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Allan Gut Anders Martin-Löf

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Postal address:

Mathematical Statistics
Dept. of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden

Internet:

http://www.math.su.se

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The topic of the present paper is a generalized St. Petersburg game in which the distribution of the payoff X is given by $P(X = sr^{k-1}) = pq^{k-1}$, $k = 1, 2, \ldots$, where $p + q = 1$, and $s, r > 0$. As for main results, we first extend Feller's classical weak law and Martin-Löf's 1985-theorem on convergence in distribution along the 2^n-subsequence. In his 2008-paper Martin-Löf's considers a truncated version of the game and the problem How much does one gain until "game over", and a variation where the player can borrow money but has to pay interest on the capital, also for the classical setting. We extend these problems to our more general setting. We close with some additional results and remarks.

Generalized St. Petersburg games revisited

Allan Gut
Uppsala University

Anders Martin-Löf
Stockholm University

Abstract

The topic of the present paper is a generalized St. Petersburg game in which the distribution of the payoff X is given by $P(X = sr^{k-1}) = pq^{k-1}$, $k = 1, 2, \dots$, where $p+q = 1$, and s, $r > 0$. As for main results, we first extend Feller's classical weak law and Martin-Löf's 1985-theorem on convergence in distribution along the 2^n-subsequence. In his 2008-paper Martin-Löf considers a truncated version of the game and the problem "How much does one gain until 'game over' ", and a variation where the player can borrow money but has to pay interest on the capital, also for the classical setting. We extend these problems to our more general setting. We close with some additional results and remarks.

1 Introduction

The classical St. Petersburg game is defined as follows: Peter throws a fair coin repeatedly until heads turns up. If this happens at trial number k he has to pay Paul 2^k ducates. The question is what the value of the game might be to Paul. Now, since the random variable X describing the payoff is governed by

$$P(X = 2^k) = \frac{1}{2^k}, \quad k = 1, 2, \dots ,$$

which has infinite expectation, we have no guidance there for what a fair price would be for Paul to participate in the game.

One variation is to set the fee as a function of the number of games, which leads to the celebrated Feller solution [3], namely, that if X, X_1, X_2, $\dots$ are i.i.d. random variables as above, and $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$, then

$$\frac{S_n}{n \log_2 n} \xrightarrow{p} 1 \quad \text{as} \quad n \to \infty, \tag{1.1}$$

where, generally, $\log_r(\cdot)$ denotes the logarithm relative to base $r > 0$. For details, see [4], Chapter X, and [5], Chapter VII (and/or [9], Section 6.4.1). More on the history of the game can be found in [11].

The present paper is devoted to the generalization in which I toss a biased coin for which $P(\text{heads}) = p$, $0 < p < 1$, repeatedly until heads appears. If this happens at trial number k you receive sr^{k-1} Euro, where s, $r > 0$, which induces the random variable

$$P(X = sr^{k-1}) = pq^{k-1}, \quad k = 1, 2, \dots . \tag{1.2}$$

Our first result is an extension of Feller's weak law (1.1) to the setting (1.2) under the assumption that $r = 1/q$. If, in additon, $s = 1/p$ the result reduces to Theorem 2.1(i) of [7], where additional references can be found. The case, $p = q = 1/2$ corresponds (of course) to the classical game.

As for convergence in distribution in the classical case, Martin-Löf [11] obtains convergence in distribution along the *geometric subsequence* 2^n to an infinitely divisible, semistable Our second result extends his theorem to the general case.

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If, in particular, $s = 1/p$ och $r = 1/q$, (some of) the results reduce to those of [7, 8], and if, in addition, $p = q = 1/2$ to the setting in [11, 12].

The results mentioned so far are stated in Section 2 and proved in Sections 4 and 5, respectively, after some preliminaries in Section 3.

In Section 6 we consider a trunctated game and the problem "How much does one gain until game over?", thereby extending the classical setting from [12]. A second model treated in the cited paper concerns the case when the player can borrow money without limit for the stakes, but has to pay interest on the capital. Our extensions to the present setting is treated in Section 7.

A final section contains some additional results and remarks.

2 Main results

Thus, let throughout $X, X_1, X_2, \ldots$ be i.i.d. random variables with

$$P(X = sr^{k-1}) = pq^{k-1}, \quad k = 1, 2, \ldots,$$

and set $S_n = \sum_{k=1}^n X_k$ and $M_n = \max_{1\le k\le n} X_k$, $n \ge 1$.

Since we are aiming at weak limits we begin by noticing that if $r < 1/q$, then $E\,X < \infty$, so that the classical strong law holds, viz.

$$\frac{S_n}{n} \overset{a.s.}{\to} \frac{sp}{1-rq} \quad \text{as} \quad n \to \infty,$$

where for the value of the limit we refer to (3.2) with $\beta = 1$ below.

In the following we therefore assume that $rq \ge 1$, and thus, in particular, that $r > 1$.

REMARK 2.1 If, in addition, $r < \sqrt{q}$, then $\operatorname{Var} X < \infty$ and a central limit theorem holds. □

Theorem 2.1 *If $r = 1/q$, then*

$$\frac{S_n}{n \log_r n} \overset{p}{\to} sp \quad \text{as} \quad n \to \infty.$$

REMARK 2.2 For $p = q = 1/2$ and $\alpha = 1$ the theorem reduces to (1.1), and for general p, $s = 1/p$, and $\alpha = 1$ to [7], Theorem 2.1(i). □

Our next theorem extends Martin-Löf's subsequence result for the classical game [11]. We remark that M, N, and uM below are not integers. We leave it to the reader to replace such quantities with the respective integer parts and to make the necessary amendments.

Theorem 2.2 *Let $N = r^n$ and $M = q^{-n}$.*
(i) *If $r = 1/q$, then, for $u > 0$,*

$$\frac{S_{uN} - spuNn}{N} = \frac{S_{uN}}{N} - spun \overset{d}{\to} Z(u) \quad \text{as} \quad n \to \infty\,,$$

where $Z(u)$ is the Lévy process defined via the characteristic function $\varphi_{Z(u)}(t) = E\exp\{itZ(u)\} = \exp\{ug(t)\}$, where

$$\begin{aligned} g(t) &= \sum_{k=-\infty}^{-1} \big(\exp\{itsr^k\} - 1 - itsr^k\big)\cdot pq^k + \sum_{k=0}^{\infty} \big(\exp\{itsr^k\} - 1\big)\cdot pq^k \\ &= \sum_{k=-\infty}^{\infty} \big(\exp\{itsr^k\} - 1 - itsr^k c_k\big)\cdot pq^k \quad \text{with} \quad c_k = 0 \text{ for } k > 0 \text{ and } c_k = 1 \text{ for } k \le 0. \end{aligned}$$

(ii) *If $r > 1/q$, then, for $u > 0$,*

$$\frac{S_{uM}}{N} \overset{d}{\to} Z(u) \quad \text{as} \quad n \to \infty\,,$$

or, equivalently,

$$\frac{1}{(rq)^n} \cdot \frac{S_{uM}}{M} \xrightarrow{d} Z(u) \quad \text{as} \quad n \to \infty\,,$$

where now $Z(u)$ is defined via the characteristic function $\varphi_{Z(u)}(t) = \exp\{ug(t)\}$ with

$$g(t) = \sum_{k=-\infty}^{\infty} \big(\exp\{itsr^k\} - 1\big) \cdot pq^k\,.$$

In complete analogy with [11] we infer that the limit law is infinitely divisible, that the corresponding Lévy measure has point masses pq^k at the points sr^k for $k \in \mathbb{Z}$, and that we are facing a compound Poisson distribution with (two-sided geometric weights).

Proofs of Theorems 2.1 and 2.2, will be given in Sections 4 and 5, respectively.

In addition, by replacing 2^m by $q^{-m} = r^m$ in the proof of [11], Theorem 2, one can show that the limit distribution in Theorem 2.2(i) is *semistable* in the sense of Lévy:

Lemma 2.1 *We have*

$$g(t) = q^m\big(g(t) + itspm\big) \quad \text{for all} \quad m \in \mathbb{Z}.$$

In particular, this illustrates the fact that we do not have a limit distribution for the full sequence (since such a limit would have been stable with index 1).

For the proof of the lemma we refer to the end of Section 5.

3 Preliminaries

In this section we collect some facts that will be used later with or without specific reference.

The following well-known relation holds between logarithms with bases r and u for $y > 0$:

$$\log_r y = \log_u(y) \cdot \log_r(u). \tag{3.1}$$

Lemma 3.1 *For X as defined in Theorem 2.1 we have*

$$E\,X^\beta \quad = \quad \begin{cases} \dfrac{s^\beta p}{1 - r^\beta q}, & \text{for} \quad r < q^{-1/\beta}\,, \\ = \infty, & \text{for} \quad r \geq q^{-1/\beta}\,. \end{cases} \tag{3.2}$$

Moreover, as $x \to \infty$,

$$P(X > x) \quad = \quad q^{[\log_r(x/s)]+1} \geq q^{\log_r(x/s)} \geq \frac{s}{x} \quad \text{as} \quad x \to \infty\,, \tag{3.3}$$

$$E\,XI\{X \leq x\} \quad \sim \quad sp\log_r(x/s) \quad \text{for} \quad r = 1/q\,. \tag{3.4}$$

Proof. Relation (3.2) follows via

$$E\,X^\beta = \sum_{k=1}^{\infty} (sr^{k-1})^\beta pq^{k-1},$$

and the tail estimate is equivalent to formula (1) in [2]. The final inequality there exploits the fact that $rq \geq 1$.

The asymptotics for the truncated first moment follows via

$$E\,XI\{X \leq x\} = \sum_{\{k:sr^{k-1} \leq x\}} sr^{k-1}pq^{k-1} \sim sp \sum_{1 \leq k \leq \log_r(x/s)+1} 1\,.$$

□

4 Proof of Theorem 2.1

Recall that $r = 1/q$. We first observe that the function $x \log_{1/q} x \in \mathcal{RV}(1)$ (that is, regularly varying with exponent 1).

Next, since by (3.3),

$$nP(X > n \log_r n) = n \cdot q^{[\alpha \log_r (sn \log_r n)]+1} \sim n \log_r n / s^{-\log_r (1/q)} \to 0 \quad \text{as} \quad n \to \infty\,,$$

and, by (3.4),

$$E\,XI\{X \leq n \log_r n\} \sim sp \cdot \log_r (sn \log_r nn),$$

so that

$$\frac{n \cdot E\,XI\{X \leq n \log_r n\}}{n \log_r n} \to 0 \quad \text{as} \quad n \to \infty,$$

the conclusion is an immediate consequence of the extension of Feller's weak law of large numbers given in [6], Theorem 1.3; cf. also [9], Theorem 6.4.2. □

5 Proof of Theorem 2.2

Theorem 2.2(i) is proved via a fairly straightforward modification of the corresponding proof in [11].

Proof of (i)

Since, $P(X = sr^{k-1}) = pq^{k-1}$, we have

$$\varphi_X(t) = E\,e^{itX} = \sum_{k=1}^{\infty} e^{itsr^{k-1}} \cdot pq^{k-1},$$

from which it follows that

$$\begin{aligned}
\varphi_{\frac{S_{uN}}{N} - uspn}(t) &= e^{-ituspn}\Big(\sum_{k=0}^{\infty} e^{i\frac{t}{N} sr^k} \cdot pq^k\Big)^{uN} = e^{-ituspn}\Big(\sum_{k=0}^{\infty} e^{itsr^{k-n}} \cdot pq^k\Big)^{uN} \\
&= e^{-ituspn}\Big(1 + \sum_{k=0}^{\infty} \big(e^{itsr^{k-n}} - 1\big) \cdot pq^k\Big)^{uN} \\
&= e^{-ituspn}\Big(1 + q^n \sum_{k=-n}^{\infty} \big(e^{itusr^k} - 1\big) \cdot pq^k\Big)^{uN} \\
&= e^{-ituspn}\Big(1 + \frac{1}{N} \sum_{k=-n}^{\infty} \big(e^{itsr^k} - 1\big) \cdot pq^k\Big)^{uN} \\
&= e^{-ituspn}\Big(1 + \frac{1}{N} \sum_{k=-n}^{-1} \big(e^{itsr^k} - 1 - itsr^k\big) \cdot pq^k \\
&\qquad\qquad + itsp\frac{n}{N} + \frac{1}{N} \sum_{k=0}^{\infty} \big(e^{itsr^k} - 1\big) \cdot pq^k\Big)^{uN}, \\
&= e^{-ituspn}\bigg(1 + \frac{1}{N}\Big\{ \sum_{k=-n}^{-1} \big(e^{itsr^k} - 1 - itsr^k\big) \cdot pq^k \\
&\qquad\qquad + itspn + \sum_{k=0}^{\infty} \big(e^{itsr^k} - 1\big) \cdot pq^k\Big\}\bigg)^{uN},
\end{aligned}$$

which converges to $e^{ug(t)}$ as $n \to \infty$. □

Proof of (ii)

The same computations with obvious modifications yield

$$\begin{aligned}\varphi_{\frac{S_{uM}}{N}}(t) &= \Big(\sum_{k=0}^{\infty} e^{i\frac{t}{N}sr^k}\cdot pq^k\Big)^{uM}\\ &= \Big(1+\sum_{k=0}^{\infty}\big(e^{itsr^{k-n}}-1\big)\cdot pq^k\Big)^{uM}\\ &= \Big(1+q^n\sum_{k=-n}^{\infty}\big(e^{itsr^k}-1\big)\cdot pq^k\Big)^{uM}\\ &= \Big(1+\frac{1}{M}\sum_{k=-n}^{\infty}\big(e^{itsr^k}-1\big)\cdot pq^k\Big)^{uM},\end{aligned}$$

which converges to $e^{ug(t)}$ as $n\to\infty$. □

Proof of Lemma 2.1

Let $m\in\mathbb{Z}$, and remember that $r=1/q$. Then

$$\begin{aligned}g(tq^m) &= \sum_{k=-\infty}^{\infty}\big(e^{itsr^{k-m}}-1-itsr^{k-m}c_k\big)\cdot pq^k\\ &= \sum_{k=-\infty}^{\infty}\big(e^{itsr^k}-1-itsr^kc_{k+m}\big)\cdot pq^{k+m}\\ &= q^m\Big(g(t)+\sum_{k=-\infty}^{\infty} itsr^k(c_k-c_{k+m})\cdot pq^k\Big)\\ &= q^m(g(t)+itspm).\end{aligned}$$

□

6 How much does one gain until "game over"?

This section extends results from [12], where the classical game was treated.

We consider a truncated version of the game in which the duration T_n of a single game is truncated to $T_n\le n$, that is, "game over" happens when $T_n>n$ for the first time. Otherwise she gains one Euro and the game continues. The following result then holds for the total gain during one such game.

Theorem 6.1 *Let G_n be the total gain until game over, and E be a standard exponential random variable.*
(i) *If $rq=1$, then*

$$r^{-n}G_n=q^nG_n\stackrel{d}{\to}qs(E-1)\qquad as\quad n\to\infty.$$

(ii) *If $rq>1$, then*

$$r^{-n}G_n\stackrel{d}{\to}\frac{ps}{r-1}(E-1)\qquad as\quad n\to\infty.$$

Remark 6.1 For $r=2$ (i) turns into $2^nG_n\stackrel{d}{\to}\mathrm{Exp}(1)$, which, if, in addition, $ps=1$ (and, hence, $p=q=1/2$), reduces to Martin-Löf's Theorem 2.1.

Remark 6.2 Note that if we, formally, set $rq=1$ in (ii), then (ii) reduces to (i). □

Proof. Let $N_n=$ the number of rounds until game over. The first observation then is that, since $P(T_n>n)=q^n$, it follows that N_n has a geometric distribution with mean q^{-n}.

The truncated gain is given by

$$\begin{aligned} P(X_n = sr^{k-1}) &= P(T_n = k) = pq^{k-1} \quad \text{for} \quad k \le n \\ P(X_n = 0) &= P(T_n > n) = q^n \end{aligned}$$

Since the fee is psr^{k-1} in round k, the net gain, that is, the true gain − the amount spent, becomes

$$V_n = \begin{cases} sr^{k-1} - p\big(s + sr + \cdots + sr^{k-1}\big) \\ = sr^{k-1} - ps\dfrac{r^k - 1}{r-1} = sr^{k-1}\dfrac{qr-1}{r-1} + \dfrac{ps}{r-1}, & \text{if} \quad T_n = k \le n \\ \\ 0 - p\big(s + sr + \cdots + sr^{n-1}\big) = -ps\dfrac{r^n - 1}{r-1}, & \text{if} \quad T_n > n. \end{cases}$$

It is now easy to check that $E\,V_n = 0$, so the game is fair.

Proof of (i)

Now, suppose that $rq = 1$. Then

$$V_n = \begin{cases} \dfrac{ps}{r-1} = qs, & \text{if} \quad T_n = k \le n \\ -qsq^{-n} + qs = qs(1 - q^{-n}), & \text{if} \quad T_n > n, \end{cases}$$

which tells us that the total gain until "game over" equals

$$G_n = qs \cdot N_n + qs(1 - q^{-n}) = qs \cdot (N_n + 1 - q^{-n})$$

Furthermore, since, as noted above, N_n has a geometric distribution with mean q^{-n}, it is well-known that

$$q^n N_n \stackrel{d}{\to} \text{Exp}(1) \quad \text{as} \quad n \to \infty,$$

from which the conclusion follows.

Proof of (ii)

This case is a bit harder, since G_n now is equal to a sum of $N_n - 1$ i.i.d. random variables corresponding to gains, thus distributed as V_n^+, say, and one final "game over"-variable, distributed as V_n^-, say. All summands are independent of N_n. This thus allows us to resort to the well-known relation for the characteristic function of a sum of a random number of i.i.d. random variables, which in our case amounts to

$$\varphi_{G_n}(t) = g_{N_n - 1}\big(\varphi_{V_n^+}(t)\big) \cdot \varphi_{V_n^-}(t) \tag{6.1}$$

where φ and g denote characteristic and (probability) generation functions, respectively.

As for $N_n - 1$, we have

$$g_{N_n - 1}(t) = \frac{q^n}{1 - (1 - q^n)t}.$$

Furthermore,

$$\begin{aligned} \varphi_{V_n^+}(t) &= \sum_{k=1}^{n} \exp\Big\{it \cdot \Big(\frac{s(qr-1)}{r-1} \cdot r^{k-1} + \frac{ps}{r-1}\Big)\Big\} \cdot \frac{pq^{k-1}}{1 - q^n} \\ &\sim 1 + \sum_{k=1}^{n} it \cdot \frac{s(qr-1)}{r-1} \cdot r^{k-1} \cdot \frac{pq^{k-1}}{1-q^n} + it \cdot \frac{ps}{r-1} \\ &= 1 + it \cdot \frac{s(qr-1)}{r-1} \cdot \frac{p}{1-q^n} \sum_{k=0}^{n-1} (qr)^k + it \cdot \frac{ps}{r-1} \\ &= 1 + it \cdot \frac{ps(qr-1)}{(r-1)(1-q^n)} \cdot \frac{(qr)^n - 1}{qr - 1} + it \cdot \frac{ps}{r-1} \\ &= 1 + it \cdot \frac{psq^n(r^n - 1)}{(r-1)(1-q^n)} \sim 1 + it \cdot \frac{ps}{r-1} \cdot (qr)^n\,, \end{aligned}$$

and

$$\varphi_{V_n^-}(t) = \exp\big\{-itps\cdot\frac{r^n-1}{r-1}\big\}.$$

An application of (6.1) therefore tells us that

$$\begin{aligned}\varphi_{G_n}(t) &\sim \frac{q^n}{1-(1-q^n)\big(1+it\cdot\frac{ps}{r-1}\cdot(qr)^n\big)}\cdot\exp\big\{-itps\cdot\frac{r^n-1}{r-1}\big\}\\ &= \frac{1}{1-it(1-q^n)\frac{ps}{r-1}r^n\big)}\cdot\exp\big\{-itps\cdot\frac{r^n-1}{r-1}\big\},\end{aligned}$$

and, hence, that

$$\varphi_{r^{-n}G_n}(t) \sim \frac{1}{1-it(1-q^n)\frac{ps}{r-1}\big)}\cdot\exp\big\{-itps\cdot\frac{1-r^{-n}}{r-1}\big\} \to \frac{1}{1-it\frac{ps}{r-1}}\cdot e^{-it\frac{ps}{r-1}} \quad \text{as} \quad n\to\infty,$$

which, in view of the continuity theorem for characteristic functions, finishes the proof of (ii). □

7 Capital with interest

Following [12] in this section we assume that the player can borrow money without restriction and that he has to pay interest on the capital with a discount factor $\gamma < 1$ per game. Once again we consider the model (1.2), where now $r = 1/q$, introducing T as the generic duration of a single game, viz.,

$$P(T=k) = P(X = sr^{k-1}) = pq^{k-1}), \quad k = 1,2,\ldots. \tag{7.1}$$

In this case the present value of the gain is

$$E\big(\gamma^T X\big) = \sum_{k=1}^{\infty}\gamma^k sr^{k-1}pq^{k-1} = \frac{sp\gamma}{1-\gamma} < \infty, \tag{7.2}$$

(which reduces to Martin-Löf's $\gamma/(1-\gamma)$ when $p = q = 1/2$ and $ps = 1$).

If an infinite number of games are played they occur att times $T_1, T_2, \ldots$ forming a renewal process with increments $\tau_k = T_k - T_{k-1}$, $k \geq 1$ (with $T_0 = 0$) having the same distribution as T. The present value of the total gain is then given by

$$V(\gamma) = \sum_{k=1}^{\infty}\gamma^{T_k}X_k = \sum_{i=1}^{\infty}\gamma^{T_{i-1}}\gamma^{\tau_k}sr^{\tau_k-1}. \tag{7.3}$$

We now want to find an asymptotic distribution of $V(\gamma)$ when $\gamma \nearrow 1$. As in [11] we scale time by a factor $N = r^n = q^{-n}$ (cf. Theorem 2.2(i)). The renewal process $\{T_k,\, k\geq 1\}$ then has a deterministic limit

$$\frac{T_{uN}}{N} \overset{a.s.}{\to} uET = \frac{u}{p} \quad \text{as} \quad N\to\infty,$$

and

$$\frac{S_{uN}}{N} - uspn = \frac{1}{N}\sum_{k=1}^{uN}(X_k - spn) \overset{a.s.}{\to} Z(u) \quad \text{as} \quad N\to\infty,$$

for fixed $u > 0$, and where $\{Z(u),\, u \geq 0\}$ is the Lévy process defined via the characteristic function

$$\varphi_{Z(u)}(t) = E\Big(e^{itZ(u)}\Big) = e^{ug(t)},$$

where, in turn, $ug(t)$ is the Lévy exponent with

$$\begin{aligned} g(t) &= \sum_{k=-\infty}^{-1}\big(\exp\{itsr^k\} - 1 - itsr^k\big)\cdot pq^k + \sum_{k=0}^{\infty}\big(\exp\{itsr^k\}-1\big)\cdot pq^k\\ &= \sum_{k=-\infty}^{\infty}\big(\exp\{itsr^k\} - 1 - itsr^kc_k\big)\cdot pq^k\,, \end{aligned} \tag{7.4}$$

where $c_k = 0$ for $k \geq 0$, and $c_k = 1$ for $k < 0$.

It follows that

$$\frac{1}{N}V(\gamma) = \frac{1}{N}\sum_{k=1}^{\infty}\gamma^{T_k}(X_k - spn) + \frac{1}{N}\sum_{k=1}^{\infty}\gamma^{T_k}spn,$$

and, setting $\gamma = \exp\{-ap/N\}$, we obtain

$$\frac{1}{N}V(\gamma) - \frac{spn}{N}\sum_{k=1}^{\infty}e^{-apT_k/N} = \frac{1}{N}\sum_{k=1}^{\infty}e^{-apT_k/N}(X_k - spn).$$

Letting $N \to \infty$ yields

$$\frac{1}{N}V(\gamma) - spn\int_0^{\infty}e^{-au}\,du \overset{a.s.}{\to} \int_0^{\infty}e^{-au}\,dZ(u),$$

i.e.,

$$\frac{1}{N}V(\gamma) - \frac{spn}{a} \overset{a.s.}{\to} \int_0^{\infty}e^{-au}\,dZ(u)\,. \tag{7.5}$$

This is interesting because of the following

Lemma 7.1 *The characteristic function of the random variable*

$$U = \int_0^{\infty}e^{-au}\,dZ(u)$$

equals

$$\varphi_U(t) = E\,e^{itU} = e^{g(t)/a},$$

with

$$g(t) = itsq + \sum_{k=-\infty}^{\infty}\int_{sr^{k-1}}^{sr^k}q^k\big(e^{itx} - 1 - itxc_k\big)\frac{dx}{x}.$$

The Lévy measure thus has a density

$$q^k\frac{dx}{x} \quad \text{for} \quad sr^{k-1} < x \leq sr^k \quad \text{and all} \quad k.$$

Proof. Exploiting formula (7.4) we find that

$$Z(u) = \sum_{k=-\infty}^{\infty}sr^kZ_k(u),$$

where $\{Z_k(u)\}$ are independent having characteristic function

$$\varphi_{Z_k}(u) = \exp\{upq^k(e^{it} - 1 - itc_k\}.$$

This tells us that

$$U = \sum_{k=-\infty}^{\infty}sr^kU_k \quad \text{with} \quad U_k = \int_0^{\infty}e^{-au}\,dZ_k(u)\,.$$

Since $\{Z_k(u),\, u \geq 0\}$ has independent increments for all k, this means, via a change of variable, that

$$\begin{aligned}\varphi_{U_k}(t) &= \exp\Big\{\int_0^{\infty}pq^k\big(e^{ite^{-au}} - 1 - ite^{-au}c_k\big)\,du\Big\}\\ &= \exp\Big\{\int_0^1\frac{pq^k}{a}\big(e^{itx} - 1 - itxc_k\}\big)\frac{dx}{x}\Big\},\end{aligned}$$

and, hence, that

$$\begin{aligned}\varphi_{U_k}(itsr^k) &= \exp\Big\{\int_0^{sr^k}\frac{pq^k}{a}\big(e^{itx}-1-itxc_k\}\big)\frac{dx}{x}\Big\}\\ &= \exp\Big\{\sum_{j=-\infty}^{k}\int_{sr^{j-1}}^{sr^j}\frac{pq^k}{a}\big(e^{itx}-1-itxc_k\}\big)\frac{dx}{x}\Big\}.\end{aligned}$$

Summing over k we then obtain

$$\varphi_U(t)=\prod_{k=-\infty}^{\infty}\varphi_{U_k}(itsr^k)=\exp\Big\{\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k}\int_{sr^{j-1}}^{sr^j}\frac{pq^k}{a}\big(e^{itx}-1-itxc_k\big)\frac{dx}{x}\Big\}. \tag{7.6}$$

Now, for fixed $j\geq 0$ we have $\sum_{k=j}^{\infty}q^k=q^j/p$, so that

$$\sum_{k=j}^{\infty}c_kq^k=\begin{cases}0=c_j, & \text{for}\quad j\geq 0,\\ \sum_{k=j}^{-1}c_kq^k=\frac{1}{q}\cdot\frac{(1/q)^{|j|}-1}{(1/q)-1}=\frac{q^j-1}{p}\cdot c_j, & \text{for}\quad j<0.\end{cases}$$

Inserting this into (7.6), and changing the order of summation, finally shows that

$$\begin{aligned}\varphi_U(t) &= \exp\Big\{\sum_{j=-\infty}^{\infty}\int_{sr^{j-1}}^{sr^j}\Big(\frac{q^j}{a}\big(e^{itx}-1\big)-\frac{itxc_j}{a}(q^j-1)\Big)\frac{dx}{x}\Big\}\\ &= \exp\Big\{\sum_{j=-\infty}^{\infty}\int_{sr^{j-1}}^{sr^j}\Big(\frac{q^j}{a}\big(e^{itx}-1-itxc_j\big)\frac{dx}{x}+\frac{itc_j}{a}\,dx\Big)\Big\}\\ &= \exp\Big\{\sum_{j=-\infty}^{\infty}\int_{sr^{j-1}}^{sr^j}\Big(\frac{q^j}{a}\big(e^{itx}-1-itxc_j\big)\frac{dx}{x}+\sum_{j=-\infty}^{-1}\big(sr^j-sr^{j-1}\big)\frac{it}{a}\Big)\Big\}\\ &= \exp\Big\{\sum_{j=-\infty}^{\infty}\int_{sr^{j-1}}^{sr^j}\Big(\frac{q^j}{a}\big(e^{itx}-1-itxc_j\big)\frac{dx}{x}+\frac{itqs}{a}\Big\}\\ &= e^{g(t)/a}\,.\end{aligned}$$

□

7.1 U is semistable

Next we prove an analog of Lemma 2.1, to the effect that the distribution of U is semistable.

Lemma 7.2 *For any integer m we have*

$$g(tq^m)=q^m\big(g(t)+itsmp\big)$$

PROOF. We first observe that

$$\begin{aligned} g(tq^m) &= (itsq)q^m + \sum_{k=-\infty}^{\infty} \int_{sr^{k-1}}^{sr^k} q^k \big(e^{itq^m x} - 1 - itq^m x c_k\big) \frac{dx}{x} \\ &= (itsq)q^m + \sum_{k=-\infty}^{\infty} \int_{sr^{k-m-1}}^{sr^{k-m}} q^k \big(e^{itx} - 1 - itxc_k\big) \frac{dx}{x} \\ &= (itsq)q^m + \sum_{k=-\infty}^{\infty} \int_{sr^{k-1}}^{sr^k} q^{k+m} \big(e^{itx} - 1 - itxc_{k+m}\big) \frac{dx}{x} \\ &= q^m \Big(g(t) + \sum_{k=-\infty}^{\infty} \int_{sr^{k-1}}^{sr^k} q^k itx(c_k - c_{k+m}) \frac{dx}{x}\Big) \\ &= q^m \Big(g(t) + \sum_{k=-\infty}^{\infty} (sr^k - sr^{k-1}) q^k it(c_k - c_{k+m})\Big) \\ &= q^m \Big(g(t) + \sum_{k=-\infty}^{\infty} itsp(c_k - c_{k+m})\Big) \\ &= q^m (g(t) + itspm). \end{aligned}$$

□

7.2 The tail of U

Our next step is to exploit the semistability for an estimate for the tail of the distribution of U.

Toward that end, set $\bar{U}_m = q^m(U - pms/a)$ and let φ_m be the characteristic function of U_m, viz.,

$$\begin{aligned} \varphi_m(t) &= e^{-itq^m pms/a} \varphi_U(q^m t) = \exp\big\{ - itq^m pms/a + q^m g(t)/a\big\} \\ &= \exp\big\{q^m g(t)/a\big\} \approx 1 + q^m g(t)/a \quad \text{for } m \text{ large}, \end{aligned}$$

from which we conclude that

$$q^{-m}(\varphi_m(t) - 1) \to g(t)/a \quad \text{as} \quad m \to \infty. \tag{7.7}$$

Now, the LHS equals the Lévy exponent corresponding to the Lévy measure $L_m(dx) = q^{-m} \cdot P(\bar{U}_m \in dx)$ and the RHS has Lévy measure $L(dx)$. Using the continuity theorem for Lévy exponents, cf. [5], Chapter XVII.2, Theorem 2, we thus conclude that

$$q^{-m} P(U_m > x) \to \int_x^\infty L(dy) = \bar{L}(x) \quad \text{as} \quad m \to \infty; \quad (x > 0), \tag{7.8}$$

and, hence, that

$$q^{-m} P(U > xq^{-m} + pms/a) \to \bar{L}(x) \quad \text{as} \quad m \to \infty; \quad (x > 0). \tag{7.9}$$

From Lemma 7.1 we remember that L has density

$$q^k \frac{dx}{x} \quad \text{for} \quad sr^{k-1} < x \le sr^k \quad \text{and all} \quad k,$$

from which we infer that

$$\begin{aligned} \bar{L}(xr^k) &= \frac{q^k}{a} \int_{xr^k}^{sr^k} \frac{dx}{x} + \sum_{j=k+1}^{\infty} \frac{q^j}{a} \int_{sr^{j-1}}^{sr^j} \frac{dx}{x} \\ &= \frac{q^k}{a} \log(s/x) + \sum_{j=k+1}^{\infty} \frac{q^j}{a} \log r \\ &= \frac{q^k}{a} \log(s/x) + \frac{q^{k+1}}{pa} \log r \\ &= \frac{q^k}{a} \big(\frac{q}{p} \log r - \log(x/s)\big) \quad \text{for} \quad qs < x < s. \end{aligned} \tag{7.10}$$

7.3 An infinite number of games

Let us now see how this can be used to analyze an infinite number of St. Petersburg games.

Consider first a single game and put, for simplicity, $s = r = 1/q$, so that $X = r^T$. The fee for playing round k in one game then is γpr^k (= the present value of the stake prior to round k). The present value of the net gain at the beginning of the game then, recalling that $rq = 1$, becomes

$$\begin{aligned}
&\gamma^T r^T - \gamma p \cdot \big(r + \gamma r^2 + \gamma^2 r^3 + \cdots + \gamma^{T-1} r^T\big) \\
&= (\gamma r)^T - \gamma pr \sum_{j=0}^{T-1} (\gamma r)^j\big) \\
&= (\gamma r)^T - \gamma pr \cdot \frac{(\gamma r)^T - 1}{\gamma r - 1} \\
&= (\gamma r)^T \cdot \Big(1 - \frac{\gamma pr}{\gamma r - 1}\Big) + \frac{\gamma pr}{\gamma r - 1} \\
&= \frac{\gamma pr}{\gamma r - 1} - \frac{1-\gamma}{\gamma r - 1} \cdot (\gamma r)^T.
\end{aligned}$$

Since T has a geometric distribution with mean $1/p$ we conclude that the expected value of this quantity eqals

$$\frac{\gamma pr}{\gamma r - 1} - \frac{1-\gamma}{\gamma r - 1} \cdot E(\gamma r)^T = \frac{\gamma pr}{\gamma r - 1} - \frac{1-\gamma}{\gamma r - 1} \cdot \frac{p\gamma r}{1-\gamma} = 0\,,$$

which tells us that the game is fair.

In analogy with (7.3) we thus conclude that the present value of the total gain equals

$$\tilde{V} = \sum_{k=1}^{\infty} \gamma^{T_{k-1}} \cdot \Big(\frac{\gamma pr}{\gamma r - 1} - \frac{1-\gamma}{\gamma r - 1} \cdot (\gamma r)^{\tau_k}\Big).$$

The asymptotic expansion with $\gamma = e^{-ap/N}$ and $N = r^n$, so that $1 - \gamma \sim ap/n$, $\gamma r - 1 \sim r - 1 = (1/q) - 1 = p/q = pr$, then turns this into

$$\tilde{V} \approx \sum_{k=1}^{\infty} \gamma^{T_i} - \frac{aq}{N} V(\gamma) \approx N \int_0^{\infty} e^{-au}\,du - aqU = \frac{N}{a} - aq\big(U + \frac{spn}{a}\big)\,.$$

This tells us that, neglecting spn/a, the ruin probability $P(\tilde{V} < 0)$ can be approximated by $P(U > N/(qa^2))$.

Finally, by exploiting formulas (7.9) and (7.10) concerning the tail of the distribution of U, recalling that $s = r$, we obtain the following approximation for this probability:

$$\begin{aligned}
P(\tilde{V} < 0) &\approx P\Big(U > \frac{N}{qa^2}\Big) \approx P\Big(U > \frac{N}{qa^2} + \frac{\log N}{r}\Big) \\
&\approx \frac{1}{N}\bar{L}(x) \approx \frac{1}{aN}\Big(\frac{q}{p}\log r - \log(x/r)\Big) \\
&= \frac{1}{aN}\Big(\frac{1}{p}\log r - \log x\Big) \quad \text{for} \quad x = \frac{1}{qa^2} \text{ and } 1 < x < r. \qquad (7.11)
\end{aligned}$$

For the special case $p = q = 1/2$ the result reduces to that of Martin-Löf, [12].

8 Some remarks

We close with some additional results and comments.

8.1 Polynomial and geometric size deviations

In this subsection we provide immediate extensions of the results from [7], Section 7 (cf. also [8]), which, in turn were inspired by [10] and [13] respectively.

Theorem 2.1 and Corollary 2.3 of Hu and Nyrhinen [10] adapted to the present setting yield the following result.

Theorem 8.1 *For any $b > 1$,*

$$\lim_{n\to\infty} \frac{\log_r P(S_n > n^b)}{\log_r n} = 1 - b\,,$$
$$\lim_{n\to\infty} \frac{\log_r P(M_n > n^b)}{\log_r n} = 1 - b\,.$$

PROOF. The only thing to check is, in the notation of [10], that $\overline{\alpha} = \underline{\alpha}$ (with $\alpha = 1$) in formulas (5) and (6) there, and this is immediate, since, by (3.3),

$$P(\log X > x) = P(X > e^x) \to -1 \quad \text{as} \quad x \to \infty.$$ □

As for geometric size deviations, we have

Theorem 8.2 *For any $\varepsilon > 0$ and $b > 1$,*

$$\lim_{n\to\infty} \frac{\log_r P(X > \varepsilon b^n)}{\log_r (\varepsilon b^n)} = -1\,, \tag{8.1}$$
$$\lim_{n\to\infty} \frac{\log_r P(S_n > \varepsilon b^n)}{n} = -\log_r b\,, \tag{8.2}$$
$$\lim_{n\to\infty} \frac{\log_r P(M_n > \varepsilon b^n)}{n} = -\log_r b\,. \tag{8.3}$$

In particular, for $b = r$ the limits in (8.2) and (8.3) equal -1.

PROOF. Relation (8.1) is an immediate consequence of (3.3), and for (8.3) we exploit [9], Lemma 4.2, to conclude that

$$\frac{1}{2} nP\big(X > \varepsilon b^n\big) \le P\big(M_n > \varepsilon b^n\big) \le nP\big(X > \varepsilon b^n\big) \quad \text{for } n \text{ large,} \tag{8.4}$$

cf. (cf. [9], p. 270), after which the remaining details are the same as in [7, 8]. □

8.2 Almost sure convergence?

In this subsection we discuss the possibility of having almost sure convergence in Theorem 2.1. Recall that $rq = 1$ here.

Since $E\,X = +\infty$, the converse of the Kolmogorov strong law tells us that this cannot be the case. However, more can be said. Namely, since, by (3.3),

$$\sum_{n=1}^{\infty} P(X > cn\log_r n) \ge \sum_{n=1}^{\infty} \frac{s}{cn\log_r n} = \infty \quad \text{for any} \quad c > 0,$$

the first Borel–Cantelli lemma tells us that

$$P(X_n > cn\log_r n \text{ i.o.}) = 1 \quad \text{for any} \quad c > 0,$$

and, hence, since $S_n \ge X_n$ for all $n \ge 1$, that

$$\limsup_{n\to\infty} \frac{S_n}{n\log_r n} = +\infty. \tag{8.5}$$

As for the limit inferior, following [1] we set $\mu(x) = \int_0^x P(X > y)\,dy$, and note, via partial integration and Lemma 3.1, that

$$\mu(x) = xP(X > x) + \int_0^x y dF_X(y) = q^{[\log_r (x/s)]+1} + sp\log_r (x/s) \sim sp\log_r (x/s) \quad \text{as} \quad x \to \infty,$$

from which we, via (3.1), also conlcude that $\mu(x) \sim \mu(x\log_2 x)$ as $x \to \infty$. An application of [1], Theorem 2 (with $\alpha = 0$ and $b_n = n\log_r n$)), therefore tells us that

$$\liminf_{n\to\infty} \frac{S_n}{n\log_r n} = ps.$$

For the case $ps = 1$ the conclusion obviously reduces to Example 4 of [1].

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Allan Gut, Department of Mathematics, Uppsala University, Box 480, SE-751 06 Uppsala, Sweden;
Email: `allan.gut@math.uu.se` URL: `http://www.math.uu.se/~allan`

Anders Martin-Löf, Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden;
Email: `andersml@math.su.se` URL: `http://www2.math.su.se/~andersml`