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A method to minimize the expected long-term cost for a policy holder of an insurance

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Abstract

What can we do in order to reduce our long-term costs if we are a policy-holder of an insurance? This is the problem this paper is about. One concept is that we could choose to pay a possible damage ourselves in order to prevent a future increase of the premium. This concept extends with a way to determine an optimal excess. We balance here the lower premium with the undertaking of paying a part of a possible damage ourselves. Finally we examine if we would win anything by investing in any damage preventive measures. Now we balance the advantages with a lower premium and lower risk for damages with the actual costs of the measures. A conclusion that could be drawn from this paper is that for at least some insurances the costs for a policy-holder could be reduced quite much if good choices are made.

Preface

This is my Master's thesis in mathematical statistics at Stockholm University. The problem this paper deals with is which choices policy-holders should make in order to reduce their long-term costs. This paper should be useful for people and companies that have an insurance. However, insurance companies could use this paper as well when they are going to determine premiums. Perhaps this paper also could be used in similar areas if some changes are made. The theory used in this paper is how to control a Markov process by making optimal choices. There are no theorems and proofs in this paper. However if one realizes that the methods in this paper are the same as in a book that describes the theory more carefully there should be no question about if the results from this paper are correct.

How should this paper be read? I think that one should first understand chapter 2 in order to be familiar with the problem and the theory. Chapter 3 should then be understood before going into the latter chapters. The chapters in this paper are very similar to each other and once one has understood the concept used it should not be too difficult to understand this paper. However, the latter chapters could be quite messy but the concept is still the same. The strategy in this paper should be possible to understand from the figures. If the mathematical formulas are hard to understand the results from the examples should hopefully be understood anyway. The assumed values in these examples may be a little unrealistic.

Why did I choose to write about this? The reason is that my supervisor recommended this theory and I wanted to write about something related to insurance. After some time I figured out the problem in this paper. The article mentioned in the reference list that my supervisor has written have been very useful for me. I would like to thank my supervisor Anders Martin-Löf for these things and for encouraging me to write about this. I would also like to thank Monica Bäfverfeldt for her valuable comments and for helping me with LaTeX-related problems.

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Chapter 1

Introduction

A problem that there has been written a lot about is how an insurance company should determine premiums for insurances. This is not the main reason for this paper even if it could be useful for that purpose as well. The main purpose of this paper is to examine what a policy-holder of an insurance can do in order to try to reduce the long-term costs. This problem has not been written about to the same extent. The concept is that we want to prevent future increases of the premium and instead reduce them. In order to do this we deal with issues like, when to claim a damage, what excess to have and which damage preventive measures to have. This should be reasonable problems to analyze. However the immediate costs of some of these actions might be too large and then it won't be profitable, not even in the long run.

We can see the problem as a Markov decision process which uses the theory of dynamic programming. We here control a Markov process by choosing optimal choices during time. We will start by giving a short summary of what a Markov decision process is in next chapter. This chapter is sort of a survey while the rest of this paper examines these things more carefully.

Chapter 2

Markov Decision Processes

A Markov decision process is a Markov process that we can influence by different decisions in frequent "decision epochs". Assume that we have a Markov process with transition probabilities given by $P(i,j)$, that means that the probability for a transition from state i to state j is given by $P(i,j)$ where $i,j = 1,2,...,N$. Also assume that a transition from state i to state j has a certain cost given by $C(i,j)$. The last assumption is that we can influence the transition probabilities and the transition costs through a control variable u. I.e. the transition probabilities are $P(i,j|u)$ instead and the transition costs are $C(i,j|u)$ when we are choosing the control u in state i, where $u(i) \in D(u(i))$. $D(u(i))$ is not necessarily the same for every i.

What can be done in order to minimize the costs for a long period of time? In this chapter we intend to minimize the total expected discounted cost during both a finite and an infinite time horizon by choosing an optimal $u(i)$ in every decision epoch. The discount factor is given by r. We begin with the finite case.

2.1 Finite time-horizon

Let $V_n(i)$ be the total expected discounted cost in n stages if an optimal policy is followed and our present state is i. In other words $V_n(i)$ is the expected present value of all costs during the next n stages. An optimal policy means that in every decision epoch we choose the control u that minimize the remaining expected discounted cost. How do we determine $V_n(i)$? We use the simple principle that the total cost is the sum of the immediate cost and future costs. Hence we must balance the the wish for low immediate cost with increased future costs. With this principle in mind $V_n(i)$ can be determined from

$$V_n(i) = \min_u \left[\sum_{j=1}^{N} P(i,j|u) \cdot C(i,j|u) + r \sum_{j=1}^{N} P(i,j|u) \cdot V_{n-1}(j) \right] \tag{2.1}$$

The first part of the right side of (2.1) is the expected immediate cost and the second part is the expected future costs. Equation (2.1) is sometimes called the equation of dynamic programming. Assume that $V_0(i)$ are given $\forall\ i$. In many cases $V_0(i)$=0. Now we can determine $V_n(i)$ from (2.1) by iteration.

Example 1

This example is sort of a survey of this paper but has some obvious limitations. Compare with section 6.2.

Imagine that we have a factory and for that factory we have an insurance against fire, burglary and similar things. The insurance has three bonus classes with different premiums. Let the state be given by (k, x). Here k is the bonus class during a year and x is the cost of the damages during the same year. We assume that there are only three possible amount of damages x_1, x_2 and x_3. The insurance company has told us that we can reduce our premium cost if we use some damage preventive measures. These measures affect the probabilities for damages and hence also the transition probabilities. We can also choose to pay the damages on our own so that we don't loose bonus and prevent an increase of the premium. The control variable u is thus given by $u = (u_1, u_2)$ where in this example

$$u_1 = \Big(1(\text{if measure 1 is used next year}), 1(\text{if measure 2 is used next year})\Big)$$

Here measure 1 is a fire alarm and measure 2 is a watchman. For example if we choose to use a firealarm but no watchman next year then $u_1 = (1, 0)$.

$$u_2 = 1(\text{if we pay the damage we had last year on our own})$$

So if we claim a damage $u_2 = 0$. The changes of bonus classes are given by

$$k \rightarrow \begin{cases} a(k,1) & \text{if } u_2 = 0 \text{ and } x = x_1 \\ a(k,2) & \text{if } u_2 = 0 \text{ and } x = x_2 \\ a(k,3) & \text{if } u_2 = 0 \text{ and } x = x_3 \end{cases}$$

Thus $a(k, j)$ is the bonus class we will come to if we were in bonus class k and claim the damage x_j. In this example we have

$$a(k,j) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix}$$

On the other hand

$$k \to b(k) \qquad \text{if } u_2 = 1 \ \forall\ x$$

Thus $b(k)$ is the bonus class we will come to if we are in bonus class k and pay the damage on our own. Then naturally

$$b(k) = \begin{pmatrix} 1 & 1 & 2 \end{pmatrix}$$

The probability for having the damage x_j when we have damage preventive measures given by u_1 is denoted $p_j(u_1) = P(X = x_j|u_1)$. Keep in mind that we know which bonus class we will come to in the end of the year when we have observed X. Thus if we claim a damage the transition probabilities are given by

$$P((k,x_i),(l,x_j)|u = (u_1,0)) =$$

$$\begin{pmatrix}
p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_1(u_1) & p_2(u_1) & p_3(u_1) \\
p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_1(u_1) & p_2(u_1) & p_3(u_1) \\
0 & 0 & 0 & 0 & 0 & 0 & p_1(u_1) & p_2(u_1) & p_3(u_1) \\
0 & 0 & 0 & p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & p_1(u_1) & p_2(u_1) & p_3(u_1) \\
0 & 0 & 0 & 0 & 0 & 0 & p_1(u_1) & p_2(u_1) & p_3(u_1)
\end{pmatrix}$$

and if we pay the damage on our own

$$P((k,x_i),(l,x_j)|u = (u_1,1)) =$$

$$\begin{pmatrix}
p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0 \\
0 & 0 & 0 & p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0 \\
0 & 0 & 0 & p_1(u_1) & p_2(u_1) & p_3(u_1) & 0 & 0 & 0
\end{pmatrix}$$

The corresponding transition costs are

$$C((k,x_i),(l,x_j)|u=(u_1,0)) =$$

$$\begin{pmatrix} c(1,u_1) & c(1,u_1) & c(1,u_1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c(2,u_1) & c(2,u_1) & c(2,u_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c(3,u_1) & c(3,u_1) & c(3,u_1) \\ c(1,u_1) & c(1,u_1) & c(1,u_1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c(3,u_1) & c(3,u_1) & c(3,u_1) \\ 0 & 0 & 0 & 0 & 0 & 0 & c(3,u_1) & c(3,u_1) & c(3,u_1) \\ 0 & 0 & 0 & c(2,u_1) & c(2,u_1) & c(2,u_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c(3,u_1) & c(3,u_1) & c(3,u_1) \\ 0 & 0 & 0 & 0 & 0 & 0 & c(3,u_1) & c(3,u_1) & c(3,u_1) \end{pmatrix}$$

and

$$C((k,x_i),(l,x_j)|u=(u_1,1)) =$$

$$\begin{pmatrix} c(1,u_1)+x_1 & c(1,u_1)+x_1 & c(1,u_1)+x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ c(1,u_1)+x_2 & c(1,u_1)+x_2 & c(1,u_1)+x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ c(1,u_1)+x_3 & c(1,u_1)+x_3 & c(1,u_1)+x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ c(1,u_1)+x_1 & c(1,u_1)+x_1 & c(1,u_1)+x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ c(1,u_1)+x_2 & c(1,u_1)+x_2 & c(1,u_1)+x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ c(1,u_1)+x_3 & c(1,u_1)+x_3 & c(1,u_1)+x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c(2,u_1)+x_1 & c(2,u_1)+x_1 & c(2,u_1)+x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c(2,u_1)+x_2 & c(2,u_1)+x_2 & c(2,u_1)+x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c(2,u_1)+x_3 & c(2,u_1)+x_3 & c(2,u_1)+x_3 & 0 & 0 & 0 \end{pmatrix}$$

$c(k,u_1) = p(k,u_1) + c(u_1)$ where $p(k,u_1)$ is the premium cost for bonus class k when we are using measures given by u_1. $c(u_1)$ is the cost for using the measures given by u_1.

We assume the following values:

$$\begin{array}{lllllllll} p_1((0,0)) & = & 0.68 & p_2((0,0)) & = & 0.21 & p_3((0,0)) & = & 0.11 \\ p_1((0,1)) & = & 0.82 & p_2((0,1)) & = & 0.14 & p_3((0,1)) & = & 0.04 \\ p_1((1,0)) & = & 0.75 & p_2((1,0)) & = & 0.17 & p_3((1,0)) & = & 0.08 \\ p_1((1,1)) & = & 0.86 & p_2((1,1)) & = & 0.11 & p_3((1,1)) & = & 0.03 \end{array}$$

$$\begin{array}{lllllllll} p(1,(0,0)) & = & 6 & p(2,(0,0)) & = & 10 & p(3,(0,0)) & = & 14 \\ p(1,(0,1)) & = & 4 & p(2,(0,1)) & = & 6 & p(3,(0,1)) & = & 8 \\ p(1,(1,0)) & = & 5 & p(2,(1,0)) & = & 8 & p(3,(1,0)) & = & 11 \\ p(1,(1,1)) & = & 3 & p(2,(1,1)) & = & 5 & p(3,(1,1)) & = & 7 \end{array}$$

$$c((0,0)) = 0 \quad c((0,1)) = 5.5 \quad c((1,0)) = 2.5 \quad c((1,1)) = 8$$

$$x_1 = 0 \quad x_2 = 10 \quad x_3 = 60$$

$$r = 0.9$$

We make the following notations for the control variable u in order to make things more clear.

$$
\begin{aligned}
((0,0),0) &= 0\\
((0,0),1) &= 1\\
((0,1),0) &= 2\\
((0,1),1) &= 3\\
((1,0),0) &= 4\\
((1,0),1) &= 5\\
((1,1),0) &= 6\\
((1,1),1) &= 7
\end{aligned}
$$

The optimal choises and corresponding values for V for different horizons and states are then:

u

state	(1,x_1)	(1,x_2)	(1,x_3)	(2,x_1)	(2,x_2)	(2,x_3)	(3,x_1)	(3,x_2)	(3,x_3)
$n=1$	0	0	2	0	2	2	0	2	2
$n=2$	0	4	2	0	2	2	4	2	2
$n=3$	0	4	2	0	1	2	4	2	2
$n=100$	0	4	2	0	1	2	4	2	2
$n=1000$	0	4	2	0	1	2	4	2	2

V

state	(1,x_1)	(1,x_2)	(1,x_3)	(2,x_1)	(2,x_2)	(2,x_3)	(3,x_1)	(3,x_2)	(3,x_3)
$n=1$	5.30	10.00	12.20	5.30	12.20	12.20	10.00	12.20	12.20
$n=2$	11.79	16.51	21.60	11.79	21.60	21.60	16.51	21.60	21.60
$n=3$	17.95	23.08	27.97	17.95	27.95	27.97	23.08	27.97	27.97
$n=100$	74.59	79.67	84.83	74.59	84.59	84.83	79.67	84.83	84.83
$n=1000$	74.59	79.67	84.83	74.59	84.59	84.83	79.67	84.83	84.83

So if we are in state $(2, x_2)$ we should choose no measures but pay the damage of 10 on our own if $n \geq 3$. V_n will then be about 84.59 for large n.

2.2 Infinite time-horizon

In the case with an infinite horizon we let $V(i)$ be the total expected discounted cost for an infinite time period if an optimal policy is followed and our present state is i. $V(i)$ will be determined from the uniqe solution of

$$V(i) = \min_u \left[\sum_{j=1}^{N} P(i,j|u) \cdot C(i,j|u) + r \sum_{j=1}^{N} P(i,j|u) \cdot V(j) \right] \tag{2.2}$$

To find the solution of (2.2) we must follow the algorithm given below.

1. Choose initial guesses $u_k(i)$, $(k = 0)$, for the control u $\forall$ i.
2. Determine the corresponding $V_k(i)$ from (2.2) by letting $u = u_k(i)$.
3. An improved control $u = u_{k+1}(i)$ can now be obtained by choosing the u $\forall$ i for which

$$\min_u \left[\sum_{j=1}^{N} P(i,j|u) \cdot C(i,j|u) + r \sum_{j=1}^{N} P(i,j|u) \cdot V_k(j) \right]$$

   is obtained.
4. We must make step 2-3 again until we have found the optimal control $u(i)$ which we have done when $V_k(i)$ has stopped decreasing $\forall$ i.

Example 1 (continued)

With this method the results are:

state	(1,x_1)	(1,x_2)	(1,x_3)	(2,x_1)	(2,x_2)	(2,x_3)	(3,x_1)	(3,x_2)	(3,x_3)
u	0	4	2	0	1	2	4	2	2
V	74.59	79.67	84.83	74.59	84.59	84.83	79.67	84.83	84.83

We see that this is exactly what we get when we have a long horizon in the finite case.

Chapter 3

A method to find the optimal claim-decision

The example from the last chapter can have more bonus classes and more possible damages but then things would be very unclear. We will now analyze things in another way. This chapter describes many of the things that will be used in the rest of this paper and should be read carefully.

Imagine that we are a policy holder of an insurance with a bonus system. It means that the premium cost depends on which bonus class we belong to. The bonus classes are $k = 1, 2, \ldots, K$ with a premium of $C(k)$ per year in bonus class k. However it's not necessary to see them as bonus classes. They could also be seen as different levels of the premium cost, which in practice is the same thing. The damage X for a year is the sum of all damages during the year and is supposed to be independent of damages from other years. The distribution of X is given below, where $F(x)$ is assumed to be a distribution of a continuous stochastistic variable.

$$F^*(x) = P(X \leq x) = P(X \leq x \mid X = 0)P(X = 0) + P(X \leq x \mid X > 0)P(X > 0) =$$

$$= p_0 + (1 - p_0)F(x)$$

In this chapter we consider the following transitions for the bonus classes. The transitions occur at the beginning of every year. If a claim is made the change of a bonus class k is

$$k \rightarrow a_k \qquad \forall\ x$$

where x is the cost of the damage. If a claim is not made, i.e. we pay the damage ourself, the change of a bonus class k is

$$k \to b_k \qquad \forall\ x$$

Of course b_k should be less than a_k when a small value of k means a more desirable bonus class than a high value of k. We assume that a_k, b_k, K and $C(k)$ are known, otherwise we have to make guesses about them. Hopefully the following analysis could give us some useful information anyway. The parameter p_0 and the distribution $F(x)$ are not known so they must be estimated.

As mentioned before this paper is about a strategy for minimizing the costs for a policy holder of an insurance. In this chapter we can influence these costs through the controlvariable u that could be chosen to a or b. The choice $u = a$ means that we claim the damage from a year which is a known amount. The choice $u = b$ means that we pay the damage on our own. These decisions are made at the end of every year. Let (k, x) be the state of the Markov decision process, where k is the bonus class we belonged to during a year and x is the amount of damages during the same year. The transition probabilities are accordingly:

$$P[(k, x) \to (u_k, y)] = F^*(dy)$$

where

$$u_k = \begin{cases} a_k & \text{if } u = a \\ b_k & \text{if } u = b \end{cases}$$

We can now start to develop the strategy.

3.1 Infinite time-horizon

The following section is mainly what is described in the article by Anders Martin-Löf, [2]. Let $V(k, x)$ be the total expected discounted costs in state (k, x) during an infinite time horizon with discount factor r under the optimal policy. $V(k, x)$ can be determined from

$$V(k, x) = \min_u \left[C(u_k, x) + r \left(p_0 \cdot V(u_k, 0) + (1 - p_0) \int_0^\infty V(u_k, y) F(dy) \right) \right]$$

where the immediate cost $C(u_k, x)$ is

$$C(u_k, x) = C(k, x|u) = \begin{cases} c(a_k) & \text{if } u = a \\ c(b_k) + x & \text{if } u = b \end{cases}$$

Here $c(k)$ is the premium cost for bonus class k. After some consideration we realize that the control variable u has the following form:

$$u = \begin{cases} b & \text{if } x \leq x(k) \\ a & \text{if } x > x(k) \end{cases}$$

$V(k,x)$ will look like figure 3.1. The line $v(k)$ corresponds to claiming a damage and the line $w(k) - x(k) + x$ corresponds to paying the damage on our own. The thick parts correspond to the optimal choice. $x(k)$ is the x-value when it doesn't matter if we choose $u = a$ or $u = b$. That is when $V(k,x)$ has the same value for the two choices. From figure 3.1 we see that when we have found the optimal values then $w(k) = v(k)$. The problem is thus to determine the optimal values for $x(k)$ and $v(k)$.

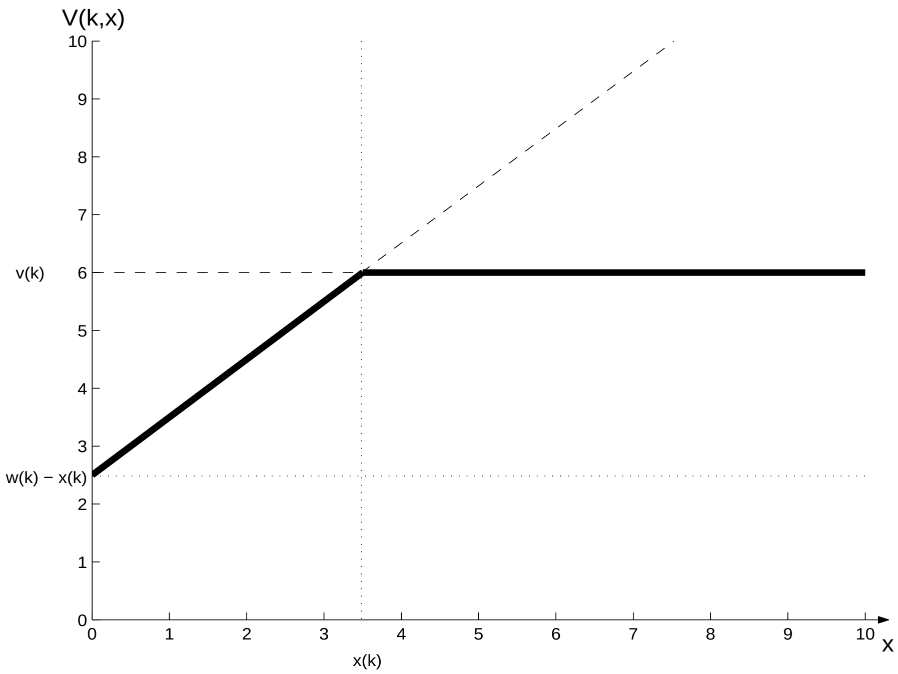

Figure 3.1: $V(k,x)$ for a fix k.

We now know that

$$V(k,x) = \begin{cases} C(b_k) + x + r\Big(p_0 \cdot V(b_k,0) + (1-p_0)\int_0^\infty V(b_k,y)F(dy)\Big) \\ \text{if } x \leq x(k) \\ C(a_k) + r\Big(p_0 \cdot V(a_k,0) + (1-p_0)\int_0^\infty V(a_k,y)F(dy)\Big) \\ \text{if } x > x(k) \end{cases} \tag{3.1}$$

We see from figure 3.1 that $V(k,x)$ is of the form

$$V(k,x) = \min[w(k) - x(k) + x; v(k)] \tag{3.2}$$

or

$$V(k,x) = \begin{cases} w(k) - x(k) + x & \text{if } x \leq x(k) \\ v(k) & \text{if } x > x(k) \end{cases} \tag{3.3}$$

for suitable choices of $w(k)$ and $v(k)$. This gives us the following expression for $V(k,0)$ and $\int_0^\infty V(k,y)F(dy)$.

$$V(k,0) = w(k) - x(k) := N(k)$$

$$\int_0^\infty V(k,y)F(dy) = \int_0^{x(k)}(w(k) - x(k) + y)F(dy) + \int_{x(k)}^\infty v(k)F(dy) =$$

$$= (w(k) - x(k))F(x(k)) + x(k)F(x(k)) - G(x(k)) + v(k)(1 - F(x(k))) =$$

$$= w(k)F(x(k)) - G(x(k)) + v(k)(1 - F(x(k))) := E(k)$$

where $G(x) = \int_0^x F(y)dy$

We have from (3.1) and (3.3) that

$$\begin{aligned} w(k) &= C(b_k) + x(k) + r\Big(p_0 \cdot N(b_k) + (1-p_0) \cdot E(b_k\Big) \\ v(k) &= C(a_k) + r\Big(p_0 \cdot N(a_k) + r \cdot (1-p_0) \cdot E(a_k\Big) \end{aligned} \tag{3.4}$$

In order to find the optimal policy we should go on like this.

1. Choose initial guesses $x_j(k)$, $(j = 0)$, for $x(k)$ $\forall$ k.
2. Determine the corresponding $w_j(k)$ and $v_j(k)$ for $w(k)$ and $v(k)$ from (3.4). This will be a system of equations.

3. Determine the updated $x(k)$ from

$$x_{j+1}(k) = x_j(k) + v_j(k) - w_j(k). \tag{3.5}$$

   The index j stands for the j-th iteration. (3.5) follows from (3.2).

4. Repete 2-3 until $x(k)$ has converged $\forall\ k$.

It could be interesting to compare this strategy with one where we always claim a damage (except when the damage is zero). In that case just put $x(k) = 0\ \forall\ k$ and solve (3.4).

Example 2

Consider a factory with insurance against damages. Assume that there are 10 bonus-classes. $F(x)$ is assumed to have a Pareto-distribution with parameters $\alpha = 60$ and $\gamma = 5$, p_0 is assumed to be 0.9. The expected value for the damage is then $0.1 \cdot 15 = 1.5$. The discount factor is set to 0.9. Finally is this assumed:

k	1	2	3	4	5	6	7	8	9	10
a_k	2	3	4	5	6	7	8	9	10	10
b_k	1	1	2	3	4	5	6	7	8	9
$C(k)$	1	1	1.2	1.2	1.5	1.5	2	2	3	3

The results are with these values:

k	1	2	3	4	5	6	7	8	9	10
$x(k)$	0.021	0.259	0.478	0.780	1.063	1.530	1.979	2.891	3.504	1.586
$v(k)$	10.04	10.28	10.52	11.06	11.58	12.59	13.56	15.48	17.06	17.06

So if we have a damage of 1 during a year when we belong to bonus class 8 we should pay the damage on our own since $1 < 2.89$ and $V(8,1) = 15.48 - 2.89 + 1 = 13.59$.

If we always claim a damage then we get these results:

k	1	2	3	4	5	6	7	8	9	10
$w(k)$	10.02	10.02	10.28	10.52	11.07	11.60	12.02	13.63	15.58	17.06
$v(k)$	10.04	10.28	10.52	11.07	11.60	12.62	13.63	15.58	17.17	17.17

Here the corresponding value is $V(8,1) = 15.58$. We see that the difference is not so big but that is mainly because p_0 is quite large.

3.2 Finite time-horizon

This section is very similar to the previous one and therefore mostly the corresponding formulas for the finite case will be given. Let $V_n(k,x)$ be the total expected cost in n

periods with discount factor r under the optimal policy.

$$V_n(k,x) = \min_u \left[C(u_k,x) + r\left(p_0 \cdot V_{n-1}(u_k,0) + (1-p_0)\int_0^\infty V_{n-1}(u_k,y)F(dy) \right) \right]$$

where

$$V_0(k,x) = 0$$

$V_n(k,x)$ will look like figure 3.2 precisely as earlier and the control variable u has the following form.

$$u = \begin{cases} b & \text{if } x \le x_n(k) \\ a & \text{if } x > x_n(k) \end{cases}$$

Hence

$$V_n(k,x) = \begin{cases} C(b_k) + x + r\Big(p_0 \cdot V_{n-1}(b_k,0) + (1-p_0)\int_0^\infty V_{n-1}(b_k,y)F(dy)\Big) \\ \text{if } x \le x_n(k) \\ C(a_k) + r\Big(p_0 \cdot V_{n-1}(a_k,0) + (1-p_0)\int_0^\infty V_{n-1}(a_k,y)F(dy)\Big) \\ \text{if } x >_n x(k) \end{cases} \tag{3.6}$$

$V_n(k,x)$ is of the form

$$V_n(k,x) = \min[w_n(k) - x_n(k) + x; v_n(k)]$$

or

$$V_n(k,x) = \begin{cases} w_n(k) - x_n(k) + x & \text{if } x \le x_n(k) \\ v_n(k) & \text{if } x > x_n(k) \end{cases} \tag{3.7}$$

$V_n(k,0)$ and $\int_0^\infty V_n(k,y)F(dy)$ are then

$$V_n(k,0) = w_n(k) - x_n(k) := N_n(k)$$

$$\int_0^\infty V_n(k,y)F(dy) = w_n(k)F(x_n(k)) - G(x_n(k)) + v_n(k)(1 - F(x_n(k))) := E_n(k)$$

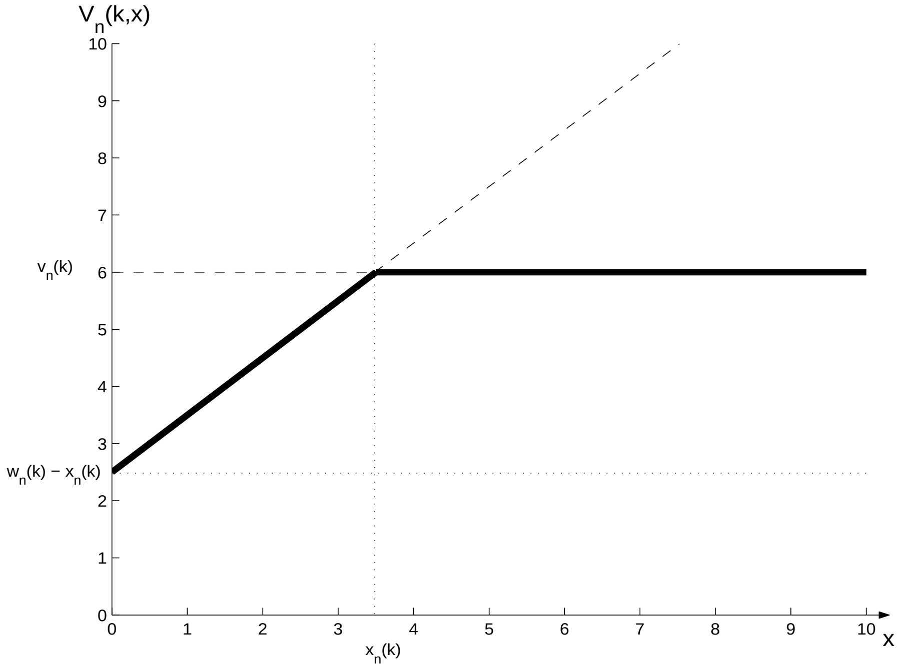

Figure 3.2: $V_n(k, x)$ for fix k and n.

We have from (3.6) and (3.7) that

$$\begin{aligned} w_n(k) &= C(b_k) + x_n(k) + r\Big(p_0 \cdot N_{n-1}(b_k) + (1 - p_0) \cdot E_{n-1}(b_k)\Big) \\ v_n(k) &= C(a_k) + r\Big(p_0 \cdot N_{n-1}(a_k) + (1 - p_0) \cdot E_{n-1}(a_k)\Big) \end{aligned} \tag{3.8}$$

In order to find the optimal policy if our present state is (k^*, x^*) and the time horizon is n^* we should go on like this.

1. Do 2-5 for $i = 1, 2, ..., n^* - 1$.
2. Choose initial guesses $x_{i_j}(k)$, $(j = 0)$, for $x_i(k)$ $\forall$ k.
3. Determine the corresponding $w_{i_j}(k)$ and $v_{i_j}(k)$ for $w_i(k)$ and $v_i(k)$ from (3.8)
4. Determine the updated $x_i(k)$ from

$$x_{i_{j+1}} = x_{i_j}(k) + v_{i_j}(k) - w_{i_j}(k).$$

5. Repete 3-4 until $x_i(k)$ has converged $\forall$ k.

6. Do 2-5 for $i = n^*$ but only for $k = k^*$.

If we always claim a damage then put $x_i(k) = 0 \; \forall \; k, i$ and make for $i = 1, 2, ..., n^* - 1$ step 3 above $\forall \; k$ but for $i = n^*$ only for $k = k^*$.

Example 2 (continued)

The results are in this case for different n^* :

k	1	2	3	4	5	6	7	8	9	10
$x_1(k)$	0.000	0.200	0.200	0.300	0.300	0.500	0.500	1.000	1.000	0.000
$v_1(k)$	1.000	1.200	1.200	1.500	1.500	2.000	2.000	3.000	3.000	3.000
$x_2(k)$	0.018	0.218	0.389	0.489	0.587	0.787	0.992	1.492	1.814	0.814
$v_2(k)$	1.918	2.118	2.307	2.607	2.894	3.394	3.886	4.886	5.700	5.700
$x_5(k)$	0.021	0.257	0.466	0.754	0.980	1.389	1.685	2.462	2.918	1.207
$v_5(k)$	4.120	4.357	4.586	5.111	5.566	6.500	7.251	8.962	10.17	10.17
$x_{20}(k)$	0.021	0.259	0.478	0.780	1.063	1.530	1.979	2.891	3.504	1.586
$v_{20}(k)$	8.822	9.060	9.300	9.840	10.36	11.37	12.34	14.26	15.85	15.85
$x_{100}(k)$	0.021	0.259	0.478	0.780	1.063	1.530	1.979	2.891	3.504	1.586
$v_{100}(k)$	10.04	10.28	10.52	11.06	11.58	12.59	13.56	15.48	17.06	17.06

We see that we get the same results as in the infinite case when n^* is large enough.

Example 3

We will now use another example. This time with a factory that has a greater risk for damages. The last example didn't show how useful this strategy can be mainly because of the high value of p_0. Let us have the following situation instead. Assume that there are 6 bonusclasses. $F(x)$ is assumed to have a Pareto-distribution with parameters $\alpha = 24$ and $\gamma = 4$, p_0 is assumed to be 0.5. In practice this company probably would have an excess but for now we assume that the company don't have that. The expected value for the damage is then $0.5 \cdot 8 = 4$. The discount factor is set to 0.9. The transitions between bonus classes and the premium costs for different bonus classes are

k	1	2	3	4	5	6
a_k	3	4	5	6	6	6
b_k	1	1	1	2	3	4
$C(k)$	1	2.5	3.5	4.5	5.5	7

The results are in this case for $n^* = 1$ and $n^* = 100$:

k	1	2	3	4	5	6
$x_1(k)$	2.500	3.500	4.500	4.500	3.500	2.500
$v_1(k)$	3.500	4.500	5.500	7.000	7.000	7.000
$x_{100}(k)$	3.470	6.621	8.642	10.33	9.023	5.872
$v_{100}(k)$	25.46	28.612	30.63	34.48	34.48	34.48

If we have a damage of 20 during a year when we belong to bonus class 4 we should claim the damage and $V_{100}(4, 20) = 34.48$.

If we always claim a damage then we get the result for $n^* = 100$:

k	1	2	3	4	5	6
$w_{100}(k)$	32.93	32.93	32.93	36.21	38.02	41.97
$v_{100}(k)$	38.02	41.97	43.79	47.07	47.07	47.07

In this case the corresponding value is $V_{100}(4, 20) = 47.07$. We see that this factory has much to win if they follow the first strategy.

Chapter 4

A method to find the optimal claim-decision with thresholds

In the previous chapter it didn't matter how big the damage was if we made a claim. A reasonable assumption is that a large damage would result in a less desirable bonus class than a small damage. We therefore extend the analysis with "thresholds". For these thresholds we will come to worse bonus classes if the damage exceeds them. From now on we will always use a definite time horizon. The reasons for this are that it is easier to analyze things and because we get the same results when n is large enough we don't have to use the infinite horizon.

4.1 One threshold

If a claim is made the change of a bonus class k is

$$k \to \begin{cases} a_{k1} & \text{if } x < x^{(1)}(k) \\ a_{k2} & \text{if } x \geq x^{(1)}(k) \end{cases}$$

This means that if x is greater than the threshold $x^{(1)}(k)$ we will come to the less favourable bonus class a_{k2}. If a claim is not made the change is

$$k \to b_k \qquad \forall\ x$$

The transition probabilities are given by

$$P[(k, x) \to (u_k(x), y)] = F^*(dy)$$

where

$$u_k(x) = \begin{cases} a_{k1} & \text{if } u = a,\ x < x^{(1)}(k) \\ a_{k2} & \text{if } u = a,\ x \geq x^{(1)}(k) \\ b_k & \text{if } u = b,\ \forall\ x \end{cases}$$

$V_n(k,x)$ can in this case be determined from

$$V_n(k,x) = \min_u \Big[C(u_k(x),x) + r\Big(p_0 \cdot V_{n-1}(u_k(x),0) + \\ +(1-p_0)\int_0^\infty V_{n-1}(u_k(x),y)F(dy)\Big)\Big]$$

where

$$C(u_k(x),x) = \begin{cases} c(a_{k1}) & \text{if } u = a,\ x < x^{(1)}(k) \\ c(a_{k2}) & \text{if } u = a,\ x \geq x^{(1)}(k) \\ c(b_k) + x & \text{if } u = b,\ \forall\ x \end{cases}$$

and

$$V_0(k,x) = 0$$

After some consideration we realize that $V_n(k,x)$ will look like figure 4.1. $x_n^1(k)$ is the first change $b \to a$ and $x_n^2(k)$ is the second change. When the iteration has converged we have that $w_n^1(k) = v_n^1(k)$, $w_n^2(k) = v_n^2(k)$ and $v_n^2(k) - x_n^2(k) = v_n^1(k) - x_n^1(k)$.
The controlvariable u has the following form:

$$u = \begin{cases} b & \text{if } x \leq x_n^1(k) \\ a & \text{if } x_n^1(k) < x < x^{(1)}(k) \\ b & \text{if } x^{(1)}(k) \leq x \leq x_n^2(k) \\ a & \text{if } x > x_n^2(k) \end{cases}$$

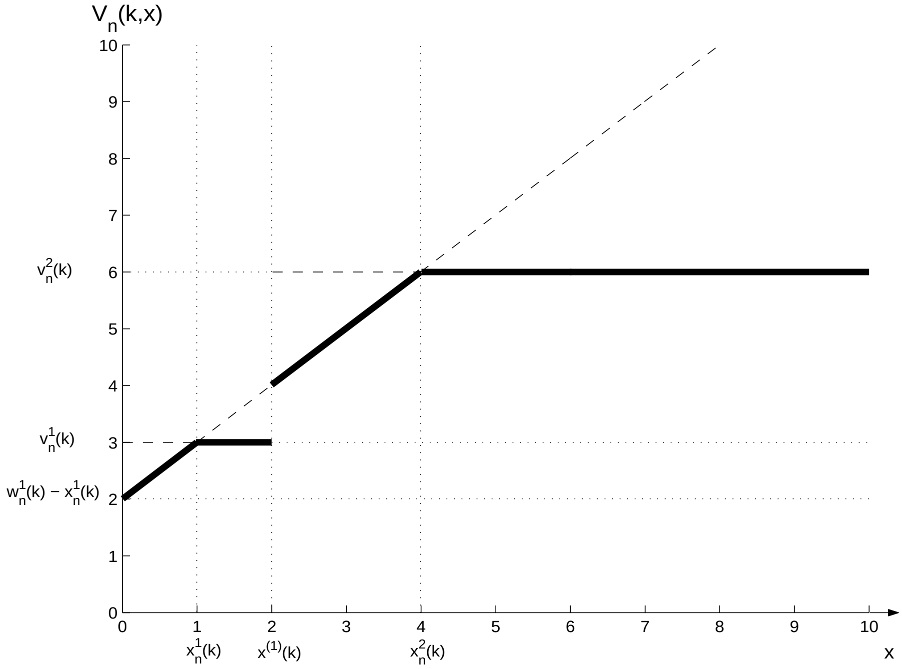

Figure 4.1: $V_n(k,x)$ with one threshold.

This gives us:

$$V_n(k,x) = \begin{cases} C(b_k) + x + r\Big(p_0 \cdot V_{n-1}(b_k,0) + (1-p_0)\int_0^\infty V_{n-1}(b_k,y)F(dy)\Big) \\ \text{ if } x \leq x_n^1(k) \\ C(a_{k1}) + r\Big(p_0 \cdot V_{n-1}(a_{k1},0) + (1-p_0)\int_0^\infty V_{n-1}(a_{k1},y)F(dy)\Big) \\ \text{ if } x_n^1(k) < x < x^{(1)}(k) \\ C(b_k) + x + r\Big(p_0 \cdot V_{n-1}(b_k,0) + (1-p_0)\int_0^\infty V_{n-1}(b_k,y)F(dy)\Big) \\ \text{ if } x^{(1)}(k) \leq x \leq x_n^2(k) \\ C(a_{k2}) + r\Big(p_0 \cdot V_{n-1}(a_{k2},0) + (1-p_0)\int_0^\infty V_{n-1}(a_{k2},y)F(dy)\Big) \\ \text{ if } x > x_n^2(k) \end{cases} \tag{4.1}$$

From figure 4.1 we see that

$$V_n(k,x) = \begin{cases} \min[w_n^1(k) - x_n^1(k) + x; v_n^1(k)] & \text{if } x < x^{(1)}(k) \\ \min[w_n^2(k) - x_n^2(k) + x; v_n^2(k)] & \text{if } x \geq x^{(1)}(k) \end{cases} \tag{4.2}$$

or

$$V_n(k,x) = \begin{cases} w_n^1(k) - x_n^1(k) + x & \text{if } x \le x_n^1(k) \\ v_n^1(k) & \text{if } x_n^1(k) < x < x^{(1)}(k) \\ w_n^2(k) - x_n^2(k) + x & \text{if } x^{(1)}(k) \le x \le x_n^2(k) \\ v_n^2(k) & \text{if } x > x_n^2(k) \end{cases} \tag{4.3}$$

for suitable $w_n^1(k)$, $w_n^2(k)$, $v_n^1(k)$ and $v_n^2(k)$.

If we look at figure 4.1 and move the threshold $x^{(1)}(k)$ to the left we will eventually (when we pass $x_n^1(k)$) have a situation that looks like figure 4.2. Now $x^{(1)}(k)$ and $x_n^1(k)$ have no significance so if we, after determined $x_n^1(k)$, see that $x^{(1)}(k) < x_n^1(k)$ we introduce a new variable called $x_n^{(1)}(k)$ and put $x_n^{(1)}(k) = 0$. We also put $x_n^1(k) = 0$. If we instead move the threshold $x^{(1)}(k)$ to the right we will eventually (when we pass $x_n^2(k)$) have a situation that looks like figure 4.3. So if $x^{(1)}(k) > x_n^2(k)$ we now put $x_n^2(k) = x^{(1)}(k)$. We can now use the following expressions:

$$V_n(k,0) = w_n^1(k) - x_n^1(k) := N_n(k)$$

and

$$\int_0^\infty V_n(k,y)F(dy) =$$

$$= \int_0^{x_n^1(k)} (w_n^1(k) - x_n^1(k) + y)F(dy) + \int_{x_n^1(k)}^{x_n^{(1)}(k)} v_n^1(k)F(dy) +$$

$$+ \int_{x_n^{(1)}(k)}^{x_n^2(k)} (w_n^2(k) - x_n^2(k) + y)F(dy) + \int_{x_n^2(k)}^\infty v_n^2(k)F(dy) =$$

$$= (w_n^1(k) - x_n^1(k)) \int_0^{x_n^1(k)} f(y)dy + \int_0^{x_n^1(k)} yf(y)dy + v_n^1(k) \int_{x_n^1(k)}^{x_n^{(1)}(k)} f(y)dy +$$

$$+ (w_n^2(k) - x_n^2(k)) \int_{x_n^{(1)}(k)}^{x_n^2(k)} f(y)dy + \int_{x_n^{(1)}(k)}^{x_n^2(k)} yf(y)dy + v_n^2(k) \int_{x_n^2(k)}^\infty f(y)dy =$$

$$= (w_n^1(k) - x_n^1(k))F(x_n^1(k)) + x_n^1(k)F(x_n^1(k)) - G(x_n^1(k)) +$$

$$+ v_n^1(k)\Big(F(x_n^{(1)}(k)) - F(x_n^1(k))\Big) + (w_n^2(k) - x_n^2(k))\Big(F(x_n^2(k)) - F(x_n^{(1)}(k))\Big) +$$

$$+ x_n^2(k)F(x_n^2(k)) - x_n^{(1)}(k)F(x_n^{(1)}(k)) - G(x_n^2(k)) + G(x_n^{(1)}(k)) + v_n^2(k)\Big(1 - F(x_n^2(k))\Big)$$

$$:= E_n(k)$$

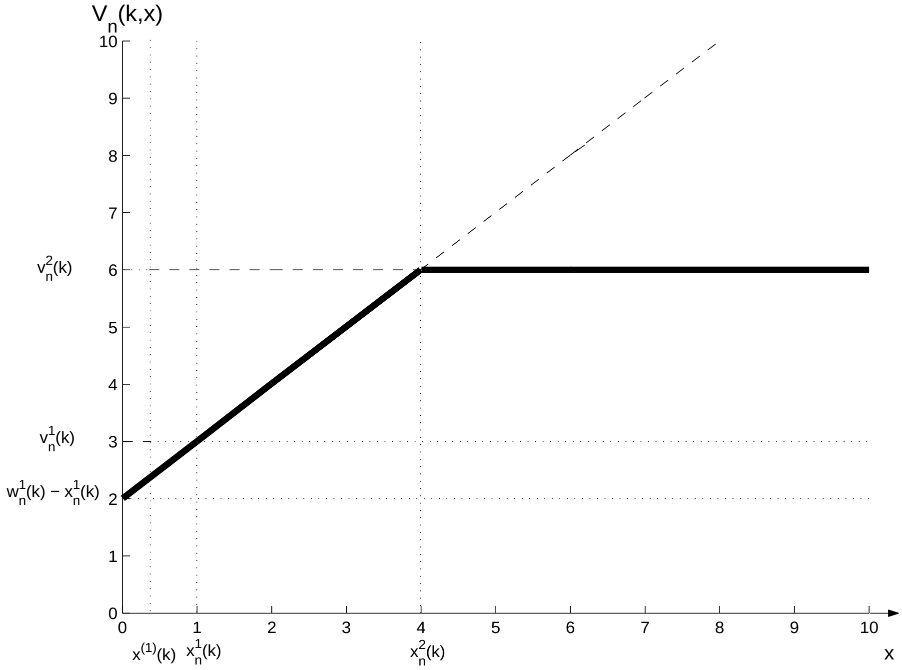

Figure 4.2: $V_n(k,x)$.

We see from (4.1) and (4.3) that

$$\begin{aligned} w_n^1(k) &= C(b_k) + x_n^1(k) + r\Big(p_0 \cdot N_{n-1}(b_k) + (1-p_0) \cdot E_{n-1}(b_k)\Big) \\ v_n^1(k) &= C(a_{k1}) + r\Big(p_0 \cdot N_{n-1}(a_{k1}) + (1-p_0) \cdot E_{n-1}(a_{k1})\Big) \\ w_n^2(k) &= C(b_k) + x_n^2(k) + r\Big(p_0 \cdot N_{n-1}(b_k) + (1-p_0) \cdot E_{n-1}(b_k)\Big) \\ v_n^2(k) &= C(a_{k2}) + r\Big(p_0 \cdot N_{n-1}(a_{k2}) + (1-p_0) \cdot E_{n-1}(a_{k2})\Big) \end{aligned} \tag{4.4}$$

In order to find the optimal policy if our present state is (k^*, x^*) and the horizon is n^* we should go on like this.

1. Do 2-5 for $i = 1, 2, ..., n^* - 1$
2. Choose initial guesses $x_{i_j}^1(k)$ and $x_{i_j}^2(k)$, $(j = 0)$, for $x_i^1(k)$ and $x_i^2(k)$ $\forall\ k$.
3. Determine the corresponding $w_{i_j}^1(k)$, $w_{i_j}^2(k)$, $v_{i_j}^1(k)$ and $v_{i_j}^2(k)$ for $w_i^1(k)$, $w_i^2(k)$, $v_i^1(k)$ and $v_i^2(k)$ from (4.4)

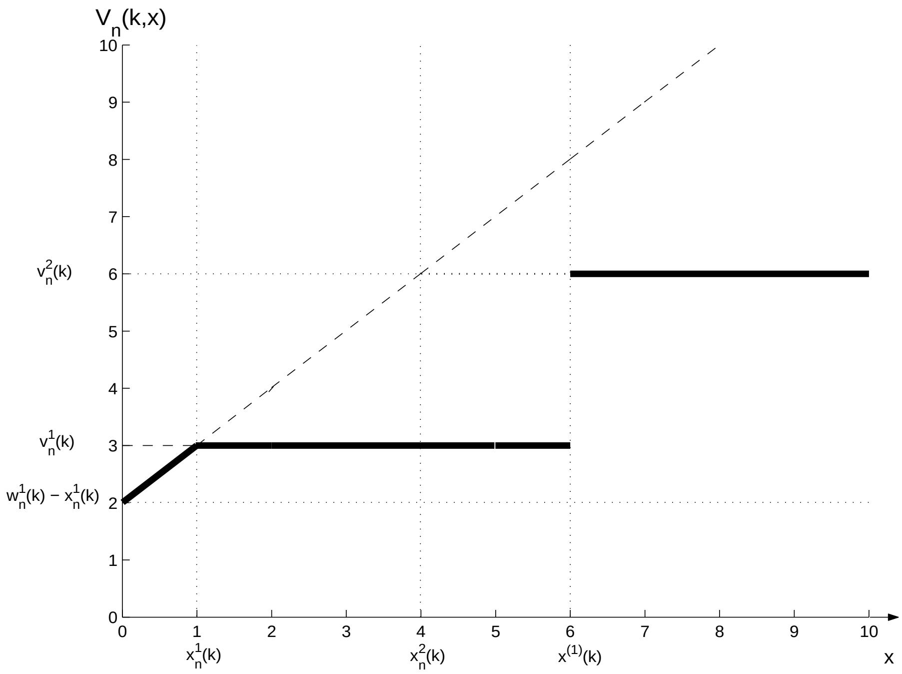

Figure 4.3: $V_n(k,x)$.

4. Determine the updated $x_i^1(k)$ and $x_i^2(k)$ from

$$\begin{aligned} x^1_{i_{j+1}}(k) &= x^1_{i_j}(k) + v^1_{i_j}(k) - w^1_{i_j}(k) \\ x^2_{i_{j+1}}(k) &= x^2_{i_j}(k) + v^2_{i_j}(k) - w^2_{i_j}(k). \end{aligned} \tag{4.5}$$

   (4.5) follows from (4.2).

5. Repete 3-4 until $x_i^1(k)$ and $x_i^2(k)$ have convergated $\forall\ k$.

6. Do 2-5 for $i = n^*$ but only for $k = k^*$.

If we always want to claim a damage just put $x_i^1(k) = 0$ and $x_i^2(k) = 0\ \forall\ k, i$. Then make for $i = 1, 2, ..., n^* - 1$ step 3 above $\forall\ k$ but for $i = n^*$ only for $k = k^*$. Then we will get the result.

Example 2 (continued)

We use the same values as before but we have to assume some values for a_{k1}, a_{k2} and $x^{(1)}(k)$.

k	1	2	3	4	5	6	7	8	9	10
a_{k1}	2	3	4	5	6	7	8	9	10	10
a_{k2}	3	4	5	6	7	8	9	10	10	10
$x^{(1)}(k)$	1	0.5	1.5	1	2	1.5	2.5	2	0	0

The results are in this case for different n^*:

k	1	2	3	4	5	6	7	8	9	10
$x_1^1(k)$	0.000	0.200	0.200	0.300	0.300	0.500	0.500	1.000	1.000	0.000
$x_1^2(k)$	0.200	0.200	0.500	0.300	0.800	0.500	1.500	2.000	0.000	0.000
$v_1^1(k)$	1.000	1.200	1.200	1.500	1.500	2.000	2.000	3.000	3.000	3.000
$v_1^2(k)$	1.200	1.200	1.500	1.500	2.000	2.000	3.000	3.000	3.000	3.000
$x_2^1(k)$	0.001	0.225	0.389	0.503	0.587	0.823	0.992	1.419	1.814	0.814
$x_2^2(k)$	0.225	0.390	0.727	0.752	1.161	1.242	1.992	2.233	1.814	0.814
$v_2^1(k)$	1.918	2.142	2.307	2.645	2.894	3.468	3.886	4.886	5.700	5.700
$v_2^2(k)$	2.142	2.307	2.645	2.894	3.468	3.886	4.886	5.700	5.700	5.700
$x_5^1(k)$	0.020	0.278	0.502	0.785	1.033	1.458	1.758	2.378	2.776	1.189
$x_5^2(k)$	0.278	0.522	1.043	1.277	2.000	2.249	3.344	3.567	2.776	1.189
$v_5^1(k)$	4.183	4.441	4.685	5.226	5.718	6.684	7.475	9.062	10.25	10.25
$v_5^2(k)$	4.441	4.685	5.226	5.718	6.684	7.475	9.062	10.25	10.25	10.25
$x_{20}^1(k)$	0.025	0.293	0.538	0.874	1.182	1.684	2.130	2.877	3.386	1.532
$x_{20}^2(k)$	0.293	0.563	1.143	1.451	2.289	2.707	3.984	4.409	3.386	1.532
$v_{20}^1(k)$	9.016	9.285	9.554	10.16	10.74	11.84	12.87	14.72	16.25	16.25
$v_{20}^2(k)$	9.285	9.554	10.16	10.74	11.84	12.87	14.72	16.25	16.25	16.25
$x_{100}^1(k)$	0.025	0.293	0.538	0.874	1.182	1.684	2.131	2.878	3.386	1.532
$x_{100}^2(k)$	0.293	0.563	1.143	1.451	2.289	2.708	3.985	4.410	3.386	1.532
$v_{100}^1(k)$	10.27	10.54	10.81	11.41	11.99	13.10	14.12	15.97	17.51	17.51
$v_{100}^2(k)$	10.54	10.81	11.41	11.99	13.10	14.12	15.97	17.51	17.51	17.51

We see that the values of $V(k,x)$ are a bit higher now which is natural. For example $V_{100}(8,1) = 15.97 - 2.88 + 1 = 14.09$. If we always want to claim a damage then the results are:

k	1	2	3	4	5	6	7	8	9	10
$w_{100}^1(k)$	10.25	10.25	10.27	10.54	10.82	11.43	12.03	13.16	14.23	16.11
$v_{100}^1(k)$	10.27	10.54	10.82	11.43	12.03	13.16	14.23	16.11	17.64	17.64
$v_{100}^2(k)$	10.54	10.82	11.43	12.03	13.16	14.23	16.11	17.64	17.64	17.64

Here $V_{100}(8,1) = 16.11$. We see again that the difference is not so big.

4.2 Several thresholds

If a claim is made the change of a bonus class k is

$$k \to a_{kt} \qquad \text{if } x^{(t-1)}(k) \le x < x^{(t)}(k),\ t = 1, 2, .., T, T+1$$

where $x^{(0)}(k) = 0,\ x^{(T+1)}(k) = \infty$ and $T =$ number of thresholds .

If x is greater than threshold t, $x^{(t)}(k)$, we will come to bonus class $a_{k(t+1)}$. If a claim is not made the change is

$$k \to b_k \qquad \forall\ x$$

The transition probabilities are given by

$$P[(k, x) \to (u_k(x), y)] = F^*(dy)$$

where

$$u_k(x) = \begin{cases} a_{kt} & \text{if } u = a, \quad x^{(t-1)}(k) \le x < x^{(t)}(k) \\ b_k & \text{if } u = b, \quad \forall\ x \end{cases}$$

$V_n(k, x)$ can be determined from

$$V_n(k, x) = \min_u \Big[C(u_k(x), x) + r\Big(p_0 \cdot V_{n-1}(u_k(x), 0) + \\ +(1 - p_0) \int_0^\infty V_{n-1}(u_k(x), y) F(dy) \Big) \Big]$$

where

$$C(u_k(x), x) = \begin{cases} c(a_{kt}) & \text{if } u = a,\ x^{(t-1)}(k) \le x < x^{(t)}(k) \\ c(b_k) + x & \text{if } u = b,\ \forall\ x \end{cases}$$

and

$$V_0(k, x) = 0$$

We now realize that $V_n(k, x)$ will look like figure 4.4. Here $x_n^t(k)$ stands for the t:th change $b \to a$. When the iteration has converged we have that $w_n^t(k) = v_n^t(k)$ and $v_n^t(k) - x_n^t(k) = v_n^{t-1}(k) - x_n^{t-1}(k)$.

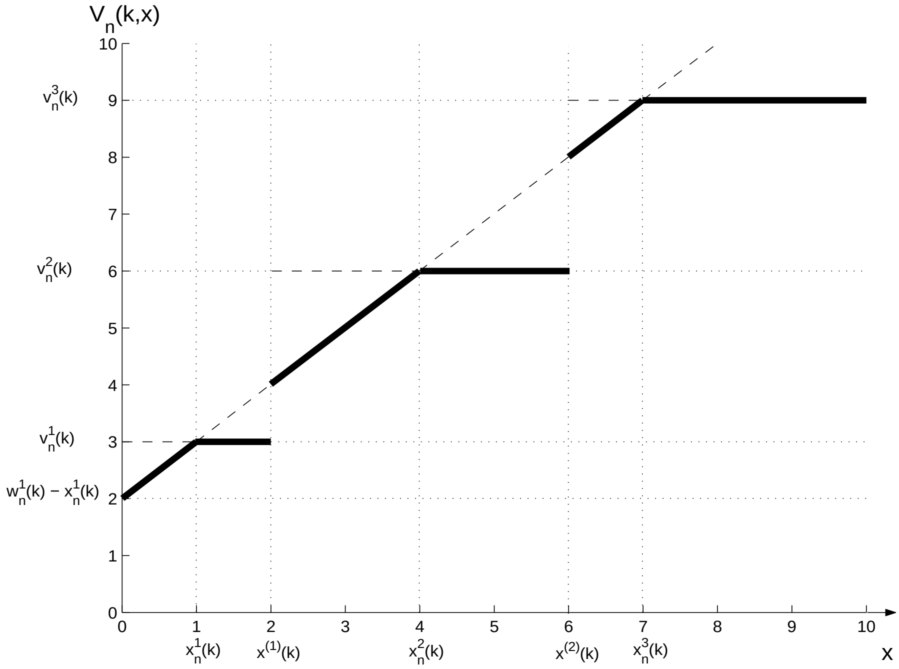

Figure 4.4: $V_n(k,x)$ with two thresholds ($T=2$).

The controlvariable u has the following form:

$$u = \begin{cases} b & \text{if } x^{(t-1)}(k) \le x \le x_n^t(k) \\ a & \text{if } x_n^t(k) < x < x^{(t)}(k) \end{cases}$$

which gives us

$$V_n(k,x) = \begin{cases} C(b_k) + x + r\Big(p_0 \cdot V_{n-1}(b_k,0) + (1-p_0)\int_0^\infty V_{n-1}(b_k,y)F(dy)\Big) \\ \text{if } x_n^{(t-1)}(k) < x \le x_n^t(k) \\ C(a_{kt}) + r\Big(p_0 \cdot V_{n-1}(a_{kt},0) + (1-p_0)\int_0^\infty V_{n-1}(a_{kt},y)F(dy)\Big) \\ \text{if } x_n^t(k) < x \le x^{(t)}(k) \end{cases} \tag{4.6}$$

From figure 4.4 we see that

$$V_n(k,x) = \min[w_n^t(k) - x_n^t(k) + x; v_n^t(k)] \text{ if } \; x^{(t)}(k) \le x < x^{(t+1)}(k)$$

or

$$V_n(k,x) = \begin{cases} w_n^t(k) - x_n^t(k) + x & \text{if } x^{(t-1)}(k) < x \leq x_n^t(k) \\ v_n^t(k) & \text{if } x_n^t(k) < x \leq x^{(t)}(k) \end{cases} \tag{4.7}$$

for suitable $w_n^t(k)$ and $v_n^t(k)$.

If we look at figure 4.4 and move the threshold $x^{(1)}(k)$ to the left and move the threshold $x^{(2)}(k)$ to the right we will have a situation that looks like figure 4.5. Now $x^{(1)}(k)$, $x_n^1(k)$ and $x_n^3(k)$ have no significance. What should be done in general for being able to use the expression $E_n(k)$ for $\int_0^\infty V_n(k,y)F(dy)$ given below? We will give a rule for this.

Rule 1

The general rule for being able to use the expression $E_n(k)$ is:

Put $x_n^{(t)}(k) := x^{(t)}(k) \ \forall \ k, n, t$
if $x_n^{(t)}(k) < x_n^t(k)$ put $x_n^t(k) = x_n^{(t-1)}(k)$ and $x_n^{(t)}(k) = x_n^{(t-1)}(k)$
if $x_n^{(t)}(k) > x_n^{t+1}(k)$ put $x_n^{t+1}(k) = x_n^{(t)}(k)$
for $t = 1, 2, ..., T$

We can now use the following:

$$V_n(k,0) = w_n^1(k) - x_n^1(k) := N_n(k)$$

and

$$\int_0^\infty V_n(k,y)F(dy) =$$

$$= \int_0^{x_n^1(k)} (w_n^1(k) - x_n^1(k) + y)F(dy) + \int_{x_n^1(k)}^{x_n^{(1)}(k)} v_n^1(k)F(dy) + ...$$

$$... + \int_{x_n^{(T)}(k)}^{x_n^{T+1}(k)} (w_n^{T+1}(k) - x_n^{T+1}(k) + y)F(dy) + \int_{x_n^{T+1}(k)}^{\infty} v_n^{T+1}(k)F(dy) =$$

$$= (w_n^1(k) - x_n^1(k))F(x_n^1(k)) + \sum_{i=1}^{T+1}[x_n^i(k)F(x_n^i(k)) - G(x_n^i(k))] +$$

$$+ \sum_{i=1}^{T}[v_n^i(k)\left(F(x_n^{(i)}(k)) - F(x_n^i(k))\right) + G(x_n^{(i)}(k)) - x_n^{(i)}(k)F(x_n^{(i)}(k))] +$$

$$+ \sum_{i=2}^{T+1}[(w_n^i(k) - x_n^i(k))(F(x_n^i(k)) - F(x_n^{(i-1)}(k)))] + v_n^{T+1}(k)\Big(1 - F(x_n^{T+1}(k))\Big)$$

$$:= E_n(k)$$

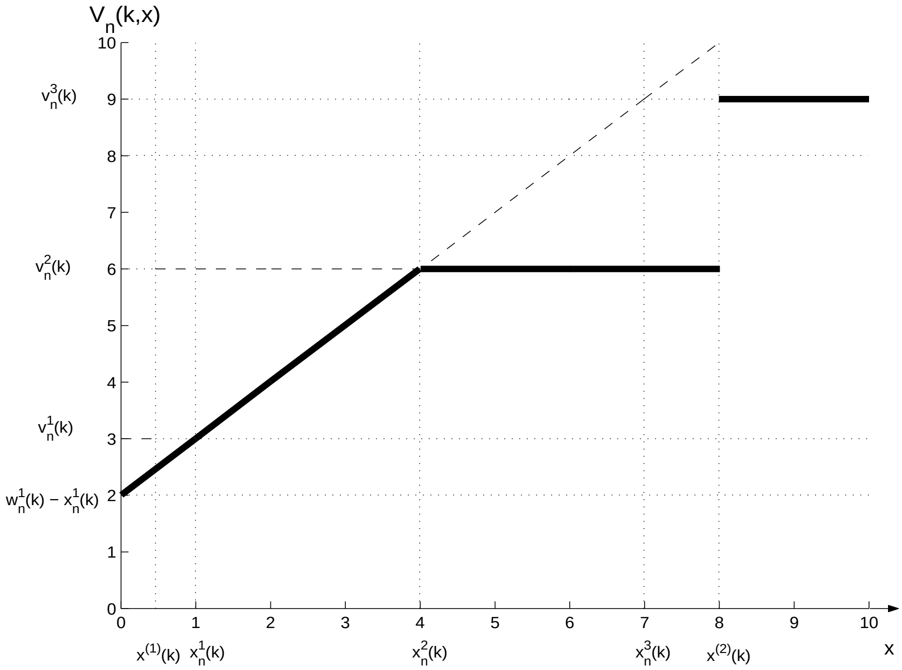

Figure 4.5: $V_n(k, x)$ with two thresholds ($T = 2$).

We see from (4.6) and (4.7) that

$$\begin{aligned} w_n^t(k) &= C(b_k) + x_n^t(k) + r\Big(p_0 \cdot N_{n-1}(b_k) + (1-p_0) \cdot E_{n-1}(b_k)\Big) \\ v_n^t(k) &= C(a_{kt}) + r\Big(p_0 \cdot N_{n-1}(a_{kt}) + (1-p_0) \cdot E_{n-1}(a_{kt})\Big) \end{aligned} \tag{4.8}$$

In order to find the optimal policy if our present state is (k^*, x^*) and the horizon is n^* we should go on like this.

1. Do 2-5 for $i = 1, 2, ..., n^* - 1$.
2. Choose initial guesses $x_{i_j}^t(k)$, $(j = 0)$, for $x_i^t(k)$ $\forall k, t$.
3. Determine the corresponding $w_{i_j}^t(k)$ and $v_{i_j}^t(k)$ for $w_i^t(k)$ and $v_i^t(k)$ from (4.8)
4. Determine the updated $x_i^t(k)$ from

$$x_{i_{j+1}}^t(k) = x_{i_j}^t(k) + v_{i_j}^t(k) - w_{i_j}^t(k).$$

5. Repete 3-4 until $x_i^t(k)$ has convergated $\forall k, t$.
6. Make 2-5 for $i = n^*$ but only for $k = k^*$.

If we always want to claim a damage just put $x_i^t(k) = 0 \;\forall\; k, t, i$. Then make for $i = 1, 2, ..., n^* - 1$ step 3 above $\forall\; k, t$ but for $i = n^*$ only for $k = k^*$. Then we will get the result.

Example 3 (continued)

We now extend with following values:

k	1	2	3	4	5	6
a_{k1}	3	4	5	6	6	6
a_{k2}	4	5	6	6	6	6
a_{k3}	5	6	6	6	6	6
b_k	1	1	1	2	3	4
$x^{(1)}(k)$	2	4	8	0	0	0
$x^{(2)}(k)$	10	15	0	0	0	0
$C(k)$	1	2.5	3.5	4.5	5.5	7

The results are in this case for $n^* = 1$ and $n^* = 100$:

k	1	2	3	4	5	6
$x_1^1(k)$	2.500	3.500	4.500	4.500	3.500	2.500
$x_1^2(k)$	3.500	4.500	6.000	4.500	3.500	2.500
$x_1^3(k)$	4.500	6.000	6.000	4.500	3.500	2.500
$v_1^1(k)$	3.500	4.500	5.500	7.000	7.000	7.000
$v_1^2(k)$	4.500	5.500	7.000	7.000	7.000	7.000
$v_1^3(k)$	5.500	7.000	7.000	7.000	7.000	7.000
$x_{100}^1(k)$	3.034	5.471	7.386	8.820	7.617	5.180
$x_{100}^2(k)$	5.471	7.386	10.65	8.820	7.617	5.180
$x_{100}^3(k)$	7.386	10.65	10.65	8.820	7.617	5.180
$v_{100}^1(k)$	31.73	34.17	36.08	39.35	39.35	39.35
$v_{100}^2(k)$	34.17	36.08	39.35	39.35	39.35	39.35
$v_{100}^3(k)$	36.08	39.35	39.35	39.35	39.35	39.35

We see for example that $V_{100}(4, 20) = 39.35$. If we always claim a damage then we get these results:

k	1	2	3	4	5	6
$w_{100}^1(k)$	39.02	39.02	39.02	41.37	43.28	45.65
$v_{100}^1(k)$	43.28	45.65	47.51	50.08	50.08	50.08
$v_{100}^2(k)$	45.65	47.51	50.08	50.08	50.08	50.08
$v_{100}^3(k)$	47.51	50.08	50.08	50.08	50.08	50.08

Here $V_{100}(4, 20) = 50.08$.

Chapter 5

A method to find the optimal claim-decision and excess

Most policy holders of insurances have an excess and we will now extend the previous method with how to decide which excess to have next year. We assume that the policy-holders can choose between a number of excesses. Naturally the limit when we will start to claim a damage will be greater than the excess. Let now the state be given by (k, x, x_s) where k and x are as before and x_s is the excess during the same year. The control variables are now $u \in [a, b]$ and $U \in D(X_s)$ where $U = X_s^1$ means that the chosen excess for next year is X_s^1. $D(X_s)$ is assumed to be a finite set.

5.1 Without thresholds

The changes of bonus classes are the same as in chapter 3 and the transition probabilities are

$$P[(k, x, x_s) \to (u_k, y, U)] = F^*(dy)$$

where u_k is the same as before. $V_n(k, x, x_s)$ can now be determined from

$$V_n(k, x, x_s) = \min_{u,U} \Big[C(u_k, x, x_s, U) + r\Big(p_0 \cdot V_{n-1}(u_k, 0, U) + \\ +(1 - p_0) \int_0^\infty V_{n-1}(u_k, y, U) F(dy)\Big)\Big]$$

where

$$V_0(k, x, x_s) = \min(x, x_s)$$

since we must pay the part of the damage that is less than the excess.

$$C(u_k, x, x_s, U) = \begin{cases} c(a_k, X_s) + \min(x, x_s) & \text{if } u = a,\ U = X_s \\ c(b_k, X_s) + x & \text{if } u = b,\ U = X_s \end{cases}$$

Here $c(k, X_s)$ is the premium when we have the excess X_s in bonus class k. For a fix choice $U = X_s$ next period $V_n(k, x, x_s|U = X_s)$ will be

$$V_n(k, x, x_s|U = X_s) = \min_u \Big[C(u_k, x, x_s\,, X_s) + r\Big(p_0 \cdot V_{n-1}(u_k, 0, X_s) + \\ +(1 - p_0)\int_0^\infty V_{n-1}(u_k, y, X_s)F(dy)\Big)\Big]$$

We realize that the controlvaraible $(u|U = X_s)$ has the form

$$(u|U = X_s) = \begin{cases} b & \text{if } x \le x_n(k, x_s, X_s) \\ a & \text{if } x > x_n(k, x_s, X_s) \end{cases}$$

where $x_n(k, x_s, X_s) > x_s$. A plot of $V_n(k, x, x_s|U = X_s)$ looks like figure 5.1.

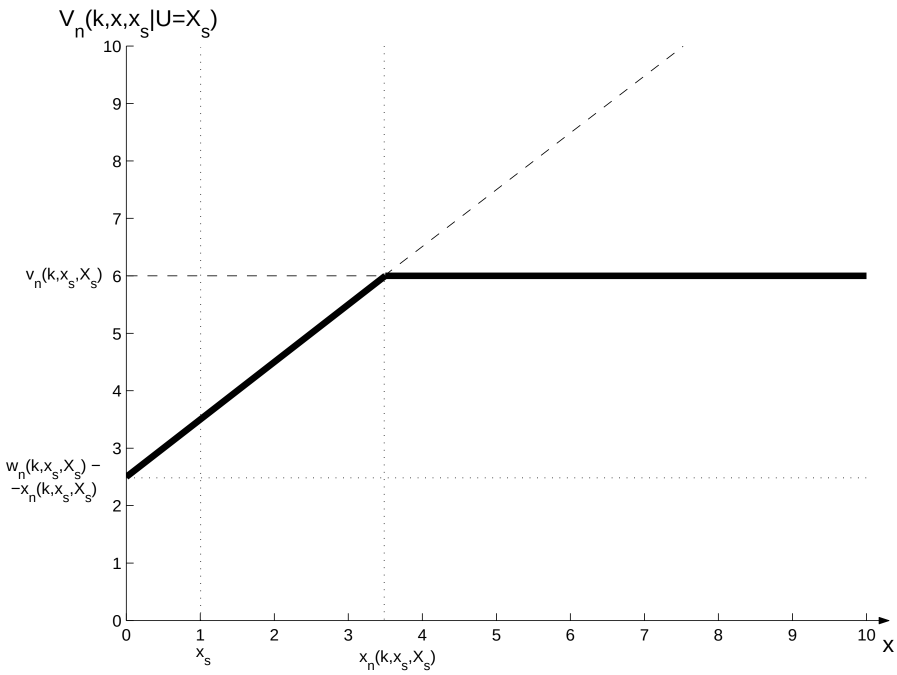

Figure 5.1: $V_n(k, x, x_s|U = X_s)$.

Now we can write:

$$V_n(k,x,x_s|U=X_s)=\begin{cases} C(b_k,X_s)+x+r\Big(\; p_0\cdot V_{n-1}(b_k,0,X_s)+ \\ \qquad\qquad\qquad\qquad (1-p_0)\int_0^\infty V_{n-1}(b_k,y,X_s)F(dy)\Big) \\ \text{ if } x\le x_n(k,x_s,X_s) \\ C(a_k,X_s)+x_s+r\Big(p_0\cdot V_{n-1}(a_k,0,X_s)+ \\ \qquad\qquad\qquad\qquad (1-p_0)\int_0^\infty V_{n-1}(a_k,y,X_s)F(dy)\Big) \\ \text{ if } x> x_n(k,x_s,X_s) \end{cases} \tag{5.1}$$

From figure 5.1 we see that $V_n(k,x,x_s|U=X_s)$ can be expressed as

$$V_n(k,x,x_s|U=X_s)=\min[w_n(k,x_s,X_s)-x_n(k,x_s,X_s)+x; v_n(k,x_s,X_s)]$$

or

$$V_n(k,x,x_s|U=X_s)=\begin{cases} w_n(k,x_s,X_s)-x_n(k,x_s,X_s)+x & \text{if } x\le x_n(k,x_s,X_s) \\ v_n(k,x_s,X_s) & \text{if } x> x_n(k,x_s,X_s) \end{cases} \tag{5.2}$$

From (5.1) and (5.2) we have

$$\begin{gathered} w_n(k,x_s,X_s)=C(b_k,X_s)+x_n(k,x_s,X_s)+ \\ +r\Big(p_0\cdot N_{n-1}(b_k,X_s)+(1-p_0)E_{n-1}(b_k,X_s)\Big) \\ v_n(k,x_s,X_s)=C(a_k,X_s)+x_s+ \\ +r\Big(p_0\cdot N_{n-1}(a_k,X_s)+(1-p_0)E_{n-1}(a_k,X_s)\Big) \end{gathered} \tag{5.3}$$

where

$$N_n(k,x_s)=V_n(k,0,x_s)$$

$$E_n(k,x_s)=\int_0^\infty V_n(k,y,x_s)F(dy)$$

When we have several choices for U in next period $V_n(k,x,x_s)$ looks like figure 5.2 where

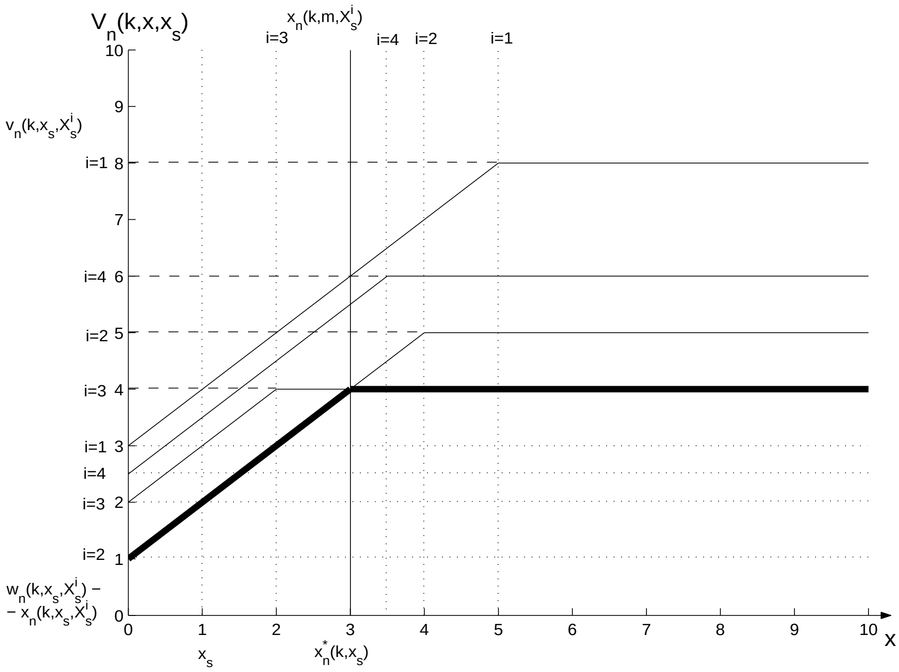

Figure 5.2: $V_n(k, x, x_s)$.

$$x_n^*(k, x_s) = \min_U \Big(v_n(k, x_s, U)\Big) - \min_U \Big(w_n(k, x_s, U) - x_n(k, x_s, U)\Big)$$

When we have studied the figure and determined $x_n^*(k, x_s)$ we can determine the following:

$$N_n(k, x_s) = V_n(k, 0, x_s) = \min_U \Big(w_n(k, x_s, U) - x_n(k, x_s, U)\Big)$$

and

$$E_n(k, x_s) = \int_0^\infty V_n(k, y, x_s) F(dy) =$$

$$= \int_0^{x_n^*(k,x_s)} (\min_U [w_n(k, x_s, U) - x_n(k, x_s, U)] + y) F(dy) +$$

$$+ \int_{x_n^*(k,x_s)}^\infty \min_U [v_n(k, x_s, U)] F(dy) =$$

$$= \min_U [w_n(k, x_s, U) - x_n(k, x_s, U)] F(x_n^*(k, x_s)) + x_n^*(k, x_s) F(x_n^*(k, x_s)) +$$

$$-G(x_n^*(k,x_s)) + \min_U[v_n^*(k,x_s)(1-F(x_n^*(k,x_s)))]$$

We see from figure 5.2 that the controlvariables are

$$(u,U) = \begin{cases} (b, \quad [U:\min(w_n(k,x_s,U) - x_n(k,x_s,U))]) & \text{if } x \leq x_n^*(k,x_s) \\ (a, \quad [U:\min(v_n(k,x_s,U))]) & \text{if } x > x_n^*(k,x_s) \end{cases}$$

If our present state is (k^*, x^*, x_s^*) and our horizon is n^* then we should go on like this to get the optimal decisions

1. Do 2-6 for $i = 1, 2, ..., n^* - 1$.

2. Choose initial guesses $x_{i_j}(k,x_s,X_s) > x_s$, $(j=0)$, for $x_i(k,x_s,X_s)$ $\forall k, x_s, X_s$

3. Determine the corresponding $w_{i_j}(k,x_s,X_s)$ and $v_{i_j}(k,x_s,X_s)$ for $w_i(k,x_s,X_s)$ and $v_i(k,x_s,X_s)$ from (5.3)

4. Update $x_i(k,x_s,X_s)$ from

$$x_{i_{j+1}}(k,x_s,X_s) = x_{i_j}(k,x_s,X_s) + v_{i_j}(k,x_s,X_s) - w_{i_j}(k,x_s,X_s)$$

5. Repete 3-4 until $x_i(k,x_s,X_s)$ has converged $\forall\ k, x_s, X_s$.

6. Determine

$$x_i^*(k,x_s),$$

$$\min_U[w_i(k,x_s,U) - x_i(k,x_s,U)] \text{ and}$$

$$\min_U(v_i(k,x_s,U))$$

7. Do 2-6 for $i = n^*$ but this time only for $k = k^*$ and $x_s = x_s^*$, but still $\forall\ X_s$. Determine also $U_w = [U : \min(w_{n^*}(k^*, x_s^*, U) - x_{n^*}(k^*, x_s^*, U))]$ and $U_v = [U : \min(v_{n^*}(k^*, x_s^*, U))]$. These give us the optimal excess.

Example 2 (continued)

We assume that we can choose among the following excesses.

$$D(X_s) \quad = \quad [0 \quad 5 \quad 10 \quad 20]$$

The premium costs for different bonus classes and excesses are

$$c(k, X_s) = \begin{pmatrix} 1.00 & 0.65 & 0.45 & 0.30 \\ 1.00 & 0.65 & 0.45 & 0.30 \\ 1.20 & 0.80 & 0.60 & 0.40 \\ 1.20 & 0.80 & 0.60 & 0.40 \\ 1.50 & 1.00 & 0.90 & 0.55 \\ 1.50 & 1.00 & 0.90 & 0.55 \\ 2.00 & 1.40 & 1.20 & 0.65 \\ 2.00 & 1.40 & 1.20 & 0.65 \\ 3.00 & 1.90 & 1.60 & 0.80 \\ 3.00 & 1.90 & 1.60 & 0.80 \end{pmatrix}$$

The results are with the following explanation of the rows: (we assume that our excess last year was 0)

$$k$$

$$x^*_{n^*}(k, 0)$$

$$U_w(k, 0)$$

$$\min_U [w_{n^*}(k, 0, U) - x_{n^*}(k, 0, U)]$$

$$U_v(k, 0)$$

$$\min_U [v_{n^*}(k, 0, U)]$$

$$n^* = 1$$

1	2	3	4	5	6	7	8	9	10
0.000	0.170	0.170	0.200	0.200	0.203	0.203	0.150	0.150	0.000
0	0	0	5	5	5	5	20	20	20
1.000	1.000	1.000	1.170	1.170	1.370	1.370	1.573	1.573	1.723
0	5	5	5	5	20	20	20	20	20
1.000	1.170	1.170	1.370	1.370	1.573	1.573	1.723	1.723	1.723

and

$$n^* = 100$$

1	2	3	4	5	6	7	8	9	10
0.018	0.205	0.367	0.551	0.712	0.821	0.912	0.976	1.018	0.475
0	0	0	5	5	5	5	20	20	20
10.02	10.02	10.03	10.22	10.40	10.77	11.11	11.59	12.03	12.57
0	5	5	5	5	20	20	20	20	20
10.03	10.22	10.40	10.77	11.11	11.59	12.03	12.57	13.04	13.04

Here $V_{100}(8,1,0)$=12.57. We see that that we can save a little bit if we choose optimal excesses because in chapter 3 we had that $V_{100}(8,1)$=13.59.

5.2 With thresholds

The changes of bonus classes are the same as in section 4.2 and the transition probabilities are

$$P[(k,x,x_s) \to (u_k(x),y,U)] = F^*(dy)$$

where $u_k(x)$ is described in section 4.2. $V_n(k,x,x_s)$ can be determined from

$$V_n(k,x,x_s) = \min_{u,U} \Big[C(u_k(x), x\ , x_s, U) + r\Big(p_0 \cdot V_{n-1}(u_k(x), 0, U) + \\ +(1-p_0)\int_0^\infty V_{n-1}(u_k(x), y, U) F(dy)\Big)\Big]$$

where

$$V_0(k,x,x_s) = \min(x,x_s)$$

and

$$C(u_k(x),x,x_s,U) = \begin{cases} c(a_{kt}, X_s) + \min(x,x_s) & \text{if } u=a,\ U=X_s,\ x^{(t-1)}(k) \le x < x^{(t)}(k) \\ c(b_k, X_s) + x & \text{if } u=a,\ U=X_s,\ \forall x \end{cases}$$

For a fix choice $U = X_s$ next period $V_n(k,x,x_s|U=X_s)$ will be

$$V_n(k,x,x_s|U=X_s) = \min_u \Big[C(u_k(x\ ), x, x_s, X_s) + r\Big(p_0 \cdot V_{n-1}(u_k(x), 0, X_s) \\ +(1-p_0)\int_0^\infty V_{n-1}(u_k(x), y, X_s) F(dy)\Big)\Big]$$

The controlvariable $(u|U = X_s)$ has the form

$$(u|U = X_s) = \begin{cases} b & \text{if } x^{(t-1)}(k) \leq x \leq x_n^t(k, x_s, X_s) \\ a & \text{if } x_n^t(k, x_s, X_s) < x < x^{(t)}(k) \end{cases}$$

A plot of $V_n(k, x, x_s|U = X_s)$ look like figure 5.3.

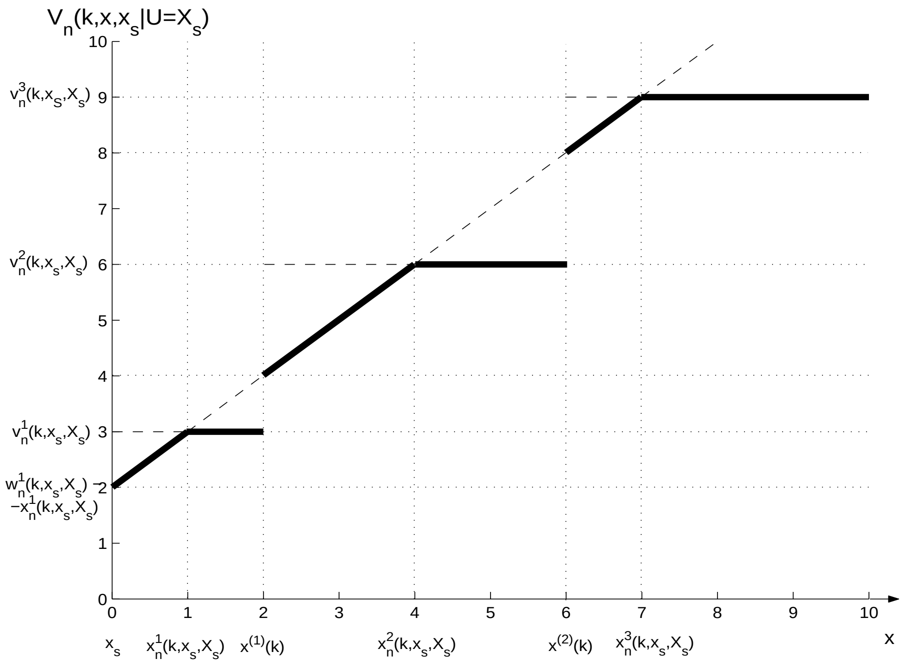

Figure 5.3: $V_n(k, x, x_s|U = X_s)$.

Now we can write:

$$V_n(k, x, x_s|U = X_s) = \begin{cases} C(b_k, X_s) + x + r\Big(p_0 \cdot V_{n-1}(b_k, 0, X_s) + \\ \qquad + (1-p_0)\int_0^\infty V_{n-1}(b_k, y, X_s)F(dy)\Big) \\ \text{if } x^{(t-1)}(k) \leq x \leq x_n^t(k, x_s, X_s) \\ C(a_{kt}, X_s) + x_s + r\Big(p_0 \cdot V_{n-1}(a_{kt}, 0, X_s) + \\ \qquad + (1-p_0)\int_0^\infty V_{n-1}(a_{kt}, y, X_s)F(dy)\Big) \\ \text{if } x_n^t(k, x_s, X_s) < x < x^{(t)}(k) \end{cases} \tag{5.4}$$

From figure 5.3 we see that

$$V_n(k,x,x_s|U=X_s)=\min[w_n^t(k,x_s,X_s)-x_n^t(k,x_s,X_s)+x;v_n^t(k,x_s,X_s)]$$
$$\text{if } x^{(t-1)}(k)\le x<x^{(t)}(k)$$

or

$$V_n(k,x,x_s|U=X_s)=\begin{cases} w_n^t(k,x_s,X_s)-x_n^t(k,x_s,X_s)+x \\ \quad\text{if } x^{(t-1)}(k)\le x\le x_n^t(k,x_s,X_s) \\ v_n^t(k,x_s,X_s) \\ \quad\text{if } x_n^t(k,x_s,X_s)<x<x^{(t)}(k)\end{cases} \tag{5.5}$$

(5.4) and (5.5) give us

$$\begin{gathered} w_n^t(k,x_s,X_s)=C(b_k,X_s)+x_n^t(k,x_s,X_s)+ \\ +r(p_0\cdot N_{n-1}(b_k,X_s)+(1-p_0)E_{n-1}(b_k,X_s) \\ v_n^t(k,x_s,X_s)=C(a_{kt},X_s)+x_s+ \\ +r(p_0\cdot N_{n-1}(a_{kt},X_s)+(1-p_0)E_{n-1}(a_{kt},X_s)\end{gathered} \tag{5.6}$$

where

$$N_n(k,x_s)=V_n(k,0,x_s)$$
$$E_n(k,x_s)=\int_0^\infty V_n(k,y,x_s)F(dy)$$

When we have several choices for U next year $V_n(k,x,x_s)$ looks like figure 5.4 where

$$x_n^{t*}(k,x_s)=\min_U\left(v_n^t(k,x_s,U)\right)-\min_U\left(w_n^t(k,x_s,U)-x_n^t(k,x_s,U)\right)$$

When we have studied the figure and determined $x_n^{t*}(k,x_s)$ we must use the following rule.

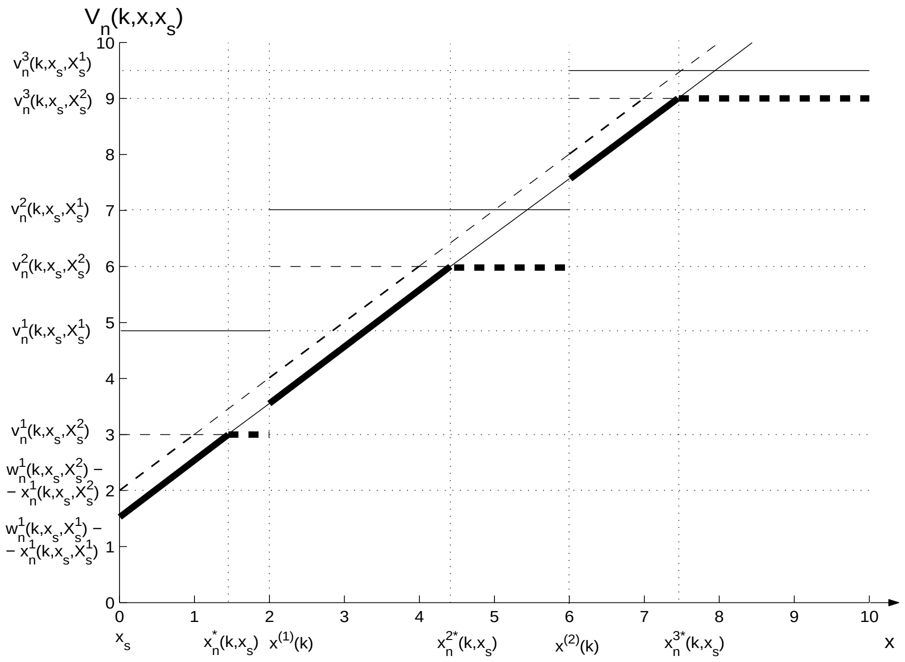

Figure 5.4: $V_n(k,x,x_s)$ with two possible excesses.

Rule 2

Put $x_n^{(t)}(k,x_s) := x^{(t)}(k)\ \forall\ k,n,t,x_s$
if $x_n^{(t)}(k,x_s) < x_n^{t*}(k,x_s)$ put $x_n^{t*}(k,x_s) = x_n^{(t-1)}(k,x_s)$ and $x_n^{(t)}(k,x_s) = x_n^{(t-1)}(k,x_s)$
if $x_n^{(t)}(k,x_s) > x_n^{(t+1)*}(k,x_s)$ put $x_n^{(t+1)*}(k,x_s) = x_n^{(t)}(k,x_s)$
for $t = 1,2,...,T,T+1$

Now we can determine the following:

$$N_n(k,x_s) = V_n(k,0,x_s) = \min_U \left(w_n^1(k,x_s,U) - x_n^1(k,x_s,U)\right)$$

and

$$E_n(k,x_s) = \int_0^\infty V_n(k,y,x_s)F(dy) =$$

$$= \int_0^{x_n^*(k,x_s)} (\min_U[w_n^1(k,x_s,U) - x_n^1(k,x_s,U)] + y)F(dy) +$$

$$+\int_{x_n^*(k,x_s)}^{x_n^{(1)}(k,x_s)} \min_U [v_n^1(k,x_s,U)]F(dy)+...$$

$$...+\int_{x_n^{(T)}(k,x_s)}^{x_n^{(T+1)*}(k,x_s)} (\min_U \; [w_n^{T+1}(k,x_s,U)-x_n^{T+1}(k,x_s,U)]+y)F(dy)+$$

$$+\int_{x_n^{(T+1)*}(k,x_s)}^{\infty} \min_U [v_n^{T+1}(k,x_s,U)]F(dy)=$$

$$=\min_U [w_n^1(k,x_s,U)-x_n^1(k,x_s,U)]F(x_n^*(k,x_s))+$$

$$+\sum_{i=1}^{T+1}\Big[x_n^{i*}(k,x_s)F(x_n^{i*}(k,x_s))-G(x_n^{i*}(k,x_s))\Big]+$$

$$+\sum_{i=1}^{T}\Big[\min_U [v_n^{i*}(k,x_s,U)](F(x_n^{(i)}(k,x_s)-F(x_n^{i*}(k,x_s)))+$$

$$+G(x_n^{(i)}(k,x_s))-x_n^{(i)}(k,x_s)F(x_n^{(i)}(k,x_s))\Big]+$$

$$+\sum_{i=2}^{T+1}\Big[\min_U (w_n^i(k,x_s,U)-x_n^i(k,x_s,U))(F(x_n^{i*}(k,)x_s)-F(x_n^{(i-1)}(k,x_s)))\Big]+$$

$$+\min_U \Big[v_n^{T+1}(k,x_s,U)(1-F(x_n^{(T+1)*}(k,x_s)))\Big]$$

The controlvariables are

$$(u,U)=\begin{cases}(b,\;[U:\min(w_n^t(k,x_s,U)-x_n^t(k,x_s,U)]) & \text{if } x_n^{(t-1)}(k,x_s)\le x\le x_n^{t*}(k,x_s)\\ (a,\;[U:\min(v_n^t(k,x_s,U)]) & \text{if } x_n^{t*}(k,x_s)<x<x_n^{(t)}(k,x_s)\\ t=1,2,..,T,T+1 & \end{cases}$$

If our present state is (k^*,x^*,x_s^*) and our horizon is n^* then we should go on like this to get the optimal decisions:

1. Do 2-6 for $i=1,2,..,n^*-1$.
2. Choose initial guesses $x_{i_j}^t(k,x_s,X_s)>x_s$, $(j=0)$, for $x_i^t(k,x_s,X_s)>x_s\;\forall\, t,k,x_s,X_s$

3. Determine the corresponding $w^t_{i_j}(k, x_s, X_s)$ and $v^t_{i_j}(k, x_s, X_s)$ for $w^t_i(k, x_s, X_s)$ and $v^t_i(k, x_s, X_s)$ from (5.6).

4. Update $x^t_i(k, x_s, X_s)$ from

$$x^t_{i_{j+1}}(k, x_s, X_s) = x^t_{i_j}(k, x_s, X_s) + v^t_{i_j}(k, x_s, X_s) - w^t_{i_j}(k, x_s, X_s)$$

5. Repete 3-4 until $x^t_i(k, x_s, X_s)$ has converged $\forall\ t, k, x_s, X_s$.

6. Determine

$$x^{t*}_i(k, x_s),$$

$$\min_U [w^t_i(k, x_s, U) - x^t_i(k, x_s, U)] \text{ and}$$

$$\min_U (v^t_i(k, x_s, U))$$

7. Do 2-6 for $i = n^*$ but this time only for $k = k^*$ and $x_s = x^*_s$ but $\forall\ X_s$. Determine also $U^t_w = [U : \min(w^t_{n^*}(k^*, x^*_s, U) - x^t_{n^*}(k^*, x^*_S, U))]$ and $U^t_v = [U : \min(v^t_{n^*}(k^*, x^*_s, U)]$

Example 3 (continued)

We assume that we can choose among the following excesses.

$$D(X_s) = [0 \quad 5 \quad 10 \quad 20]$$

The premium costs for different bonus classes and excesses are

$$c(k, X_s) = \begin{pmatrix} 1.00 & 0.70 & 0.60 & 0.55 \\ 2.50 & 1.70 & 1.50 & 1.40 \\ 3.50 & 2.40 & 1.90 & 1.80 \\ 4.50 & 3.00 & 2.50 & 2.10 \\ 5.50 & 3.50 & 3.00 & 2.80 \\ 7.00 & 4.20 & 3.50 & 2.90 \end{pmatrix}$$

The results are the following: (we assume that our excess last year was 0)

$n^* = 1$

k	1	2	3	4	5	6
$x_1^*(k,0)$	2.472	3.072	3.572	2.772	1.800	1.200
$x_1^{2*}(k,0)$	3.072	3.572	4.272	2.772	1.800	1.200
$x_1^{3*}(k,0)$	3.572	4.272	4.272	2.772	1.800	1.200
$U_w^1(k,0)$	0	0	0	0	3	3
$U_w^2(k,0)$	0	0	0	0	3	3
$U_w^3(k,0)$	0	0	0	0	3	3
$\min_U[w_1^1(k,0,U) - x_1^1(k,0,U)]$	1.000	1.000	1.000	2.500	3.472	4.072
$U_v^1(k,0)$	3	3	3	3	3	3
$U_v^2(k,0)$	3	3	3	3	3	3
$U_v^3(k,0)$	3	3	3	3	3	3
$\min_U[v_1^1(k,0,U)]$	3.472	4.072	4.572	5.272	5.272	5.272
$\min_U[v_1^2(k,0,U)]$	4.072	4.572	5.272	5.272	5.272	5.272
$\min_U[v_1^3(k,0,U)]$	4.572	5.272	5.272	5.272	5.272	5.272

$n^* = 100$

k	1	2	3	4	5	6
$x_{100}^*(k,0)$	2.352	4.239	5.373	5.891	5.084	3.197
$x_{100}^{2*}(k,0)$	4.239	5.373	7.437	5.891	5.084	3.197
$x_{100}^{3*}(k,0)$	5.373	7.437	7.437	5.891	5.084	3.197
$U_w^1(k,0)$	0	0	0	3	8	8
$U_w^2(k,0)$	0	0	0	3	8	8
$U_w^3(k,0)$	0	0	0	3	8	8
$\min_U[w_{100}^1(k,0,U) - x_{100}^1(k,0,U)]$	23.36	23.36	23.36	24.91	25.722	7.60
$U_v^1(k,0)$	8	8	3	8	8	8
$U_v^2(k,0)$	8	3	8	8	8	8
$U_v^3(k,0)$	3	8	8	8	8	8
$\min_U[v_{100}^1(k,0,U)]$	25.72	27.60	28.74	30.80	30.80	30.80
$\min_U[v_{100}^2(k,0,U)]$	27.60	28.74	30.80	30.80	30.80	30.80
$\min_U[v_{100}^3(k,0,U)]$	28.74	30.80	30.80	30.80	30.80	30.80

Here $V_{100}(4,20,0)$=30.80. We see that that we can save a lot if we choose optimal excesses because in chapter 4 we had that $V_{100}(4,20)$=39.35.

Chapter 6

A method to find the optimal claim-decision and preventive measures

This chapter deals with the issue whether some damage preventive measures should be taken in action or not. The benefits that are reductions on the premiums and decreasing risks for damages should be balanced with the actual costs of the measures. We denote the chosen measures for next year by $M = (M_1, M_2, ..., M_N)$ where

$$\begin{aligned} M_i &= 0 \quad \text{if measure } i \text{ will not be used next year} \\ M_i &= 1 \quad \text{if measure } i \text{ will be used next year} \end{aligned}$$

With measures given by M the corresponding distribution for the damages is

$$F^{M*}(x) = p_0(M) + (1 - p_0(M))F^M(x)$$

6.1 Without thresholds and excess

The changes of bonus classes are the same as in chapter 3. Let (k, x, m) be the state of the Markov decision process, where $m = (m_1, m_2, ..., m_N)$ and

$$\begin{aligned} m_i &= 0 \quad \text{if measure } i \text{ have not been used before} \\ m_i &= 1 \quad \text{if measure } i \text{ have been used before} \end{aligned}$$

This is for handling the once-for-all costs for every measure. The control variables are $u \in [a, b]$ and $\overline{U} \in D(M)$. The transition probabilities are

$$P[(k, x, m) \to (u_k, y, \max(m, \overline{U}))] = F^{\overline{U}*}(dy)$$

where

$$\max(m, M) = (\max(m_1, M_1), \max(m_2, M_2), ..., \max(m_N, M_N))$$

$V_n(k, x, m)$ can be determined from

$$V_n(k, x, m) = \min_{u,\overline{U}} \Big[C(u_k, x\ , m, \overline{U}) + r\Big(p_0(\overline{U}) \cdot V_{n-1}(u_k, 0, \max(m, \overline{U})) + \\ +(1 - p_0(\overline{U})) \int_0^\infty V_{n-1}(u_k, y, \max(m, \overline{U}))F^{\overline{U}}(dy)\Big)\Big]$$

where

$$V_0(k, m, x) = 0$$

and

$$C(u_k, x, m, \overline{U}) = \begin{cases} c(a_k, m, M) & \text{if } u = a,\ \overline{U} = M \\ c(b_k, m, M) + x & \text{if } u = b,\ \overline{U} = M \end{cases}$$

$c(k, m, M)$ is the premium in bonusclass k with measures given by M + the cost for M + the once-for-all cost for the M_i that havn't been used before. For a fix choice $\overline{U} = M$ next year $V_n(k, x, m|\overline{U} = M)$ will be

$$V_n(k, x, m|\overline{U} = M) = \min_u [C(u_k, x, m, M) + r\Big(p_0(M) \cdot V_{n-1}(u_k, 0, \max(m, M)) + \\ +(1 - p_0(M)) \int_0^\infty V_{n-1}(u_k, y, \max(m, M))F^M(dy)\Big)\Big]$$

We see that the control variable $(u|\overline{U} = M)$ has the form

$$(u|\overline{U} = M) = \begin{cases} b \text{ if } x \leq x_n(k, m, M) \\ a \text{ if } x > x_n(k, m, M) \end{cases}$$

A plot of this situation looks like figure 6.1. Now we can write:

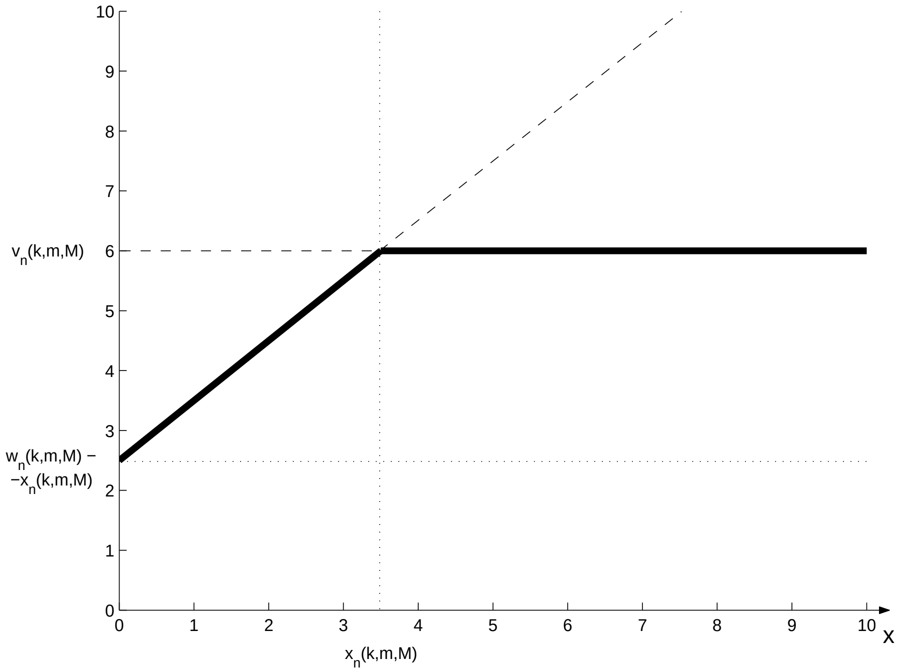

Figure 6.1: $V_n(k, x, m|\overline{U} = M)$.

$$V_n(k, x, m|\overline{U} = M) = \begin{cases} C(b_k, m, M) + x + r\Big(p_0(M) \cdot V_{n-1}(b_k, 0, \max(m, M)) + \\ \qquad + (1 - p_0(M)) \int_0^\infty V_{n-1}(b_k, y, \max(m, M)) F^M(dy)\Big) \\ \text{if } x \leq x_n(k, m, M) \\ C(a_k, m, M) + r\Big(p_0(M) \cdot V_{n-1}(a_k, 0, \max(m, M)) + \\ \qquad + (1 - p_0(M)) \int_0^\infty V_{n-1}(a_k, y, \max(m, M)) F^M(dy)\Big) \\ \text{if } x > x_n(k, m, M) \end{cases} \tag{6.1}$$

From figure 6.1 we realize that

$$V_n(k, x, m|\overline{U} = M) = \min[w_n(k, m, M) - x_n(k, m, M) + x; v_n(k, m, M)]$$

or

$$V_n(k, x, m|\overline{U} = M) = \begin{cases} w_n(k, m, M) - x_n(k, m, M) + x & \text{if } x \leq x_n(k, m, M) \\ v_n(k, m, M) & \text{if } x > x_n(k, m, M) \end{cases} \tag{6.2}$$

(6.1) and (6.2) give us

$$w_n(k,m,M) = C(b_k,m,M) + x_n(k,m,M) + r\Big(p_0(M)\cdot N_{n-1}(b_k,\max(m,M))+$$
$$+(1-p_0(M))E^M_{n-1}(b_k,\max(m,M))\Big)$$
$$v_n(k,m,M) = C(a_k,m,M) + r\Big(p_0(M)\cdot N_{n-1}(a_k,0,\max(m,M)+$$
$$+(1-p_0(M))E^M_{n-1}(a_k,\max(m,M))\Big) \tag{6.3}$$

where

$$N_n(k,m) = V_n(k,0,m)$$
$$E^M_n(k,m) = \textstyle\int_0^\infty V_n(k,y,m)F^M(dy)$$

When we have several choices for $\overline{U}$ in next period $V_n(k,x,m)$ looks like figure 6.2 where

$$x^*_n(k,m) = \min_{\overline{U}}(v_n(k,m,\overline{U}) - \min_{\overline{U}}(w_n(k,m,\overline{U}) - x_n(k,m,\overline{U}))$$

When we have studied the figure and determined $x^*_n(k,m)$ can we determine

$$N_n(k,m) = V_n(k,0,m) = \min_{\overline{U}}(w_n(k,m,\overline{U}) - x_n(k,m,\overline{U}))$$

$$E^M_n(k,m) = \int_0^\infty V_n(k,y,m)F^M(dy) =$$

$$= \int_0^{x^*_n(k,m)} (\min_{\overline{U}}[w_n(k,m,\overline{U}) - x_n(k,m,\overline{U})] + y)F^M(dy)+$$

$$+\int_{x^*_n(k,m)}^\infty \min_{\overline{U}}(v_n(k,m,\overline{U}))F^M(dy) =$$

$$= \min_{\overline{U}}[w_n(k,m,\overline{U}) - x_n(k,m,\overline{U})]\cdot F^M(x^*_n(k,m)) + x^*_n(k,m)\cdot F^M(x^*_n(k,m))+$$

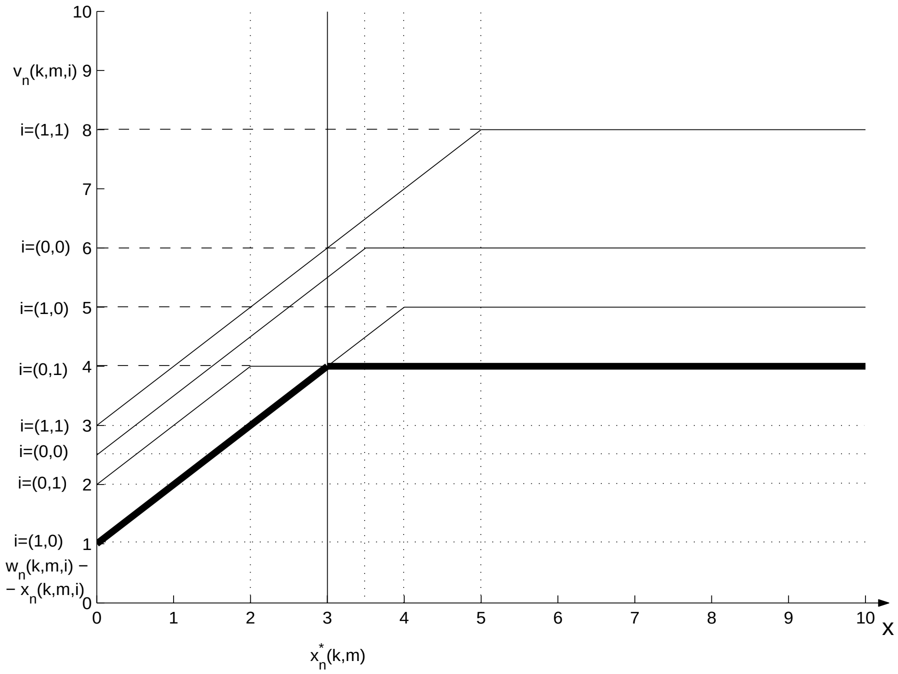

Figure 6.2: $V_n(k,x,m)$.

$$+(1-F^M(x_n^*(k,m))\cdot\min_{\overline{U}}[v_n(k,m,\overline{U})]-G^M(x_n^*(k,m))$$

where $G(x)=\int_0^x F(y)dy$
The controlvariables are

$$(u,U)=\begin{cases}(b,\ [\overline{U}:\min(w_n(k,m,\overline{U})-x_n(k,m,\overline{U}))]) & \text{if } x\le x_n^*(k,m))\\(a,\ [\overline{U}:\min(v_n(k,m,\overline{U}))]) & \text{if } x> x_n^*(k,m))\end{cases}$$

If our current state is (k^*,x^*,m^*) and our horizon is n^* then we should go on like this to get the optimal decisions:

1. Do 2-6 for $i=1,2,..,n^*-1$.
2. Choose initial guesses $x_{i_j}(k,m,M)$, $(j=0)$, for $x_i(k,m,M)\ \forall\ k,m,M$.
3. Determine the corresponding $w_{i_j}(k,m,M)$ and $v_{i_j}(k,m,M)$ for $w_i(k,m,M)$ and $v_i(k,m,M)$ from (6.3).
4. Update $x_i(k,m,M)$ by

$$x_{i_{j+1}}(k,m,M)=x_{i_j}(k,m,M)+v_{i_j}(k,m,M)-w_{i_j}(k,m,M).$$

5. Repete 3-4 until $x_i(k,m,M)$ has converged.

6. Determine

$$x_i^*(k,m),$$

$$\min_{\overline{U}}[w_i(k,m,\overline{U}) - x_i(k,m,\overline{U})] \text{ and}$$

$$\min_{\overline{U}}(v_i(k,m,\overline{U}))$$

7. Do 2-6 for $i = n^*$ but this time only for $k = k^*$ and $m = m^*$ but still $\forall\ M$. Determine $\overline{U}_w = [\overline{U} : \min(w_{n^*}(k^*,m^*,\overline{U}) - x_{n^*}(k^*,m^*,\overline{U}))]$ and $\overline{U}_v = [\overline{U} : \min(v_{n^*}(k^*,m^*,\overline{U})]$.

Example 2 (continued)

If we install some fire protection including fire alarm and other protective measures against fire we could get some reduction on the premium. The installation cost for this is 0.7 and the service cost is 0.5 every year. We could also have a watchmen that costs 0.5. The premium costs for different bonus classes and measure are

$$c(k,(0,0),M) = \begin{pmatrix} 1.00 & 1.27 & 1.95 & 2.28 \\ 1.00 & 1.27 & 1.95 & 2.28 \\ 1.20 & 1.40 & 2.05 & 2.35 \\ 1.20 & 1.40 & 2.05 & 2.35 \\ 1.50 & 1.60 & 2.20 & 2.50 \\ 1.50 & 1.60 & 2.20 & 2.50 \\ 2.00 & 2.10 & 2.60 & 2.80 \\ 2.00 & 2.10 & 2.60 & 2.80 \\ 3.00 & 2.90 & 3.20 & 3.30 \\ 3.00 & 2.90 & 3.20 & 3.30 \end{pmatrix}$$

Remember that after the fire protection have been installed the installation costs will be removed from these costs. We assume a Pareto distribution with the following parameters for different measures:

$$
\begin{aligned}
p_0((0,0)) &= 0.90\\
\alpha((0,0)) &= 60\\
\beta((0,0)) &= 5\\
p_0((0,1)) &= 0.92\\
\alpha((0,1)) &= 50\\
\beta((0,1)) &= 5\\
p_0((1,0)) &= 0.94\\
\alpha((1,0)) &= 40\\
\beta((1,0)) &= 5\\
p_0((1,1)) &= 0.95\\
\alpha((1,1)) &= 35\\
\beta((1,1)) &= 5
\end{aligned}
$$

The results are with the following explanation of the rows: (we assume that our excess last year was 0)

$$k$$

$$x^*_{n^*}(k,(0,0))$$

$$\overline{U}_w(k,(0,0))$$

$$\min_{\overline{U}}[w_{n^*}(k,(0,0),U) - x_{n^*}(k,(0,0),U)]$$

$$\overline{U}_v(k,(0,0))$$

$$\min_{\overline{U}}[v_{n^*}(k,(0,0),U)]$$

$$n^* = 1$$

1	2	3	4	5	6	7	8	9	10
0.000	0.200	0.200	0.300	0.300	0.500	0.500	0.900	0.900	0.000
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,1)
1.000	1.000	1.000	1.200	1.200	1.500	1.500	2.000	2.000	2.900
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,1)	(0,1)	(0,1)
1.000	1.200	1.200	1.500	1.500	2.000	2.000	2.900	2.900	2.900

$n^* = 100$

1	2	3	4	5	6	7	8	9	10
0.013	0.149	0.386	0.725	1.018	1.565	1.983	2.544	2.71	1.13
(0,0)	(0,0)	(0,0)	(0,1)	(0,0)	(0,1)	(0,0)	(0,0)	(0,1)	(1,0)
10.01	10.01	10.02	10.16	10.41	10.88	11.43	12.45	13.41	14.99
(0,0)	(0,1)	(0,0)	(0,1)	(0,0)	(0,0)	(0,1)	(1,0)	(1,0)	(1,0)
10.02	10.16	10.41	10.88	11.43	12.45	13.41	14.99	16.12	16.12

Now $V_{100}(8, 1, (0, 0)) = 12.45 + 1 = 13.45$, which is a little bit less than 13.59 from section 3.1.

6.2 With thresholds but without excess

The changes of bonus classes are the same as in section 4.2. The transition probabilities are

$$P[(k, x, m) \to (u_k(x), y, \max(m, \overline{U}))] = F^{\overline{U}*}(dy)$$

where $u_k(x)$ is described in section 4.2. $V_n(k, x, m)$ can be determined from

$$V_n(k, x, m) = \min_{u,\overline{U}} \Big[C(u_k(x)\,, x, m, \overline{U}) + r\Big(p_0(\overline{U}) \cdot V_{n-1}(u_k(x), 0, \max(m, \overline{U}) \\ +(1 - p_0(\overline{U})) \int_0^\infty V_{n-1}(u_k(x), y, \max(m, \overline{U}))F^{\overline{U}}(dy)\Big)\Big]$$

where

$$V_0(k, x, m) = 0$$

For a fix choice $\overline{U} = M$ next period $V_n(k, x, m)$ will be

$$V_n(k, x, m|\overline{U} = M) = \min_{u} \Big[c(u_k(x), x, m, M) + r\Big(p_0(M) \cdot V_{n-1}(u_k(x), 0, \max(m, M)) + \\ +(1 - p_0(M)) \int_0^\infty V_{n-1}(u_k(x), y, \max(m, M))F^M(dy)\Big)\Big]$$

We know that the controlvariable $(u|\overline{U} = M)$ has the form

$$(u|\overline{U} = M) = \begin{cases} b \text{ if } x^{(t-1)}(k) \le x \le x_n^t(k, m, M) \\ a \text{ if } x_n^t(k, m, M) < x < x^{(t)}(k) \end{cases}$$

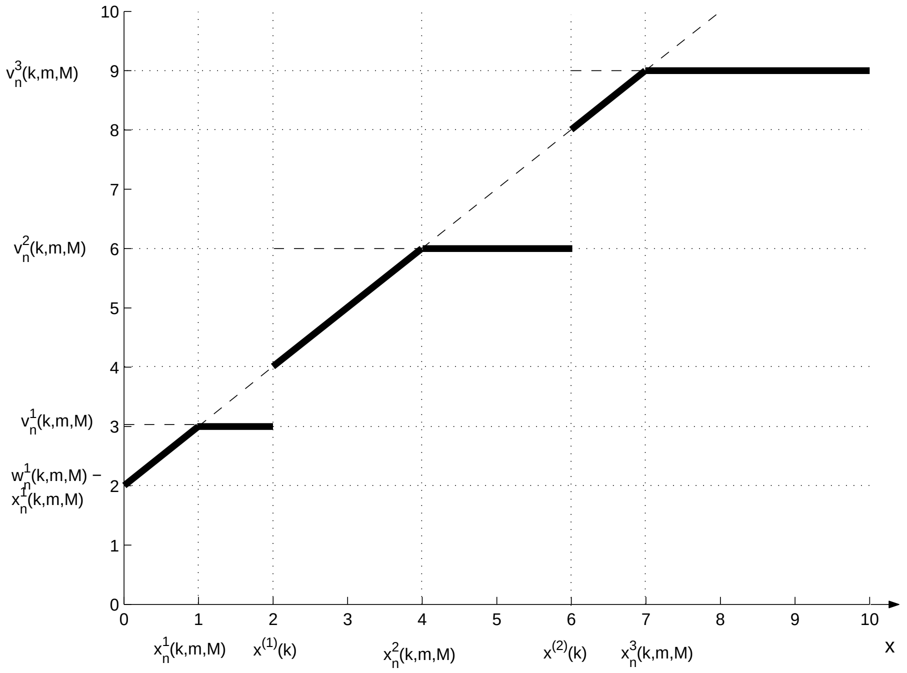

Figure 6.3: $V_n(k, x, m|\overline{U} = M)$.

A plot of $V_n(k, x, m|\overline{U} = M)$ looks like figure 6.3. Now we can write:

$$V_n(k, x, m|\overline{U} = M) = \begin{cases} C(b_k, m, M) + x + r\Big(p_0(M) \cdot V_{n-1}(b_k, 0, \max(m, M) + \\ \qquad + (1 - p_0(M)) \int_0^\infty V_{n-1}(b_k, y, \max(m, M))F^M(dy)\Big) \\ \text{ if } x^{(t-1)}(k) \le x \le x_n^t(k, m, M) \\ C(a_{kt}, m, M) + r\Big(p_0(M) \cdot V_{n-1}(a_{kt}, 0, \max(m, M)) + \\ \qquad + (1 - p_0(M)) \int_0^\infty V_{n-1}(a_{kt}, y, \max(m, M))F^M(dy)\Big) \\ \text{ if } x_n^t(k, m, M) < x < x^{(t)}(k) \end{cases} \tag{6.4}$$

From figure 6.3 we realize that

$$V_n(k, x, m|\overline{U} = M) = \min[w_n^t(k, m, M) - x_n^t(k, m, M) + x; v_n^t(k, m, M)]$$

$$\text{if } x^{(t)}(k) \le x < x^{(t+1)}(k)$$

or

$$V_n(k,x,m|\overline{U}=M) =$$
$$= \begin{cases} w_n^t(k,m,M) - x_n^t(k,m,M) + x & \text{if } x^{(t-1)}(k) \le x \le x_n^t(k,m,M) \\ v_n^t(k,m,M) & \text{if } x_n^t(k,m,M) < x < x^{(t)}(k) \end{cases} \tag{6.5}$$

(6.4) and (6.5) give us

$$\begin{aligned} w_n^t(k,m,M) &= C(b_k,m,M) + x_n^t(k,m,M) + r\Big(p_0(M)\cdot N_{n-1}(b_k,\max(m,M)) + \\ &\quad +(1-p_0(M))E_{n-1}^M(b_k,\max(m,M))\Big) \\ v_n^t(k,m,M) &= C(a_{kt},m,M) + r\Big(p_0(M)\cdot N_{n-1}(a_{kt},\max(m,M)) + \\ &\quad +(1-p_0(M))E_{n-1}^M(a_{kt},\max(m,M))\Big) \end{aligned} \tag{6.6}$$

where

$$N_n(k,m) = V_n(k,0,m)$$
$$E_n^M(k,m) = \int_0^\infty V_n(k,y,m)F^M(dy)$$

When we have several choices for $\overline{U}$ next year $V_n(k,x,m)$ look like figure 6.4 where

$$x_n^{t*}(k,m) = \min_{\overline{U}}(v_n^t(k,m,\overline{U}) - \min_{\overline{U}}(w_n^t(k,m,\overline{U}) - x_n^t(k,m,\overline{U}))$$

When we have studied the figure and determined $x_n^{t*}(k,m)$ we must use the following rule:

Rule 3

Put $x_n^{(t)}(k,m) := x^{(t-1)}(k) \forall\ k,m,n,t$
if $x_n^{(t)}(k,m) < x_n^{t*}(k,m)$ put $x_n^{t*}(k,m) = x_n^{(t-1)}(k,m)$ and $x_n^{(t)}(k,m) = x_n^{(t-1)}(k,m)$
if $x_n^{(t)}(k,m) > x_n^{(t+1)*}(k,m)$ put $x_n^{(t+1)*}(k,m) = x_n^{(t)}(k,m)$
for $t = 1,2,...,T,T+1$

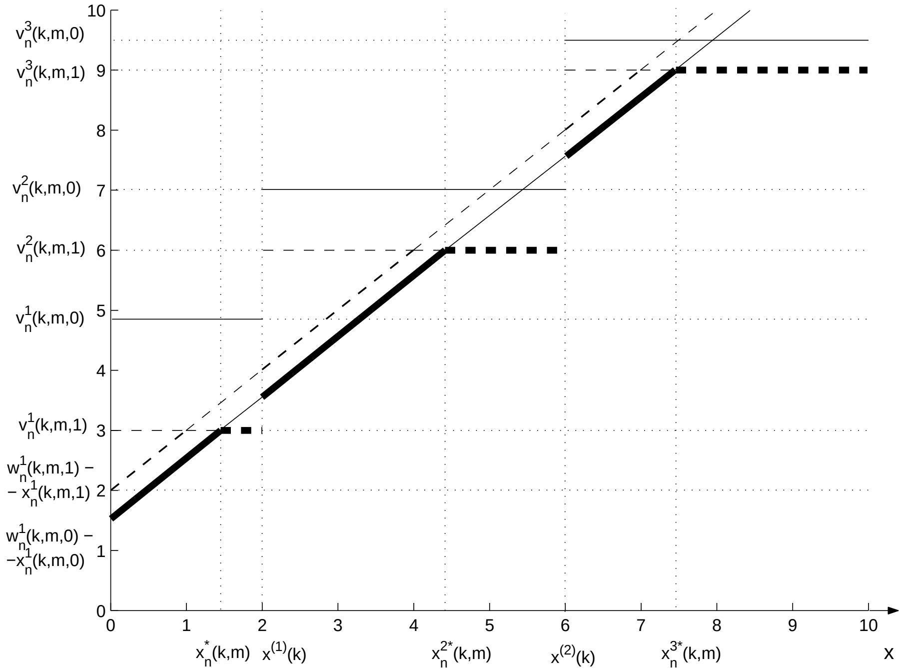

Figure 6.4: $V_n(k,x,m)$ with one possible measure.

Now we can determine the following:

$$N_n(k,m) = V_n(k,0,m) = \min_{\overline{U}}(w_n^1(k,m,\overline{U}) - x_n^1(k,m,\overline{U}))$$

$$E_n^M(k,m) = \int_0^{\infty} V_n(k,y,m)F^M(dy) =$$

$$= \int_0^{x_n^*(k,m)} (\min_{\overline{U}}[w_n^1(k,m,\overline{U}) - x_n^1(k,m,\overline{U})] + y)F^M(dy)+$$

$$+ \int_{x_n^*(k,m)}^{x_n^{(1)}(k,m)} \min_{\overline{U}}(v_n^1(k,m,\overline{U}))F^M(dy) + ...$$

$$+... \int_{x_n^{(T)}(k,m)}^{x_n^{(T+1)*}(k,m)} (\min_{\overline{U}}[w_n^{T+1}(k,m,\overline{U}) - x_n^{T+1}(k,m,\overline{U})] + y)F^M(dy)+$$

$$+ \int_{x_n^{(T+1)*}(k,m)}^{\infty} \min_{\overline{U}}(v_n^{T+1}(k,m,\overline{U}))F^M(dy) =$$

$$= \min_{\overline{U}}[w_n^1(k,m,\overline{U}) - x_n^1(k,m,\overline{U})]F^M(x_n^*(k,m)) +$$

$$+ \sum_{i=1}^{T+1}\Big[x_n^{t*}(k,m)F^M(x_n^{i*}(k,m)) - G^M(x_n^{i*}(k,m))\Big] +$$

$$+ \sum_{i=1}^{T}\Big[\min_{\overline{U}} v_n^{i*}(k,m)(F^M(x_n^{(i)}(k,m) - F^M(x_n^{i*}(k,m))) +$$

$$+ G^M(x_n^{(i)}(k,m)) - x_n^{(i)}(k,m)F^M(x_n^{(i)}(k,m))\Big] +$$

$$+ \sum_{i=2}^{T+1}\Big[\min_{\overline{U}}(w_n^i(k,m,\overline{U}) - x_n^i(k,m,\overline{U}))(F^M(x_n^{i*}(k,m)) - F^M(x_n^{(i-1)}(k,m)))\Big] +$$

$$+ \min_{\overline{U}}\Big[v_n^{T+1}(k,m,\overline{U})(1 - F^M(x_n^{(T+1)*}(k,m)))\Big]$$

The controlvariables are

$$(u,U) = \begin{cases} (b,\ [\overline{U} : \min(w_n^t(k,m,\overline{U}) - x_n^t(k,m,\overline{U}))]) & \text{if } x_n^{(t-1)}(k,m) \le x \le x_n^{t*}(k,m) \\ (a,\ [\overline{U} : \min(v_n^t(k,m,\overline{U}))]) & \text{if } x_n^{t*}(k,m) < x < x_n^{(t)}(k,m) \end{cases}$$

If our present state is (k^*, x^*, m^*) and the horizon is n^* then we should go on like this to get the optimal decisions:

1. Do 2-6 for $i = 1, 2, .., n^* - 1$.
2. Choose initial guesses $x_{i_j}^t(k,m,M)$, $(j = 0)$, for $x_i^t(k,m,M)\ \forall\ t,k,m,M$.
3. Determine the corresponding $w_{i_j}^t(k,m,M)$ and $v_{i_j}^t(k,m,M)$ for $w_i^t(k,m,M)$ and $v_i^t(k,m,M)$ from (6.6)
4. Update $x_i^t(k,m,M)$ from
$$x_{i_{j+1}}^t(k,m,M) = x_{i_j}^t(k,m,M) + v_{i_j}^t(k,m,M) - w_{i_j}^t(k,m,M).$$
5. Repete 3-4 until $x_i^t(k,m,M)$ has converged.

6. Determine
$$x_i^{t*}(k,m),$$
$$\min_{\overline{U}}[w_i^t(k,m,\overline{U}) - x_i^t(k,m,\overline{U})] \text{ and}$$
$$\min_{\overline{U}}(v_i^t(k,m,\overline{U}))$$
7. Do 2-6 for $i = n^*$ but this time only for $k = k^*$ and $m = m^*$ but still $\forall\ M$. Determine also $\overline{U}_w^t = [\overline{U} : \min(w_{n^*}^t(k^*,m^*,\overline{U}) - x_{n^*}^t(k^*,m^*,\overline{U}))]$ and $\overline{U}_v^t = [\overline{U} : \min(v_{n^*}^t(k^*,m^*,\overline{U}))]$.

Example 1 (continued)

We will now see that the analysis from this chapter gives the same optimal choices as in chapter 1 for an almost identical example. The data are given by

k	1	2	3
a_{k1}	2	3	3
a_{k2}	3	3	3
b_k	1	1	2
$x^{(1)}(k)$	35	0	0

Our possible measures are

$$m_1 = \text{firealarm}$$
$$m_2 = \text{watchmen}$$

We assume that the once-for-all costs of both measures are 0. We again assume an Pareto-distribution and the parameters are for different measures:

$$p_0((0,0)) = 0.68$$
$$\alpha((0,0)) = 81$$
$$\beta((0,0)) = 4$$
$$p_0((0,1)) = 0.82$$
$$\alpha((0,1)) = 63$$
$$\beta((0,1)) = 4$$
$$p_0((1,0)) = 0.75$$
$$\alpha((1,0)) = 78$$
$$\beta((1,0)) = 4$$
$$p_0((1,1)) = 0.86$$
$$\alpha((1,1)) = 62$$
$$\beta((1,1)) = 4$$

The costs are

$$c(k,M) = \begin{pmatrix} 6 & 9.5 & 7.5 & 11 \\ 10 & 11.5 & 10.5 & 13 \\ 14 & 13.5 & 13.5 & 15 \end{pmatrix}$$

For $n^* = 100$ the results are

k	1	2	3
$x^*_{100}(k)$	4.759	10.97	6.212
$x^{2*}_{100}(k)$	10.97	10.97	6.212
$\overline{U}^1_w(k)$	(0,0)	(0,0)	(1,0)
$\min\ (w^1_{100}(k,\overline{U}^1_w(k)) - x^1_{100}(k,\overline{U}^1_w(k)))$	76.49	76.49	81.25
$\overline{U}^1_v(k)$	(1,0)	(0,1)	(0,1)
$v^1_{100}(k,\overline{U}^1_v)$	81.25	87.46	87.46
$\overline{U}^2_w(k)$	(0,0)	(0,0)	(1,0)
$\overline{U}^2_v(k)$	(0,1)	(0,1)	(0,1)
$v^2_{100}(k,\overline{U}^2_v)$	87.46	87.46	87.46

If we compare the results with the results from the first chapter we see that the optimal choices are the same. For example

$V_{100}(1,0) = 76.49 \qquad \overline{U}(1,0) = (0,0)$

$V_{100}(1,10) = 81.25 \qquad \overline{U}(1,10) = (1,0) \qquad u = a$

$V_{100}(2,10) = 76.49 + 10 = 86.49 \qquad \overline{U}(2,10) = (0,0) \qquad u = b$

$V_{100}(2,60) = 87.46 \qquad \overline{U}(2,60) = (0,1) \qquad u = a$

6.3 With thresholds and excess

Let the state be (k, x, x_s, m) and the control variables $u \in [a,b]$, $U \in D(X_s)$ and $\overline{U} \in D(M)$. The transition probabilities are then

$$P[(k,x,x_s,m) \to (u_k(x), y, U, \max(m,\overline{U}))] = F^{\overline{U}*}(dy)$$

$V_n(k,x,x_s,m)$ can be determined from

$$V_n(k,x,x_s,m) =$$

$$= \min_{u,U,\overline{U}} \Big[c(u_k(x), x, x_s, m, U, \overline{U}) + r\Big(p_0(\overline{U}) \cdot V_{n-1}(u_k(x), 0, U, \max(m, \overline{U})) +$$

$$+ (1 - p_0(\overline{U})) \int_0^\infty V_{n-1}(u_k(x), y, U, \max(m, \overline{U})) F^{\overline{U}}(dy) \Big) \Big]$$

where

$$V_0(k, x, x_s, m) = \min(x, x_s)$$

and

$$C(u_k(x), x, x_s, m, U, \overline{U}) =$$

$$\begin{cases} c(a_{kt}, m, X_s, M) + \min(x, x_s) & \text{if } u = a, U = X_s, \overline{U} = M \text{ and } x^{(t-1)}(k) \leq x < x^{(t)}(k) \\ c(b_k, m, X_s, M) + x & \text{if } u = b, U = X_s, \overline{U} = M, \forall\ x \end{cases}$$

For fix choices $U = X_s$ and $\overline{U} = M$ next year $V_n(k, x, x_s, m | U = X_s, \overline{U} = M)$ will be

$$V_n(k, x, x_s, m | U = X_s, \overline{U} = M) =$$

$$= \min_{u} \Big[C(u_k(x), m, X_s, M) + \min(x, x_s) +$$

$$+ r\Big(p_0(M) \cdot V_{n-1}(u_k(x), 0, X_s, \max(m, M)) +$$

$$+ (1 - p_0(M)) \textstyle\int_0^\infty V_{n-1}(u_k(x), y, X_s, \max(m, M)) F^M(dy) \Big) \Big]$$

We see that the control variable $(u|U = X_s\overline{U} = M)$ has the form

$$(u|U = X_s, \overline{U} = M) = \begin{cases} b \text{ if } x^{(t-1)}(k) \leq x \leq x_n^t(k, m, M) \\ a \text{ if } x_n^t(k, m, M) < x < x^{(t)}(k) \end{cases}$$

A typical plot of $V_n(k, x, x_s, m | U = X_s, \overline{U} = M)$ looks like figure 6.5.

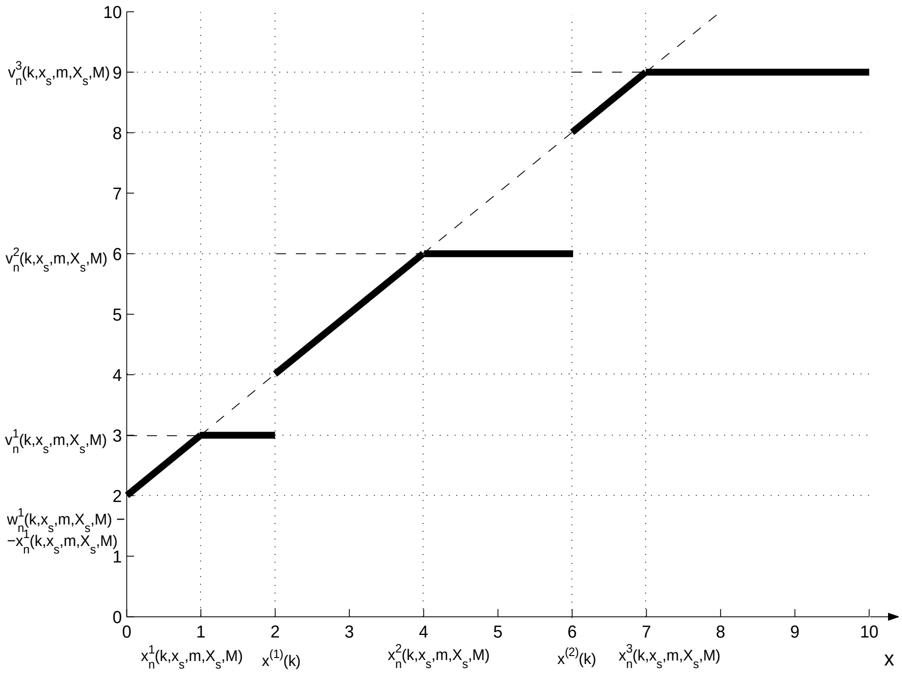

Figure 6.5: $V_n(k, x, x_s, m|U = X_s, \overline{U} = M))$.

Now we can write

$$V_n(k, x, x_s, m|U = X_s, \overline{U} = M) =$$

$$= \begin{cases} c(b_k, m, X_s, M) + x+ \\ \quad + r\Big(p_0(M) \cdot V_{n-1}(b_k, 0, X_s, \max(m, M))+ \\ \quad + (1 - p_0(M)) \int_0^\infty V_{n-1}(b_k, y, X_s, \max(m, M))F^M(dy)\Big) \\ \text{ if } x^{(t-1)}(k) \le x \le x_n^t(k, x_s, m, X_s, M) \\ c(a_{kt}, m, X_s, M) + x_s+ \\ \quad + r\Big(p_0(M) \cdot V_{n-1}(a_{kt}, 0, X_s, \max(m, M))+ \\ \quad + (1 - p_0(M)) \int_0^\infty V_{n-1}(a_{kt}, y, X_s, \max(m, M))F^M(dy)\Big) \\ \text{ if } x_n^t(k, x_s, m, X_s, M) < x < x^{(t)}(k) \end{cases} \tag{6.7}$$

From figure 6.5 we realize that

$$V_n(k, x, x_s, m|U = X_s, \overline{U} = M) =$$

$$\min[w_n^t(k, m, X_s, M) - x_n^t(k, m, X_s, M) + x; v_n^t(k, m, X_s, M)]$$

$$\text{if } x^{(t)}(k) \leq x < x^{(t+1)}(k)$$

or

$$V_n(k, x, x_s, m|U = X_s, \overline{U} = M) =$$

$$= \begin{cases} w_n^t(k, x_s, m, X_s, M) - x_n^t(k, x_s, m, X_s, M) + x \\ \text{ if } x^{(t-1)}(k) \leq x \leq x_n^t(k, x_s, m, X_s, M) \\ v_n^t(k, x_s, m, X_s, M) \\ \text{ if } x_n^t(k, x_s, m, X_s, M) < x < x^{(t)}(k) \end{cases} \tag{6.8}$$

(6.7) and (6.8) give us

$$w_n^t(k, x_s, m, X_s, M) = c(b_k, m, X_s, M) + x_n^i(k, x_s, m, X_s, M) +$$

$$+ r\Big(p_0(M) \cdot N_{n-1}(b_k, X_s, \max(m, M)) +$$

$$+ (1 - p_0(M))E_{n-1}^M(b_k, X_s, \max(m, M))\Big) \tag{6.9}$$

$$v_n^t(k, X_s, m, X_s, M) = c(a_{kt}, m, X_s, M) + r\Big(p_0(M) \cdot N_{n-1}(a_{kt}, X_s, \max(m, M)) +$$

$$+ (1 - p_0(M))E_{n-1}^M(a_{kt}, X_s, \max(m, M))\Big)$$

where

$$N_n(k, x_s, m) = V_n(k, 0, x_s, m)$$

$$E_n^M(k, x_s, m) = \textstyle\int_0^\infty V_n(k, y, x_s, m)F^M(dy)$$

When we have several choices for $\overline{U}$ and U next year $V_n(k, x, x_s, m)$ looks like figure 6.6 when we have two possible excesses and one possible measure.

Here

$$x_n^{t*}(k, x_s, m) =$$

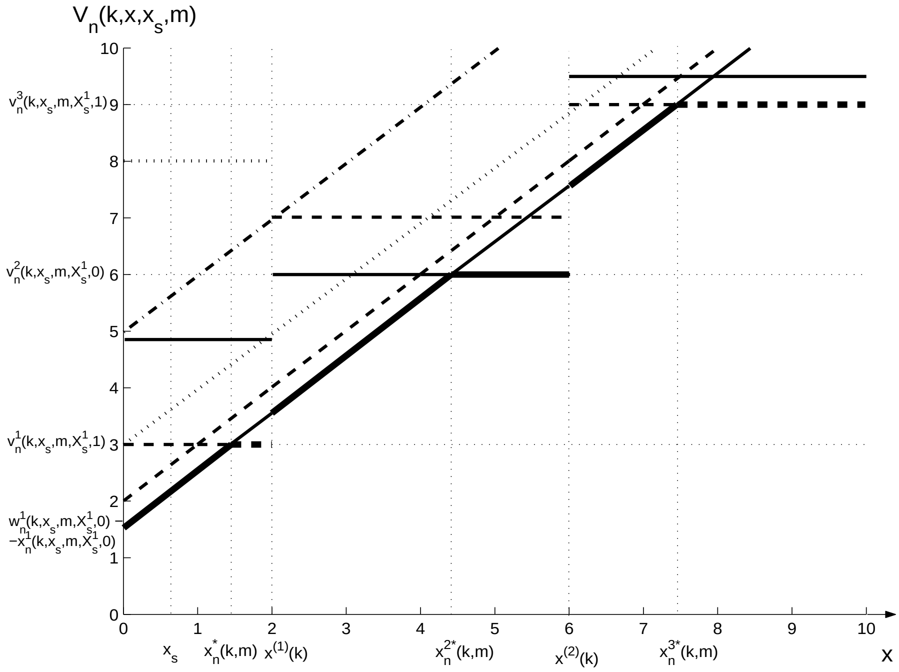

Figure 6.6: $V_n(k, x, x_s, m)$.

$$= \min_{U,\overline{U}}(v_n^t(k, x_s, m, U, \overline{U}) - \min_{\overline{U}}(w_n^t(k, x_s, m, U, \overline{U}) - x_n^t(k, x_s, m, U, \overline{U}))$$

When we have studied the figure and determined $x_n^{t*}(k, x_s, m)$ we must use the following rule:

Rule 4

Put $x_n^{(t)}(k, x_s, m) := x^{(t)}(k) \ \forall \ k, m, n, t, x_s$
if $x_n^{(t)}(k, x_s, m) < x_n^{t*}(k, x_s, m)$
 put $x_n^{t*}(k, x_s, m) = x_n^{(t-1)}(k, x_s, m)$ and $x_n^{(t)}(k, x_s, m) = x_n^{(t-1)}(k, x_s, m)$
if $x_n^{(t)}(k, x_s, m) > x_n^{(t+1)*}(k, x_s, m)$ put $x_n^{(t+1)*}(k, x_s, m) = x_n^{(t)}(k, x_s, m)$
for $t = 1, 2, ..., T, T + 1$

Now we can determine the following:

$$N_n(k, x_s, m) = V_n(k, 0, x_s, m) = \min_{U,\overline{U}}(w_n^1(k, x_s, m, U, \overline{U}) - x_n^1(k, x_s, m, U, \overline{U}))$$

and

$$E_n^M(k, x_s, m) = \int_0^\infty V_n(k, y, x_s, m) F^M(dy) =$$

$$= \int_0^{x_n^*(k,x_s,m)} (\min_{U,\overline{U}}[w_n^1(k,x_s,m,U,\overline{U}) - x_n^1(k,x_s,m,U,\overline{U})] + y)F^M(dy) +$$

$$+ \int_{x_n^*(k,x_s,m)}^{x_n^{(1)}(k,x_s,m)} \min_{U,\overline{U}}(v_n^1(k,x_s,m,U,\overline{U}))F^M(dy) + ...$$

$$... + \int_{x_n^{(T)}(k,x_s,m)}^{x_n^{(T+1)*}(k,x_s,m)} (\min_{U,\overline{U}}[w_n^{T+1}(k,x_s,m,U,\overline{U}) - x_n^{T+1}(k,m,x_s,U,\overline{U})] + y)F^M(dy) +$$

$$+ \int_{x_n^{(T+1)*}(k,m,x_s)}^{\infty} \min_{U,\overline{U}}(v_n^{T+1}(k,m,x_s,U,\overline{U}))F^M(dy) =$$

$$= \min_{U,\overline{U}}[w_n^1(k,m,x_s,U,\overline{U}) - x_n^1(k,m,x_s,U,\overline{U})]F^M(x_n^*(k,m,x_s)) +$$

$$+ \sum_{i=1}^{T+1} \Big[x_n^{i*}(k,m,x_s)F^M(x_n^{i*}(k,m,x_s)) - G^M(x_n^{i*}(k,m,x_s)) \Big] +$$

$$+ \sum_{i=1}^{T} \Big[\min_{U,\overline{U}} v_n^{i*}(k,m,x_s)(F^M(x_n^{(i)}(k,x_s,m) - F^M(x_n^{i*}(k,x_s,m))) +$$

$$+ G^M(x_n^{(i)}(k,x_s,m)) - x_n^{(i)}(k,x_s,m)F^M(x_n^{(i)}(k,x_s,m)) \Big] +$$

$$+ \sum_{i=2}^{T+1} \Big[\min_{U,\overline{U}}(w_n^i(k,m,x_s,U,\overline{U}) - x_n^i(k,m,x_s,U,\overline{U})) \cdot$$

$$\cdot (F^M(x_n^{i*}(k,m,x_s)) - F^M(x_n^{(i-1)}(k,x_s,m))) \Big] +$$

$$+ \min_{U,\overline{U}} \Big[v_n^{T+1}(k,m,x_s,U,\overline{U})(1 - F^M(x_n^{(T+1)*}(k,m,x_s))) \Big]$$

The controlvariables are

$$(u, U, \overline{U}) =$$

$$= \begin{cases} (b,\ [U,\overline{U} : \min(w_n^t(k,x_s,m,U,\overline{U}) - x_n^t(k,x_s,m,U,\overline{U}))]) \\ \ \text{if } x_n^{(t-1)}(k,x_s,m) \le x \le x_n^{t*}(k,x_s,m) \\ (a,\ [U,\overline{U} : \min(v_n^t(k,x_s,m,U,\overline{U}))]) \\ \ \text{if } x_n^{t*}(k,m,x_s) < x < x_n^{(t)}(k,x_s,m) \end{cases}$$

If our present state is (k^*, x^*, x_s^*, m^*) and our horizon is n^* then we should go on like this to get the optimal decisions:

1. Do 2-6 for $i = 1, 2, .., n^* - 1$.
2. Choose initial guesses, $(j = 0)$, $x_{i_j}^t(k, x_s, m, X_s, M) > x_s$ for $x_i^t(k, x_s, m, X_s, M)$ $\forall\ t, k, x_s, m, X_s M$
3. Determine the corresponding $w_{i_j}^t(k, x_s, m, X_s, M)$ and $v_{i_j}^t(k, x_s, m, X_s, M)$ for $w_i^t(k, x_s, m, X_s, M)$ and $v_i^t(k, x_s, m, X_s, M)$ from (6.9)
4. Update $x_i^t(k, x_s, m, X_s, M)$ from

$$x_{i_{j+1}}^t(k, x_s, m, X_s, M) =$$

$$= x_{i_j}^t(k, x_s, m, X_s, M) + v_{i_j}^t(k, x_s, m, X_s, M) - w_{i_j}^t(k, x_s, m, X_s, M)$$

5. Repete 3-4 until $x_i^t(k, x_s, m, X_s, M)$ has converged.
6. Determine

$$x_i^{t*}(k, x_s, m),$$

$$\min_{U,\overline{U}}[w_i^t(k, x_s, m, U, \overline{U}) - x_i^t(k, x_s, m, U, \overline{U})] \text{ and}$$

$$\min_{U,\overline{U}}(v_i^t(k, x_s, m, U, \overline{U}))$$

7. Do 2-6 for $i = n^*$ but this time only for $k = k^*$, $x_s = x_s^*$ and $m = m^*$ but still $\forall\ X_s, M$. Determine also $(U_w^t, \overline{U}_w^t) = [(U, \overline{U}) : \min(w_{n^*}^t(k^*, x_s^*, m^*, U, \overline{U}) - x_{n^*}^t(k^*, x_s^*, m^*, U, \overline{U}))]$ and $(U_v^t, \overline{U}_v^t) = [(U, \overline{U}) : \min(v_{n^*}^t(k^*, x_s^*, m^*, U, \overline{U}))]$

Example 3 (continued)

Let us assume that we have the following excesses to choose among:

$$D(X_s) = [0\ 3\ 8\ 20]$$

We can also use the following damage preventive measures:

$$m_1 = \text{firealarm}$$
$$m_2 = \text{watchmen}$$

The installation cost of the fire alarm is 2. The cost for the alarm is 0.6 and for the watchmen 0.9. The corresponding parameters for the different measures are:

$$\begin{aligned} p_0((0,0)) &= 0.50 \\ \alpha((0,0)) &= 24 \\ \beta((0,0)) &= 4 \end{aligned}$$

$$\begin{aligned} p_0((0,1)) &= 0.75 \\ \alpha((0,1)) &= 19 \\ \beta((0,1)) &= 4 \end{aligned}$$

$$\begin{aligned} p_0((1,0)) &= 0.80 \\ \alpha((1,0)) &= 18 \\ \beta((1,0)) &= 4 \end{aligned}$$

$$\begin{aligned} p_0((1,1)) &= 0.90 \\ \alpha((1,1)) &= 14 \\ \beta((1,1)) &= 4 \end{aligned}$$

The costs are:

$$c(k,(0,0),X_s,(0,0)) = \begin{pmatrix} 1.00 & 0.70 & 0.60 & 0.55 \\ 2.50 & 1.70 & 1.50 & 1.40 \\ 3.50 & 2.40 & 1.90 & 1.80 \\ 4.50 & 3.00 & 2.50 & 2.10 \\ 5.50 & 3.50 & 3.00 & 2.80 \\ 7.00 & 4.20 & 3.50 & 2.90 \end{pmatrix}$$

$$c(k,(0,0),X_s,(0,1)) = \begin{pmatrix} 1.82 & 1.49 & 1.46 & 1.41 \\ 3.25 & 2.55 & 2.25 & 2.10 \\ 4.00 & 2.80 & 2.50 & 2.32 \\ 4.70 & 3.30 & 3.10 & 2.70 \\ 5.40 & 4.10 & 3.60 & 3.30 \\ 6.70 & 4.40 & 3.90 & 3.40 \end{pmatrix}$$

$$C(k,(0,0),X_s,(1,0)) = \begin{pmatrix} 3.40 & 3.18 & 3.10 & 3.07 \\ 4.80 & 4.10 & 3.90 & 3.75 \\ 5.50 & 4.40 & 4.10 & 4.00 \\ 5.70 & 4.95 & 4.50 & 4.30 \\ 6.40 & 5.50 & 4.90 & 4.60 \\ 7.00 & 5.70 & 5.20 & 4.90 \end{pmatrix}$$

$$C(k,(0,0),X_s,(1,1)) = \begin{pmatrix} 4.25 & 4.02 & 3.98 & 3.95 \\ 5.35 & 5.00 & 4.73 & 4.50 \\ 5.90 & 5.20 & 4.78 & 4.70 \\ 6.20 & 5.40 & 5.35 & 5.00 \\ 6.40 & 5.50 & 5.40 & 5.20 \\ 6.70 & 5.90 & 5.50 & 5.40 \end{pmatrix}$$

The installation cost of the fire alarm can be removed from the costs above after we have installed it. If we had excess 0 last year then the results for $n^*=1$ and $n^*=100$ are:

$$n^* = 1$$

k	1	2	3	4	5	6
$x_1^*(k,0,(0,0))$	1.250	1.550	2.250	1.500	1.100	0.800
$x_1^{2*}(k,0,(0,0))$	1.550	2.250	2.350	1.500	1.100	0.800
$x_1^{3*}(k,0,(0,0))$	2.250	2.350	2.350	1.500	1.100	0.800
$U_w^1(k,0,(0,0))$	20	20	20	20	20	20
$\overline{U}_w^1(k,0,(0,0))$	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
$w_1^1(k,0,(0,0),U_w^1,\overline{U}_w^1)-x_1^1(..)$	0.550	0.550	0.550	1.400	1.800	2.100
$U_v^1(k,0,(0,0))$	20	20	20	20	20	20
$\overline{U}_v^1(k,0,(0,0))$	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
$v_1^1(k,0,(0,0),U_v^1,\overline{U}_v^1)$	1.800	2.100	2.800	2.900	2.900	2.900
$U_w^2(k,0,(0,0))$	20	20	20	20	20	20
$\overline{U}_w^2(k,0,(0,0))$	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
$U_v^2(k,0,(0,0))$	20	20	20	20	20	20
$\overline{U}_v^2(k,0,(0,0))$	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
$v_1^2(k,0,(0,0),U_v^2,\overline{U}_v^2)$	2.100	2.800	2.900	2.900	2.900	2.900
$U_w^3(k,0,(0,0)$	20	20	20	20	20	20
$\overline{U}_w^3(k,0,(0,0))$	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
$U_v^3(k,0,(0,0))$	20	20	20	20	20	20
$\overline{U}_v^3(k,0,(0,0))$	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
$v_1^3(k,0,(0,0),U_v^3,\overline{U}_v^3)$	2.800	2.900	2.900	2.900	2.900	2.900

and

$n^* = 100$

k	1	2	3	4	5	6
$x^*_{100}(k, 0, (0, 0))$	1.215	2.444	2.934	3.244	3.033	1.804
$x^{2*}_{100}(k, 0, (0, 0))$	2.444	2.934	4.248	3.244	3.033	1.804
$x^{3*}_{100}(k, 0, (0, 0))$	2.934	4.248	4.248	3.244	3.033	1.804
$U^1_w(k, 0, (0, 0))$	0	0	0	8	8	20
$\overline{U}^1_w(k, 0, (0, 0))$	(0,0)	(0,0)	(0,0)	(1,0)	(1,0)	(1,0)
$w^1_{100}(k, 0, (0, 0), U^1_w, \overline{U}^1_w) - x^1_{100}(..)$	19.48	19.48	19.48	20.48	20.69	21.92
$U^1_v(k, 0, (0, 0))$	8	20	20	20	20	20
$\overline{U}^1_v(k, 0, (0, 0))$	(1,0)	(1,0)	(1,0)	(1,1)	(1,1)	(1,1)
$v^1_{100}(k, 0, (0, 0), U^1_v, \overline{U}^1_v)$	20.69	21.92	22.41	23.72	23.72	23.72
$U^2_w(k, 0, (0, 0))$	0	0	0	8	8	20
$\overline{U}^2_w(k, 0, (0, 0))$	(0,0)	(0,0)	(0,0)	(1,0)	(1,0)	(1,0)
$U^2_v(k, 0, (0, 0))$	20	20	20	20	20	20
$\overline{U}^2_v(k, 0, (0, 0))$	(1,0)	(1,0)	(1,1)	(1,1)	(1.1)	(1,1)
$v^2_{100}(k, 0, (0, 0), U^2_v, \overline{U}^2_v)$	21.92	22.41	23.72	23.72	23.72	23.72
$U^3_w(k, 0, (0, 0)$	0	0	0	8	8	20
$\overline{U}^3_w(k, 0, (0, 0))$	(0,0)	(0,0)	(0,0)	(1,0)	(1,0)	(1,0)
$U^3_v(k, 0, (0, 0))$	20	20	20	20	20	20
$\overline{U}^3_v(k, 0, (0, 0))$	(1,0)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)
$v^3_{100}(k, 0, (0, 0), U^3_v, \overline{U}^3_v)$	22.41	23.72	23.72	23.72	23.72	23.72

For example if we are in bonus class 4 and have a damage of 20 we see that $V_{100}(4, 20, 0, (0, 0))$=23.72. This is much lower than $V_{100}(4, 20)$=50.08 from section 4.2 where we had no excess and no damagepreventive measures and always claimed a damage. Hence, we see that we can save a lot of money.

Chapter 7

Conclusion

The strategy developed in this paper should be possible to use in real life, at least for some insurances. Of course some information that are assumed in this paper to be given are not given in real life. For example the increase of premiums are probably to unpredictable for many insurances for certain damages. However, if good guesses about these things are made the strategy described in this paper should be a good guidance to get a lower long-term cost.

This paper is probably most applicable for companies that insure their business. The reason for this is that they could have negotiations with the insurance company and therefore they could get more information. The premiums for companies are probably also more dependent on how big the costs of their damages have been. For new policy-holders this paper could be a help for deciding which insurance company to have. They might be in bonus classes with high premiums and want to advance to better bonus classes quickly without too high immediate costs. Certain insurance companies might have low premiums for new customers while others have high premiums in the beginning but might give more reduction later. The insurance companies might also not give the same reductions for different damage preventive measures and excesses. These things could be balanced to see which insurance company is likely to be most profitable in the long run.

If a policy-holder has a better knowledge about the risk for damages for different measures than the insurance company this could be useful for the policy-holder. Then the best combination of preventive measures and excesses could be chosen. Since there are a lot of different premiums for different measures and excesses to determine for the insurance company, this should be a problem for them. Another problem for the insurance company is what to do with new customers so that they will keep them without loosing money. Maybe they have to accept a loss for new customers in order to keep more customers in the future. If they make clear about how the premiums are determined maybe they could attract more customers. This is problems for the insurance company that this paper could be a help for. This paper might also be used for deciding if reinsurance should be made. Maybe there should also be a co-operation between the insurance company and security firms that are providing the damage preventive measures.

If it would be possible to claim a part of a damage the optimal strategy with thresholds would look like figure 7.1 instead. This situation is not treated in this paper but should be very similar to those which are treated. Since I don't have much knowledge about the insurance business maybe some things should have been analyzed in different ways. However this is how I think things work. Maybe some things should be developed more. For example if the distribution of damages are likely to change from year to year one could model the damages as $F_n^*(x)$ instead. An easy way to handle a general increase for the premiums is to choose a higher discount factor.

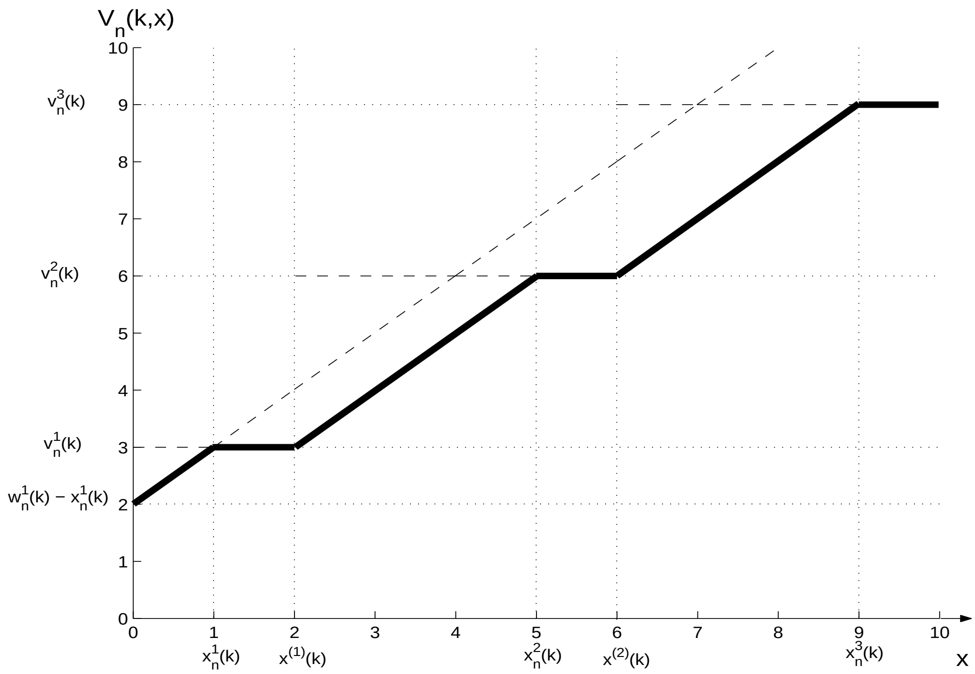

Figure 7.1: $V_n(k, x)$.

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