necessary. Both situations will cost a company huge money, because the first possibility will force an insurer to make credits, and in the case of temporary surplus, we have to keep them under the mattress to fit the future payments. A possible compensation for immunization is to have a buffer capital for such happenings.

For an asset portfolio consists of more than interest rate instruments, for instance, shares and properties, ALM (Asset-Liability Model) seems to be a reasonable approach to deal with the problem of mismatch. ALM is a modeling trying to capture the stochastic uncertainties involved in the assets and liabilities. It provides the insurers a solution in which they have a dynamic control over the development of their assets and liabilities, provided that the structure of the assets and liabilities are unchanged through the ages. However, the reality for a life insurer is much more complicated in the sense that the actuarial assumptions do change over time. There is limitation in ALM to keep up a correspondence to such uncertainties. A possible compensation for the ALM is to build up a realistic model for the in- and out- cash flows in a life insurance company. This model will reflect all uncertainties hid behind the assets and insurance liabilities, e.g. the financial, operational and insurance risks.

7.2 THE RESTRICTED ASSETS

Building up a realistic model is out of this work because the object of this paper is protecting the guaranteed undertakings from the interest rate risk with a high degree of certainty, which means the asset portfolio is not designed for a better return but matching the guaranteed undertakings. A portfolio consists of only governments bonds should be suitable for this purpose. It would be easy for us to solve the problem if the method of immunization can be applied in this case, but unfortunately this approach is not available due to the short maturities (almost 15 years for the longest one) that Swedish government bonds have.

Our strategy for solving the mismatch problem is creating an asset portfolio whose cash flows are distributed more like our insurance liabilities; furthermore the portfolio is expected to have more tolerance to the interest rate fluctuations. The point of departure is testing reasonable combinations of bonds that could satisfy our requirements above. We decided to have an asset portfolio that consists of government bonds SO-1037, 1040, 1045, 1049 and 1047 with the consideration of their different long durations. Based on the tests we finally managed up an asset portfolio where the magnitudes of each bond is given to 2400, 1800, 400, 300 and 225 standard respectively. Table 7.2-1 listed the present value of our assets that are designed to cover our determined technical provisions. The present value calculation is carried out on the basis of three estimated yield curves. Two scenarios - buffer capital and solvency ratio are introduced to describe the positions of our restricted assets and technical provisions. The value of our buffer capital is obtained by adding the present values of the assets and provisions. Solvency ratio is equal to the absolute value of the division between the realistic value of the restricted assets and technical provisions. *PV* stands here for the realistic values.

Table7.2-1: Realistic values of restricted assets and technical provisions calculated on the basis of different yield curves. Unit value is milliard Swedish kronor.

	Mars 31	April 28	August 1
YTM(10)	3.6310%	3.8421%	3.8130%
***PV* (provisions)**	-3,804	-3,768	-3,572
***PV* (restricted assets)**	5,583	5,580	5,543
SO-1037	2400	2400	2400
SO-1040	1800	1800	1800
SO-1045	400	400	400
SO-1049	300	300	300
SO-1047	225	225	225
Buffer capital	1,780	1,812	1,971
Solvency ratio	**1.47**	**1.48**	**1.55**

Based on the earlier studies we know that the changes in the yield curves made the life insurance companies vulnerable due to the long duration for their insurance liabilities. With the help of Extended Nelson-Siegel modeling, we are capable to carry out a stress test on the yield curves aiming to prognosticating the development of our assets and liabilities corresponding to the changes in interest rates.

In the stress test, we made a series of paralleled shifts on each of original yield curves estimated on the different time points. The magnitude of each shift is 5 basis points

(5bp = 0.05%). Our created asset portfolio passed the stress tests by a good tolerance for a broad paralleled shift (from -150bp to +150bp) on the yield curves.

A part of results of stress tests is showed in the tables below in which we found that each +50bp stressed on these yield curves leads to a reduction by about 20 percent for the solvency ratio; each -50bp stressed for the curves gives an increment by about 16 percent for the solvency ratio. The most important signal for a life insurance company is the solvency ratio getting close to the crucial level 1 when the interest rate going down with 1%.

Table7.2-2: Paralleled shift by ± 50bp stressed on the different yield curves. Unit value is milliard Swedish kronor.

	+50bp			**-50bp**		
	Mars 31	***April 28***	***August 1***	***Mars 31***	***April 28***	***August 1***
YTM(10)	4,1310%	4,3421%	4,3130%	3,1310%	3,3421%	3,3130%
Paralleled shift	0,5000%	0,5000%	0,5000%	-0,5000%	-0,5000%	-0,5000%
PV(provisions)	-3, 344	-3, 309	-3, 146	-4, 341	-4, 304	-4, 068
PV(restricted assets)	5, 535	5, 535	5, 495	5, 633	5, 628	5, 593
SO-1037	2400	2400	2400	2400	2400	2400
SO-1040	1800	1800	1800	1800	1800	1800
SO-1045	400	400	400	400	400	400
SO-1049	300	300	300	300	300	300
SO-1047	225	225	225	225	225	225
Buffer capital	2,191	2,225	2,349	1, 292	1, 323	1, 525
Solvency ratio	**1,66**	**1,67**	**1,75**	**1,30**	**1,31**	**1,37**

Table7.2-3: Paralleled shift by 100bp stressed on the different yield curves. Unit value is milliard Swedish kronor.

	+100bp			**-100bp**		
	Mars 31	***April 28***	***August 1***	***Mars 31***	***April 28***	***August 1***
YTM (10)	4,6310%	4,8421%	4,8130%	2,6310%	2,8421%	2,8130%
Paralleled shift	1,0000%	1,0000%	1,0000%	-1,0000%	-1,0000%	-1,0000%
PV (provision)	-2, 949	-2, 916	-2, 780	-4, 972	-4, 935	-4, 649
PV (restricted assets)	5, 489	5, 490	5, 449	5, 685	5, 677	5, 644
SO-1037	2400	2400	2400	2400	2400	2400
SO-1040	1800	1800	1800	1800	1800	1800
SO-1045	400	400	400	400	400	400
SO-1049	300	300	300	300	300	300
SO-1047	225	225	225	225	225	225
Buffer capital	2, 540	2, 575	2, 670	0, 714	0, 742	0, 995
Solvency ratio	**1,86**	**1,88**	**1,96**	**1,14**	**1,15**	**1,21**

7.3 SENSITIVITY TO THE INTEREST RATE RISK

Even though the method of immunization cannot be applied in our case, it is still important for an actuary to control how the restricted assets, technical provisions and buffer capital behave corresponding to the changes in interest rate.

We already know from section '*Term structure of interest rate*', duration and convexity are two powerful risk measures that can be used to assess the interest rate risk. Duration provides the information about the slope of the line tangent to the price -yield curve at a certain point, and its magnitude is ($-D_{\mathrm{mod}\,ified} * P$). With the help of convexity we can get even better approximation for the price-yield curve because it provides the relative curvature at a given point on the curve.

However in this section, we will leave these two well known financial measures behind and introduce another interest rate risk measures, e.g. delta yield and gamma yield. Generally delta value shows the sensitivity of the present value to the changes in the main source of risk of an instrument. In our case the delta yield indicates the changes in the present value corresponding to each one basis point shift made on the yield curves. It should be noted that the day count fraction in our case is Act/365. The delta yield can either refer to an upward or a downward shift of the curves. The general formulas for delta yield and gamma yield are given below.

$$\Delta_{yield} = \left[PV(r+h) - PV(r)\right] * scale$$

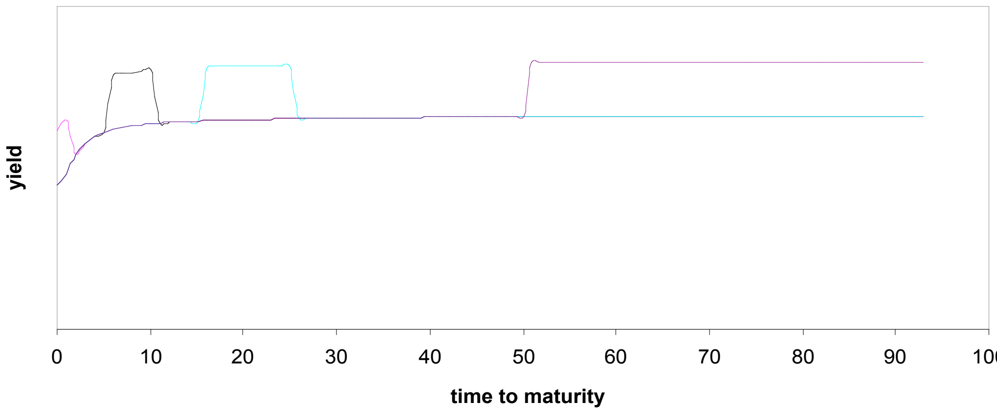
where $h = \pm 0.00001$ and $scale = 1000$.

$$\Gamma_{yield} = \left[\partial(y+h) - \partial(y)\right] = \left[PV(y+2h) - 2PV(y+h) + PV(y)\right] * scale$$

where $h = \pm 0.00001$ and $scale = 1000$.

According to the definitions and the formulas of delta yield and gamma yield; we realized that these values are reflected the shifts made on the entire yield curves. However, what a life insurance company really desires to know is how much capital will be exposed to the changes in the short or long interest rate, respectively. So instead of being satisfied with the calculations of delta and gamma yield, we want to go further to investigate what will happen with the restricted assets, technical provisions and buffer capital if a shift made on a segment of the yield curves. As we will see later, the yield curve delta and gamma can be broken down into different time buckets, and we should not be surprised either if the sum of delta/gamma yields corresponding to the different time buckets represents a one basis point shift on the entire curve. There are different kind of shifting in the time buckets, for instance, rectangle shift, triangle shift and smooth shift. Choosing which kind of shift depends on the purpose of study. In our case, the rectangle shift was chosen with the consideration of simple application and understanding of this kind of shift. Figure 7.3-1is an example for the rectangle shift, there we made four time buckets in order 0-1, 5-10, 15-25 and longer than 50 years.

Figure 7.3-1: Rectangle shift with different time buckets on the yield curve.



We start our analysis with calculating delta yield and gamma yield for our restricted assets, technical provisions and buffer capital respectively. The yield curve that used for this study is the yield curve estimated on August 1, 2006. The magnitude of step h was decided to be equal to $\pm 0{,}001\%$. The results are summarized in the following tables and figures. We can readily see from the figures that delta yield and gamma yield move symmetrically around the point where h is equal to 0, delta yield and gamma yield for the restricted assets and technical provisions have different sign, which is not a surprise because we already know that the assets and insurance liabilities will move to the different directions when interest rate go up or down with a certain percentage. Delta yield and gamma yield show once again that the technical provisions have higher sensitivity to the fluctuations in interest rate due to its long duration. The value of delta yield and gamma yield for the buffer capital is quite near to the values for the technical provisions, which indicates that the insurance liabilities have a dominate effect in the economical status. We also realized that the asset portfolio we managed up is not high qualified even though such a portfolio passed the stress tests with good solvency ratios. Table 7.3-1 listed the values of delta yield and gamma yield for the restricted assets, technical provisions and buffer capital.

Table 7.3-1: Value of delta yield and gamma yield for the restricted assets, technical provisions and buffer capital

		Restricted capital		Technical provision		Buffer capital	
Step h	**scale**	***delta***	***gamma***	***delta***	***Gamma***	***delta***	***gamma***
-0.015%	-1.5	-2,200,349	3,549	20,690,021	-142,534	18,489, 672	-138,984
-0.014%	-1.4	-1,916,680	2,693	18,020,539	-108,160	16,103,859	-105,466
-0.013%	-1.3	-1,652,588	2,003	15,535,731	-80,413	13,883,143	-78,411
-0.012%	-1.2	-1,408,072	1,454	13,235,513	-58,382	11,827,441	-56,928
-0.011%	-1.1	-1,183,129	1,027	11,119,799	-41,221	9,936,670	-40,195
-0.010%	***-1***	***-977,757***	***701***	***9,188,505***	***-28,155***	***8,210,748***	***-27, 454***
-0.009%	-0.9	-791,955	460	7,441,546	-18,472	6,649,591	-18,012
-0.008%	-0.8	-625,720	287	5,878,838	-11,532	5,253,118	-11,245

-0.007%	-0.7	-479,050	168	4,500,294	-6,760	4,021,244	-6,592
-0.006%	-0.6	-351,942	91	3,305,831	-3,649	2,953,889	-3,558
-0.005%	-0.5	-244,396	44	2,295,364	-1,760	2,050,968	-1,716
-0.004%	-0.4	-156,408	18	1,468,807	-721	1,312,400	-703
-0.003%	-0.3	-87,976	6	826,077	-228	738,101	-222
-0.002%	-0.2	-39,099	1	367,089	-45	327,990	-44
-0.001%	-0.1	-9,774	0.07	91,758	-3	81,984	-3
0.000%	***0***	***0***	***0.00***	***0***	***0***	***0***	***0***
0.001%	0.1	-9,774	0.07	91,730	-3	81,956	-3
0.002%	0.2	-39,093	1	366,864	-45	327,770	-44
0.003%	0.3	-87,957	6	825,317	-228	737,360	-222
0.004%	0.4	-156,363	18	1,467,005	-721	1,310,643	-703
0.005%	0.5	-244,308	44	2,291,844	-1,760	2,047,536	-1,716
0.006%	0.6	-351,791	91	3,299,749	-3,649	2 947 959	-3,558
0.007%	0.7	-478,809	168	4,490,637	-6,760	4,011,828	-6,592
0.008%	0.8	-625,361	287	5,864,422	-11,532	5,239,061	-11,245
0.009%	0.9	-791,444	460	7,421,022	-18,472	6,629,578	-18,012
0.010%	***1***	***-977,056***	***701***	***9,160,351***	***-28,155***	***8,183,294***	***-27, 454***
0.011%	1.1	-1,182,196	1,027	11,082,325	-41,221	9,900,129	-40,195
0.012%	1.2	-1,406,860	1,454	13,186,861	-58,382	11,780,001	-56,928
0.013%	1.3	-1,651,048	2,003	15,473,875	-80,413	13,822,827	-78,411
0.014%	1.4	-1,914,756	2,693	17,943,282	-108,160	16,028,526	-105,466
0.015%	1.5	-2,197,983	3,549	20,594,999	-142,534	18,397,016	-138,984

In the analysis of shifting segments of the yield curve, we did the following restrictions to simplify the calculations. We decided to make a shift with magnitude one basis point, and moreover we chose the underlying capital in our analysis to be the net cash flows that indicate how much capitals are available at each point of time. We summarized our results in table 7.3-2. If we compare the result with the values that we obtained for the buffer capital, (delta yield =**8 183 294** and gamma yield **= -27 454**, see table 7.3-1), we realized these values are quite close to each other. It verified that our statement made earlier in this section, the sum of different time buckets should represent the shift on the whole yield curve.

Table7.3-2: Investigation summary for sifting segments of the yield curve with one basis point

Time bucket (year)	Delta yield	Gamma yield
0-1	-19,701	4
1-2	364,118	106
2-5	-217,063	120
5-10	5,702	-23
10-15	637,292	-911
15-25	2,124,773	-4,560
25-50	5,476,851	-19,365
>50	514,578	-2,808
Total value	**8,158,313**	**27,437**

Based on table 7.3-1, we made up figures 7.3-2 where we put together the delta yield for the restricted assets, technical provisions and buffer capital. In figure 7.3-3 we put together their gamma yield. The purpose for doing so is to describe the financial risk status for a life insurance company. It is useful for an actuary to have such information besides the balance sheet, so he or she can possibly have a whole picture of the economic growth for his or her company. As we mentioned in the very beginning of this paper, the cornerstone of the Traffic Light System is to protect policyholders from the financial uncertainties and in our case especially to protect them from the interest rate fluctuations.

Figure 7.3-2: Delta yield for the restricted assets, technical provisions and buffer capital. Unit value is million Swedish kronor

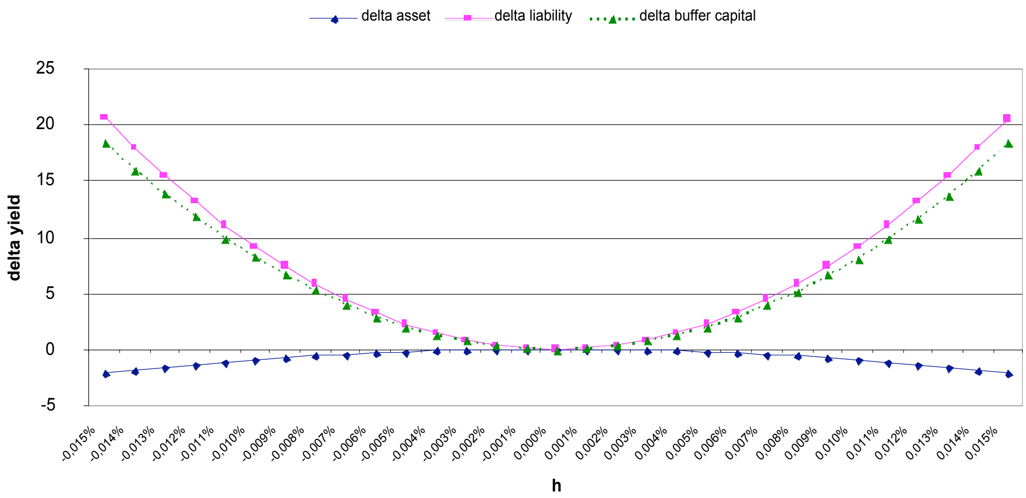
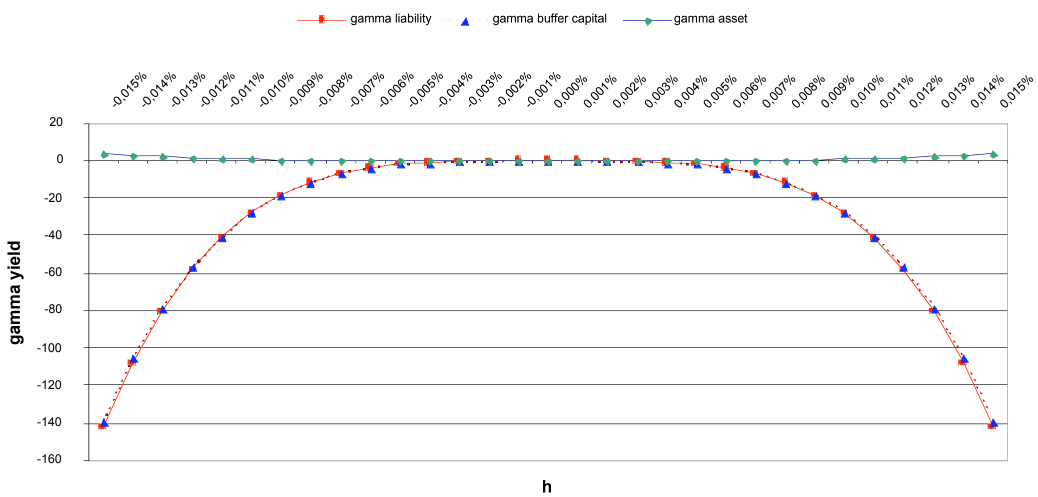


Figure7.3-3: Gamma yield for the restricted assets, technical provisions and buffer capital. Unite value is thousands Swedish kronor



8 CONCLUSION

Every insurer desires to have the suitable balance between the assets and liabilities aiming to improve growth, profit and risk control. An insurer may be exposed to many different types of risks, and the unique one amongst them is insurance risk. It requires an insurer to utilize a specific risk-control method that differs from the general financial management. As a response to the IORP Directive, the Traffic Light System is assigned by the Swedish Financial Supervisory Authority to obtain more complete control over the life insurers in Sweden.

This paper examines the impact of interest rate risk on the life insurance assets and liabilities. The main work has been estimating the yield curves and sensitivity testing. Stress test of interest rates has been applied for this purpose in order to achieve a suitable balance between the assets and liabilities in such a way that fluctuations in interest rates by shifts do not affect the financing of liabilities.

The technical provisions have been valued at their realistic value, which means at every given point of time the technical provisions have been discounted on the basis of a yield curve estimated by Extended Nelson Siegel model. Four yield curves (estimated on the different point of time and underlying instruments) have been applied in our analysis. The results delivered from these curves showed that the estimates are time dependent. Furthermore as a consequence of the dependency, the estimates have a great impact on the realistic value of our technical provisions. It motivates the requirement from the Swedish authority about estimating reasonable discount rates.

As we mentioned in the beginning of this paper, our focus is interest rate risk. That means mismatch between life insurance assets and liabilities caused by interest rate fluctuations is the issue for this paper. Several stress tests have been applied for investigating the problem. Based on the portfolio assigned for the restricted assets, we carried out stress tests to see if such a portfolio will exceed the required solvency ratio. The results showed that the created portfolio has good tolerance for the paralleled shift (from -150 bp up to +150 bp) on the entire yield curves. However, the solvency ratio will get close to the crucial level when the interest rates are going down with one percentage.

Stress tests are considered as a tool for examining what might happen in a particular stress scenario. However, it should be noted that stress tests do not predict what will happen. Moreover, stress tests are not suitable approaches for an insurer who expects to apply techniques that are appropriate for the whole risk profile and the business undertaken. We have mentioned before in section '*Life insurance mathematics*', we believe a complex realistic modeling should be an appropriate tool for the life insurers.

Furthermore, for the purpose of analyzing the interest rate risk, we decided to have simple structure for both insurance assets and liabilities. However, with the consideration of life insurers' reality, the technical provisions might be calculated in such a way that they could reflect the reality of the insurance contracts. In the case of the restricted assets, their simple structure made extremely difficult for insurers to obtain a suitable balance aiming to improve their risk control. A following work after this paper could be to create a realistic portfolio with more complex structure for the restricted assets, shares and interest rate swaps could be considered in such a case.

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10 APPENDIX

- The yield function $R(m,\Theta,t)$ in Nelson-Siegel model

$$R(m,\Theta,t)=\frac{1}{m}\int_0^m f(x,\Theta,t)dx$$

$$=\frac{1}{m}\int_0^m\left[\beta_{0,t}+\beta_{1,t}\cdot\exp(-x/\tau_{1,t})+\beta_{2,t}\cdot\left(\frac{x}{\tau_{1,t}}\exp(-x/\tau_{1,t})\right)\right]dx$$

$$=\frac{1}{m}\int_0^m\beta_{0,t}dx$$

$$+\frac{1}{m}\int_0^m\left[\beta_{1,t}\cdot\exp(-x/\tau_{1,t})\right]dx$$

$$+\frac{1}{m}\int_0^m\left[\beta_{2,t}\cdot\left(\frac{x}{\tau_{1,t}}\exp(-x/\tau_{1,t})\right)\right]dx$$

$$=\left\{y=\frac{x}{\tau_{1,t}},dx=\tau_{1,t}dy\right\}$$

$$=\beta_{0,t}+\frac{\beta_{1,t}}{m}\int_0^{\frac{m}{\tau_{1,t}}}\tau_{1,t}e^{-y}dy+\frac{\beta_{2,t}}{m}\int_0^{\frac{m}{\tau_{1,t}}}\tau_{1,t}\cdot y\cdot e^{-y}dy$$

$$=\beta_{0,t}+\frac{\beta_{1,t}\cdot\tau_{1,t}}{m}\int_0^{\frac{m}{\tau_{1,t}}}e^{-y}dy+\frac{\beta_{2,t}\cdot\tau_{1,t}}{m}\int_0^{\frac{m}{\tau_{1,t}}}y\cdot e^{-y}dy$$

$$=\beta_{0,t}+\beta_{1,t}\cdot\frac{\tau_{1,t}}{m}\cdot\left|-e^{-y}\right|_0^{\frac{m}{\tau_{1,t}}}+\beta_{2,t}\cdot\frac{\tau_{1,t}}{m}\cdot\left[\left|-ye^{-y}\right|_0^{\frac{m}{\tau_{1,t}}}+\int_0^{\frac{m}{\tau_{1,t}}}-e^{-y}dy\right]$$

$$=\beta_{0,t}+\beta_{1,t}\cdot\left(\frac{1-\exp(m/\tau_{1,t})}{m/\tau_{1,t}}\right)+\beta_{2,t}\cdot\left(\frac{1-\exp(m/\tau_{1,t})}{m/\tau_{1,t}}-\exp(m/\tau_{1,t})\right)$$

- The form of Extension Temporary Pension in a discrete context

$$\frac{N(65)-N(70)}{D_t(60)}=\frac{N(65)-N(66)}{D_t(60)}+\frac{N(66)-N(67)}{D_t(60)}+\cdots+\frac{N(69)-N(70)}{D_t(60)}$$

$$=\frac{\int_{65}^{66}D(t)dt}{D_t(60)}+\frac{\int_{66}^{67}D(t)dt}{D_t(60)}+\cdots+\frac{\int_{69}^{70}D(t)dt}{D_t(60)}\quad\text{due to the def. of }N(x)$$

$$=\{\text{restricted condition}\}$$

$$
\begin{aligned}
&= \frac{D(65)}{D_t(60)} + \frac{D(66)}{D_t(60)} + \cdots + \frac{D(69)}{D_t(60)} \\
&= \frac{l(65)\cdot e^{-\delta\cdot 65}}{l_t(60)\cdot e^{-\delta\cdot 60}} + \frac{l(66)\cdot e^{-\delta\cdot 66}}{l_t(60)\cdot e^{-\delta\cdot 60}} + \cdots + \frac{l(69)\cdot e^{-\delta\cdot 69}}{l_t(60)\cdot e^{-\delta\cdot 60}} \qquad \text{due to the def. of } D(x) \\
&= \frac{l(65)}{l_t(60)}\cdot e^{-\delta\cdot 5} + \frac{l(66)}{l_t(60)}\cdot e^{-\delta\cdot 6} + \cdots + \frac{l(69)}{l_t(60)}\cdot e^{-\delta\cdot 9}
\end{aligned}
$$