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Abstract

In the report "Risk-Free Interest Rates - Extrapolation Method" published by EIOPA it is stated on p. 13 that the so-called W-functions of the Smith-Wilson extra/ interpolation method can be interpreted as covariances to an integrated Ornstein-Uhlenbeck yield curve model. The authors have not seen a formal motivation of this fact, hence they have investigated under what assumptions that the statement is valid. In the present note it is concluded that the statement is true given that the underlying O-U process is *scaled*, has a certain parametrisation and a stochastic starting point with a certain expected value and variance. Moreover, the entire extra/ interpolation method can be interpreted as the conditional expectation of a simple yield curve model *driven by* an integrated O-U process. The proposed method does not rely on an explicit O-U process assumption, but rather applies to a wider class of Gaussian processes.

Keywords: Stochastic yield curve model, Gaussian processes, Interpolation, Solvency II

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1 Introduction

The introduction of the Solvency II insurance regulation will most likely have a massive impact on the European insurance industry. In this vast, detailed regulation various techniques and methods are described which are supposed to be used for e.g. valuation of liabilities and assets, risk calculations etc. One of these many methods, which is the focus of the current note, the proposal to be used for calculating the risk-free interest rate, is the so-called Smith-Wilson method. The Smith-Wilson method is an extra/ interpolation method, which is based on a curve fitting procedure involving a certain bond price model. The technique is described in a research note by Smith and Wilson from 2001, see [6]. Following the notation of EIOPA in [1], the pricing function at maturity t with N zero coupon bond prices with maturities $u_1, \ldots, u_N$ as input is given by

$$P(t) := e^{-UFRt} + \sum_{j=1}^{N} \zeta_j W(t, u_j), \quad t \geq 0, \tag{1}$$

where UFR is the "Ultimate Forward Rate" and where

$$W(t, u_j) := e^{-UFR(t+u_j)} \left\{ \alpha(t \wedge u_j) - 0.5 e^{-\alpha(t \vee u_j)} \left(e^{\alpha(t \wedge u_j)} - e^{-\alpha(t \wedge u_j)} \right) \right\} \tag{2}$$

or alternatively

$$W(t, u_j) := e^{-UFR(t+u_j)} \left\{ \alpha(t \wedge u_j) - e^{-\alpha(t \vee u_j)} \sinh(\alpha(t \wedge u_j)) \right\}$$

where α is a mean reversion parameter determining the rate of convergence to the UFR. The ζ_j's are obtained by solving the linear equation system defined by (1) and (2) given by the specific time points $\{u_j\}_{j=1}^{N}$. Discussion of advantages and disadvantages with this method is an ongoing project, and some views are summarised in [1] by EIOPA. Some advantages are that the method is analytic, easy to implement (read: in Excel), and provides a perfect fit to market data. Some disadvantages are that the method depends heavily on the parameters α and UFR, which are very hard to estimate/ set, the resulting curve is not necessarily monotone, and it may become negative at some part in the extrapolated region.

The main objective of this note is to get a better understanding of a statement made by EIOPA in footnote 16 on p. 13 in [1], which states that "*The function W(t,u) can be interpreted as the covariance function of an Integrated Ornstein-Uhlenbeck yield curve model*" and give a reference to [3]. Even in the light of [3] it is not entirely clear under which assumptions on the underlying Ornstein-Uhlenbeck process (O-U process) that the statement holds.

The main conclusions of the current note are the following

(a) the statement made by EIOPA is only true under certain assumptions on the underlying O-U process,

(b) the resulting pricing function given by (1) and (2) with the fitted ζ_j's can be interpreted as the expected value of a simple yield curve model *driven by* an integrated O-U process conditional on the values of this process in a set of pre-specified points,

(c) the proposed method does not rely on an explicit O-U process assumption, but rather applies to a wider class of Gaussian processes.

Note that Smith and Wilson used a different approach in their paper [6], and we provide *one* probabilistic model which gives the Smith-Wilson curve as output.

The disposition of the note is as follows: in section 2 we define (integrated) O-U processes and list a number of properties and section 3 connects integrated O-U processes to the Smith-Wilson method and provides a motivation to how the entire Smith-Wilson formula can be seen as a conditional expected value of a simple stochastic yield curve model. All straightforward but tedious derivations have been bundled off into the technical appendix A.

2 (Integrated) Ornstein-Uhlenbeck processes

For a more elaborate treatment of the subject of Ornstein-Uhlenbeck processes, we refer to e.g. [4] or [2].

2.1 Ornstein-Uhlenbeck processes

We will use the following stochastic differential representation of O-U processes:

$$dX(t) = (b - \beta X(t))dt + \sigma dB(t), \ \beta > 0, \tag{3}$$

where $B(t)$ is standardised Brownian motion, or the alternative integral representation given by:

$$X(t) = X(0)e^{-\beta t} + \frac{b}{\beta}(1 - e^{-\beta t}) + \sigma \int_0^t e^{-\beta(t-s)} dB(s), \tag{4}$$

where $X(0)$ is a stochastic variable *independent* of $dB(s)$. Straightforward computations then yields the following properties, see appendix A:

$$\begin{aligned}
\mathbb{E}[X(t)] &= \mathbb{E}[X(0)]e^{-\beta t} + \frac{b}{\beta}(1 - e^{-\beta t}) \\
\text{Var}(X(t)) &= \text{Var}(X(0))e^{-2\beta t} + \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right) \\
\text{Cov}(X(s), X(t)) &= \text{Var}(X(0))e^{-\beta(s+t)} + \frac{\sigma^2}{2\beta}e^{-\beta(s+t)}\left(e^{2\beta(s\wedge t)} - 1\right).
\end{aligned}$$

2.2 Integrated Ornstein-Uhlenbeck processes

In appendix A it is shown that

$$\int_0^t X(s)ds$$

has the following properties

$$\begin{aligned}
m(t) &:= \mathbb{E}\left[\int_0^t X(s)ds\right] \\
&= \frac{1}{\beta}\left(\mathbb{E}[X(0)] - \frac{b}{\beta}\right)\left(1 - e^{-\beta t}\right) + \frac{b}{\beta}t \qquad (5) \\
v(t) &:= \text{Var}\left(\int_0^t X(s)ds\right) \\
&= \frac{1}{\beta^2}\left(\text{Var}(X(0)) - \frac{\sigma^2}{\beta}\right)\left(1 - e^{-\beta t}\right)^2 + \frac{\sigma^2}{\beta^3}\left(\beta t - e^{-\beta t}\sinh(\beta t)\right)
\end{aligned}$$

and

$$\begin{aligned}
c(s,t) &:= \text{Cov}\left(\int_0^s X(y)dy, \int_0^t X(z)dz\right) \\
&= \frac{1}{\beta^2}\left(\text{Var}(X(0)) - \frac{\sigma^2}{\beta}\right)\left(1 - e^{-\beta(s\vee t)}\right)\left(1 - e^{-\beta(s\wedge t)}\right) \\
&+ \frac{\sigma^2}{\beta^3}\left(\beta(s\wedge t) - e^{-\beta(s\vee t)}\sinh(\beta(s\wedge t))\right). \qquad (6)
\end{aligned}$$

In the next section we will relate the covariances given above in (6) to the W-functions given by (2).

3 On the connection between integrated Ornstein-Uhlenbeck processes and the Smith-Wilson method

By comparing the expressions for the W-functions given by (2) with the covariance functions of the integrated O-U process from (6), we immediately get that these expressions only are equal up to *a scaling* of the covariance functions. The scaling constant corresponds to

$$\sigma^2 e^{-\alpha(s+t)}/\alpha^3$$

that is

$$\frac{\alpha^3 W(s,t)}{\sigma^2 e^{-\alpha(s+t)}} := c(s,t).$$

By choosing $\beta := \alpha$, $\sigma^2 := \alpha^3$ and $\mathrm{Var}(X(0)) := \alpha^2$ we obtain that $\sigma^2/\alpha^3 = 1$. Note that this will not hold unless $\mathrm{Var}(X(0)) \neq 0$. It is however not possible to cancel the factor $\exp\{-\alpha(s+t)\}$ without violating underlying assumptions of the O-U process. But, recall that the statement actually refers to an "*... integrated O-U yield curve model*". We let

$$D(t) := \exp\left\{-\int_0^t X(s)ds\right\}, \tag{7}$$

where $X(s)$ is an O-U process defined by (4) (i.e. a Vasiček short rate model, but integrated over the *time to maturity*). That is, (7) represents a stochastic discount factor. For our purposes this is meaningful, since it holds that

$$P(t) := \mathbb{E}[D(t)],$$

under the risk-neutral probability measure. Moreover, recall that discount factors often are modelled deterministically using curves within the exponential-polynomial family; a family of curves which includes both the Nelson-Siegel and the Svensson model, see e.g. Section 3.3.3. on p. 49 and onwards in [2]. Hence, from (4) we see that the choice of using an O-U process to model the evolution of the discount factors can be interpreted as using an exponential-polynomial model with added noise. If we carry on with the calculations, introduce

$$X^*(t) := -m(t) + \int_0^t X(s)ds, \tag{8}$$

with $m(t)$ from (5). That is, $\mathbb{E}[X^*(t)] = 0$. Thus, by combining (7) and (8), a Taylor expansion around 0 yields

$$D(t) \approx e^{-m(t)}(1 - X^*(t)) =: D^*(t). \tag{9}$$

If the underlying O-U process is parametrised according to

$$\begin{aligned}\beta &:= \alpha \\ \sigma^2 &:= \alpha^3 \\ \mathrm{Var}(X(0)) &:= \alpha^2 \\ b &:= \alpha UFR \\ \mathbb{E}[X(0)] &:= UFR,\end{aligned}$$

we get that

$$D^*(t) := e^{-UFRt}(1 - X^*(t)),$$

and that

$$\begin{aligned}\mathrm{Cov}(D^*(s), D^*(t)) &= \mathrm{Cov}\left(e^{-UFRs}X^*(s), e^{-UFRt}X^*(t)\right) \\ &= e^{-UFR(s+t)}c(s,t), \qquad (10)\end{aligned}$$

where the last equality follows from the definition of X^*. Thus, we see that (10) is equal to (2) under the above assumptions, as stated in the footnote by EIOPA.

In the derivations of the various properties of (integrated) O-U processes which we have used, we have made use of the fact that (integrated) O-U processes are Gaussian, see appendix A. Due to this fact, we know that, by construction, the distribution of the position of an (integrated) O-U process at time points $t_1, \ldots, t_k$, say, is multivariate normal. Recall that for a multivariate normal distribution, the following holds:

Theorem 1. *Let $\underline{Z} = \{Z_1, \ldots, Z_k\}$ be multivariate normal with mean value*

$$\underline{m} = (\underline{m}_a, \underline{m}_b)',$$

where $\underline{m}$ has dimensions $(j, k-j)'$, and covariance matrix given by

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_{aa} & \mathbf{C}_{ab} \\ \mathbf{C}_{ba} & \mathbf{C}_{bb} \end{pmatrix}.$$

Then it holds that

$$\mathbb{E}[\underline{Z}_a | \underline{Z}_b = \underline{z}_b] = \underline{m}_a + \mathbf{C}_{ab}\mathbf{C}_{bb}^{-1}(\underline{z}_b - \underline{m}_b). \qquad (11)$$

♠

In our setting it holds that

$$\underline{D}^* \in \mathcal{N}(\underline{m}, \mathbf{C}),$$

where

$$\begin{aligned}
\underline{D}^* &= (D^*(t), D^*(u_1), \dots, D^*(u_N))', \\
m_a &= e^{-UFRt}, \\
\underline{m}_b &= (e^{-UFRu_1}, \dots, e^{-UFRu_N})', \\
\underline{C}_{ab} &= \{e^{-UFR(t+u_j)} c(t, u_j)\}_{j=1}^N, \\
\mathbf{C}_{bb} &= \{e^{-UFR(u_i+u_j)} c(u_i, u_j)\}_{i,j=1}^N.
\end{aligned}$$

Thus, applying Theorem 1 to the setting above immediately gives us that (1) can be written on the form (11) according to

$$P(t) := \mathbb{E}[D^*(t) | (D^*(u_1), \dots, D^*(u_N))' = \underline{p}'] = m_a + \underline{C}_{ab} \mathbf{C}_{bb}^{-1} (\underline{p} - \underline{m}_b)$$

where the ζ's are given by

$$\underline{\zeta} = (\zeta_1, \dots, \zeta_N)' = \mathbf{C}_{bb}^{-1} (\underline{p} - \underline{m}_b).$$

Hence, we model *stochastic discount factors* and make no distinction between observed *zero coupon bond prices* and observed *discount factors* (that is $\underline{p} =$ observed market bond prices). Consequently the fair price of unobserved bond prices under the (implicit) risk-neutral probability measure are given by the expected value of the corresponding stochastic discount factors conditional on observed bond prices.

Concluding remarks

In the above we have considered only zero coupon bond prices as input, but it is straightforward to generalise the method to the situation with coupon paying bonds. It is important to note that the argumentation using Theorem 1 does not require the underlying process to be of Ornstein-Uhlenbeck type. This reasoning could be applied to a much wider class of Gaussian processes.

One can also note that since the Smith-Wilson method can be interpreted as a conditional expectation to a certain Gaussian process more information can easily be deduced, e.g. the conditional variance, given that the underlying Gaussian process assumption is regarded plausible. As has been pointed out earlier the present note provides *one* stochastic model which generates the Smith-Wilson method.

Another point which we have not touched upon is the dimensionality argument used in the original derivation of the Smith-Wilson method from [6]. Their approach is based on calculus of variations minimising slope and curvature of the resulting yield curve. We do not have any simple probabilistic arguments leading up to their parametrisation. This is a subject for future research.

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A Derivations

Some properties of O-U processes

Throughout this section we will use the definition of an O-U process given by (3) or (4).

In the derivations below we will make extensive use of the following property of classical Brownian motion:

$$\int_0^t f(s)dB(s) \in N\left(0, \int_0^t |f(s)|^2 ds\right),$$

assuming a finite variance together with Fubini's theorem, motivating interchange of integrations. By using this we get that:

$$\begin{aligned}
\mathbb{E}[X(t)] &= x_0 e^{-\beta t} + \frac{b}{\beta}(1 - e^{-\beta t}) \\
\mathrm{Var}(X(t)) &= \mathrm{Var}\left(\sigma e^{-\beta t} \int_0^t e^{\beta s} dB(s)\right) \\
&= \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right)
\end{aligned}$$

$$
\begin{aligned}
\operatorname{Cov}(X(s), X(t)) &= \mathbb{E}\left[\left(\sigma e^{-\beta s}\int_0^s e^{\beta y} dB(y)\right)\left(\sigma e^{-\beta t}\int_0^t e^{\beta z} dB(z)\right)\right] \\
&= \sigma^2 e^{-\beta(s+t)}\mathbb{E}\left[\left(\int_0^{s\wedge t} e^{\beta y} dB(y)\right)^2\right] \\
&+ \sigma^2 e^{-\beta(s+t)}\mathbb{E}\left[\left(\int_0^{s\wedge t} e^{\beta y} dB(y)\right)\left(\int_{s\wedge t}^{s\vee t} e^{\beta z} dB(z))\right)\right] \\
&= \sigma^2 e^{-\beta(s+t)}\mathbb{E}\left[\left(\int_0^{s\wedge t} e^{\beta y} dB(y)\right)^2\right] \\
&= \frac{\sigma^2}{2\beta} e^{-\beta(s+t)}\left(e^{2\beta(s\wedge t)} - 1\right)
\end{aligned}
$$

Some properties of integrated O-U processes

Following the same method of proof as the previous section yields:

$$
\begin{aligned}
m(t) &:= \mathbb{E}\left[\int_0^t X(s) ds\right] \\
\{\text{Fubini}\} &= \int_0^t \mathbb{E}[X(s)] ds \\
&= \frac{1}{\beta}\left(x_0 - \frac{b}{\beta}\right)\left(1 - e^{-\beta t}\right) + \frac{b}{\beta} t
\end{aligned}
$$

$$
\begin{aligned}
v(t) &:= \operatorname{Var}\left(\int_0^t X(s)ds\right) \\
&= \operatorname{Cov}\left(\int_0^t X(y)dy, \int_0^t X(z)dz\right) \\
\{\text{Fubini}\} &= \int_0^t \int_0^t \operatorname{Cov}(X(y), X(z))dydz \\
&= \int_0^t \int_0^t \frac{\sigma^2}{2\beta} e^{-\beta(y+z)} \left(e^{2\beta(y\wedge z)} - 1\right) dydz \\
&= \int_0^t \int_0^z \frac{\sigma^2}{2\beta} e^{-\beta(y+z)} \left(e^{2\beta(y\wedge z)} - 1\right) dydz \\
&+ \int_0^t \int_z^t \frac{\sigma^2}{2\beta} e^{-\beta(y+z)} \left(e^{2\beta(y\wedge z)} - 1\right) dydz \\
&= \frac{\sigma^2}{2\beta} \int_0^t e^{-\beta z} \int_0^z \left(e^{\beta y} - e^{-\beta y}\right) dydz \\
&+ \frac{\sigma^2}{2\beta} \int_0^t \left(e^{\beta z} - e^{-\beta z}\right) \int_z^t e^{-\beta y} dydz \\
&= \frac{\sigma^2}{\beta^3} \left(\beta t - \left(1 - e^{-\beta t}\right) - \frac{1}{2}\left(1 - e^{-\beta t}\right)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
c(s,t) &:= \operatorname{Cov}\left(\int_0^s X(y)dy, \int_0^t X(z)dz\right) \\
\{\text{Fubini}\} &= \int_0^s \int_0^t \operatorname{Cov}(X(y), X(z))dydz \\
&= \int_0^{s\vee t} \int_0^{s\wedge t} \operatorname{Cov}(X(y), X(z))dydz \\
&= \int_0^{s\wedge t} \int_0^{s\wedge t} \operatorname{Cov}(X(y), X(z))dydz \\
&+ \int_{s\wedge t}^{s\vee t} \int_0^{s\wedge t} \operatorname{Cov}(X(y), X(z))dydz \\
&= v(s\wedge t) + \underbrace{\int_{s\wedge t}^{s\vee t} \int_0^{s\wedge t} \operatorname{Cov}(X(y), X(z))dydz}_{=(\star)} = (\star\star)
\end{aligned}
$$

expanding $(\star)$ gives us

$$
\begin{aligned}
(\star) &= \int_{s\wedge t}^{s\vee t}\int_0^{s\wedge t} \mathrm{Cov}(X(y),X(z))dydz \\
&= \int_{s\wedge t}^{s\vee t}\int_0^{s\wedge t} \frac{\sigma^2}{2\beta}e^{-\beta(y+z)}\left(e^{2\beta(y\wedge z)}-1\right)dydz \\
&= \frac{\sigma^2}{2\beta}\int_{s\wedge t}^{s\vee t}\int_0^{s\wedge t} e^{-\beta(y+z)}\left(e^{2\beta y}-1\right)dydz \\
&= \frac{\sigma^2}{\beta^3}\left(\frac{1}{2}\left(1-e^{-\beta(s\wedge t)}\right)^2-\frac{1}{2}e^{-\beta(s\vee t)}\left(e^{\beta(s\wedge t)}+e^{-\beta(s\wedge t)}\right)+e^{-\beta(s\vee t)}\right)
\end{aligned}
$$

By combining $(\star)$ and $(\star\star)$ and expanding $v(s\wedge t)$ yields

$$
\begin{aligned}
c(s,t) &= \frac{\sigma^2}{\beta^3}\left(\beta(s\wedge t)-\left(1-e^{-\beta(s\wedge t)}\right)-\frac{1}{2}\left(1-e^{-\beta(s\wedge t)}\right)^2\right) \\
&+ \frac{\sigma^2}{\beta^3}\left(\frac{1}{2}\left(1-e^{-\beta(s\wedge t)}\right)^2-\frac{1}{2}e^{-\beta(s\vee t)}\left(e^{\beta(s\wedge t)}+e^{-\beta(s\wedge t)}\right)+e^{-\beta(s\vee t)}\right) \\
&= \frac{\sigma^2}{\beta^3}\left(\beta(s\wedge t)-\frac{1}{2}e^{-\beta(s\vee t)}\left(e^{\beta(s\wedge t)}+e^{-\beta(s\wedge t)}\right)\right) \\
&- \frac{\sigma^2}{\beta^3}\left(1-e^{-\beta(s\vee t)}-e^{-\beta(s\wedge t)}\right) \\
&= \frac{\sigma^2}{\beta^3}\left(\beta(s\wedge t)-e^{-\beta(s\vee t)}\sinh(\beta(s\wedge t))\right) \\
&- \frac{\sigma^2}{\beta^3}\left(1-e^{-\beta(s\vee t)}\right)\left(1-e^{-\beta(s\wedge t)}\right)
\end{aligned}
$$

Integrated O-U processes with stochastic starting point

Let us consider

$$
X(t) = X(0)e^{-\beta t}+\frac{b}{\beta}(1-e^{-\beta t})+\sigma\int_0^t e^{-\beta(t-s)}dB(s) \qquad (\ddagger)
$$

where $X(0)$ is a stochastic variable *independent* of $dB(s)$. It then follows that

$$
\begin{aligned}
\mathbb{E}[X(t)] &= \mathbb{E}[X(0)]e^{-\beta t}+\frac{b}{\beta}(1-e^{-\beta t}) \\
\mathrm{Var}(X(t)) &= \mathrm{Var}(X(0))e^{-2\beta t}+\frac{\sigma^2}{2\beta}\left(1-e^{-2\beta t}\right) \\
\mathrm{Cov}(X(s),X(t)) &= \mathrm{Var}(X(0))e^{-\beta(s+t)}+\frac{\sigma^2}{2\beta}e^{-\beta(s+t)}\left(e^{2\beta(s\wedge t)}-1\right).
\end{aligned}
$$

If we skip directly to the main point, it holds that

$$
\begin{aligned}
\tilde{c}(s,t) &:= \operatorname{Cov}\left(\int_0^s X(y)dy, \int_0^t X(z)dz\right) \\
&= \int_0^s \int_0^t \operatorname{Var}(X(0)) e^{-\beta(y+z)} dydz + c(s,t) \\
&= \operatorname{Var}(X(0)) \frac{1}{\beta^2} \left(\left(\int_0^{s\wedge t} e^{-\beta y} dy \right)^2 + \int_{s\wedge t}^{s\vee t} \int_0^{s\wedge t} e^{-\beta(y+z)} dydz \right) + c(s,t) \\
&= \operatorname{Var}(X(0)) \frac{1}{\beta^2} \left(\left(1 - e^{-\beta(s\wedge t)}\right)^2 + \left(1 - e^{\beta(s\wedge t)}\right) \left(e^{-\beta(s\wedge t)} - e^{-\beta(s\vee t)}\right) \right) \\
&+ c(s,t) \\
&= \operatorname{Var}(X(0)) \frac{1}{\beta^2} \left(1 - e^{-\beta(s\wedge t)}\right) \left(1 - e^{-\beta(s\vee t)}\right) + c(s,t)
\end{aligned}
$$