Mathematical Statistics
Stockholm University

The winner takes it all

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Research Report 2013:6

ISSN 1650-0377

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Research Report **2013:6**,
http://www.math.su.se/matstat

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Maria Deijfen* Remco van der Hofstad†

June 2013

Abstract

We study competing first passage percolation on graphs generated by the configuration model. At time 0, vertex 1 and vertex 2 are infected with the type 1 and the type 2 infection, respectively, and an uninfected vertex then becomes type 1 (2) infected at rate λ_1 (λ_2) times the number of edges connecting it to a type 1 (2) infected neighbor. Our main result is that, if the degree distribution is a power-law with exponent $\tau \in (2, 3)$, then, as the number of vertices tends to infinity, one of the infection types will almost surely occupy all but a finite number of vertices. Furthermore, which one of the infections wins is random and both infections have a positive probability of winning regardless of the values of λ_1 and λ_2. The picture is similar with multiple starting points for the infections.

Keywords: Random graphs, configuration model, first passage percolation, competing growth, coexistence, continuous-time branching process.

MSC 2010 classification: 60K35, 05C80, 90B15.

1 Introduction

Consider a graph generated by the configuration model with random i.i.d. degrees, that is, given a finite number n of vertices, each vertex is independently assigned a random number of half-edges according to a given probability distribution and the half-edges are then paired randomly to form edges (see below for more details). Independently assign two exponentially distributed passage times $X_1(e)$ and $X_2(e)$ to each edge e in the graph, where $X_1(e)$ has parameter λ_1 and $X_2(e)$ parameter λ_2, and let two infections controlled by these passage times compete for space on the graph. More precisely, at time 0, vertex 1 is infected with the type 1 infection, vertex 2 is infected with the type 2 infection and all other vertices are uninfected. The infections then spread via nearest neighbors in the graph in that the time that it takes for the type 1 (2) infection to traverse an edge e and invade the vertex at the other end is given by $X_1(e)$ ($X_2(e)$). Furthermore, once a vertex becomes type 1 (2) infected, it stays type 1 (2) infected forever and it also becomes immune to the type 2 (1) infection. Note that, since the vertices are exchangeable in the configuration model, the process is equivalent in distribution to the process obtained by infecting two randomly chosen vertices at time 0.

We shall impose a condition on the degree distribution that guarantees that the underlying graph has a giant component that comprises almost all vertices. According to the above

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dynamics, almost all vertices will then eventually be infected. We are interested in asymptotic properties of the process as $n \to \infty$. Specifically, we are interested in comparing the fraction of vertices occupied by the type 1 and the type 2 infections, respectively, when the degree distribution is a power law with exponent $\tau \in (2,3)$, that is, when the degree distribution has finite mean but infinite variance. Our main result is roughly that the probability that both infection types occupy positive fractions of the vertex set is 0 for all choices of λ_1 and λ_2. Moreover, the winning type will in fact conquer all but a finite number of vertices. A natural guess is that asymptotic coexistence is possible if and only if the infections have the same intensity – which for instance is the case for first passage percolation on $\mathbb{Z}^d$ and on random regular graphs; see Section 1.3 – but this is hence not the case in our setting.

1.1 The configuration model

Let $D_1, \ldots, D_n$ denote the degrees of the vertices in the graph. These are i.i.d. random variables, and we shall throughout assume that

(A1) $\mathbb{P}(D \geq 2) = 1$;

(A2) there exists a $\tau \in (2,3)$ and constants $c_2 \geq c_1 > 0$ such that, for all $x > 0$,

$$c_1 x^{-(\tau-1)} \leq \mathbb{P}(D > x) \leq c_2 x^{-(\tau-1)}. \tag{1}$$

For some results, the assumption (A2) will be strengthened to

(A2') there exist $\tau \in (2,3)$ and $c_D \in (0,\infty)$ such that $\mathbb{P}(D > x) = c_D x^{-(\tau-1)}(1 + o(1))$.

As described above, the graph is constructed in that each vertex i is assigned D_i half-edges, and the half-edges are then paired randomly: first we pick two half-edges at random and create an edge out of them, then we pick two half-edges at random from the set of remaining half-edges and pair them into an edge, etc. If the total degree happens to be odd, then we add one half-edge at vertex n (clearly this will not affect the properties of the model asymptotically). The construction can give rise to self-loops and multiple edges between vertices, but these imperfections will be relatively rare when n is large; see [14, 16].

It is well-known that the critical point for the occurrence of a giant component – that is, a component comprising a positive fraction of the vertices as $n \to \infty$ – in the configuration model is given by $\nu := \mathbb{E}[D(D-1)]/\mathbb{E}[D] = 1$; see e.g. [17, 20, 21]. The quantity ν is the reproduction mean in a branching process with offspring distribution $D^\star - 1$ where $D^\star$ is a size-biased version of a degree variable. More precisely, with $(p_d)_{d\geq 1}$ denoting the degree distribution, the offspring distribution is given by

$$p_d^\star = \frac{(d+1)p_{d+1}}{\mathbb{E}[D]}. \tag{2}$$

Such a branching process approximates the initial stages of the exploration of the components in the configuration model, and the asymptotic relative size of the largest component in the graph is given by the survival probability of the branching process [17, 20, 21]. When the degree distribution is a power-law with exponent $\tau \in (2,3)$, as stipulated in (A2), it is easy to see that $\nu = \infty$ so that the graph is always supercritical. Moreover, the assumption (A1) implies that the survival probability of the branching process is 1 so that the asymptotic fraction of vertices in the giant component converges to 1.

1.2 Main result

Consider two infections spreading on a realization of the configuration model according to the dynamics described in the beginning of the section, that is, an uninfected vertex becomes type 1 (2) infected at rate λ_1 (λ_2) times the number of edges connecting it to type 1 (2) infected neighbors. First note that, by time-scaling and symmetry, we may assume that $\lambda_1 = 1$ and $\lambda_2 = \lambda > 1$. Let $N_i(n)$ denote the final number of type i infected vertices, and write $\bar{N}_i(n) = N_i(n)/n$ for the final fraction of type i infected vertices. Clearly $\bar{N}_2(n) = 1 - \bar{N}_1(n)$, so it is enough to consider $\bar{N}_1(n)$. The following is our main result:

Theorem 1.1 (The winner takes it all). *Fix $\lambda \geq 1$ and write $\mu = 1/\lambda$.*

(a) *The fraction $\bar{N}_1(n)$ of type 1 infected vertices converges in probability to the indicator variable $\mathbb{1}_{\{V_1 < \mu V_2\}}$ as $n \to \infty$, where V_1 and V_2 are two i.i.d. proper random variables with support on $\mathbb{R}^+$.*

(b) *Assume (A2'). On the event that $V_1 < \mu V_2$, the number $N_2(n)$ of type 2 infected vertices converges to a proper random variable N_2. Similarly, on the event that $V_1 > \mu V_2$, the number $N_1(n)$ of type 2 infected vertices converges to a proper random variable N_1.*

Remark 1.1 (Explosion times). *The variables V_i ($i = 1, 2$) are explosion times of a certain continuous-time branching process with infinite mean. The process is started from D_i individuals, representing the edges of vertex i, and will be characterized in more detail in Section 2. In part (b), the limiting random variables N_i ($i = 1, 2$) have explicit characterizations involving the (almost surely finite) extinction time of a certain Markov process; see Section 4. In fact, the proof will reveal that the limiting number of vertices that is captured by the losing type is equal to 1 with strictly positive probability, which is the smallest possible value. Thus, the ABBA quote 'The winner takes it all. The loser standing small...' could not be more appropriate.*

Roughly stated, the theorem implies that coexistence between the infection types is never possible. Instead, one of the infection types will invade all but a finite number of vertices and, regardless of the relation between the intensities, both infections have a positive probability of winning. The proof is mainly based on ingredients from [3], where standard first passage percolation (that is, first passage percolation with one infection type and exponential passage times) on the configuration model is analyzed.

Let us first give a short heuristic explanation: With high probability (whp), the initially infected vertex 1 and vertex 2 will not be located very close to each other in the graph and hence the infection types will initially evolve without interfering with each other. This means that the initial stages of the spread of each one of the infections can be approximated by a continuous-time branching process, which has infinite mean when the degree distribution has infinite variance (because of size biasing). These two processes will both explode in finite time, and the type that explodes *first* is random and asymptotically equal to 1 precisely when $V_1 < \mu V_2$. Theorem 1.1 follows from the fact that the type with the smallest explosion time will get a lead that is impossible to catch up with for the other type. More specifically, the type that explodes first will whp occupy all hubs in the graph shortly after the time of explosion, while the other type occupies only a finite number of vertices. From the hubs the exploding type will then rapidly invade the rest of the graph before the other type makes any substantial progress at all.

We next investigate the setting where we start the competition from several vertices chosen uniformly at random.

Theorem 1.2 (Multiple starting points)**.** *Fix $\lambda \geq 1$ and write $\mu = 1/\lambda$. Also fix integers $k_1, k_2 \geq 1$, and start with k_1 type 1 infected vertices and k_2 type 2 infected vertices chosen uniformly at random from the vertex set.*

(a) *The fraction $\bar{N}_1(n)$ of type 1 infected vertices converges in probability to the indicator variable $\mathbb{1}_{\{V_{1,k_1} < \mu V_{2,k_2}\}}$ as $n \to \infty$, where V_{1,k_1} and V_{2,k_2} are two independent proper random variables with support on $\mathbb{R}^+$.*

(b) *Assume (A2'). On the event that $V_{1,k_1} < \mu V_{2,k_2}$, the number $N_2(n)$ of type 2 infected vertices converges to a proper random variable N_2.*

(c) *Assume (A2'). For every $k_1, k_2 \geq 1$, we have that $\mathbb{P}(V_{1,k_1} < \mu V_{2,k_2}) \in (0,1)$. Moreover, for fixed $\alpha \in (0, \infty)$, as $k \to \infty$,*

$$\mathbb{P}(V_{1,k} < \mu V_{2,\alpha k}) \to \mathbb{P}(Y_1 < \mu \alpha^{3-\tau} Y_2) \in (0,1), \tag{3}$$

where Y_1, Y_2 are two i.i.d. random variables with distribution

$$Y = \int_0^\infty \frac{1}{1+Q_t} dt, \tag{4}$$

for a stable subordinator $(Q_t)_{t\geq 0}$ with $\mathbb{E}[\mathrm{e}^{-sQ_t}] = \mathrm{e}^{-\sigma s^{\tau-2} t}$ for some $\sigma = \sigma(c_D)$.

Remark 1.2. *The variable V_{i,k_i} has the distribution of the explosion time of a continuous-time branching process with the same reproduction rules as in the case with a single initial type i vertex, but now the number of individuals that the process is started from is distributed as $D_1 + \ldots + D_{k_i}$ and represents the total degree of the k_i initial type i vertices. The scaling of the explosion time of the branching process started from k individuals for large k is investigated in more detail in Lemma 4.3.*

In Theorem 1.2, we see that the fastest species does not necessarily win even when it has twice as many starting points, but it does when $\alpha \to \infty$, that is, when starting from a much larger number of vertices than the slower species. We only prove Theorem 1.2 in the case where $k_1 = k_2 = 1$, in which case it reduces to Theorem 1.1. The case where $(k_1, k_2) \neq (1,1)$ is similar. Hence only the proof of (3) in Theorem 1.2(c) is provided in detail; see Section 4.

1.3 Related work and open problems

First passage percolation on various types of discrete probabilistic structures has been extensively studied; see e.g. [4, 5, 9, 12, 19, 24]. The classical example is when the underlying structure is taken to be the $\mathbb{Z}^d$-lattice. The case with exponential passage times is then often referred to as the Richardson model and the main focus of study is the growth and shape of the infected region [6, 18, 22, 23]. The Richardson model has also been extended to a two-type version that describes a competition between two infection types; see [10]. Infinite coexistence then refers to the event that both infection types occupies infinite parts of the lattice, and it is conjectured that this has positive probability if and only if the infections have the same intensity. The if-direction was proved for $d = 2$ in [10] and for general d independently in [7] and [13]. The only-if-direction remains unproved, but convincing partial results can be found in [11].

As for the configuration model, the area of network modeling has been very active the last decade and the configuration model is one of the most studied models. One of its main advantages is that it gives control over the degree distribution, which is an important quantity in a network with great impact on global properties. As mentioned, first passage percolation with exponential edge weights on the configuration model has been analyzed in [3]. The results there revolve around the length of the time-minimizing path between two vertices and the time that it takes to travel along such a path. In [5], these results are extended to all continuous edge-weight distributions under the assumption of finite variance degrees.

Recently, in [1], competing first passage percolation has been studied on so-called random regular graphs, which can be generated by the configuration model with constant degree, that is, with $\mathbb{P}(D = d) = 1$ for some d. The setup in [1] allows for a number of different types of starting configurations, and the main result relates the asymptotic fractions occupied by the respective infection types to the sizes of the initial sets and the intensities. When the infections are started from two randomly chosen vertices, coexistence occurs with probability 1 if the infections have the same intensity, while, when one infection is stronger than the other, the stronger type wins, as one might expect. The somewhat counterintuitive result in the present paper is hence a consequence of large variability in the degrees. We conjecture that the result formulated here remains valid precisely when the explosion time of the corresponding continuous-time branching process is finite. See [8] for a discussion of explosion times for age-dependent branching processes.

A natural continuation of the present work is to study the case when $\tau > 3$, that is, when the degree distribution has finite variance. We conjecture that the result is then the same as for constant degrees as described above. Another natural extension is to investigate other types of distributions for the passage times. The results may then well differ from the exponential case. For instance, ongoing work on the case with constant passage times (possibly different for the two species) and $\tau \in (2, 3)$ indicates that the fastest species always wins, and that there is no coexistence even when the passage times are equal [2]. It would also be interesting to allow for more general types of starting configurations. Would it for instance help a weaker type if the number of vertices that it is started from is taken to be some power of n? Finally we mention the possibility of investigating whether the results generalize to other graph structures with simlar degree distribution, e.g. inhomogeneous random graphs and graphs generated by preferential attachment mechanisms.

2 Preliminaries

In this section we summarize the results on one-type first passage percolation from [3] that we shall need. Theorem 1.1(a) and 1.1(b) are then proved in Section 3 and 4, respectively. Also, the proof of the asymptotic characterization (3) is given in Section 4.

Let each edge in a realization of the configuration model independently be equipped with an exponential passage time with mean 1. In summary, it is shown in [3] that, when the degree distribution satisfies (A1) and (A2), the asymptotic minimal time between vertex 1 and vertex 2 is given by $V_1 + V_2$, where V_1 and V_2 are i.i.d. random variables indicating the explosion time of an infinite mean continuous-time branching process that approximates the initial stages of the flow through the graph starting from vertex 1 and 2 respectively; see below. The result follows roughly by showing that the sets of vertices that can be reached from vertex 1 and 2, respectively, within time t are whp disjoint up until the time when the associated branching processes explode, and that they then hook up, creating a path between 1 and 2.

Exploration of first-passage percolation on the configuration model. To be a bit more precise, we first describe a natural stepwise procedure for exploring the graph and the flow of infection through it starting from a given vertex v. Let $\mathrm{SWG}_m^{(v)}$ denote the graph consisting of the set of explored vertices and edges after m steps, where SWG stands for Smallest-Weight Graph. Write $\mathcal{S}_m^{(v)}$ for the set of unexplored half-edges emanating from vertices in $\text{SWG}_m^{(v)}$ and define $S_m^{(v)} := |\mathcal{S}_m^{(v)}|$. Finally, let $\mathcal{F}_m^{(v)}$ denote the set of half-edges belonging to vertices in the complement of $\text{SWG}_m^{(v)}$. When there is no risk of confusion, we will often omit the superscript v in the notation. Set $\text{SWG}_1 = \{v\}$, so that $S_1 = D_v$. Given SWG_m, the graph SWG_{m+1} is constructed as follows:

1. Pick a half-edge at random from the set $\mathcal{S}_m$. Write x for the vertex that this half-edge is attached to, and note that $x \in \text{SWG}_m$.

2. Pick another half-edge at random from $\mathcal{S}_m \cup \mathcal{F}_m$ and write y for the vertex that this half-edge is attached to.

3. If $y \notin \text{SWG}_m$ – that is, if the second half-edge is in $\mathcal{F}_m$ – then SWG_{m+1} consists of SWG_m along with the vertex y and the edge (x, y). If n is large and m is much smaller than n, this will be the most likely scenario.

4. If $y \in \text{SWG}_m$ – that is, if the second half-edge is in $\mathcal{S}_m$ – then SWG_{m+1} consist of SWG_m along with the edge (x, y). This means that we have detected a cycle in the graph.

The above procedure can be seen as a discrete-time representation of the flow through the graph observed at the times when the infection traverses a new edge: Each unexplored half-edge emanating from a vertex that has already been reached by the flow has an exponential passage time with mean 1 attached to it. In step 1 we pick such a half-edge at random, which is equivalent to picking the one with the smallest passage time. Then, in step 2, we check which other half-edge that the chosen half-edge is connected to. This identifies the vertex at the other end of the edge. If this vertex has not yet been reached by the flow, it is added to the explored graph along with the connecting edge in step 3. If the vertex has already been reached by the flow, only the edges is added in step 4, creating a cycle.

As for the number of unexplored half-edges emanating from explored vertices, this is increased by the forward degree of the added vertex minus 1 in case a vertex is added, and decreased by 2 in case a cycle is detected. Hence, defining

$$B_i = \begin{cases} \text{the forward degree of the added vertex if a vertex is added in step } i; \\ -1 \text{ if a cycle is created in step } i, \end{cases}$$

we have for $m \geq 2$ that

$$S_m = D_v + \sum_{i=2}^{m} (B_i - 1).$$

Denote the time that it takes for the flow to grow to m edges by T_m and let $(E_i)_{i=1}^{\infty}$ be a sequence of i.i.d. Exp(1)-variables. The time for traversing the edge that is added in the ith step is the minimum of S_i i.i.d. exponential variables with mean 1 and thus it has the same distribution as E_i/S_i. Hence

$$T_m \stackrel{d}{=} \sum_{i=1}^{m} \frac{E_i}{S_i}. \tag{5}$$

Write $\mathcal{V}(G)$ for the vertex set of a graph G and define

$$R_m = \inf\{j \colon |\mathcal{V}(\text{SWG}_j)| \geq m\}, \tag{6}$$

that is, R_m is the step when the mth vertex is added to the explored graph. Since no vertex is added in a step where a cycle is created, we have that $R_m \geq m$. However, if n is large and m is small in relation to n, it is unlikely to encounter cycles in the early stages of the exploration process and thus $R_m \approx m$ for small m. Hence, we should be able to replace m by R_m above and still obtain quantities with similar behavior. Indeed, Proposition 2.1 below states that T_{R_m} (the time until the flow has reached m vertices) and T_m have the same limiting distribution as $n \to \infty$ as long as m is not too large.

Passage times for smallest-weight paths. To identify the limiting distribution of T_m, note that, as long as no cycles are encountered, the exploration graph is a tree and its evolution can therefore be approximated by a continuous-time branching process. The root is the starting vertex v, which dies immediately and leaves behind D_v children, corresponding to the D_v neighbors of v that are targeted by unexplored half-edges emanating from v. All individuals (=targeted vertices) then live for an Exp(1)-distributed amount of time, independently of each other, and when the ith individual dies it leaves behind $\widetilde{B}_i$ children, where $(\widetilde{B}_i)_{i\geq 1}$ is an i.i.d. sequence with distribution (2). Indeed, as long as no cycles are created, the offspring of a given individual is the forward degree of the corresponding vertex, and the forward degrees of explored vertices are asymptotically independent with the size-biased distribution specified in (2). The number of alive individuals after $m \geq 2$ steps is given by

$$\widetilde{S}_m = D_v + \sum_{i=2}^{m} (\widetilde{B}_i - 1)$$

and hence the time when the total offspring in the approximating branching process reaches size m is equal in distribution to $\sum_{i=1}^{m} E_i/\widetilde{S}_i$. In [3] it is shown that the branching process approximation remains valid for $m = m_n \to \infty$ as long as m_n does not grow too fast with n. Define

$$a_n = n^{(\tau-2)/(\tau-1)}.$$

It turns out that "does not grow too fast" means roughly that $m_n = o(a_n)$.

Write $X(u \leftrightarrow v)$ for the passage time between the vertices u and v, that is, $X(u \leftrightarrow v) = T_{m(u,v)}$ with $m(u,v) = \inf\{m : v \in \mathrm{SWG}_m^{(u)}\}$. The relevant results from [3] are summarized in the following proposition. Here, part (a) is essential in proving part (b), and part (d) follows by combining parts (b) and (c). For details we refer to [3]: Part (a) is Proposition 4.7, part (b) is Proposition 4.6(b), where the characterization of V is made explicit in (6.14) in the proof, part (c) is Proposition 4.9 and, finally, part (d) is Theorem 3.2(b).

Proposition 2.1 (Bhamidi, van der Hofstad, Hooghiemstra (2010)). *Consider first passage percolation on a graph generated by the configuration model with a degree distribution that satisfies (A1) and (A2).*

(a) *There exists a $\rho > 0$ such that the sequence $(B_i)_{i\geq 1}$ can be coupled to the i.i.d. sequence $(\widetilde{B}_i)_{i\geq 1}$ with law (2) in such a way that $(B_i)_{i=2}^{n^\rho} = (\widetilde{B}_i)_{i=2}^{n^\rho}$ whp.*

(b) *Let $\bar{m}_n$ be such that $\log(\bar{m}_n/a_n) = o(\sqrt{\log n})$ and assume that $m = m_n \to \infty$ is such that $m_n \leq \bar{m}_n$. As $n \to \infty$, the times T_m and T_{R_m} both converge in distribution to a proper random variable V, where*

$$V \stackrel{d}{=} \sum_{i=1}^{\infty} \frac{E_i}{\widetilde{S}_i}.$$

The law of V is interpreted as the explosion time of the approximating branching process.

(c) *For $m = m_n \ll a_n$ and any two vertices u and v, the two exploration graphs $\mathrm{SWG}_u(m)$ and $\mathrm{SWG}_v(m)$ are whp disjoint, implying that the corresponding limiting variables V_u and V_v are independent. Furthermore, at time $m = \Theta(a_n)$, the graph $\mathrm{SWG}_m^{(u)} \cup \mathrm{SWG}_m^{(v)}$ becomes connected.*

(d) *The passage time $X(u \leftrightarrow v)$ converges in distribution to a random variable distributed as $V_u + V_v$.*

Coupling of competition to first passage percolation. We now return to the setting with two infection types that are imposed at time 0 at the vertices 1 and 2 and then spread at rate 1 and $\lambda \geq 1$, respectively. Recall that $\mu = 1/\lambda$. The following coupling of the two infection types will be used in the rest of the paper: Each edge $e = (u, v)$ is equipped with an exponentially distributed random variable $X(e)$ with mean 1. The infections then evolve in that, if u is type 1 (2) infected, then the time until the infection reaches v via the edge (u, v) is given by $X(u, v)$ ($\mu X(u, v)$) and, if vertex v is uninfected at that point, it becomes type 1 (2) infected. The resulting process clearly has the same distribution as the original process. It also has the property that, if the passage time for type 1 along a given path is T, then the passage time for type 2 along the same path is μT.

3 Proof of Theorem 1.1(a)

In this section we prove Theorem 1.1(a). Recall that the randomness in the process is represented by one single Exp(1)-variable per edge, as described above. All random times that appear in the sequel are based on these variables and are then multiplied by μ to obtain the corresponding quantities for the type 2 infection. Following the notation in the previous section, we write $V_1 = \lim_{n\to\infty} T_{a_n}^{(1)}$ and $V_2 = \lim_{n\to\infty} T_{a_n}^{(2)}$, where V_i are characterized in Proposition 2.1(b).

Proposition 3.1. *Fix $\mu \leq 1$ and let U be a vertex chosen uniformly at random from the vertex set. As $n \to \infty$,*

$$\mathbb{P}\left(U \textit{ is type 1 infected}, T_{a_n}^{(1)} < \mu T_{a_n}^{(2)}\right) \to \mathbb{P}(V_1 < \mu V_2)$$

and

$$\mathbb{P}\left(U \textit{ is type 2 infected}, T_{a_n}^{(1)} > \mu T_{a_n}^{(2)}\right) \to \mathbb{P}(V_1 > \mu V_2).$$

With this proposition at hand, Theorem 1.1(a) follows easily from Markov's inequality:

Proof of Theorem 1.1(a). We start by writing

$$|\bar{N}_1(n) - \mathbb{1}_{\{V_1 < \mu V_2\}}| \leq |\mathbb{1}_{\{T_{a_n}^{(1)} < \mu T_{a_n}^{(2)}\}} - \mathbb{1}_{\{V_1 < \mu V_2\}}| + |\bar{N}_1(n) - \mathbb{1}_{\{T_{a_n}^{(1)} < \mu T_{a_n}^{(2)}\}}|. \tag{7}$$

By Proposition 2.1(d), the first term converges to 0. As for the second term, by Markov's inequality and Proposition 3.1, we have for any $\varepsilon > 0$ that

$$\mathbb{P}(|\bar{N}_1(n) - \mathbb{1}_{\{T_{a_n}^{(1)} < \mu T_{a_n}^{(2)}\}}| > \varepsilon) \leq \frac{1}{\varepsilon}\mathbb{P}(U \text{ is type 1 infected}, T_{a_n}^{(1)} \geq \mu T_{a_n}^{(2)}) \to 0.$$

Thus, it follows that $\bar{N}_1(n) \xrightarrow{\mathbb{P}} \mathbb{1}_{\{V_1 < \mu V_2\}}$, as desired. □

Let $\varepsilon_n = c(\log\log n)^{-1}$ for some constant c and define $A_n = \{T_{a_n}^{(1)} + \varepsilon_n < \mu T_{a_n}^{(2)} - \varepsilon_n\}$. In order to prove Proposition 3.1, we will show that

$$\mathbb{P}(U \text{ is type 1 infected} \,|A_n) \to 1. \tag{8}$$

With $B_n = \{T_{a_n}^{(1)} - \varepsilon_n > \mu T_{a_n}^{(2)} + \varepsilon_n\}$, analogous arguments can be applied to show that $\mathbb{P}(U \text{ is type 2 infected}|B_n) \to 1$. Since $\varepsilon_n \to 0$ and $\mathbb{P}(T_{a_n}^{(1)} < \mu T_{a_n}^{(2)}) \to \mathbb{P}(V_1 < \mu V_2)$ and $\mathbb{P}(T_{a_n}^{(1)} > \mu T_{a_n}^{(2)}) \to \mathbb{P}(V_1 > \mu V_2)$, Proposition 3.1 follows from this.

The proof of (8) is divided in three parts, specified in Lemma 3.2-3.4 below. Recall that $X(u \leftrightarrow v)$ denotes the passage time between the vertices u and v.

Lemma 3.2. *For a uniformly chosen vertex U, $\mathbb{P}\big(X(1 \leftrightarrow U) < b_n\big) \to 1$ for all $b_n \to \infty$.*

Proof. Just note that, by Proposition 2.1(d), the passage time between vertices 1 and U converges to a proper random variable. □

To formulate the second lemma, with $\mathrm{CM}_n(\boldsymbol{D})$ denoting the underlying graph obtained from the configuration model, let $\mathrm{CM}_n(\boldsymbol{D})\backslash\{u\colon D_u \geq s\}$ denote the same graph but where vertices with degree larger than or equal to s do not take part in the spread of the infection, that is, the vertices are still present in the network but are declared immune to the infection.

Lemma 3.3. *Let the vertex U be chosen uniformly at random from the vertex set. There exist $b_n \to \infty$ such that $\mathbb{P}\Big(\mu X(2 \leftrightarrow U)$ in $\mathrm{CM}_n(\boldsymbol{D})\backslash\{v\colon D_v \geq (\log n)^\sigma\} \geq b_n\Big) \to 1$ for any $\sigma < (3-\tau)^{-1}$.*

Combining Lemma 3.2 and 3.3, it follows that the randomly chosen vertex U is whp type 1 infected if, for some $\sigma < (3-\tau)^{-1}$, all vertices with degree at least $(\log n)^\sigma$ are type 1 infected at some finite time point. The last lemma states that, conditionally on A_n, this is indeed the case.

Lemma 3.4. *For $\sigma < (3-\tau)^{-1}$ sufficiently close to $(3-\tau)^{-1}$, conditionally on A_n, whp all vertices with degree larger than or equal to $(\log n)^\sigma$ are type 1 infected at time $T_{a_n}^{(1)} + \varepsilon_n$.*

It remains to prove Lemma 3.3 and Lemma 3.4. We begin with Lemma 3.3, which is the easier one.

Proof of Lemma 3.3. According to Proposition 2.1(b) and (d), the passage time $X(2 \leftrightarrow U)$ is whp at most $T_{n^\rho}^{(2)} + T_{n^\rho}^{(U)} + \varepsilon_n$ for some $\varepsilon_n \downarrow 0$, where ρ is the exponent of the exact coupling in Proposition 2.1(a). If only vertices with degree smaller than $(\log n)^\sigma$ are active, then whp

$$T_{n^\rho}^{(U)} \stackrel{d}{=} \sum_{k=1}^{n^\rho} \frac{E_k}{\widetilde{S}_k^{(\mathrm{truc})}},$$

where

$$\widetilde{S}_k^{(\mathrm{truc})} = D_U \cdot \mathbb{1}_{\{D_U \leq (\log n)^\sigma\}} + \sum_{i=2}^{k} (\widetilde{B}_i - 1) \cdot \mathbb{1}_{\{\widetilde{B}_i \leq (\log n)^\sigma\}}$$

for an i.i.d. sequence $(\widetilde{B}_i)_{i=2}^{n^\rho}$ with distribution (2), that is, a power law with exponent $\tau - 1$. Let $f(n) \sim g(n)$ denote that $c \leq f(n)/g(n) \leq c'$ in the limit as $n \to \infty$ (whp when $f(n)$ is random), where $c \leq c'$ are strictly positive constants. Often we will be able to take $c = c'$, meaning that

$f(n)/g(n)$ converges to c (in probability when $f(n)$ is random), but the more general definition is needed to handle the assumption (A2) on the degree distribution. We calculate that

$$\mathbb{E}[(\widetilde{B}_i - 1)\cdot \mathbb{1}_{\{\widetilde{B}_i \le (\log n)^\sigma\}}] \sim \sum_{j=1}^{(\log n)^\sigma} j^{-(\tau-2)} \sim (\log n)^{\sigma(3-\tau)},$$

and that

$$\mathrm{Var}((\widetilde{B}_i - 1)\cdot \mathbb{1}_{\{\widetilde{B}_i \le (\log n)^\sigma\}}) \sim \mathbb{E}[(\widetilde{B}_i)^2 \cdot \mathbb{1}_{\{\widetilde{B}_i \le (\log n)^\sigma\}}] \sim \sum_{j=1}^{(\log n)^\sigma} j^{(3-\tau)} \sim (\log n)^{\sigma(4-\tau)},$$

so that $\mathbb{E}[\widetilde{S}_{n^\rho}^{(\mathrm{truc})}] \sim n^\rho (\log n)^{\sigma(3-\tau)}$ and $\mathrm{Var}(\widetilde{S}_{n^\rho}^{(\mathrm{truc})}) \sim n^\rho(\log n)^{\sigma(4-\tau)}$. Furthermore, trivially

$$T_{n^\rho}^{(U)} \ge \sum_{k=n^\rho/2}^{n^\rho} \frac{E_k}{k}\cdot\frac{k}{\widetilde{S}_k^{(\mathrm{truc})}}.$$

We now claim that whp $\widetilde{S}_k^{(\mathrm{truc})} \le Ck(\log n)^{\sigma(3-\tau)}$ for all $k \in [n^\rho/2, n^\rho]$ and some constant C. To see this, note that $\widetilde{S}_{k+1}^{(\mathrm{truc})} \ge \widetilde{S}_k^{(\mathrm{truc})}$ so that it suffices to show that

$$\mathbb{P}\left(\widetilde{S}_{n^\rho}^{(\mathrm{truc})} > C\,(n^\rho/2)\,(\log n)^{\sigma(3-\tau)}\right) \to 0.$$

With C chosen such that $Cn^\rho(\log n)^{\sigma(3-\tau)} \ge 3\mathbb{E}[\widetilde{S}_{n^\rho}^{(\mathrm{truc})}]$, this is a consequence of Chebyshev's inequality, since

$$\mathbb{P}\left(\widetilde{S}_{n^\rho}^{(\mathrm{truc})} > \frac{1}{2}Cn^\rho(\log n)^{\sigma(3-\tau)}\right) \le \mathbb{P}\left(\widetilde{S}_{n^\rho}^{(\mathrm{truc})} > \frac{3}{2}\mathbb{E}[\widetilde{S}_{n^\rho}^{(\mathrm{truc})}]\right) \le \frac{\mathrm{Var}(\widetilde{S}_{n^\rho}^{(\mathrm{truc})})}{(\mathbb{E}[\widetilde{S}_{n^\rho}^{(\mathrm{truc})}]/2)^2} \sim \frac{(\log n)^{\sigma(\tau-2)}}{n^\rho},$$

where the right-hand side clearly converges to 0. It follows that, whp,

$$T_{n^\rho}^{(U)} \ge \frac{1}{C(\log n)^{\sigma(3-\tau)}} \sum_{k=n^\rho/2}^{n^\rho} E_k/k$$

where $\sum_{k=n^\rho/2}^{n^\rho} E_k/k \sim \log n$. If $\sigma < 1/(3-\tau)$, then $\kappa := 1-\sigma(3-\tau) > 0$ and the desired conclusion follows with $b_n = c(\log n)^\kappa$. □

In order to prove Lemma 3.4, we will need the following bound, derived in [15, (4.36)].

Lemma 3.5 (van der Hofstad, Hooghiemstra, Znamenski (2007))**.** *Let Γ and Λ be two disjoint vertex sets and write $\Gamma \not\leftrightarrow \Lambda$ for the event that no vertex in Γ is connected to a vertex in Λ. Write D_Γ and D_Λ for the total degree of the vertices in Γ and Λ, respectively, and L_n for the total degree of all vertices. Furthermore, let $\mathbb{P}_n$ be the conditional probability of the configuration model given the degree sequence $(D_i)_{i=1}^n$. Then,*

$$\mathbb{P}_n(\Gamma \not\leftrightarrow \Lambda) \le \mathrm{e}^{-D_\Gamma D_\Lambda/(2L_n)}. \tag{9}$$

Proof of Lemma 3.4. Fix a vertex w with $D_w \ge (\log n)^\sigma$, write $D_{\max} = \max_u D_u$ for the maximal degree, and denote $\mathcal{V}_n^{\max} = \{u\colon D_u = D_{\max}\}$. We will show that

$$\mathbb{P}(X(w \leftrightarrow \mathcal{V}_n^{\max}) > \varepsilon_n/2) = o\left(n^{-1}\right) \tag{10}$$

and

$$\mathbb{P}(X(1 \leftrightarrow \mathcal{V}_n^{\max}) > T_{a_n}^{(1)} + \varepsilon_n/2) = o(1). \tag{11}$$

Lemma 3.4 follows from this by noting that

$$\begin{aligned}
&\mathbb{P}(\exists w\colon D_w \geq (\log n)^\sigma,\ w \text{ is not type 1 at time } T_{a_n}^{(1)} + \varepsilon_n) \\
&\quad \leq \mathbb{P}(\exists w\colon D_w \geq (\log n)^\sigma,\ w \text{ is not type 1 at time } T_{a_n}^{(1)} + \varepsilon_n, X(1 \leftrightarrow \mathcal{V}_n^{\max}) \leq T_{a_n}^{(1)} + \varepsilon_n/2) \\
&\qquad + \mathbb{P}(X(1 \leftrightarrow \mathcal{V}_n^{\max}) > T_{a_n}^{(1)} + \varepsilon_n/2) \\
&\quad \leq n\mathbb{P}(D_w \geq (\log n)^\sigma, X(w \leftrightarrow \mathcal{V}_n^{\max}) > \varepsilon_n/2) + \mathbb{P}(X(1 \leftrightarrow \mathcal{V}_n^{\max}) > T_{a_n}^{(1)} + \varepsilon_n/2) = o(1).
\end{aligned}$$

To prove (10), we will construct a path $v_0, \dots, v_m$ with $v_0 = w$ and $v_m \in \mathcal{V}_n^{\max}$ and with the property that the passage time for the edge (v_i, v_{i+1}) is at most $(\log D_{v_i})^{-1}$, while $D_{v_i} \geq (\log n)^{\alpha_i}$ where α_i grows exponentially in i. The total passage time along the path is hence

$$\sum_{i=1}^m \frac{1}{\log D_{v_i}} = \sum_{i=1}^m \frac{1}{\log\left((\log n)^{\alpha_i}\right)} = \frac{1}{\log\log n} \sum_{i=1}^m \frac{1}{\alpha_i} = O\left(\frac{1}{\log\log n}\right), \tag{12}$$

as desired.

Say that an edge emanating from a vertex u is *fast* if its passage time is at most $(\log D_u)^{-1}$ and write M_u for the number of such edges. Note that

$$\mathbb{E}[M_u \mid D_u] = D_u[1 - \mathrm{e}^{-1/\log D_u}] = \frac{D_u}{\log D_u}\left[1 + O\left(\frac{1}{\log D_u}\right)\right]$$

and that, by standard concentration inequalities,

$$\mathbb{P}(M_u \leq D_u/[2\log D_u]) \leq \mathrm{e}^{-cD_u/\log D_u}.$$

Indeed, conditionally on $D_u = d$, we have that $M_u \overset{d}{=} \mathrm{Bin}(d, 1/\log d)$ and, for any p, it follows from standard large deviation techniques that

$$\mathbb{P}(\mathrm{Bin}(d,p) \leq pd/2) \leq \mathrm{e}^{-pd(1-\log 2)/2}, \tag{13}$$

see e.g. [14, Corollary 2.18]. In particular, if $D_u \geq (\log n)^\sigma$ and $\sigma > 1$, we obtain that

$$\mathbb{P}\Big(\exists u\colon D_u \geq (1-\log 2)(\log n)^\sigma, M_u \leq D_u/[2\log D_u]\Big) \leq n\mathrm{e}^{-(\log n)^\sigma/[2\log((\log n)^\sigma)]} = o(1). \tag{14}$$

Thus, we may assume that $M_u > D_u/[2\log(D_u)]$ for any u with $D_u \geq (\log n)^\sigma$.

Write $\Lambda_i = \{u\colon D_u \geq \eta_i\}$, where η_i will be defined below and shown to equal $(\log n)^{\alpha_i}$ for an exponentially growing sequence (α_i). Furthermore, let $\Gamma(u)$ denote the set of fast half-edges from a vertex u and write $|\Gamma(u)| = D_u^{\mathrm{fast}}$. We now construct the aforementioned path connecting w and $\mathcal{V}_n^{\max}$ iteratively, by setting $v_0 := w$ and then, given v_i, defining $v_{i+1} \in \Lambda_{i+1}$ to be the vertex with smallest index such that a half-edge in $\Gamma(v_i)$ is paired to a half-edge incident to v_{i+1}. We need to show that, with sufficiently high probability, such vertices exist all the way up until we have reached $\mathcal{V}_n^{\max}$. This will follow basically by observing that, for any vertex $u_i \in \Lambda_i$, we have by Lemma 3.5 that

$$\mathbb{P}_n(\Gamma(u_i) \not\leftrightarrow \Lambda_{i+1}) \leq \mathrm{e}^{-D_{u_i}^{\mathrm{fast}} D_{\Lambda_{i+1}}/(2L_n)} \tag{15}$$

and then combining this with suitable estimates of the exponent.

First we define the sequence $(\eta_i)_{i\geq 1}$. To this end, let $\eta_1 = (\log n)^{\sigma}$ and define η_i for $i \geq 2$ recursively as

$$\eta_{i+1} = \Big(\frac{\eta_i}{\log n}\Big)^{(1-\delta)/(\tau-2)}, \tag{16}$$

where $\delta \in (0,1)$ will be determined later on. To identify $(\eta_i)_{i\geq 1}$, write $\eta_i = (\log n)^{\alpha_i}$ and check that $(\alpha_i)_{i\geq 1}$ satisfy the recursion

$$\alpha_{i+1} = \frac{1-\delta}{\tau-2}\alpha_i - \frac{1-\delta}{\tau-2}. \tag{17}$$

As a result,

$$\begin{aligned} \alpha_i &= \alpha_1\Big(\frac{1-\delta}{\tau-2}\Big)^{i-1} - \sum_{j=1}^{i-1}\Big(\frac{1-\delta}{\tau-2}\Big)^{j} \\ &= \alpha_1\Big(\frac{1-\delta}{\tau-2}\Big)^{i-1} - \frac{\Big(\frac{1-\delta}{\tau-2}\Big)^{i}-1}{1-\frac{1-\delta}{\tau-2}} \\ &\geq \Big[\alpha_1 - \frac{1}{1-\frac{\tau-2}{1-\delta}}\Big]^{i}, \end{aligned} \tag{18}$$

which grows exponentially as long as $\alpha_1 = \sigma > (1-\delta)/(3-\tau)$, that is, $\delta > 1-\sigma(3-\tau)$.

We next proceed to estimate the exponent in (15). Let $\Lambda(s) = \{j\colon D_j \geq s\}$ – so that hence $\Lambda_i = \Lambda(\eta_i)$ – and note that

$$D_{\Lambda(s)} \stackrel{d}{=} \sum_u D_u \cdot \mathbb{1}_{\{D_u\geq s\}} \geq s\sum_u \mathbb{1}_{\{D_u\geq s\}} \geq s\cdot\frac{n\mathbb{P}(D\geq s)}{2} \sim ns^{-(\tau-2)}.$$

where the last inequality holds with probability $1-o(n^{-a})$ for any $a>0$ as long as s is much smaller than the maximal degree, e.g. $s \leq n^{(1-\delta/2)/(\tau-1)}$ for some $\delta > 0$ – this follows from (13) by noting that $\sum_u \mathbb{1}_{\{D_u\geq s\}}$ is binomially distributed. Also recall that we may assume that $|\Gamma(u)| \geq D_u/[2\log(D_u)]$ for every u with $D_u \geq (\log n)^{\sigma}$. Hence, for every vertex $u_i \in \Lambda_i$ and as long as $\eta_i \leq n^{(1-\delta/2)/(\tau-1)}$,

$$\mathbb{P}(\Gamma(u_i) \not\leftrightarrow \Lambda_{i+1}) \leq \exp\{-c(\eta_i/\log(\eta_i))\eta_{i+1}^{-(\tau-2)}\} + o(n^{-a}). \tag{19}$$

Using (16) it follows that

$$\mathbb{P}(\Gamma(u_i) \not\leftrightarrow \Lambda_{i+1}) \leq \exp\{-c(\eta_i^{\delta}/\log(\eta_i))\cdot(\log n)^{1/(1-\delta)}\} + o(n^{-a}), \tag{20}$$

which is $o(n^{-a})$ for any $a>0$, since $1/(1-\delta) > 1$ and $\eta_i^{\delta}/\log(\eta_i)$ is uniformly bounded from below as $\eta_i \to \infty$. Taking $a > 3$, this implies that, as long as $\eta_i \leq n^{(1-\delta/2)/(\tau-1)}$,

$$\mathbb{P}_n(\exists i \text{ and } u_i \in \Lambda_i\colon \Gamma(u_i) \not\leftrightarrow \Lambda_{i+1}) = o(n^{-1}). \tag{21}$$

Hence, as long as $\eta_i \leq n^{(1-\delta/2)/(\tau-1)}$, the probability that the construction of the path $(v_i)_{i\geq 1}$ fails in some step is $o(n^{-1})$.

Let $i^* = \max\{i\colon \eta_i \leq n^{(1-\delta/2)/(\tau-1)}\}$ be the largest i for which η_i is small enough to guarantee that the failure probability is suitably small. The path $v_0,\ldots,v_{i^*}$ then has the property that $D_{v_i} \geq (\log n)^{\alpha_i}$ and the passage time on the edge (v_i, v_{i+1}) is at most $(\log D_{v_i})^{-1}$, as required. To complete the proof of (10), it remains to show that, with probability $1-o(n^{-1})$, the vertex

v_{i^*} has an edge with vanishing weight connecting to a vertex in $\mathcal{V}_n^{\max}$. To this end, note that, by construction

$$n^{\frac{(1-\delta/2)(\tau-2)}{(1-\delta)(\tau-1)}} \leq D_{v_{i^*}} \leq n^{\frac{(1-\delta/2)}{(\tau-1)}}. \tag{22}$$

Furthermore, $D_{\max} \geq n^{(1-\delta/4)/(\tau-1)}$ with probability $1-o(n^{-1})$, since

$$\mathbb{P}(D_{\max} \geq x) \leq 1-(1-cx^{-(\tau-1)})^n, \tag{23}$$

which decays stretched exponentially for $x = n^{(1-\delta/4)/(\tau-1)}$. Define $\gamma = [(1-\delta/2)(\tau-2)]/[(1-\delta)(\tau-1)]$ and $\xi = (1-\delta/4)/(\tau-1)$ and let H denote the number of (multiple) edges between v_{i^*} and $\mathcal{V}_n^{\max}$, assuming that $D_{v_{i^*}} = n^\gamma$ and $D_{\max} = n^\xi$. Then H is hypergeometrically distributed with

$$\mathbb{E}[H] = n^\gamma \cdot \frac{n^\xi}{n-n^\gamma} \sim n^{\gamma+\xi-1},$$

where

$$\gamma+\xi-1 = \frac{\delta}{4(1-\delta)(\tau-1)}[\delta+2(\tau-1)-1],$$

which is positive as soon as $\delta > 1-2(\tau-2)$. To bound $\mathbb{P}(H \leq \mathbb{E}[H]/4)$, let H' be a binomial random variable with parameters $p = (n^\xi - n^\gamma)/(n-n^\gamma)$ and n^γ, where we note that $\xi > \gamma$ for $\delta < 1-(\tau-2)$. Then H and H' can be coupled so that $\mathbb{P}(H' \leq H) = 1$, and furthermore $\mathbb{E}[H']/\mathbb{E}[H] \uparrow 1$, so that $\mathbb{E}[H] \leq 2\mathbb{E}[H']$ for large n. Using (13), it follows that

$$\mathbb{P}\left(H \leq \mathbb{E}[H]/4\right) \leq \mathbb{P}\left(H' \leq \mathbb{E}[H']/2\right) \leq \mathrm{e}^{-cn^\gamma(n^\xi-n^\gamma)/(n-n^\gamma)} \sim \mathrm{e}^{-cn^{\gamma+\xi-1}}.$$

Hence, with probability $1-o(n^{-1})$, the vertex v_{i^*} is connected to $\mathcal{V}_n^{\max}$ by at least $\mathbb{E}[H]/4 \sim n^{\gamma+\xi-1}$ edges. Let $(E_i)_{i\geq 1}$ be an i.i.d. sequence of Exp(1)-variables. The probability that among the edges connecting v_{i^*} and $\mathcal{V}_n^{\max}$ there is at least one with passage time at most $(\log n)^{-1}$ is bounded from above by

$$\mathbb{P}\left(\min_{i=1}^{n^{\gamma+\xi-1}} E_i \leq \frac{1}{\log n}\right) = 1-\mathbb{P}\left(E_i > \frac{1}{\log n}\right)^{n^{\xi+\gamma-1}} = 1-\mathrm{e}^{-n^{\gamma+\xi-1}/\log n} = 1-o(n^{-1}).$$

This completes the proof of (10).

To prove (11), first note that it follows from [3, Lemma A.1], that the number of infected vertices at time $T_{a_n}^{(1)}$ is whp larger than b_n for any b_n with $b_n/a_n \to 0$, and that, by Proposition 2.1(a), there exist $\rho > 0$ such that the degrees $(B_i)_{i=2}^{n^\rho}$ of the n^ρ first vertices that were infected are whp equal to an i.i.d. collection $(\widetilde{B}_i)_{i=2}^{n^\rho}$ with distribution (2). A calculation analogous to (23) yields that $\max\{B_2, \ldots, B_{n^\rho}\} \geq n^{\rho(1-\delta)/(\tau-2)}$ whp for any $\delta \in (0,1)$. The vertex with maximal degree at time $T_{a_n}^{(1)}$ can now be connected to $\mathcal{V}_n^{\max}$ by a path constructed in the same way as in the proof of (10). Note that in this case we have $\eta_1 = n^{\rho(1-\delta)/(\tau-2)}$, which gives $\eta_i = n^{\rho\zeta^i}/(\log n)^{\zeta^{i-1}}$ with $\zeta = (1-\delta)/(\tau-2)$. This means that the bound on the passage time for the path is of the order $(\log n)^{-1}$, which is even smaller than the required $(\log\log n)^{-1}$. □

4 Proof of Theorem 1.1(b)

In this section, we prove Theorem 1.1(b). We now explore the first passage percolation from the two vertices 1 and 2 simultaneously. Let $T_{R_m}^{(1,2)}$ denote the time when the SWG from these two vertices consists of m vertices (recall the definition (6) of R_m). Furthermore, write I and II for the winning and the losing type, respectively, that is, $I = 1+\mathbb{1}_{\{V_1>\mu V_2\}}$ and $II = 1+\mathbb{1}_{\{V_1\leq\mu V_2\}}$.

Our first result is that $T^{(1,2)}_{R a_n}$ converges to the minimum of the explosion times V_1 and μV_2 of the one-type exploration processes, and that the asymptotic number N^*_{II} of vertices that are then occupied by the losing type is finite. In the rest of the section we then prove that the asymptotic number N^{**}_{II} of vertices occupied by the losing type after time $T^{(1,2)}_{R a_n}$ is also almost surely finite.

Lemma 4.1. *Let* $N^{**}_{II}(n) = \max\{m \colon T^{(II)}_{R m} \leq T^{(1,2)}_{R m}\}$. *Then, as* $n \to \infty$,

$$(T^{(1,2)}_{R a_n}, N^*_{II}(n)) \xrightarrow{d} (V_1 \wedge (\mu V_2), N^*_{II}), \tag{24}$$

where

$$N^*_{II} = \max\{m \colon \sum_{j=1}^{m} E_j / S^{(II)}_j \leq V_1 \wedge (\mu V_2)\}. \tag{25}$$

Proof. By Proposition 2.1(c), the set of type 1 and type 2 infected vertices, respectively, are whp disjoint at the time $T^{(1,2)}_{R a_n}$ when the winning type reaches size a_n, that is, none of the infection types has then tried to occupy a vertex that was already taken by the other type. Up to that time, the exploration processes starting from vertex 1 and 2, respectively, hence behaves like in the corresponding one-type processes. The asymptotic distributions of $T^{(1,2)}_{R a_n}$ and $N^*_{II}(n)$ follow from the characterization (5) of the time T_m in a one-type process and the convergence result in Proposition 2.1(c). □

The next result describes how vertices are being found by the winning species. Write $\bar{N}^{(t,k)}_I(n)$ for the fraction of vertices that have degree k and that have been captured by the winning type at time $T^{(1,2)}_{R a_n} + t$, that is,

$$\bar{N}^{(t,k)}_I(n) = \#\{v \colon D_v = k \text{ and } v \text{ is infected by type } I \text{ at time } T^{(1,2)}_{R a_n} + t\}/n.$$

The essence of the result is that $\bar{N}^{(t,k)}_I(n)$ develops in the same way as in a one-type process with the winning type. Indeed, $T^{(1,2)}_{R a_n}$ can be interpreted as the time when the super-vertices have been found by the winning type and, after this time, the winning type will start finding vertices very quickly, which will make it hard for the losing type to spread. We denote the mean passage time per edge for the winning type by μ_I (that is, μ_I is equal to 1 or μ depending on whether $I = 1$ or $I = 2$) and define

$$V(k) = \sum_{j=1}^{\infty} E_j / S_j(k),$$

where

$$S_j(k) = k + \sum_{i=1}^{j} (B_i - 1).$$

Proposition 4.2 (Number of fixed degree winning type vertices at fixed time)**.** *As* $n \to \infty$,

$$\bar{N}^{(t,k)}_I(n) \xrightarrow{\mathbb{P}} \mathbb{P}(\mu_I V(k) \leq t) \mathbb{P}(D = k).$$

The proof of Proposition 4.2 is deferred to the end of this section. We first complete the proof of Theorem 1.1(b) subject to it. To this end, we grow the SWG of the losing type from size $N^*_{II}(n)$ onwards. At this moment, whp the losing type has not yet tried to occupy a vertex that was already taken by the winning type. However, when we grow the SWG further, then the winning kind will grow very quickly due to its explosion. We will show that the growth of the losing kind is thus delayed to the extent that it will only conquer finitely many vertices.

An important tool in proving this rigorously is a stochastic process (S'_m) keeping track of the number of half-edges incident to the SWG of the losing type.

Recall that, by the construction of the exploration process described in Section 2, the quantity $S^{(I)}_{R_j}$ represents the number of half-edges incident to the SWG of the losing type when the SWG contains precisely j vertices. Write $R_{N^*_{II}(n)} = R^*_n$ and define $S'_0 = S^{(I)}_{R^*_n}$ and $T'_m = 0$. The sequences $(T'_m)_{m\geq 1}$ and $(S'_m)_{m\geq 1}$ are then constructed recursively in that $T'_m - T'_{m-1} = E_m/S'_{m-1}$ for an i.i.d. sequence $(E_m)_{m\geq 1}$ of exponential variables with parameter 1, and

$$S'_m - S'_{m-1} = D'_m - 1, \qquad \text{with} \qquad D'_m = \widetilde{B}_m I_m, \tag{26}$$

where, conditionally on $\widetilde{B}_m$ and T'_{m-1}, the indicator I_m is Bernoulli with success probability $\mathbb{P}(V(\widetilde{B}_m) > T'_{m-1} \mid \widetilde{B}_m, T'_{m-1})$. Here, the sequence $(\widetilde{B}_m)$ is i.i.d. with distribution (2).

We claim that the process (S'_m) keeps track of the asymptotic number of half-edges incident to the SWG of the losing type. To understand this, assume that there are S'_m half-edges incident to the SWG after the $(N^*_{II}(n) + m)$th growth. The minimal edge weight then has distribution $T'_m - T'_{m-1} = E'_m/S'_{m-1}$. When we pair the half-edge with this minimal weight, the conditional probability of attaching it to a vertex that is of the winning type at time T'_{m-1} and that has degree k given T'_{m-1} is, by Proposition 4.2, close to

$$\frac{k\bar{N}_I^{(T'_{m-1},k)}(n)}{L_n} \approx \frac{k\mathbb{P}(D=k)}{\mathbb{E}[D]}\mathbb{P}(\mu_I V(k) \leq T'_{m-1} \mid T'_{m-1}).$$

As a result, with $D^\star$ denoting a size-biased version of a degree variable D, the probability that the half-edge is attached to a vertex of degree k that does not have the winning type at time T'_{m-1} is close to

$$\frac{k\mathbb{P}(D=k)}{\mathbb{E}[D]}\mathbb{P}(\mu_I V(k) > T'_{m-1} \mid T'_{m-1}) = \mathbb{P}(D^\star = k)\mathbb{P}(\mu_I V(D^\star) > T'_{m-1} \mid T'_{m-1}, D^\star = k).$$

When this happens, the number of half-edges incident to the SWG of the losing type is increased by $k-1$. On the other hand, when the half-edge is attached to a vertex of the winning kind, then the number of losing type half-edges decreases by 1. Putting this together and using that $B \stackrel{d}{=} D^\star - 1$ explains (26).

Recall that the total asymptotic number of losing type vertices is denoted by N_{II}. This number can now be expressed as

$$N_{II} = N^*_{II} + N^{**}_{II},$$

where N^*_{II} is defined in Lemma 4.1 and $N^{**}_{II} := \max\{S'_m \geq 1\}$. Indeed, the losing type cannot grow any further after the point when (S'_m) hits 0. To prove Theorem 1.1, it hence suffices to show that the random variable N^{**}_{II} is finite almost surely. Note that the sequence $(\widetilde{B}_m)_{m\geq 1}$ that determines the step sizes D'_m in the recursion (26) has infinite mean, which implies that many of its values are large. This is the problem that we need to overcome in showing that N^{**}_{II} is finite. In order to do this, we first need to investigate $V(k)$ and some related quantities in more detail.

Lemma 4.3 (Asymptotics for $V(k)$ for large k)**.** *Assume that (A2') holds. As $k \to \infty$,*

$$k^{3-\tau}V(k) \xrightarrow{d} \int_0^\infty 1/(1+Q_t)dt,$$

where $(Q_t)_{t\geq 0}$ is a $(\tau-2)$-stable motion. Further,

$$\mathbb{E}\Big[\int_0^\infty 1/(1+Q_t)dt\Big] < \infty. \tag{27}$$

Proof. Recall that $V(k) = \sum_{j=1}^{\infty} E_j/S_j(k)$, where $S_j(k) = k + \sum_{i=1}^{j}(B_i - 1)$. Since B_i is in the domain of attraction of a stable law with exponent $\tau - 2$, we have that $(S_{tk^{\tau-2}}(k)/k)_{t\geq 0} \xrightarrow{d} (1 + Q_t)_{t\geq 0}$, where $(Q_t)_{t\geq 0}$ is a stable subordinator with exponent $\tau - 2$. Thus,

$$k^{3-\tau}V(k) \xrightarrow{d} \int_0^{\infty} 1/(1 + Q_t)dt =: Y. \tag{28}$$

As for the expectation of the integral random variable Y, we use Fubini to write

$$\begin{aligned} \mathbb{E}\Big[\int_0^{\infty} 1/(1 + Q_t)dt\Big] &= \int_0^{\infty} \mathbb{E}[1/(1 + Q_t)]dt \\ &= \int_0^{\infty}\int_0^{\infty} \mathbb{E}\big[\mathrm{e}^{-s(1+Q_t)}\big]dsdt \\ &= \int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-s}\mathrm{e}^{-\sigma t s^{\tau-2}}dsdt, \end{aligned} \tag{29}$$

where we have used that $\mathbb{E}[\mathrm{e}^{-sQ_t}] = \mathrm{e}^{-\sigma t s^{\tau-2}}$ for some $\sigma > 0$. We continue to compute this as

$$\mathbb{E}\Big[\int_0^{\infty} 1/(1 + Q_t)dt\Big] = \frac{1}{\sigma}\int_0^{\infty} \mathrm{e}^{-s}s^{-(\tau-2)}ds < \infty, \tag{30}$$

since $\tau - 2 \in (0, 1)$. □

Lemma 4.3 allows us to prove (3) in Theorem 1.2(c):

Proof of (3) *in Theorem 1.2(c).* We note that $V_{i,k} \overset{d}{=} V_i(A_{i,k})$, where $A_{i,k} = \sum_{j=1}^{k} D_{i,j}$ and $(D_{i,j})_{i,j\geq 1}$ are i.i.d. random variables with the same distribution as D. When $k \to \infty$, we have that $A_{i,k}/k \xrightarrow{\mathbb{P}} \mathbb{E}[D]$. As a result, $(\mathbb{E}[D]k)^{3-\tau}V_{1,k} \xrightarrow{d} Y_1$, while $(\mathbb{E}[D]k)^{3-\tau}V_{2,\alpha k} \xrightarrow{d} \alpha^{\tau-3}Y_2$, where Y_1, Y_2 are i.i.d. copies of Y. Hence,

$$\mathbb{P}(V_{i,k} < \mu V_{2,\alpha k}) = \mathbb{P}\Big((\mathbb{E}[D]k)^{3-\tau}V_{1,k} < \mu(\mathbb{E}[D]k)^{3-\tau}V_{2,\alpha k}\Big) \to \mathbb{P}(Y_1 < \mu\alpha^{\tau-3}Y_2). \tag{31}$$

□

Next we investigate the tail behavior of the random variables Q_y and $Y = \int_0^{\infty} 1/(1 + Q_t)dt$ in more detail.

Lemma 4.4 (Tail probabilities of Q_y and Y). *There exists a $\gamma = \gamma(\sigma, \tau) > 0$ such that*

$$\mathbb{P}(Q_y \leq u) \leq \mathrm{e}^{-\gamma y^{1/(3-\tau)}/u^{(4-\tau)/(3-\tau)}}, \tag{32}$$

and there exists a κ such that

$$\mathbb{P}(Y \geq y) \leq \mathrm{e}^{-\kappa y^{1/(3-\tau)}}. \tag{33}$$

Proof. For (32), we use the exponential Chebychev inequality to obtain that, for every $s \geq 0$

$$\mathbb{P}(Q_y \leq u) = \mathbb{P}(\mathrm{e}^{-sQ_y} \geq \mathrm{e}^{-su}) \leq \mathrm{e}^{su}\mathbb{E}[\mathrm{e}^{-sQ_y}] = \mathrm{e}^{su}\mathrm{e}^{-\sigma y s^{\tau-2}}.$$

Minimizing over $s \geq 0$ gives $s = (\sigma(\tau - 2)y/u)^{1/(3-\tau)}$, and substitution of s yields the claim in (32).

For (33), we fix $A > 1$ to be chosen later on, and condition on $Q_y \leq A - 1$ or $Q_y \geq A - 1$. This yields

$$\mathbb{P}(Y \geq y) = \mathbb{P}(Y \geq y, Q_{y/2} \leq A - 1) + \mathbb{P}(Y \geq y, Q_{y/2} > A - 1).$$

The first probability is by (32) bounded by

$$\mathbb{P}(Y \ge y, Q_{y/2} \le A-1) \le \mathbb{P}(Q_{y/2} \le A-1) \le \mathrm{e}^{-\gamma' y^{1/(3-\tau)}/(A-1)^{(4-\tau)/(3-\tau)}},$$

where $\gamma' = \gamma 2^{-1/(3-\tau)}$. For $\mathbb{P}(Y \ge y, Q_{y/2} > A-1)$, we note that $Q_u \ge 0$, so that $1/(1+Q_t) \le 1$ for every $t \le y/2$ and the process $(Q_{t+y/2} - Q_{y/2})_{t\ge 0}$ has the same law as $(Q_t)_{t\ge 0}$. Thus, on the event that $Q_{y/2} > A-1$,

$$Y \le \int_0^{y/2} 1dt + \int_{y/2}^\infty \frac{1}{1+Q_t}dt \le \int_0^{y/2} 1dt + \int_{y/2}^\infty \frac{1}{A+(Q_{t+y/2}-Q_{y/2})}dt.$$

Hence

$$\begin{aligned}\mathbb{P}(Y \ge y, Q_{y/2} > A-1) &\le \mathbb{P}\Big(\int_{y/2}^\infty \frac{1}{A+(Q_{t+y/2}-Q_{y/2})}dt \ge y/2\Big)\\ &= \mathbb{P}\Big(\int_0^\infty \frac{1}{A+Q_t}dt \ge y/2\Big).\end{aligned}$$

Further, since for every $C>0$, the law of $(CQ_t)_{t\ge0}$ is the same as that of $(Q_{tC^{\tau-2}})$, we see that

$$\begin{aligned}\int_0^\infty \frac{1}{A+Q_t}dt &= \frac{1}{A}\int_0^\infty \frac{1}{1+Q_t/A}dt \overset{d}{=} \frac{1}{A}\int_0^\infty \frac{1}{1+Q_{t/A^{\tau-2}}}dt\\ &= A^{\tau-3}\int_0^\infty \frac{1}{1+Q_t}dt = A^{\tau-3}Y.\end{aligned}$$

Thus, we obtain that

$$\mathbb{P}(Y \ge y) \le \mathrm{e}^{-\gamma' y^{1/(3-\tau)}/(A-1)^{(4-\tau)/(3-\tau)}} + \mathbb{P}(Y \ge A^{3-\tau}y/2).$$

Taking A such that $A^{3-\tau}/2 = 2$, this leads to

$$\mathbb{P}(Y \ge y) \le \mathrm{e}^{-\kappa' y^{1/(3-\tau)}} + \mathbb{P}(Y \ge 2y). \tag{34}$$

Iteration of (34) leads to (33). □

With these estimates at hand we are now ready to prove Theorem 1.1(b).

Proof of Theorem 1.1(b). Recall the construction of the process $(S'_m)_{m\ge0}$ in the recursion (26). As described above, the process keeps track of the number of half-edges incident to losing type vertices after the explosion of the winning type. Also recall that the asymptotic number of vertices captured by the losing type after this time is given by $N_{II}^{**} = \max\{m : S'_m \ge 1\}$. We need to show that $N_{II}^{**} < \infty$ almost surely.

We first claim that $\mathbb{E}[D'_m \mid N_{II}^*, D'_1] < \infty$ for $m \ge 3$. To this end, recall that $T'_m - T'_{m-1} = E_m/S'_{m-1}$, where $(E_m)_{m\ge1}$ is an i.i.d. sequence of Exp(1)-variables. Conditionally on N_{II}^*, D'_1, we thus have that

$$T'_{m-1} \ge E'_1/N_{II}^* + E'_2/(N_{II}^* + D'_1 - 1).$$

It suffices to investigate the case where $N_{II}^* + D'_1 - 1 \ge 1$, since otherwise $N_{II}^{**} = 2$. Then, for $k \ge 1$ large, we split

$$\begin{aligned}\mathbb{E}[D_m \mid N_{II}^*, D''_1] = {} & \mathbb{E}[\widetilde{B}_m \mathbb{P}(V(\widetilde{B}_m) > T'_{m-1} \mid \widetilde{B}_m, T'_{m-1})\mathbb{1}_{\{\widetilde{B}_m \le k\}} \mid N_{II}^*, D'_1]\\ &+ \mathbb{E}[\widetilde{B}_m \mathbb{P}(Y \ge T'_{m-1}\widetilde{B}_m^{3-\tau} \mid \widetilde{B}_m, T'_{m-1})\mathbb{1}_{\{\widetilde{B}_m > k\}} \mid N_{II}^*, D'_1].\end{aligned}$$

The first term is bounded by k, the second is, for large k, dominated by

$$\mathbb{E}[\widetilde{B}_m \exp\{-\kappa(T'_{m-1})^{1/(3-\tau)}\widetilde{B}_m\}\mathbb{1}_{\{\widetilde{B}_m>k\}} \mid N^*_{II}, D'_1].$$

Now, $T'_{m-1} \geq (E_1+E_2)/Z$, where $Z = N^*_{II} \vee (N^*_{II} + D'_1 - 1)$. Therefore,

$$\begin{aligned}
&\mathbb{E}[\widetilde{B}_m \exp\{-\kappa(T'_{m-1})^{1/(3-\tau)}\widetilde{B}_m\}\mathbb{1}_{\{\widetilde{B}_m>k\}} \mid N^*_{II}, D'_1]\\
&\quad\leq \mathbb{E}[\widetilde{B}_m \exp\{-\kappa Z^{1/(3-\tau)}(E_1+E_2)^{1/(3-\tau)}\widetilde{B}_m\}\mathbb{1}_{\{\widetilde{B}_m>k\}} \mid N^*_{II}, D'_1].
\end{aligned}$$

Using that

$$\mathbb{E}\big[\exp\{-a(E_1+E_2)^{1/(3-\tau)}\}\big] = \int_0^\infty u\mathrm{e}^{-au^{1/(3-\tau)}}du = ca^{-2(3-\tau)},$$

we thus arrive at

$$\begin{aligned}
&\mathbb{E}[\widetilde{B}_m \exp\{-\kappa(T'_{m-1})^{1/(3-\tau)}\widetilde{B}_m\}\mathbb{1}_{\{\widetilde{B}_m>k\}} \mid N^*_{II}, D'_1]\\
&\quad\leq O(1)Z^{-2}\mathbb{E}[\widetilde{B}_m\widetilde{B}_m^{-2(3-\tau)}\mathbb{1}_{\{\widetilde{B}_m>k\}} \mid N^*_{II}D'_1] = O(1)Z^{-2}\mathbb{E}[\widetilde{B}_m^{2\tau-5}\mathbb{1}_{\{\widetilde{B}_m>k\}}].
\end{aligned}$$

We compute that

$$\mathbb{E}[\widetilde{B}_m^{2\tau-5}] \leq O(1)\sum_{\ell=1}^\infty \ell^{2\tau-5}\ell^{-(\tau-1)} = O(1)\sum_{\ell=1}^\infty \ell^{\tau-4} < \infty,$$

since $\tau \in (2,3)$. Thus, indeed $\mathbb{E}[D'_m \mid N^*_{II}, D'_1] < \infty$, and also $\mathbb{E}[D'_m] < \infty$.

We next extend this argument to show that $\mathbb{E}[D'_m \mid N^*_{II}, D'_1] \to 0$ as $m \to \infty$. We use the fact that $m \mapsto T'_{m-1}$ is stochastically increasing, and, since $\mathbb{E}[D'_m \mid N^*_{II}, D'_1] < \infty$, we obtain that $T'_{m-1} \xrightarrow{a.s.} \infty$. Thus, $\mathbb{P}(V(\widetilde{B}_m) > T'_{m-1} \mid \widetilde{B}_m, T'_{m-1})$ tends to zero unless $\widetilde{B}_m$ is large. By monotone convergence, $\mathbb{E}[D'_m \mid N^*_{II}, D'_1] \xrightarrow{\mathbb{P}} 0$. As a result, since $\mathbb{E}[D'_m] < \infty$, by dominated convergence also $\mathbb{E}[D'_m] \to 0$.

Since $S'_m - S'_{m-1} = D'_m - 1$, we have that $S'_m - S'_{m-1} \xrightarrow{d} -1$. This in turn implies that $(S'_m)_{m\geq 0}$ hits zero in finite time, so that $N^{**}_{II} = \max\{m\colon S'_m \geq 1\} < \infty$ a.s. Indeed, take m_0 so large that $\mathbb{E}[D'_m] \leq \varepsilon$ for every $m \geq m_0$. Then S'_{m_0} is some finite random variable. In order for $S'_m \geq 1$ for every $m \geq m_0$ to occur, we need to have that $\sum_{j=m_0}^m D'_m \geq (m-m_0) + S'_{m_0}$ for every $m \geq m_0$. Take $m \geq 2(m_0 \wedge S'_{m_0})$, which we can do a.s. by taking m sufficiently large. Then, $\sum_{j=m_0+1}^m D'_m \geq (m-m_0) + S'_{m_0}$ implies that $\sum_{j=m_0+1}^m D'_m \geq (m-m_0)/2$. By the Markov inequality, the probability of this event is at most

$$\mathbb{P}\Big(\sum_{j=m_0+1}^m D'_m \geq (m-m_0)/2\Big) \leq \frac{2}{m-m_0}\sum_{j=m_0+1}^m \mathbb{E}[D'_m] \leq 2\varepsilon.$$

The above is true for arbitrary $\varepsilon > 0$, so that $\mathbb{P}(S'_m \geq 1\ \forall m) = 0$. □

We finish by proving Proposition 4.2.

Proof of Proposition 4.2. Let U be a randomly chosen vertex and write $\mathbb{1}_U^{(t,k)}$ for the indicator taking the value 1 when vertex U has degree k and is occupied by the winning type I at time $T^{(1,2)}_{R_{a_n}} + t$. Note that

$$\bar{N}_I^{(t,k)}(n) = \mathbb{E}[\mathbb{1}_U^{(t,k)} \mid \mathrm{CM}_n(\boldsymbol{D})]. \tag{35}$$

We aim at using a second moment method on $\bar{N}_I^{(t,k)}(n)$, and start by showing that $\mathbb{E}[\mathbb{1}_U^{(t,k)}] \to \mathbb{P}(V(k) \leq t)\mathbb{P}(D=k)$. First note that

$$\mathbb{P}(\mathbb{1}_U^{(t,k)} = 1) = \mathbb{P}(U \text{ is infected by type } I \text{ at time } T^{(1,2)}_{R_{a_n}} + t | D_U = k)\mathbb{P}(D_U = k), \tag{36}$$

so that it suffices to show that the first factor above converges to $\mathbb{P}(V(k) \leq t)$. To this end, assume that $V_1 < \mu V_2$ – that is, assume that the winning type is $I = 1$ – and recall that $X(1 \leftrightarrow U)$ denotes the passage time between vertices 1 and U in a one-type process with only type 1 infection. It follows from the analysis in [3], summarized in Proposition 2.1, that $X(1 \leftrightarrow U)$ converges in distribution to $V_1 + V(k)$: First we grow the SWG from vertex 1 to size a_n. The time when this occurs $T_{a_n}^{(1)}$ converges in distribution to V_1. We then grow the SWG from U until it hits the SWG from vertex 1. This occurs when it has size $C_n \sim a_n$ and the time it takes to reach this size converges to $V(k)$ – indeed, $V(k)$ describes the asymptotic explosion time for an exploration process started at a vertex with degree k. Hence,

$$\mathbb{P}(X(1 \leftrightarrow U) \leq T_{a_n}^{(1)} + t | D_U = k) \to \mathbb{P}(V(k) \leq t). \tag{37}$$

We need to show that the presence of type 2 infection started from vertex 2 does not affect this convergence result.

Write $\mathrm{SWG}^{(u)}(s)$ for the one-type SWG from vertex u at time s, that is, $\mathrm{SWG}^{(u)}(s)$ consists of the vertices and edges that have been reached by the flow from vertex u at time s. Let $\varepsilon_n = (\log\log n)^{-1}$ and note that, if $V_1 < \mu V_2$, then $T_{a_n}^{(1)} + \varepsilon_n < \mu T_{a_n}^{(2)} - \varepsilon$ for n large. By Lemma 4.1, the number of type 2 infected vertices at time $T_{a_n}^{(1)}$ then converges to an almost surely finite random variable. Furthermore, the probability that any additional vertices become type 2 infected in the time interval $(T_{a_n}^{(1)}, T_{a_n}^{(1)} + \varepsilon_n)$ converges to 0, since $\varepsilon_n \to 0$. Hence, whp $\mathrm{SWG}^{(1)}(T_{a_n}^{(1)} + \varepsilon_n) \cap \mathrm{SWG}^{(2)}(T_{a_n}^{(1)} + \varepsilon_n) = \varnothing$. Also, by Lemma 3.4, the type 1 infection has whp occupied all vertices with degree larger than $(\log n)^\sigma$ at time $T_{a_n}^{(1)} + \varepsilon_n$.

Now consider the SWG from vertex U, where whp $U \notin \mathrm{SWG}^{(2)}(T_{a_n}^{(1)} + \varepsilon_n)$. Without the presence of the type 2 infection, this will hit $\mathrm{SWG}^{(1)}(T_{a_n}^{(1)} + \varepsilon_n)$ when it has reached size $C_n \sim a_n$ and the time for this converges to $V(k)$. We claim that whp it does not hit the type 2 infection before this happens. This follows from Lemma 3.3: The passage time from any vertex in $\mathrm{SWG}^{(2)}(T_{a_n}^{(1)} + \varepsilon_n)$ to U, not using vertices with degree larger than $(\log n)^\sigma$ – indeed, these are already occupied by the type 1 infection and hence not available – is whp larger than b_n, where $b_n \to \infty$. Hence, the passage time from any type 2 vertex to U is whp larger than $2V(k) + \varepsilon$ for any $\varepsilon > 0$. This means that whp the type 2 infection does not reach any of the vertices along the minimal weight path between $\mathrm{SWG}^{(1)}(T_{a_n}^{(1)} + \varepsilon_n)$ and U before time $V(k) + \varepsilon$. Indeed, if it would, then there would be a path between vertex 2 and U that avoids high-degree vertices and that has passage time less than $2V(k) + \varepsilon$.

It follows that, if $V_1 < \mu V_2$, then the passage time between vertex 1 and U behaves asymptotically the same as in a one-type process with only type 1 infection, as desired.

We continue by studying the second moment of $\bar{N}_I^{(t,k)}(n)$. By (35),

$$\mathbb{E}[\bar{N}_I^{(t,k)}(n)^2] = \mathbb{E}\Big[\mathbb{E}[\mathbb{1}_U^{(t,k)} \mid \mathrm{CM}_n(\boldsymbol{D})]^2\Big] = \mathbb{E}[\mathbb{1}_{U_1}^{(t,k)} \mathbb{1}_{U_2}^{(t,k)}], \tag{38}$$

where U_1, U_2 are two independent uniform random vertices. We rewrite this as

$$\mathbb{E}[\bar{N}_I^{(t,k)}(n)^2] = \mathbb{P}(\mathbb{1}_{U_1}^{(t,k)} = \mathbb{1}_{U_2}^{(t,k)} = 1) = \mathbb{P}(\mathbb{1}_{U_1}^{(t,k)} = \mathbb{1}_{U_2}^{(t,k)} = 1 \mid D_{U_1} = D_{U_2} = k)\mathbb{P}(D_{U_1} = D_{U_2} = k).$$

By the fact that the degrees are i.i.d., it follows that

$$\mathbb{P}(D_{U_1} = D_{U_2} = k) = \left(1 - \frac{1}{n}\right)\mathbb{P}(D = k)^2 + \frac{1}{n}\mathbb{P}(D = k) \to \mathbb{P}(D = k)^2.$$

Further,

$$\begin{aligned} &\mathbb{P}(\mathbb{1}_{U_1}^{(t,k)} = \mathbb{1}_{U_2}^{(t,k)} = 1 \mid D_{U_1} = D_{U_2} = k) \\ &\quad = \mathbb{P}(U_1, U_2 \text{ both infected by type } I \text{ at time } T_{R a_n}^{(1,2)} + t | D_{U_1} = D_{U_2} = k). \end{aligned} \tag{39}$$

The above conditional probability converges to $\mathbb{P}(V(k) \leq t)^2$ – this follows from the steps below (37) – and we have already shown that $X(1 \leftrightarrow U_1)$ converges in distribution to $V_1 + V_1(k)$. In the same way, we can construct the SWG from vertex U_2 to see that $X(1 \leftrightarrow U_2)$ converges in distribution to $V_1 + V_2(k)$, where $V_2(k)$ is independent of $V_1(k)$. Indeed, the SWG from both vertex 1 and U_1 are asymptotically negligible compared to the entire graph, and therefore hardly change the distribution of the SWG from U_2. As a result,

$$\begin{aligned} &\mathbb{P}(U_1, U_2 \text{ both infected by type } I \text{ at time } T^{(1,2)}_{R_{a_n}} + t | D_{U_1} = D_{U_2} = k) \\ &\quad \to \mathbb{P}(V_1(k) \leq t, V_2(k) \leq k) = \mathbb{P}(V(k) \leq t)^2, \end{aligned} \tag{40}$$

the latter due to independence. We conclude that $\mathbb{E}[\bar{N}_I^{(t,k)}(n)^2] = \mathbb{E}[\bar{N}_I^{(t,k)}(n)]^2 + o(1)$, so that $\mathrm{Var}(\bar{N}_I^{(t,k)}(n)^2) \to 0$. As a result, since also $\mathbb{E}[\bar{N}_I^{(t,k)}(n)] \to \mathbb{P}(V(k) \leq t)\mathbb{P}(D = k)$, we arrive at

$$\bar{N}_I^{(t,k)}(n) \xrightarrow{\mathbb{P}} \mathbb{P}(V(k) \leq t)\mathbb{P}(D = k), \tag{41}$$

as required. □

Acknowledgement. The work of MD was supported in part by the Swedish Research Council (VR), and the work of RvdH was supported in part by the Netherlands Organisation for Scientific Research (NWO).

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