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Abstract

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Key words: American type option, Markov log-price process, Geometric random walk, Gaussian random walk, Compound Gaussian random walk, Reward function, Binomial approximation, Trinomial approximation, Skeleton approximation, Convergence of rewards, Rate of convergence, MATLAB implementation

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1. Introduction

The standard binomial approximation was introduced by Cox, Ross, and Rubinstein (1979) for European options and has won widespread acceptance by its simplicity and efficiency.

As well known, closed-form formulas for rewards of American type options exist only in some special cases. We refer to works by Kim (1990), Dayanik and Karatzas (2003), Detemple (2005), Peskir and Shiryaev (2006), Zhang and Li (2010), Zhao and Wong (2012), where one can also find further references.

Approximation methods are usually used by researchers and practitioners for computing rewards and prices for American type options. These methods can mainly be subdivided into the categories of stochastic approximation lattice methods, integro-differential approximations and Monte Carlo based approximation methods. We refer here to books, surveys and papers containing results of comparison studies for American type contingent claims such as Broadie and Detemple (1996, 2004), Glasserman (2003), Higham (2004), Achdou and Pironneau (2005), Detemple (2005), Jiang (2005), Bender, Kolodko, and Schoenmakers (2006), Pressacco, Gaudenzi, Zanette, and Ziani (2008), Ahn, Bae, Koo, and Lee (2011), Pascucci (2011), and Zhu (2011).

The subject of the present paper relates to stochastic approximation lattice methods for American type options.

Results of convergence studies for reward approximations for American options can be found in Amin and Khanna (1994), Broadie and Detemple (1996), Leisen and Reimer (1996), Cutland, Kopp, Willinger, and Wyman (1997), Lamberton (1998), Leisen (1998), Mulinacci and Pratelli (1998), Jiang and Dai (1999, 2004), Prigent (2003), Nieuwenhuis and Vellekoop (2004), Dupuis and Wang (2005), Jönsson (2005), Qian, Xu, Jiang, and Bian (2005), Maller, Solomon, and Szimayer (2006), Silvestrov, Jönsson, and Stenberg (2006, 2008, 2009), Stenberg (2006), Coquet and Toldo (2007), Lundgren, Silvestrov, and Kukush (2008), Joshi (2009), Vellekoop and Nieuwenhuis (2009), Lundgren (2010), Lundgren and Silvestrov (2011), Li and Xing (2011), Silvetsrov and Lundgren (2011), and Zhang and Wang (2011).

Some results on rate of convergence for reward approximations for European and American options are given in Broadie and Detemple (1996), Leisen and Reimer (1996), Lamberton (1998), Leisen (1998), Jiang and Dai (1999), Heston and Zhou (2000), Kukush and Silvestrov (2001), Liang, Hu, Jiang, and Bian (2007), Liang (2008), Liang, Hu, and Jiang (2010), Dolinsky (2011), and Kwon and Lee (2011).

This paper continues the line of research realised in Jönsson, Kukush, and

Silvestrov (2002, 2004, 2005), Jönsson (2005), Silvestrov, Jönsson, and Stenberg (2006, 2008, 2009), Stenberg (2006), Lundgren, Silvestrov, and Kukush (2008), Silvestrov, Jönsson, and Stenberg (2008, 2009), Lundgren and Silvestrov (2009, 2010), Lundgren (2010), Silvestrov and Lundgren (2011), and Silvestrov (2013).

We present some new results on convergence of lattice approximations for rewards for American type options with general possibly discontinuous pay-off functions for log-price processes represented by inhomogeneous in time random walks, as well as results of experimental studies of rates of convergence for the corresponding approximation algorithms.

2. Approximation and Convergence for American Type Options

Let's consider a family of log-price processes, which depend on some perturbation parameter $\varepsilon \geq 0$ and are defined by the following stochastic transition dynamic relation,

$$Y_{\varepsilon,n+1} = Y_{\varepsilon,n} + W_{\varepsilon,n+1}, \ n = 0, 1, \dots, \tag{1}$$

where: (a) $W_{\varepsilon,n}, \ n = 1, 2, \dots$ is a sequence of real-valued, independent random variables, and (b) $Y_{\varepsilon,0} = y_0$ is a real-valued constant.

We do prefer to operate with the log-price processes, which have simpler additive structure of increments, in comparison with the corresponding price processes $X_{\varepsilon,n} = e^{Y_{\varepsilon,n}}, n = 0, 1, \dots$, which have more complicated multiplicative structure of increments.

Let $\mathcal{F}_{\varepsilon,n} = \sigma[Y_{\varepsilon,0}, \dots, Y_{\varepsilon,n}], \ n = 0, 1, \dots$ be a natural filtration generated by the log-price process $Y_{\varepsilon,n}$.

Let us also denote by $\mathcal{M}_{\varepsilon,n,N}$ the class of all Markov stopping times $\tau_{\varepsilon,n}$ for the process $Y_{\varepsilon,n}$ such that $n \leq \tau_{\varepsilon,n} \leq N$.

We also introduce a pay-off function $g(n, y)$, which is a real-valued Borel measurable function defined for $(n, y) \in \mathbb{N} \times \mathbb{R}$, where $\mathbb{N} = \{0, 1, \dots\}$.

An American type option is a contract, in which an option holder has the right, but not the obligation, to execute the contract at any stopping time $\tau_{\varepsilon,0} \in \mathcal{M}_{\varepsilon,0,N}$ and to receive, in this case, the pay-off $g(\tau_{\varepsilon,0}, Y_{\varepsilon,\tau_{\varepsilon,0}})$. The parameter N is called the maturity of the option.

One of the goals for an option holder is to find so called reward functions $\phi_{\varepsilon,n}(y)$ for the option contract defined by the following relation, for $n = 0, \dots, N$,

$$\phi_{\varepsilon,n}(y) = \sup_{\tau_{\varepsilon,n} \in \mathcal{M}_{\varepsilon,n,N}} \mathsf{E}_{y,n} g(\tau_{\varepsilon,n}, Y_{\varepsilon,\tau_{\varepsilon,n}}), \ y \in \mathbb{R}. \tag{2}$$

Here and henceforth, $\mathsf{P}_{y,n}$ and $\mathsf{E}_{y,n}$ denote, respectively, conditional probability and expectation under condition $Y_{\varepsilon,n} = y$.

Let us assume that the following condition holds for the log-price processes, for some $\beta \geq 0$:

A$[\beta]$: $\overline{\lim}_{\varepsilon \to 0} \max_{1 \leq n \leq N} \mathsf{E} e^{\pm \beta W_{\varepsilon,n}} < K$, for some $1 < K < \infty$.

Condition **A**$[\beta]$ guarantees that there exists $\varepsilon_0 > 0$ such that the expression under sign of $\overline{\lim}$ is less than K, for every $\varepsilon \in (0, \varepsilon_0]$.

We also impose the following condition on the pay-off function $g(n, y)$, which is assumed to hold for some $\gamma \geq 0$:

B$[\gamma]$: $\max_{0 \leq n \leq N} \sup_{y \in \mathbb{R}} \frac{|g(n,y)|}{1 + L'' e^{\gamma |y|}} < L'$, where $0 < L' < \infty$ and $0 \leq L'' < \infty$.

Standard examples of pay-off functions for call and put option contracts are, respectively, $g(n, y) = e^{-rn} \max(0, e^y - S) = e^{-rn}[e^y - S]_+$ and $g(n, y) = e^{-rn} \max(0, S - e^y) = e^{-rn}[S - e^y]_+$. Here, $S, r > 0$ are positive constants, which are a strike price and a risk-free interest rate respectively.

Condition **B**$[\gamma]$ means that we study options with pay-off functions which have not more than polynomial rate of growth in argument of $e^{|y|}$. For example, in the case of the standard call option, condition **B**$[1]$ holds. In the case of the standard put option contract, the pay-off function is bounded in y and condition **B**$[0]$ holds.

The following theorem guarantees, under the above conditions, that the reward functions defined in relation (2) are finite for any ε small enough.

Theorem 1. *Let conditions* **A**$[\beta]$ *and* **B**$[\gamma]$ *hold for some parameters* $0 \leq \gamma \leq \beta < \infty$. *Then there exist constants* $0 \leq M', M'' < \infty$ *such that the following inequalities hold for* $\varepsilon \in [0, \varepsilon_0]$ *and any* $y \in \mathbb{R}, n = 0, \ldots, N$,

$$\begin{aligned} |\phi_{\varepsilon,n}(y)| &= \sup_{\tau_{\varepsilon,n} \in \mathcal{M}_{\varepsilon,n,N}} \mathsf{E}_{y,n} |g(\tau_{\varepsilon,n}, Y_{\tau_{\varepsilon,n}})| \\ &\leq \mathsf{E}_{y,n} \max_{n \leq r \leq N} |g(r, Y_{\varepsilon,r})| \leq M' + M'' e^{\gamma |y|}. \end{aligned} \tag{3}$$

The above theorem is a slight modification of the corresponding result given in Silvestrov, Jönsson, and Stenberg (2009, 2010) and Silvestrov (2013), where one can find the explicit expressions for constants M', M''.

Let us now impose the following condition of convergence in distribution for jumps of log-price processes:

C: $W_{\varepsilon,n} \xrightarrow{d} W_{0,n}$ as $\varepsilon \to 0$, for $n = 1, \ldots, N$.

We do not assume that the pay-off function $g(n, y)$ is continuous in the argument y. Let us denote by $\mathbb{Y}_{g,n}$ the set of continuity for function $g(n, y)$ in y, for every $n = 0, 1, \ldots$.

Finally, let us assume that the following condition holds:

D: $\mathsf{P}\{y + W_{0,n} \in \overline{\mathbb{Y}}_{g,n}\} = 0$, for $y \in \mathbb{R}, n = 1, \ldots, N$.

Note that in the case, where function $g(n, y)$ is continuous in y for every $n = 1, \ldots, N$, and, thus, sets $\overline{\mathbb{Y}}_{g,n} = \emptyset, n = 1, \ldots, N$, condition **D** automatically holds. Also, if the sets $\overline{\mathbb{Y}}_{g,n}, n = 1, \ldots, N$ are at most countable, condition **D** holds if the distributions of random variables $W_{0,n}, n = 1, \ldots, N$ have no atoms. Finally, if $L(\overline{\mathbb{Y}}_{g,n}) = 0, n = 1, \ldots, N$, where $L(A)$ is the Lebesgue measure on real line, then condition **D** holds if distributions of random variables $W_{0,n}, n = 1, \ldots, N$ are absolutely continuous with respect to the Lebesgue measure.

Note that condition **D** admits discontinuous and very irregular pay-off functions.

The following theorem is a direct corollary of results given in Silvestrov, Jönsson, and Stenberg (2009, 2010), Lundgren and Silvestrov (2009, 2010), Silvestrov and Lundgren (2011), and Silvestrov (2013).

Theorem 2. *Let conditions* $\mathbf{A}[\beta]$ *and* $\mathbf{B}[\gamma]$ *hold for some parameters* $0 < \gamma < \beta < \infty$ *or* $\gamma = \beta = 0$*, and also conditions* **C** *and* **D** *hold. Then, the following relation holds for any* $y \in \mathbb{Y}_{g,n},\ n = 0, \ldots, N$*,*

$$\phi_{\varepsilon,n}(y) \to \phi_{0,n}(y) \text{ as } \varepsilon \to 0. \tag{4}$$

It is worth to point out a natural generalisation of Theorem 2 for the case, where the initial state of the log-price processes $Y_{\varepsilon,0}$ is a random variable independent of jump sequence $W_{\varepsilon,n}, n = 1, 2, \ldots$. In such case, conditions of Theorem 2, plus analogous to $\mathbf{A}[\beta]$ and **C** conditions imposed on random variables $Y_{\varepsilon,0}$, plus condition $\mathbf{D}_0$: $\mathsf{P}\{Y_{0,n} \in \overline{\mathbb{Y}}_{g,0}\} = 0$, imply the following convergence relation, $\phi_\varepsilon = \mathsf{E}\phi_{\varepsilon,n}(Y_{\varepsilon,0}) \to \phi_0 = \mathsf{E}\phi_{0,n}(Y_{0,0})$ as $\varepsilon \to 0$.

3. Lattice Approximations for American Type Options

Let's assume now that the following condition holds:

E: $W_{\varepsilon,n}, n = 1, 2, \ldots$ are, for every $\varepsilon \in (0, \varepsilon_0]$, independent discrete random variables taking, respectively, values $l\delta_\varepsilon,\ l = -r_{\varepsilon,n}, -r_{\varepsilon,n} + 1, \ldots, r_{\varepsilon,n}$, for $n = 1, 2, \ldots$, where δ_ε are positive real numbers and $r_{\varepsilon,n}, n = 1, 2, \ldots$ are positive integer numbers.

In this case, the conditional distribution of the random variable $Y_{\varepsilon,m}$, under condition $Y_{\varepsilon,n} = y$, is concentrated in points $y + l\delta_\varepsilon,\ l = -r_{\varepsilon,n,m}, -r_{\varepsilon,n,m} + 1, \ldots, r_{\varepsilon,n,m}$, for every $0 \le n \le m \le N$, where $r_{\varepsilon,n,m} = \sum_{k=n+1}^{m} r_{\varepsilon,k},\ 0 \le n \le m \le N$.

Using condition **E**, we get

$$\begin{aligned}
|\phi_{\varepsilon,n}(y)| &\leq \sup_{\tau_{\varepsilon,n} \in \mathcal{M}_{\varepsilon,n,N}} \mathsf{E}_{y,n} |g(\tau_{\varepsilon,n}, Y_{\tau_{\varepsilon,n}})| \\
&\leq \mathsf{E}_{y,n} \max_{n \leq m \leq N} |g(m, Y_{\varepsilon,m})| \\
&= \sum_{m=n}^{N} \sum_{l_m = -r_{\varepsilon,n,m}}^{r_{\varepsilon,n,m}} \max_{n \leq m \leq N} |g(m, y + l_m \delta_\varepsilon)| \\
&\quad \times \mathsf{P}\{Y_{\varepsilon,m} = y + l_m \delta_\varepsilon, n \leq m \leq N\} < \infty. \qquad (5)
\end{aligned}$$

In this case, the reward functions $\phi_{\varepsilon,n}(y)$ take finite values without any additional assumptions.

The following theorem is a variant of the corresponding results from Chow, Robbins and Siegmund (1971) and Shiryaev (1976).

Theorem 3. *Let log-price processes $Y_{\varepsilon,n}$ satisfy condition* **E**. *Then, the following recurrence backward relations hold, for every $y \in \mathbb{R}, 0 \leq n \leq N$ and $\varepsilon \in (0, \varepsilon_0]$,*

$$\begin{cases}
\phi_{\varepsilon,N}(y + l\delta_\varepsilon) = g(N, y + l\delta_\varepsilon), \ l = -r_{\varepsilon,n,N}, \ldots, r_{\varepsilon,n,N}, \\
\phi_{\varepsilon,m}(y + l\delta_\varepsilon) = \max\Big(g(m, y + l\delta_\varepsilon), \\
\qquad \sum_{k=-r_{\varepsilon,m+1}}^{r_{\varepsilon,m+1}} \phi_{\varepsilon,m+1}(y + l\delta_\varepsilon + k\delta_\varepsilon)\mathsf{P}\{W_{\varepsilon,m+1} = k\delta_\varepsilon\}\Big), \\
l = -r_{\varepsilon,n,m}, \ldots, r_{\varepsilon,n,m}, \ \text{where } m = N-1, \ldots, n.
\end{cases} \qquad (6)$$

Note that we assume that parameter δ_ε representing the step of the greed, where the distributions of random jumps $W_{\varepsilon,n}$ are concentrated, does not depend on n. This implies that the so-called recombining condition holds for process $Y_{\varepsilon,n}$. Due to this condition, if $Y_{\varepsilon,n} = y$ then, for every $0 \leq n \leq m < \infty$, the random variable $Y_{\varepsilon,m}$ take values $y + l\delta_\varepsilon, l = -r_{\varepsilon,n,m}, \ldots, r_{\varepsilon,n,m}$.

If, for example, parameters $r_{\varepsilon,n} \leq r_\varepsilon$, then the random variable $Y_{\varepsilon,m}$ has the number of possible values $L_{\varepsilon,n,m} \leq 2mr_\varepsilon + 1$, with not more than linear rate of growth in m.

If the recombining condition would not hold (that could be the case, if parameter δ_e would depend on n), then the random variable $Y_{\varepsilon,m}$ could possess a very large number of possible values, up to the extreme one, $L_{\varepsilon,n,m} = \prod_{k=n+1}^{m}(2r_{\varepsilon,k} + 1)$. In this case, $L_{\varepsilon,n,m} \geq e^{m \ln 3}$ would have the exponential rate of growth in m.

In the recombining case, the above reward backward recurrence algorithm is computationally very effective even for very large values of parameter N. In the non-recombining case, the algorithm is computationally not effective even for moderate values of parameter N.

We would also like to mention some transformations that let one to simplify the above backward algorithm presented in Theorem 3. These are elimination of the deterministic trend and standardisation of the initial value for the log-price process $Y_{\varepsilon,n}$.

The process $Y_{\varepsilon,n}$ can be represented in the following form $Y_{\varepsilon,n} = y_{\varepsilon,n} + \tilde{Y}_{\varepsilon,n}, n = 0, 1, \ldots$, where (a) $y_{\varepsilon,n} = y_0 + w_{\varepsilon,1} + \cdots + w_{\varepsilon,n}, n = 0, 1, \ldots$ is a non-random trend function; (b) $\tilde{Y}_{\varepsilon,n+1} = \tilde{Y}_{\varepsilon,n} + \tilde{W}_{\varepsilon,n+1}, n = 0, 1, \ldots$ is a random walk with the initial value $\tilde{Y}_{\varepsilon,0} = 0$ and independent centred random jumps $\tilde{W}_{\varepsilon,n} = W_{\varepsilon,n} - w_{\varepsilon,n}, n = 1, 2, \ldots$.

It is obvious that both price processes $Y_{\varepsilon,n}$ and $\tilde{Y}_{\varepsilon,n}$ generate the same natural filtration $F_{\varepsilon,n} = \sigma[Y_{\varepsilon,n}, \ldots, Y_{\varepsilon,N}] = \sigma[\tilde{Y}_{\varepsilon,n}, \ldots, \tilde{Y}_{\varepsilon,N}],\ n = 0, 1, \ldots$ and, therefore, they have the same classes of stopping times $\mathcal{M}_{\varepsilon,n,N}, 0 \leq n \leq N$.

In this case, one can use the following transformation formula for the reward functions $\phi_{\varepsilon,n}(y), y \in \mathbb{R}, n = 0, 1, \ldots, N$,

$$\begin{aligned} \phi_{\varepsilon,n}(y) &= \sup_{\tau_{\varepsilon,n} \in \mathcal{M}_{\varepsilon,n,N}} \mathsf{E}_{y,n} g(\tau_{\varepsilon,n}, Y_{\varepsilon,\tau_{\varepsilon,n}}) \\ &= \sup_{\tau_{\varepsilon,n} \in \mathcal{M}_{\varepsilon,n,N}} \mathsf{E}_{y,n} g(\tau_{\varepsilon,n}, y_{\varepsilon,n} + \tilde{Y}_{\varepsilon,\tau_{\varepsilon,n}}) \\ &= \tilde{\phi}_{\varepsilon,n}(y) = \sup_{\tau_{\varepsilon,n} \in \mathcal{M}_{\varepsilon,n,N}} \mathsf{E}_{y,n} \tilde{g}(\tau_{\varepsilon,n}, \tilde{Y}_{\varepsilon,\tau_{\varepsilon,n}}). \end{aligned} \tag{7}$$

where $\tilde{g}_\varepsilon(n, y)$ is a new pay-off function defined for $y \in \mathbb{R}, n = 0, 1, \ldots$ by the following formula,

$$\tilde{g}_\varepsilon(n, y) = g(n, y_{\varepsilon,n} + y). \tag{8}$$

Now, the backward algorithm given in Theorem 3 can be applied. In this case, the log-price processes $Y_{\varepsilon,n}, n = 0, 1, \ldots$ should be replaced by the log-price process $\tilde{Y}_{\varepsilon,n}$ and the pay-off function $g(n, y)$ should be replaced by the transformed pay-off function $\tilde{g}_\varepsilon(n, y)$. According relation (7) the corresponding reward functions $\phi_{\varepsilon,n}(y)$ are invariant with respect to the above transformation. We use the above elimination transformation in what follows.

4. A basic approximation algorithm

The following theorem is a corollary of Theorem 2.

Theorem 4. *Let conditions* $\mathbf{A}[\beta]$ *and* $\mathbf{B}[\gamma]$ *hold for parameters* $0 < \gamma < \beta < \infty$ *or* $\gamma = \beta = 0$, *and also conditions* **C**, **D**, *and* **E** *hold. Then, the following relation holds for any* $y \in \mathbb{Y}_{n,g}$ $n = 0, \ldots, N$,

$$\phi_{\varepsilon,n}(y) \to \phi_{0,n}(y) \text{ as } \varepsilon \to 0. \tag{9}$$

It is worth to note that condition **C** usually imposes some natural additional conditions on parameters penetrating condition **E**: (a) $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, and (b) $\delta_\varepsilon r_{\varepsilon,n} \to \infty$ as $\varepsilon \to 0$, for $n = 1, 2, \ldots$.

The following two steps approximation algorithm for computing the reward functions $\phi_{0,n}(y)$ can be designed by using Theorems 3 and 4.

At the first step, based on application of Theorem 3, one should compute the reward function $\phi_{\varepsilon,n}(y)$ at points $y + l\delta_\varepsilon, l = -r_{\varepsilon,n,m}, \ldots, r_{\varepsilon,n,m}, n \leq m \leq N$ using backward recurrence algorithm given in Theorem 3.

At the second step, based on Theorem 4, one should repeat computing of values $\phi_{\varepsilon_k,n}(y)$ for sequential values $\varepsilon_1 > \cdots > \varepsilon_k \to 0$ as $k \to \infty$, until the values $\phi_{\varepsilon_k,n}(y)$ become stabilised with appropriate small relative errors $\Delta_k(n, y) = |\frac{\phi_{\varepsilon_k,n}(y) - \phi_{\varepsilon_{k-1},n}(y)}{\phi_{\varepsilon_k,n}(y)}|$ for reasonably long subsequence of values for parameter k.

It would be nice to have upper bounds for rates of convergence in the asymptotical relation (9). In principle, such bound in the form of $\mathrm{O}(\cdot)$ can be obtained. In practice, such bounds would have rather psychological than a real value. The bounds with explicit constants would be required for the practical use. However, the reward functions have very nonlinear character. It should be expected to be difficult to get values for such constants admissible for the practical use. Thus, one should accept the using of usual engineering approach described above, in the use of convergence relations of type (9).

We would like also to point out that backward computational algorithms let one also investigate approximative structure of optimal stopping time-space domains. Indeed, the backward algorithm presented in Theorem 3 includes computing of maxima of stopping pay-off and optimal expected continuation reward in points $y + l\delta_\varepsilon, l = -r_{\varepsilon,n,m}, \ldots, r_{\varepsilon,n,m}, 0 \leq n \leq m \leq N$ and, thus, classify these points as points, which do or do not belong to the corresponding optimal stopping domains for approximating log-price process $Y_{\varepsilon,n}$.

The question about convergence of the optimal stopping time-space domains for the log-price processes $Y_{\varepsilon,n}$ to the optimal stopping time-space domains for the log-price process $Y_{0,n}$ as $\varepsilon \to 0$ do require separate studies.

We just refer here to works by Lamberton (1993, 1998), Leisen (1998), Jiang and Dai (1999), Kukush and Silvestrov (2000, 2004), Jönsson, Kukush, and Silvestrov (2002, 2004, 2005), Prigent (2003), Dupuis and Wang (2005), Jönsson (2005), Lundgren (2010).

5. Binomial and trinomial approximation models

Let us assume that the log-price process $Y_{0,n}$ defined by the stochastic transition dynamic relation (1) is a Gaussian random walk, i.e., $W_{0,n},\ n = 1, 2, \ldots$ are independent Gaussian random variables with $\mathsf{E}W_{0,n} = \mu_n$ and $VarW_{0,n} = \sigma_n^2 > 0$, for $n = 1, 2, \ldots$.

In the trinomial approximation model, the log-price process $Y_{\varepsilon,n}$, defined for every $\varepsilon > 0$ by the stochastic transition dynamic relation (1), is a trinomial random walk, i.e., $W_{\varepsilon,n} = X_{\varepsilon,n,1} + \cdots + X_{\varepsilon,n,r_{\varepsilon,n}}, n = 1, 2, \ldots$, where $X_{\varepsilon,n,k}, n, k = 1, 2, \ldots$ are independent random variables taking, for every $n = 1, 2, \ldots$ values $\delta_\varepsilon, 0$ and $-\delta_\varepsilon$ with probabilities, respectively, $p_{\varepsilon,n,+}, p_{\varepsilon,n,\circ}$ and $p_{\varepsilon,n,-}$. Here, jump values $\delta_\varepsilon > 0$, probabilities $p_{\varepsilon,n,+}, p_{\varepsilon,n,\circ}, p_{\varepsilon,n,-} \geq 0$ and $p_{\varepsilon,n,+} + p_{\varepsilon,n,\circ} + p_{\varepsilon,n,-} = 1$, and parameters $r_{\varepsilon,n}, n = 1, 2, \ldots$ are positive integer numbers.

Note that the above approximation trinomial model reduces to the approximation binomial model if we choose probabilities $p_{\varepsilon,n,\circ} \equiv 0, n = 1, 2, \ldots$. That is why the binomial model do not require a separate consideration.

By the definition, random variables $W_{\varepsilon,n}, n = 1, 2, \ldots$ are, for every $\varepsilon > 0$, trinomial random variables taking values $l\delta_\varepsilon, l = -r_{\varepsilon,n}, \ldots, r_{\varepsilon,n}$ with probabilities,

$$\mathsf{P}\{W_{\varepsilon,n} = l\delta_\varepsilon\} = \sum_{l_+ - l_- = l,\ l_\circ = r_{\varepsilon,n} - l,\ l_+, l_\circ, l_- \geq 0} \frac{r_{\varepsilon,n}!}{l_+! l_\circ! l_-!} p_{\varepsilon,n,+}^{l_+} p_{\varepsilon,n,\circ}^{l_\circ} p_{\varepsilon,n,-}^{l_-}. \quad (10)$$

Taking into account the elimination transformation described in Section 4, we assume, in what follows, that $\mu_n = 0, n = 1, \ldots, N$.

In order to fit parameters, we should provide the asymptotic fitting of the moments for random variables $W_{\varepsilon,n}$ and $W_{0,n}$, for every $n = 1, \ldots, N$,

$$\begin{cases} \mathsf{E}W_{\varepsilon,n} = r_{\varepsilon,n}(\delta_\varepsilon p_{\varepsilon,n,+} - \delta_\varepsilon p_{\varepsilon,n,-}) \to 0 \text{ as } \varepsilon \to 0, \\ \mathsf{E}W_{\varepsilon,n}^2 = r_{\varepsilon,n}\delta_\varepsilon^2(1 - p_{\varepsilon,n,\circ}) \to \sigma_n^2 \text{ as } \varepsilon \to 0, \\ n = 1, 2, \ldots, N. \end{cases} \quad (11)$$

It is readily seen that the asymptotic fitting relations (11) are satisfied if we chose parameters δ_ε and $r_{\varepsilon,n}, n = 1, \ldots, N$ in the following form,

$$\delta_\varepsilon = \frac{1}{\sqrt{r_\varepsilon}},\ 0 < p_{\varepsilon,n,\circ} = p_{n,\circ} < 1,\ p_{\varepsilon,n,\pm} = \frac{1 - p_{n,\circ}}{2},\ r_{\varepsilon,n} = [\frac{r_\varepsilon \sigma_n^2}{1 - p_{n,\circ}}], \quad (12)$$

where r_ε is a positive real numbers such that $r_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Indeed, in this case the following relation holds, for $n = 1, \ldots, N$,

$$\mathsf{E}W_{\varepsilon,n}^2 = r_{\varepsilon,n}\delta_\varepsilon^2(1 - p_{n,\circ}) = [\frac{r_\varepsilon \sigma_n^2}{1 - p_{n,\circ}}]\frac{1}{r_\varepsilon}(1 - p_{n,\circ}) \to \sigma_n^2, \text{ as } \varepsilon \to 0. \quad (13)$$

Condition **D** can be replaced in this case by the following condition:

$\mathbf{D}'$: $L(\overline{\mathbb{Y}}_{g,n}) = 0, n = 1, \ldots, N$.

The following theorem follows from Theorem 2.

Theorem 5. *Let the log-price processes $Y_{\varepsilon,n}$ be constructed by using the trinomial approximation scheme described above and with parameters given by relation* (12). *Also, let condition $\mathbf{B}[\gamma]$ holds for some parameter $\gamma \geq 0$ and condition $\mathbf{D}'$ holds. Then, the following relation holds for any $y \in \mathbb{Y}_{g,n}, n = 0, \ldots, N$,*

$$\phi_{\varepsilon,n}(y) \to \phi_{0,n}(y) \text{ as } \varepsilon \to 0. \tag{14}$$

Proof. The random variable $W_{0,n}$ has the normal distribution with parameters 0 and σ_n^2, and, thus, $\mathsf{E}e^{\pm\beta W_{0,n}} = e^{\frac{\beta^2\sigma_n^2}{2}} < \infty$ for any $n = 1, \ldots, N$.

Also, random variable $W_{\varepsilon,n}$ has the trinomial distribution with parameters given in relation (12), for every $\varepsilon > 0$ and $n = 1, \ldots, N$. Thus, for any $n = 1, \ldots, N$ and $\beta \geq 0$, we get using Taylor expansion for the corresponding moment generating function,

$$\begin{aligned}
\mathsf{E}e^{\pm\beta W_{\varepsilon,n}} &= \left(e^{\pm\beta\frac{1}{\sqrt{r_\varepsilon}}}\frac{1-p_{n,\circ}}{2} + p_{n,\circ} + e^{\mp\beta\frac{1}{\sqrt{r_\varepsilon}}}\frac{1-p_{n,\circ}}{2}\right)^{[\frac{r_\varepsilon\sigma_n^2}{1-p_{n,\circ}}]} \\
&= \left(1 + \frac{(1-p_{n,\circ})\beta^2}{2r_\varepsilon} + o\left(\frac{1}{r_\varepsilon}\right)\right)^{[\frac{r_\varepsilon\sigma_n^2}{1-p_{n,\circ}}]} \\
&\to e^{\frac{\beta^2\sigma_n^2}{2}} < \infty \text{ as } \varepsilon \to 0.
\end{aligned} \tag{15}$$

By continuity theorem for moment generating functions, relation (15) also implies that condition $\mathbf{C}$ holds.

Relation (15) implies that condition $\mathbf{A}[\beta]$ holds for any parameter $\beta \geq 0$. Therefore, one can always chose $\beta > \gamma$ if $\gamma > 0$ or $\beta = \gamma$ if $\gamma = 0$.

Condition $\mathbf{D}'$ implies condition $\mathbf{D}$ to hold, since random variables $W_{0,n}, n = 1, 2, \ldots$ are Gaussian. $\square$.

Now, approximation algorithm based on application of Theorems 3 and 5 can be applied for approximative computing of reward functions $\phi_{0,n}(y)$.

Note that, in this case, trinomial probabilities given by relation (10) should be used in the backward recurrence relations (6). These probabilities can be effectively computed with the use of well-known numerical procedures.

6. A skeleton approximation model

Let us assume that the log-price process $Y_{0,n}$ is a random walk given by the stochastic transition dynamic relation (1).

Let $h_{\varepsilon,n}(y)$ be, for every $\varepsilon > 0$ and $n = 0, 1, \ldots$ so-called skeleton functions defined by the following relation,

$$h_{\varepsilon,n}(y) = \sum_{l=-r_{\varepsilon,n}}^{r_{\varepsilon,n}} l\delta_\varepsilon \mathbf{I}(y \in A_{\varepsilon,n,l}), \tag{16}$$

where

$$A_{\varepsilon,n,l} = \begin{cases} (-\infty,\ \delta_\varepsilon(-r_{\varepsilon,n} + \frac{1}{2})] & \text{if } l = -r_{\varepsilon,n}, \\ (\delta_\varepsilon(l - \frac{1}{2}),\ \delta_\varepsilon(l + \frac{1}{2})] & \text{if } -r_{\varepsilon,n} < l < r_{\varepsilon,n}, \\ (\delta_\varepsilon(r_{\varepsilon,n} - \frac{1}{2}),\ \infty) & \text{if } l = r_{\varepsilon,n}. \end{cases} \tag{17}$$

where (a) $\delta_\varepsilon > 0$ are positive real numbers; (b) $r_{\varepsilon,n}, n = 1, \ldots$ are positive integer numbers.

As before, the log-price process $Y_{\varepsilon,n}$ is given, for every $\varepsilon \geq 0$, by the transition dynamic relation (1), but here we assume $W_{0,n}, n = 1, \ldots$ are independent random variables, while $W_{\varepsilon,n}, n = 1, \ldots$ are, for every $\varepsilon > 0$, discrete random variables with distributions defined by the following relation,

$$W_{\varepsilon,n} = h_{\varepsilon,n}(W_{0,n}),\ n = 1, 2, \ldots. \tag{18}$$

By the definition, random variables $W_{\varepsilon,n}, n = 1, \ldots, N$ are, for every $\varepsilon > 0$, random variables taking values $l\delta_\varepsilon, l = -r_{\varepsilon,n}, \ldots, r_{\varepsilon,n}$ with probabilities, which are interval probabilities for random variables $W_{0,n}, n = 1, \ldots, N$,

$$\mathsf{P}\{W_{\varepsilon,n} = l\delta_\varepsilon\} = \mathsf{P}\{W_{0,n} \in A_{\varepsilon,n,l}\}. \tag{19}$$

Finally, let us assume that parameters δ_ε and $r_{\varepsilon,n}, n = 1, \ldots, N$ are chosen, for every $\varepsilon > 0$ according the following formula,

$$\delta_\varepsilon = \frac{1}{\sqrt{r_\varepsilon}}, \quad r_{\varepsilon,n} = [r_\varepsilon \sigma_n^2], \tag{20}$$

where $\sigma_n > 0, n = 1, \ldots, N$ are some scaling parameters, and $0 < r_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Condition $\mathbf{A}[\beta]$ can be replaced in this case by the following condition, assumed to hold for some $\beta \geq 0$:

$\mathbf{A}'[\beta]$: $\max_{0 \leq n \leq N} \mathsf{E} e^{\pm \beta W_{0,n}} < K'$, where $1 < K' < \infty$.

The following theorem presents the version of Theorem 2 for the skeleton approximation model.

Theorem 6. *Let the log-price processes $Y_{\varepsilon,n}$ be constructed as in the skeleton approximation model described above and with parameters given by relation* (20). *Also, let conditions $\mathbf{A}'[\beta]$ and $\mathbf{B}[\gamma]$ hold, for some parameters*

$0 < \gamma < \beta < \infty$ *or* $\gamma = \beta = 0$, *and also condition* **D** *holds. Then, the following relation holds for any* $y \in \mathbb{Y}_{g,n}$, $n = 0, \ldots, N$,

$$\phi_{\varepsilon,n}(y) \to \phi_{0,n}(y) \text{ as } \varepsilon \to 0. \quad (21)$$

Proof. Let us define intervals $A_{\varepsilon,n} = [(-r_{\varepsilon,n} + \frac{1}{2})\delta_\varepsilon,\ (r_{\varepsilon,n} - \frac{1}{2})\delta_\varepsilon]$.

The following inequality takes place, for every $\varepsilon > 0$ and $y \in \mathbb{R}, n = 0, 1, \ldots, N$,

$$|h_{\varepsilon,n}(y)| \leq |y|\mathbf{I}(y \notin A_{\varepsilon,n}) + (|y| + \delta_\varepsilon)\mathbf{I}(y \in A_{\varepsilon,n}) \leq |y| + \delta_\varepsilon. \quad (22)$$

Using this inequality, we get, for every $\varepsilon > 0, n = 0, 1, \ldots, N$ and $\beta \geq 0$ penetrating condition $\mathbf{A}'[\beta]$,

$$\begin{aligned} \mathsf{E}e^{\pm\beta W_{\varepsilon,n}} &= \mathsf{E}e^{\pm\beta h_{\varepsilon,n}(W_{0,n})} \leq e^{\beta\delta_\varepsilon}\mathsf{E}e^{\beta|W_{0,n}|} \\ &\leq e^{\beta\delta_\varepsilon}(\mathsf{E}e^{\beta W_{0,n}} + \mathsf{E}e^{-\beta W_{0,n}}) < \infty. \end{aligned} \quad (23)$$

Relation (23) and condition $\mathbf{A}'[\beta]$ imply that condition $\mathbf{A}[\beta]$ holds.

Also, the following inequality takes place, for every $\varepsilon > 0, y \in \mathbb{R}, n = 0, 1, \ldots, N$,

$$\begin{aligned} |h_{\varepsilon,n}(y) - y| &\leq ((-r_{\varepsilon,n} + \frac{1}{2})\delta_\varepsilon - y)\mathbf{I}(y \leq (-r_{\varepsilon,n} + \frac{1}{2})\delta_\varepsilon) \\ &+ \delta_\varepsilon\mathbf{I}(y \in A_{\varepsilon,n}) + (y - (r_{\varepsilon,n} - \frac{1}{2})\delta_\varepsilon)\mathbf{I}((r_{\varepsilon,n} - \frac{1}{2})\delta_\varepsilon \leq y). \end{aligned} \quad (24)$$

Inequality (24) implies that $|h_{\varepsilon,n}(y) - y| \to 0$ as $\varepsilon \to 0$, for every $y \in \mathbb{R}, n = 0, 1, \ldots, N$. It follows from this relation that, for every $n = 0, 1, \ldots, N$,

$$W_{\varepsilon,n} = h_{\varepsilon,n}(W_{0,n}) \xrightarrow{a.s.} W_{0,n} \text{ as } \varepsilon \to 0. \quad (25)$$

Relation (25) implies that condition **C** holds.

Therefore, Theorem 2 can be applied that yields relation (21). $\square$.

Now, approximation algorithm based on sequential application of Theorems 3 and 6 can be applied for approximative computing of reward functions $\phi_{0,n}(y)$.

Note that, in this case, probabilities given by relation (19) should be used in the backward recurrence relations (6). These probabilities can be effectively computed not only for the case, where $Y_{0,n}$ is a Gaussian random walk. For example, it can be done for various models, where $Y_{0,n}$ is a Lévy random walk, in particular, for many discrete time analogues of jump diffusion processes.

In this paper, we consider two examples of skeleton approximations, for log-price processes represented by Gaussian and compound Gaussian random walks. In both cases, random variables $W_{0,n}, n = 1, 2, \ldots$ have distributions absolutely continuous with respect to the Lebesgue measure on real line, and, thus, condition **D** can be replaced by simpler condition **D′**, in Theorem 6 applied to these models.

In the Gaussian case, the above skeleton approximation can be compared with the binomial and trinomial approximations presented in Section 5.

7. Rate of convergence for binomial and trinomial approximations for log-price processes represented by Gaussian random walk

In this section, we test the rate of convergence for the binomial and trinomial approximation models.

We assume that the log-price process $Y_{0,n}$ is a homogeneous in time Gaussian random walk with parameters of jumps μ and σ.

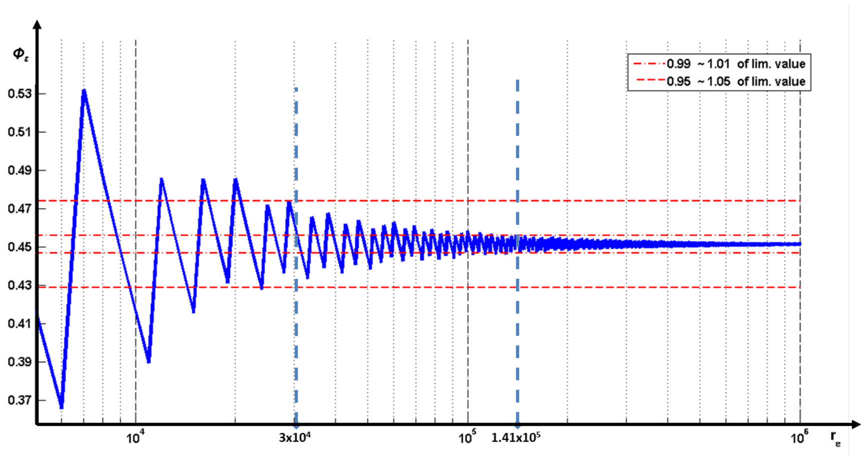

Figure 1: The binomial approximation for a Gaussian model

The pay-off function is a standard pay-off function for call option, $g(n, y) = e^{-rn}[e^y - S]_+$.

We choose option parameters in "year" units, such that the maturity time $T = 0.25$ corresponds to a quarter of a year. We imbed the model in discrete time assuming that the option can be executed at moments $\frac{lT}{N}, l = 0, 1, \ldots, N$. This means, in fact, that we consider a Bermudian option, if the continuous time framework is used.

We take the initial value of the price process as $x_0 = 10$, which corresponds to the initial value $y_0 = \ln 10$ for the log-price process. We also take the yearly values for the trend, $\mu_Y = -20\%$, and for the volatility, $\sigma_Y = 30\%$.

We also take the risk free yearly interest rate $r_Y = 5\%$ and the strike price $S = 10$.

We perform numerical experiments for two models with two different values of maturity parameter $N = 100$ and $N = 1000$, which approximately correspond in calendar time, respectively, to one day and one hour, for one time period $\frac{T}{N}$.

According the above remarks, the one-period parameters μ, r and σ^2 should be re-calculated from the corresponding yearly values by multiplying the values μ_Y, r_Y and σ_Y^2 respectively by factor TN^{-1}, respectively, for $N = 100$ and $N = 1000$.

Computations have been performed with double precision in MATLAB R07B, on a laptop with moderate characteristics that are 1.3 GHZ Intel Mobile Core 2 Duo SU7300 CPU and 4 GB of internal memory. The operation system is Windows 7 Home Premium 64 bit.

We would also like to point out that the execution speed is improved significantly, by using vectors and matrix calculation in MATLAB computations, similar to those presented in Desmond (2002).

Let us first present the results of numerical experiments for the binomial approximation model. Figure 1 summarises the results of computations for this model, for the case $N = 100$.

This figure shows how the reward value changes with an increasing parameter r_ε. Y-axle shows the values of the reward function $\phi_\varepsilon = \phi_{\varepsilon,0}(y_0)$, and on X-axle shows the values of r_ε in the log-scale $\log_{10} r_\varepsilon$. In this way we can get better overview of convergence for ϕ_ε.

We choose the sequence of $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_k > \cdots$ such that the corresponding sequence of parameters $1000 = r_{\varepsilon_0} < r_{\varepsilon_1} < \cdots < r_{\varepsilon_k} < \cdots$ has the step $\Delta r_{\varepsilon_k} = r_{\varepsilon_k} - r_{\varepsilon_{k-1}} = 1000, k = 1, 2, \ldots$. The neighbour points in the sequence $(r_{\varepsilon_k}, \phi_{\varepsilon_k,0}(y_0)), k = 1, 2, \ldots$ have been connected by intercepts of strait lines in order to improve visualisation of graphics.

N; Precision	100; 5%	100; 1%	BAV for $\phi_{0,0}(y_0)$
Reward value	0.463035	0.454129	0.451404
r_ε	3×10^4	1.41×10^5	2.0×10^6
$r_{\varepsilon,n}$	7	32	450
Time (sec)	0.08	0.17	11.9
N; Precision	1000; 5%	1000; 1%	BAV for $\phi_{0,0}(y_0)$
Reward value	0.466016	0.454962	0.451609
r_ε	3.4×10^5	1.54×10^6	1.0×10^7
$r_{\varepsilon,n}$	8	35	225
Time (sec)	1.06	8.86	517

Table 1: Computing time for 5% and 1% precision in the binomial approximation for a Gaussian model

The benchmark approximative value (BAV) for $\phi_{0,0}(y_0)$ is achieved by taking the large value of $r_\varepsilon = 2 \times 10^6$. This benchmark approximative value, 0.451404, is shown with 6 digits after point that reduces the rounding error to the negligible level of 0.05 %. As shown in the Figure 1, values of r_ε roughly larger, respectively, than 3.0×10^4 or 1.41×10^5 guarantee that the deviation of computed reward values from the benchmark approximative value are less than, respectively, $\pm 5\%$ or $\pm 1\%$ of the benchmark approximative value. Finally, when r_ε moves toward 10^6, the reward values are stabilised near the above benchmark approximative value with the deviation within the $\pm 0.1\%$ limits. This is consistent with the convergence relation given in Theorem 5.

Table 1 shows the real computational times needed to get the corresponding reward values, with 5% and 1% precision in the binomial approximation model, respectively, with parameters $N = 100$ and $N = 1000$.

Let us also explain the choice of trend parameter μ_Y. In this case, the risk neutral value of trend, satisfying risk neutral condition $\mu^*_Y = r_Y - \sigma^2_Y/2$, is $\mu^*_Y = 0.5\%$.

It is well known that for values of $\mu_Y \geq \mu^*_Y$ the optimal stopping strategy will be $\tau \equiv N$.

$\mu_Y\%$	BAV for $\phi_{0,0}(y_0)$	BAV for $\mathsf{E}g(N, e^{Y_{0,N}})$	Δ	$\Delta\%$
0.5	0.658308	0.658308	0.000000	0.00
0.0	0.651294	0.651294	0.000000	0.00
- 10.0	0.534689	0.521423	0.013266	2.49
- 20.0	0.451396	0.410706	0.040690	9.01
- 30.0	0.385771	0.318020	0.067751	17.56

Table 2: American and European type expected rewards for the binomial model

Table 2 shows the difference between the benchmark approximative values for the reward function $\phi_{0,0}(y_0)$ and the expected reward $\mathsf{E}g(N, e^{Y_{0,N}})$ (for the simplest European type stopping time $\tau \equiv N$), for a series of values $\mu_Y \leq \mu^*_Y$. In this case parameter $N = 100$.

The coresponding benchmark approximative values are computed for the value of parameter $r_\varepsilon = 2 \times 10^6$, which stabilise the reward values $\phi_{\varepsilon,0}(y_0)$ and $\mathsf{E}g(N, e^{Y_{\varepsilon,N}})$ near the corresponding benchmark approximative values with deviations within the $\pm 0.05\%$ limits.

It is worth to note that the expected rewards $\mathsf{E}g(N, e^{Y_{\varepsilon,N}})$ corresponding to the stopping time $\tau \equiv N$ are computed with the use of the backward recurrence algorithm for American reward functions presented in Theorem 3, but with the modified pay-off function $\hat{g}(n, e^y) = g(n, e^y)I(n = N)$.

The results presented in Table 2 show that the early execution, which makes difference between American and European type options, begins to play a meaningful role for negative values of μ_Y about -10%.

In particular, the benchmark approximative value for the reward function $\phi_{0,0}(y_0)$ exceeds the benchmark approximative value for the expected reward $\mathsf{E}g(N, e^{Y_{0,N}})$, in the case where $\mu_Y = -20\%$, for about 10%. This is about twice larger than the 5% lower accuracy limit used in our numerical experiments.

It is not out of the picture to note that negative values of the trend parameter have the similar effects as could be caused by implementation dividends in the underlying model.

The value $\mu_Y = -20\%$ chosen as the basic value for the trend parameter in presentation of results of our numerical experimental studies. The results are analogous for other values of μ_Y.

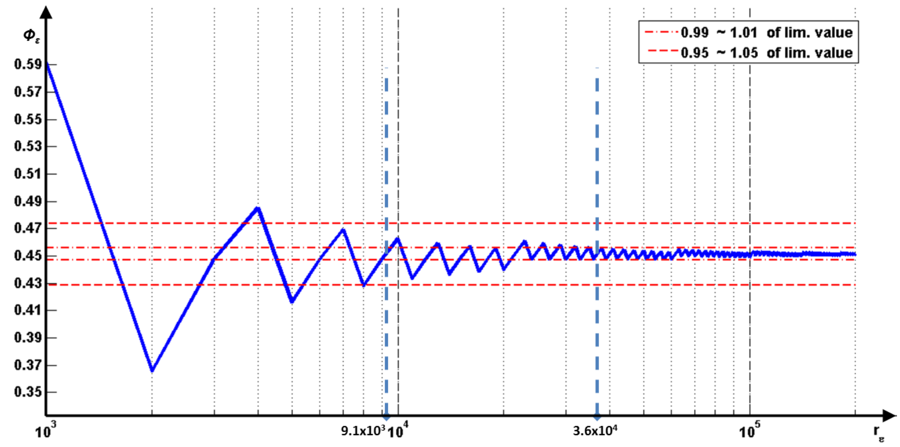

Figure 2: The trinomial approximation for a Gaussian model

Let us now present the results of numerical experiments for the trinomial approximation model.

Figure 2 summarise the results of computations for the case $N = 100$ for the trinomial approximation model in the same way as in Figure 1 for the binomial approximation model.

Below, we show results for the standard case with probability of zero-jump equal to 0.666666. Calculations we did show that variation of this parameter in the limits separated of extreme values 0 and 1, for example in the interval $[0.1, 0.9]$ does not affect significantly the results of computations.

Table 3 shows the real computational times used for computing of the corresponding reward values, with 5% and 1% precision in the trinomial approximation model, respectively, with parameters $N = 100$ and $N = 1000$.

N; Precision	100; 5%	100; 1%	BAV for $\phi_{0,0}(y_0)$
Reward value	0.447541	0.447541	0.451408
r_ε	9.0×10^3	3.6×10^4	7.2×10^5
$r_{\varepsilon,n}$	6	24	474
Time (sec)	0.22	0.36	78.5
N; Precision	1000; 5%	1000; 1%	BAV for $\phi_{0,0}(y_0)$
Reward value	0.447926	0.447936	0.451714
r_ε	9×10^4	3.6×10^5	2.0×10^6
$r_{\varepsilon,n}$	6	24	135
Time (sec)	1.81	15	810

Table 3: Computing time for 5% and 1% precision in the trinomial approximation for a Gaussian model

We also evaluated, at which level computational rounding errors can penetrate the computed reward values. MATLAB can perform computations with double or single precision, i.e., respectively, with 16 or 8 floating digits. The computations described above have been performed with the double precision. We repeated the same computations with the single precision. The result was that at least six digits after point were the same in both cases, even for the model with $N = 1000$. This means that the rounding errors were at the level less than 0.01% that is at the negligible level for computations of rewards with 1% precision.

8. Rate of convergence for skeleton approximations for log-price processes represented by Gaussian random walk

N; Precision	100; 5%	100; 1%	BAV for $\phi_{0,0}(y_0)$
Reward value	0.440469	0.452112	0.451451
r_ε	1.6×10^4	2.9×10^4	2.0×10^6
$r_{\varepsilon,n}$	4	7	450
Time (sec)	0.11	0.16	62.3
N; Precision	1000; 5%	1000; 1%	BAV for $\phi_{0,0}(y_0)$
Reward value	0.433613	0.448058	0.451925
r_ε	1.8×10^5	2.7×10^5	1.0×10^7
$r_{\varepsilon,n}$	4	6	225
Time (sec)	1.6	2.42	2.26×10^3

Table 4: Computing time for 5% and 1% precision in the skeleton approximation for a Gaussian model

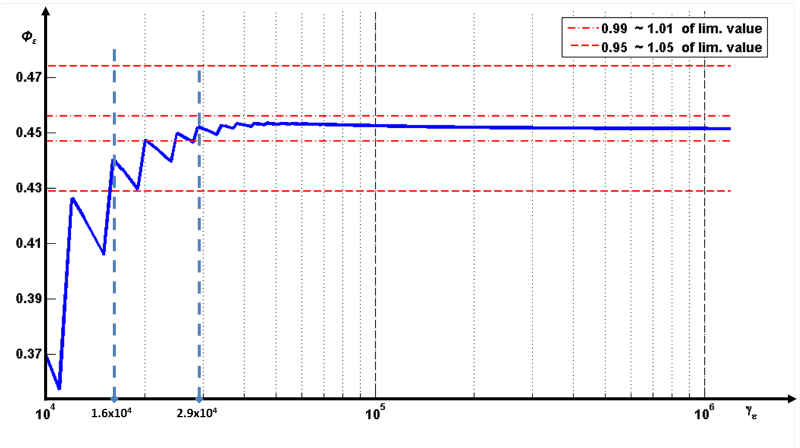

Figure 3: The skeleton approximation for a Gaussian model

Let us now present the results of numerical experiments for the skeleton approximation model.

In order to compare the computational results for skeleton approximation model with those for binomial-trinomial approximations we consider the same model of log-price process represented by Gaussian random walk with the same parameters μ and σ as for the binomial-trinomial model. We also choose the scale parameters $\sigma_n = \sigma, n = 1, 2, \ldots$ in the skeleton approximation model.

Figure 3 summarise the results of computations for the case $N = 100$ for the skeleton approximation model in the same way, as Figure 1 makes this for the binomial approximation model.

Table 4 shows the real computational times needed to get the corresponding reward values, with 5% and 1% precision in the skeleton approximation model, respectively, with parameters $N = 100$ and $N = 1000$.

9. Rate of convergence for skeleton approximations for log-price processes represented by compound Gaussian random walks

The great advantage of skeleton approximations is their universality in comparison with binomial–trinomial approximations. The latter approximations can be used only for Gaussian models, while the skeleton approximation can also be used for wide classes of non-Gaussian models.

In this section, we test the rate of convergence for the skeleton approximation model for the case, where the underlying log-price process $Y_{0,n}$ is a compound Gaussian random walk. This means that the random jumps $W_{0,n}, n = 1, 2, \ldots$ are independent random variables that can be represented in the following form,

$$W_{0,n} = W'_{0,n} + \sum_{k=1}^{N_n} W''_{0,n,k}, n = 1, 2, \ldots, \tag{26}$$

where: (a) $W'_{0,n}, W''_{0,n,k}, N_n, n, k = 1, 2, \ldots$ are independent random variables; (b) $W'_{0,n}$ is a normal random variable with a mean value μ'_n and a variance σ'^2_n, for $n = 1, 2, \ldots$; (c) $W''_{0,n,k}, k = 1, 2, \ldots$ are normal random variables with a mean value μ'_n and a variance σ''^2_n, for $n = 1, 2, \ldots$; (d) N_n is a Poisson random variable with parameter λ_n, for $n = 1, 2, \ldots$.

In this case the random variables $W_{\varepsilon,n} = h_{\varepsilon,n}(W_{0,n}), n = 1, \ldots$ are, for every $\varepsilon > 0$, random variables taking values $l\delta_\varepsilon, l = -r_{\varepsilon,n}, \ldots, r_{\varepsilon,n}$ with probabilities, which are compound Gaussian interval probabilities for random variables $W_{0,n}, n = 1, \ldots$ given by the following formula,

$$\begin{aligned} \mathsf{P}\{W_{\varepsilon,n} = l\delta_\varepsilon\} &= \mathsf{P}\{W_{0,n} \in A_{\varepsilon,n,l}\} \\ &= \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} e^{-\lambda_n} \mathsf{P}\{y_0 + \mu'_n + k\mu''_n + \sqrt{\sigma'^2_n + k\sigma''^2_n} \cdot W \in A_{\varepsilon,n,l}\}, \end{aligned} \tag{27}$$

where W is a standard normal random variable with parameters 0 and 1.

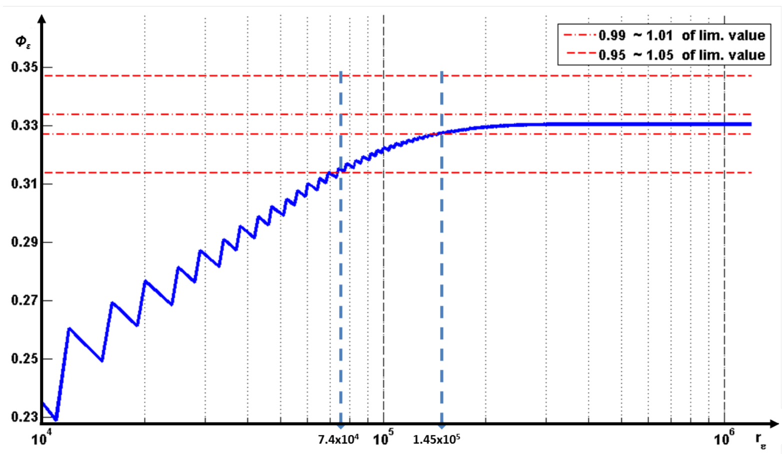

Figure 4: The skeleton approximation for a compound Gaussian model

An usual assumption is that the mean value μ'_n and the variance σ'^2_n for the Gaussian jump component $W'_{0,n}$ are comparable by values, respectively, with the mean value $\lambda\mu''_n$ and the variance $\lambda\sigma''^2_n$ for the compound Gaussian component $\sum_{k=1}^{N_n} W''_{0,n,k}$ and also that the intensity of jumps λ_n is comparatively small.

N; Precision	100; 5%	100; 1%	BAV for $\phi_{0,0}(y_0)$
Reward value	0.315473	0.327509	0.330528
r_ε	7.4×10^4	1.45×10^5	2.0×10^6
$r_{\varepsilon,n}$	17	33	450
Time (sec)	0.25	0.3	68.5
N; Precision	1000; 5%	1000; 1%	BAV for $\phi_{0,0}(y_0)$
Reward value	0.319221	0.330769	0.333658
r_ε	7.8×10^5	1.54×10^6	1.0×10^7
$r_{\varepsilon,n}$	18	35	225
Time (sec)	10.6	37.9	2.38×10^3

Table 5: Computing time for 5% and 1% precision in the skeleton approximation for a compound Gaussian model

μ_Y%	BAV for $\phi_{0,0}(y_0)$	BAV for $\mathsf{E}g(N, e^{Y_{0,N}})$	Δ	Δ%
0.5	0.663645	0.663645	0.000000	0.00
0.0	0.649564	0.649564	0.000000	0.00
- 10.0	0.448729	0.408980	0.039749	8.86
- 20.0	0.33053	0.240423	0.090107	27.26
- 30.0	0.252259	0.131195	0.121064	47.99

Table 6: American and European type expected rewards for a compound Gaussian model

We perform numerical experiments for two models with the values of maturity parameter $N = 100$ and $N = 1000$.

In order to be able to compare the results of numerical computations we choose the same standard pay-off functions $g(n, y) = e^{-rn}[e^y - S]_+$ and parameters μ_Y, σ_Y^2, y_0 and T, S, r_Y, as in Sections 7 and 8, then re-calculate the one-period parameters μ, r and σ^2 from the corresponding yearly values by multiplying the values μ_Y, r_Y and σ_Y^2 by factor TN^{-1}, respectively, for $N = 100$ and $N = 1000$.

Finally, we take parameters $\mu'_n = \mu', \sigma'^2_n = \sigma'^2, n = 1, 2, \ldots$, $\mu''_n = \mu'', \sigma''^2_n = \sigma''^2, n = 1, 2, \ldots$, and $\lambda_n = \lambda, n = 1, 2, \ldots$, such that

$$\mu' = \lambda\mu'' = \frac{\mu}{2}, \ \sigma'^2 = \lambda\sigma''^2 = \frac{\sigma^2}{2}. \tag{28}$$

We also choose the value of parameter $\lambda = \frac{1}{10}$ that automatically implies the relations $\mu'' = 10\mu'$ and $\sigma''^2 = 10\sigma'^2$.

Finally, we choose scale parameters $\sigma_n = \sigma, n = 1, 2, \ldots$ in the above skeleton approximation model.

We omit the details connected with truncation of series in formula (27) and mention only that the truncation of all terms for $k \geq 5$ cause changes in the corresponding reward values at the negligible level of 0.01%.

Figure 4 summarise the results of computations for the case $N = 100$ for the skeleton approximation model in the same way, as Figure 1 makes this for the binomial approximation model.

Table 5 shows the real computational times needed to get the corresponding reward values, with 5% and 1% precision in the skeleton approximation model, respectively, with parameters $N = 100$ and $N = 1000$.

Both Figure 4 and Table 5 present results obtained for parameter $\mu_Y = -20\%$.

Table 6 shows, how this parameter impacts the benchmark approximative values for the reward function $\phi_{0,0}(y_0)$ and the expected reward $\mathsf{E}g(N, e^{Y_{0,N}})$ for the simplest European type stopping time $\tau \equiv N$, in the case of compound Gaussian model. Here, parameter $N = 100$.

10. Comparison of numerical results and conclusion

Numerical results analogous to those described above have been also obtained for other values of parameters for log-price processes represented by Gaussian and compound Gaussian random walks and parameters of call and put type options.

We refer to the Appendix where one can find the MATLAB programs used in our experimental studies.

Let us make some short concluding remarks.

The lattice approximation models possess very good smoothing properties. As show the results presented in the paper, these approximations well converge even for very irregular discontinuous pay-off functions.

The approximation algorithms based on these models also well converge for much more general discrete and continuous time multivariate modulated Markov type log-price processes. Here, we just refer to recent works by Silvestrov, Jönsson and Stenberg (2009, 2010), Lundgren and Silvestrov (2009, 2010), Silvestrov and Lundgren (2011), and Silvestrov (2013).

Comparison of binomial–trinomial and skeleton approximations based on numerical experiments show that all approximations have appropriate computing times. The binomial model has slightly shorter computing times for computing reward values with given precision.

However, we would like to mention that trinomial approximations also can be useful, for example, for multivariate and inhomogeneous in time models. In such models, binomial approximations may possess not enough free parameters required for exact or asymptotic fitting of parameters. One can find the corresponding examples in Lundgren (2010), Lundgren and Silvestrov (2009, 2010), Silvestrov and Lundgren (2011), and Silvestrov (2013).

We can also conclude that the skeleton approximations have better stabilisation properties than others.

As mentioned above, skeleton approximations have a principal advantage in comparison with binomial–trinomial approximations. The latter approximations can be used only for Gaussian models, while the skeleton approximation can also be used for wide classes of non-Gaussian models.

Theoretical results presented in Sections 2 – 6 show that the above stochastic approximations possess good convergence properties even for discontinuous and very irregular pay-off functions. We shall present the results of experimental studies for the corresponding convergence rates in future publications.

11. Appendix: MATLAB codes

Here we present the core functional MATLAB programs used in our experimental studies. These functional programs are:

- ProbMatrixBinoFuc
- ProbMatrixTrioFuc
- BinoSumCallFuc
- TriSumCallFuc
- SkeletonCallFuc
- Skeleton model for compound Gaussian

• ProbMatrixBinoFuc:

```
function PMatrix = ProbMatrixBinoFuc(P,N)
    % P=1/2, N is the time steps
    % \delta_t= T/N;

P_Plus = P;
P_Minus = P;

PMatrix{1,1} = 1;             % To build a Probability Cell Array

for i = 2:(N+1)
```

```
         tempMatrix = PMatrix{i-1,1};
     tempMatrixPlus = [tempMatrix 0]*P_Plus;
    tempMatrixMinus = [0 tempMatrix]*P_Minus;

    PMatrix{i,1} = tempMatrixPlus + tempMatrixMinus;

end

PMatrix(1,:) = [];          % To get rid of the first row of the
                            % Pmatrix cell arry, which constains
                            % just numberal 1.
```

- **ProbMatrixTrioFuc:**

```
function PMatrix = ProbMatrixTrioFuc(P,N)
    % P_0=2/3,  the probability of no changing;
    % N is the time steps, \delta_t= T/N.

    P_0 = P;                 % Probability for no-changing.
P_Minus = (1-P_0)/2;         % Probability for "down" movement.
 P_Plus = (1-P_0)/2;         % Probability for "up" movement.

PMatrix{1,1} = 1;            % To build a Probability Cell Array

for i = 2:(N+1)

    tempMatrix = PMatrix{i-1,1};

     tempMatrixPlus = [tempMatrix 0 0]*P_Plus;
        tempMatrix0 = [0 tempMatrix 0]*P_0;
    tempMatrixMinus = [0 0 tempMatrix]*P_Minus;

    PMatrix{i,1} = tempMatrixPlus + tempMatrix0 + tempMatrixMinus;
end

PMatrix(1,:) = [];           % To get rid of the first row of the
                             % Pmatrix cell arry, which constains
                             % just numberal 1.
```

- **BinoSumCallFuc:**

```
function OptionPrice = BinoSumCallFuc(gamma,N,Stock_0,strikeP,...
                                   mu,sigma,alfa,PMatrix)

s0 = Stock_0;               % The Initial Stock Price.
K = strikeP;                % Strike Price.

delta = 1/sqrt(gamma);      % Jump magnitude.

muM = (mu)*ones(N,1);       % Trend for Stock Price Process.
sigmaM = (sigma)*ones(N,1); % Volatility of Stock Price.
alfaM = (alfa)*ones(N,1);   % Risk Free Interest Rate.

%% Changing Probilities for every time interval.
gamma_n = round(sigmaM.*sigmaM.*gamma);   % Num. of Nodes in
                                          % one Changing Unit.

for i = 1:N
  SubProbability{i,1} = PMatrix{gamma_n(i),1};
end

%% Total Possible Jumping Numbers.
M = sum(gamma_n);
dpowers = -delta*((0:M)');
upowers = delta*((M:-1:0)');

%%Option Price at maturity, 'N'.
priceN = max(s0*exp(dpowers+upowers+sum(muM))-K,0);  %Call Option

%% Re-trace to get option value at time zero
for i = N:-1:1

    expPrice = 0;

    prob = SubProbability{i,1};

    % to calculation exp. value of Option Price at moment 'i-1'.
    for j = 1:(gamma_n(i)+1)
        expPrice = expPrice+prob(j)*priceN(j:(M-gamma_n(i)+j));
    end
```

```
    % to calculation Stock-price at moment 'i-1'.
    M = sum(gamma_n(1:(i-1)));
    dpowers = -delta*((0:M)');
    upowers = delta*((M:-1:0)');
    Si = s0*exp(dpowers+upowers+(i-1)*muM(i));

    % max(excercised option price at moment (i-1), present value
    % of expected option value at moment (i-1)). So this is
    % option price at moment 'i-1'.

    priceN = max(max(Si-K,0),expPrice*exp(-alfaM(i)));
end

OptionPrice = priceN;
```

- **TriSumCallFuc:**

```
function OptionPrice = TriSumCallFuc(gamma,N,Stock_0,StrikeP,mu,...
                                  sigma,alfa,PMatrix,P_2)

s0 = Stock_0;                 % The Initial Stock Price.
K = StrikeP;                  % Strike price.
PP = PMatrix;

delta = sqrt(1/gamma);        % Jump magnitude.

muM = (mu)*ones(N,1);         % Trend for Stock Price Process.
sigmaM = (sigma)*ones(N,1);   % Volatility of Stock Price.
alfaM = (alfa)*ones(N,1);     % Risk Free Interest Rate.

%% Changing Probilities for every time interval.
gamma_n = round(sigmaM.*sigmaM.*gamma/(1-P_2));

for i = 1:N
  SubProbability{i,1} = PP{gamma_n(i),1};
end

%% Total Possible Jumping Numbers.
M = sum(gamma_n);
```

```
powers = delta*((M:-1:-M)');


%%Option Price at maturity, 'N'.
priceN = max(s0*exp(powers+sum(muM))-K,0);

%% Re-trace to get option value at time zero
for i = N:-1:1

    expPrice = 0;

    prob = SubProbability{i,1}';

    % to calculation exp. value of Option Price at moment 'i-1'.
    for j = 1:(2*gamma_n(i)+1)
        expPrice = expPrice+prob(j)*priceN(j:(2*M-2*gamma_n(i)+j));
    end

    % to calculation Stock-price at moment 'i-1'.
    M = sum(gamma_n(1:(i-1)));
    powers = delta*((M:-1:-M)');
    Si = s0*exp(powers+sum(muM(1:(i-1))));

    % max(excercised option price at moment (i-1), present value
    % of expected option value at moment (i-1)). So this is
    % option price at moment 'i-1'.

    priceN = max(max(Si-K,0),expPrice*exp(-alfaM(i)));
end

OptionPrice = priceN;
```

- **SkeletonCallFuc:**

```
function OptionPrice = SkeletonCallFuc(gamma,N,Stock_0,strikeP,...
                                    mu,sigma,alfa)

s0 = Stock_0;                  % The Initial Stock Price.
K = strikeP;                   % Strike price
```

```
endPlus = inf;
endMinus = -inf;

delta = 1/sqrt(gamma);       % Jump magnitude.

muM = (mu)*ones(N,1);        % Trend for Stock Price Process.
sigmaM = (sigma)*ones(N,1);  % Volatility of Stock Price.
alfaM = (alfa)*ones(N,1);    % Risk Free Interest Rate.

%% Changing Probilities for every time interval.
gamma_n = round(sigmaM.*sigmaM.*gamma);
for i = 1:N

    A = (gamma_n(i)-0.5):-1:(-gamma_n(i)+0.5);
    D = [endPlus A endMinus];

    for j = 1:(2*gamma_n(i)+1)
        temp1 = D(j)*delta/sigmaM(i);
        temp2 = D(j+1)*delta/sigmaM(i);
        temp3 = normcdf([temp1 temp2]);
        Prob(j) = temp3(1)-temp3(2);
    end

SubProbability{i,1} = Prob;

end

%% Total Possible Jumping Numbers.
M = sum(gamma_n);
powers = delta*((M:-1:-M)');

%%Option Price at maturity, 'N'.
priceN = max(s0*exp(powers+sum(muM))-K,0);
%% Re-trace to get option value at time zero
for i = N:-1:1

    expPrice = 0;

    prob = SubProbability{i,1}';

    % to calculation exp. value of Option Price at moment 'i-1'.
```

```
    for j = 1:(2*gamma_n(i)+1)
        expPrice = expPrice+prob(j)*priceN(j:(2*M-2*gamma_n(i)+j));
    end

    % to calculation Stock-price at moment 'i-1'.
    M = sum(gamma_n(1:(i-1)));
    powers = delta*((M:-1:-M)');
    Si = s0*exp(powers+sum(muM(1:(i-1))));

    % max(excercised option price at moment (i-1), present value
    % of expected option value at moment (i-1)). So this is
    % option price at moment 'i-1'.

    priceN = max(max(Si-K,0),expPrice*exp(-alfaM(i)));

end

OptionPrice = priceN;
```

• Skeleton model for compound Gaussian:

```
function OptionPrice=SkePoisCallFuc(gamma,N,Stock_0,strikeP,mu,
                                sigma,alfa,muV,sigmaV,itemPower)

s0=Stock_0;
K=strikeP;

gamma_n=round(sigma^2*gamma);
gamma_N=(gamma_n)*ones(N,1);

muM=(mu)*ones(N,1);
sigmaM=(sigma)*ones(N,1);
alfaM=(alfa)*ones(N,1);
delta=1/sqrt(gamma);

temp5=[];

A=(gamma_n-0.5):-1:(-gamma_n+0.5);

D=[inf A -inf];
```

```
for j=1:(2*gamma_n+1)

    temp5=[];
    temp1=D(j)*delta;
    temp2=D(j+1)*delta;

    temp3=normcdf(temp1*ones(1,length(muV)), muV, sigmaV);
    temp4=normcdf(temp2*ones(1,length(muV)), muV, sigmaV);
    temp5=itemPower.*(temp3-temp4);

    Prob(j)= sum(temp5);

end

for i=1:N
    SubProbability{i,1}=Prob;
end

%% Total jumps
M=sum(gamma_N);

powers = delta*((M:-1:-M)');

%%Reward Value at N, maturity
priceN =max(s0*exp(powers+sum(muM))-K,0);

%% Re-trace to get option value at time zero
for i = N:-1:1

    expPrice=0;
    prob=SubProbability{i,1}';

    % to calculation expected value at moment n
    for j=1:(2*gamma_N(i)+1)
        expPrice=expPrice+prob(j)*priceN(j:(2*M-2*gamma_N(i)+j));
    end

    % to calculation Stock-price at moment (n-1)
    M=sum(gamma_N(1:(i-1)));
    powers = delta*((M:-1:-M)');
```

```
    Si=s0*exp(powers+sum(muM(1:(i-1))));

    % max(excercised option price at moment (n-1), present value
    % of expected option value at moment n
    priceN = max(max(Si-K,0),expPrice*exp(-alfaM(i)));

end

OptionPrice=priceN;
```

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