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Mikael Petersson
Dmitrii Silvestrov

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Postal address:
Mathematical Statistics
Dept. of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden

Internet:
http://www.math.su.se/matstat

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Asymptotic Expansions for Perturbed Discrete Time Renewal Equations and Regenerative Processes

Mikael Petersson and Dmitrii Silvestrov

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Abstract

This report contains two papers. In the first paper exponential asymptotic expansions for solutions of a perturbed discrete time renewal equation are given. Applications to studies of quasi- and pseudo-stationary ergodic theorems for distributions of perturbed discrete time regenerative processes with regenerative stopping times are discussed. Theoretical results are also illustrated by examples related to queuing systems and risk processes. In the second paper nonlinearly perturbed discrete time regenerative processes with regenerative stopping times are considered. We define the quasi-stationary distributions for such processes and present conditions for their convergence. Under some additional conditions, the quasi-stationary distributions can be expanded in an asymptotic power series with respect to the perturbation parameter. We give an explicit recurrence algorithm for calculating the coefficients of this asymptotic expansion. Applications to a perturbed alternating regenerative process with absorption are presented.

Exponential Expansions for Perturbed Discrete Time Renewal Equations

Mikael Petersson
mikpe@math.su.se

Dmitrii Silvestrov
silvestrov@math.su.se

Department of Mathematics
Stockholm University, Sweden

Abstract

Exponential asymptotic expansions for solutions of a perturbed discrete time renewal equation are given. Applications to studies of quasi- and pseudo-stationary ergodic theorems for distributions of perturbed discrete time regenerative processes with regenerative stopping times are discussed. Theoretical results are also illustrated by examples related to queuing systems and risk processes.

Key words and phrases: Renewal equation, nonlinear perturbation, exponential expansion, regenerative process, risk process.

1 Introduction

Let $q^{(\varepsilon)}(n), n = 0, 1, \ldots,$ be a sequence of real numbers and let $f^{(\varepsilon)}(n), n = 0, 1, \ldots,$ be a discrete probability distribution which may be improper but not concentrated at zero, depending on parameter $\varepsilon \geq 0$. Consider, for every $\varepsilon \geq 0$ a discrete time renewal equation,

$$x^{(\varepsilon)}(n) = q^{(\varepsilon)}(n) + \sum_{k=0}^{n} x^{(\varepsilon)}(n-k) f^{(\varepsilon)}(k),\ n = 0, 1, \ldots \tag{1}$$

As known, the renewal equation (1) has a unique solution.

Suppose that the distribution $f^{(\varepsilon)}(n)$ and the function $q^{(\varepsilon)}(n)$ converges in some natural sense to $f^{(0)}(n)$ and $q^{(0)}(n)$, respectively, as $\varepsilon \to 0$. Then for

$\varepsilon > 0$, equation (1) can be interpreted as a perturbed version of the renewal equation obtained by setting $\varepsilon = 0$.

In the non-perturbed case, the classical renewal theorem given in Erdös, Feller, Pollard (1949) states that if $f^{(0)}(n)$ is a non-periodic, proper distribution with finite expectation $m_1^{(0)} = \sum_{k=0}^{\infty} kq^{(0)}(k) < \infty$ and $\sum_{k=0}^{\infty} |q^{(0)}(k)| < \infty$, then

$$x^{(0)}(n) \to \frac{1}{m_1^{(0)}} \sum_{k=0}^{\infty} q^{(0)}(k) \text{ as } n \to \infty. \tag{2}$$

This result can also be found in Feller (1950) together with an introduction to discrete time renewal theory.

This theorem plays an important role in ergodic theorems for discrete time Markov type processes and queuing and reliability applications, due to the well known fact that the distribution at moment n for such processes usually satisfies a renewal equation.

In the present paper, we present results about the asymptotic behaviour of the solution $x^{(\varepsilon)}(n)$ of a perturbed discrete time renewal equation as $\varepsilon \to 0$ and $n \to \infty$ simultaneously. We consider two cases of so-called quasi-stationary and pseudo-stationary asymptotics, where the limiting distribution $f^{(0)}(k)$ may be, respectively, improper or proper. We improve the asymptotic relation (2) to the much more advanced form of an exponential asymptotic expansion. We also illustrate theoretical results by examples related to queuing systems and risk processes.

In the pseudo-stationary case, our results generalize results given in Englund, Silvestrov (1997). At the same time, our results are discrete time analogues of results given in the book Gyllenberg, Silvestrov (2008) for perturbed continuous time renewal equations, where one can also find an extended bibliography of works in the area.

Some examples of more recent works related to perturbed renewal equations in continuous time and applications to risk theory are Kartashov (2009), Blanchet, Zwart (2010) and Ni (2010).

Since the results in this paper has applications to discrete time Markov chains, we also mention that some results on asymptotic expansions for perturbed Markov chains can be found in, for example, Latouche (1988), Hassin, Haviv (1992), Khasminskii, Yin, Zhang (1996), Yin, Zhang (2003), Altman, Avrachenkov, Núñes-Queija (2004) and Yin, Nguyen (2009).

2 Asymptotic Results

This section presents asymptotic results for the perturbed renewal equation in discrete time with a possibly improper limiting distribution. Without loss of generality it can be assumed that $n = n^{(\varepsilon)}$ is a function of ε such that $n^{(\varepsilon)} \to \infty$ as $\varepsilon \to 0$. Let $f^{(\varepsilon)}$ denote the defect of the distribution $f^{(\varepsilon)}(n)$,

$$f^{(\varepsilon)} = 1 - \sum_{n=0}^{\infty} f^{(\varepsilon)}(n).$$

Furthermore, let $\phi^{(\varepsilon)}(\rho)$ be the moment generating function for the distribution $f^{(\varepsilon)}(n)$,

$$\phi^{(\varepsilon)}(\rho) = \sum_{n=0}^{\infty} e^{\rho n} f^{(\varepsilon)}(n), \ \ \rho \geq 0. \tag{3}$$

We assume that the following conditions on the probability distributions $f^{(\varepsilon)}(n)$ hold.

A: **(a)** $f^{(\varepsilon)}(n) \to f^{(0)}(n)$ as $\varepsilon \to 0$, $n = 0, 1, \ldots$, where the limiting distribution is non-periodic and not concentrated in zero.

(b) $f^{(\varepsilon)} \to f^{(0)} \in [0, 1)$ as $\varepsilon \to 0$.

B: There exists $\delta_c > 0$ such that

(a) $\overline{\lim}_{\varepsilon \to 0} \phi^{(\varepsilon)}(\delta_c) < \infty$, for some $\delta_c > 0$.

(b) $\phi^{(0)}(\delta_c) > 1$.

We define the characteristic equation of the probability distribution $f^{(\varepsilon)}(n)$ by

$$\phi^{(\varepsilon)}(\rho) = 1. \tag{4}$$

The root of equation (4) is crucial for the asymptotic behaviour of $x^{(\varepsilon)}(n)$.

As will be shown, there exists a unique non-negative root $\rho^{(\varepsilon)}$ of equation (4) for sufficiently small ε if **A** and **B** hold. Thus, under conditions **A** and **B**, the following conditions on the functions $q^{(\varepsilon)}(n)$ are well defined.

C: **(a)** $\overline{\lim}_{\varepsilon \to 0} |q^{(\varepsilon)}(n)| < \infty$, for $n = 0, 1, \ldots$

(b) $\sum_{n=0}^{\infty} e^{\rho^{(\varepsilon)} n} q^{(\varepsilon)}(n) \to \sum_{n=0}^{\infty} e^{\rho^{(0)} n} q^{(0)}(n)$ as $\varepsilon \to 0$.

(c) $\overline{\lim}_{\varepsilon \to 0} \sum_{n=0}^{\infty} e^{(\rho^{(0)} + \delta) n} |q^{(\varepsilon)}(n)| < \infty$, for some $\delta > 0$.

Henceforth we will use the following notations.

$$\tilde{m}_1^{(\varepsilon)} = \sum_{n=0}^{\infty} n e^{\rho^{(\varepsilon)} n} f^{(\varepsilon)}(n), \quad \tilde{x}^{(\varepsilon)}(\infty) = \frac{1}{\tilde{m}_1^{(\varepsilon)}} \sum_{n=0}^{\infty} e^{\rho^{(\varepsilon)} n} q^{(\varepsilon)}(n).$$

Theorem 1. *Assume that conditions* **A** *and* **B** *hold. Then*

(i) *There exists a unique nonnegative root $\rho^{(\varepsilon)}$ of the equation (4) for all sufficiently small ε and $\rho^{(\varepsilon)} \to \rho^{(0)} < \delta_c$ as $\varepsilon \to 0$.*

(ii) *If, in addition, condition* **C** *is satisfied, then for any non-negative integer-valued function $n^{(\varepsilon)} \to \infty$ as $\varepsilon \to 0$, the following asymptotic relation holds.*

$$\frac{x^{(\varepsilon)}(n^{(\varepsilon)})}{\exp(-\rho^{(\varepsilon)} n^{(\varepsilon)})} \to \tilde{x}^{(0)}(\infty) \text{ as } \varepsilon \to 0.$$

If the limiting distribution is proper, that is $f^{(0)} = 0$, then $\rho^{(0)} = 0$ and Theorem 1 reduces to the corresponding result in Englund, Silvestrov (1997).

In particular, the asymptotic result in theorem 1 is good in the sense that the limit does not depend on in which way $n^{(\varepsilon)}$ tends to infinity. However, a drawback is that the relation involves $\rho^{(\varepsilon)}$, which is only given implicitly as the solution to a nonlinear equation. Next we present how some additional perturbation conditions guarantee that $\rho^{(\varepsilon)}$ can be expanded in an asymptotic power series with respect to ε. These conditions require that certain asymptotical expansions hold for the following mixed power-exponential moment generating functions,

$$\phi^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} f^{(\varepsilon)}(n), \ \rho \geq 0, \ r = 0, 1, \ldots \tag{5}$$

Before proceeding it is useful to note that if r is a nonnegative number, then $\phi^{(\varepsilon)}(\rho^{(0)}, r)$ and $\phi^{(\varepsilon)}(\rho^{(\varepsilon)}, r)$ are finite for all ε small enough. Indeed, if $0 < \beta < \delta_c$ then $c_r = \sup_{n \geq 0} n^r e^{-(\delta_c - \beta)n} < \infty$ and

$$\phi^{(\varepsilon)}(\beta, r) \leq c_r \sum_{n=0}^{\infty} e^{\delta_c n} f^{(\varepsilon)}(n) = c_r \phi^{(\varepsilon)}(\delta_c).$$

It follows from condition **B** that the left hand side is finite for all sufficiently small ε, say $\varepsilon \leq \varepsilon_1$. Since $\rho^{(\varepsilon)} \to \rho^{(0)} < \delta_c$, we can choose $\beta < \delta_c$ and $\varepsilon_2 = \varepsilon_2(\beta)$ such that $\rho^{(\varepsilon)} < \beta$ for all $\varepsilon \leq \varepsilon_2$. From this it follows that if $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$, then, for all $0 \leq \varepsilon \leq \varepsilon_0$,

$$\phi^{(\varepsilon)}(\rho^{(0)}, r) < \infty, \ \phi^{(\varepsilon)}(\rho^{(\varepsilon)}, r) < \infty.$$

We now assume that the following perturbation condition holds for some positive integer k:

$\mathbf{P^{(k)}}$: $\phi^{(\varepsilon)}(\rho^{(0)}, r) = \phi^{(0)}(\rho^{(0)}, r) + b_{1,r}\varepsilon + \cdots + b_{k-r,r}\varepsilon^r + o(\varepsilon^{k-r})$ for $r = 0, \ldots, k$, where $|b_{n,r}| < \infty$, $n = 1, \ldots, k-r$, $r = 0, \ldots, k$.

Note that if $f^{(0)} = 0$, then $\phi^{(\varepsilon)}(\rho^{(0)}, 0) = 1 - f^{(\varepsilon)}$ and $\phi^{(\varepsilon)}(\rho^{(0)}, r) = \sum_{n=0}^{\infty} n^r f^{(\varepsilon)}(n)$ for positive integers r, so in this case $\mathbf{P^{(k)}}$ reduces to conditions on the defect and the usual moments.

Henceforth we will denote $b_{0,r} = \phi^{(0)}(\rho^{(0)}, r)$.

The following theorem is the main result in the present paper.

Theorem 2. *Suppose that conditions* **A**, **B** *and* $\mathbf{P^{(k)}}$ *hold. Then*

(i) *The root $\rho^{(\varepsilon)}$ of equation (4) has the asymptotical expansion*

$$\rho^{(\varepsilon)} = \rho^{(0)} + a_1\varepsilon + \cdots + a_k\varepsilon^k + o(\varepsilon^k). \tag{6}$$

The coefficients $a_1, \ldots, a_k$ are given by the recurrence formulas $a_1 = -b_{1,0}/b_{0,1}$ and for $n = 2, \ldots, k$,

$$a_n = -\frac{1}{b_{0,1}}\Bigg(b_{n,0} + \sum_{q=1}^{n-1} b_{n-q,1}a_q + \sum_{m=2}^{n}\sum_{q=m}^{n} b_{n-q,m} \cdot \sum_{n_1,\ldots,n_{q-1}\in D_{m,q}} \prod_{p=1}^{q-1} \frac{a_p^{n_p}}{n_p!}\Bigg), \tag{7}$$

where $D_{m,q}$ is the set of all nonnegative integer solutions to the system

$$n_1 + \cdots + n_{q-1} = m, \quad n_1 + \cdots + (q-1)n_{q-1} = q. \tag{8}$$

(ii) *If $b_{l,0} = 0, l = 1, \ldots, r$, for some $1 \le r \le k$, then $a_1, \ldots, a_r = 0$. If $b_{l,0} = 0, l = 1, \ldots, r-1$, and $b_{r,0} < 0$ hold for some $1 \le r \le k$, then $a_1, \ldots, a_{r-1} = 0$ and $a_r > 0$.*

(iii) *If, in addition, condition* **C** *holds and $n^{(\varepsilon)} \to \infty$ as $\varepsilon \to 0$ in such a way that $\varepsilon^r n^{(\varepsilon)} \to \lambda_r \in [0, \infty)$ for some $1 \le r \le k$, then*

$$\frac{x^{(\varepsilon)}(n^{(\varepsilon)})}{\exp(-(\rho^{(0)} + a_1\varepsilon + \cdots + a_{r-1}\varepsilon^{r-1})n^{(\varepsilon)})} \to \frac{\tilde{x}^{(0)}(\infty)}{e^{\lambda_r a_r}} \ \text{as } \varepsilon \to 0.$$

In order to prove Theorems 1 and 2, we shall use the following result from Englund, Silvestrov (1997), which is a modification of an earlier result by Kalashnikov (1978).

Let us impose the following conditions on the distributions $f^{(\varepsilon)}(n)$ and functions $q^{(\varepsilon)}(n)$:

A′: **(a)** $f^{(\varepsilon)}(n) \to f^{(0)}(n)$ as $\varepsilon \to 0$, $n = 0, 1, \dots$, where the limiting distribution is proper, non-periodic and not concentrated in zero.

(b) $m_1^{(\varepsilon)} \to m_1^{(0)} < \infty$ as $\varepsilon \to 0$.

C′: **(a)** $\overline{\lim}_{\varepsilon\to 0} |q^{(\varepsilon)}(n)| < \infty$, for $n = 0, 1, \dots$

(b) $\lim_{n\to\infty} \overline{\lim}_{\varepsilon\to 0} \sum_{k=n}^{\infty} |q^{(\varepsilon)}(k)| = 0$.

(c) $\sum_{n=0}^{\infty} q^{(\varepsilon)}(n) \to \sum_{n=0}^{\infty} q^{(0)}(n)$ as $\varepsilon \to 0$.

Lemma 1. *Assume that* **A′** *and* **C′** *hold. If* $n^{(\varepsilon)} \to \infty$ *in such a way that* $f^{(\varepsilon)} n^{(\varepsilon)} \to \lambda \in [0, \infty]$ *as* $\varepsilon \to 0$*, then*

$$x^{(\varepsilon)}(n^{(\varepsilon)}) \to \exp(-\lambda/m_1^{(0)}) \frac{1}{m_1^{(0)}} \sum_{k=0}^{\infty} q^{(0)}(k) \text{ as } \varepsilon \to 0.$$

3 Proofs

Proof of Theorem 1. Condition **B** implies that $\phi^{(\varepsilon)}(\delta_c) < \infty$ for ε sufficiently small, say $\varepsilon \le \varepsilon_1$. From condition **A** it follows that there exists ε_2 such that the distributions $f^{(\varepsilon)}(n)$ are not concentrated at zero for $\varepsilon \le \varepsilon_2$. From this it can be concluded that for every fixed $\varepsilon \le \min\{\varepsilon_1, \varepsilon_2\}$, the function $\phi^{(\varepsilon)}(\rho)$ is nonnegative, continuous and strictly increasing on the interval $[0, \delta_c]$. Since $\phi^{(0)}(0) = 1 - f^{(0)} \le 1$ and $\phi^{(0)}(\delta_c) > 1$ it follows that there exists a unique root $\rho^{(0)} \in [0, \delta_c)$ to the equation $\phi^{(0)}(\rho) = 1$. For any $\beta \in (\rho^{(0)}, \delta_c]$,

$$\begin{aligned}\underline{\lim}_{\varepsilon\to 0} \phi^{(\varepsilon)}(\beta) &\ge \lim_{N\to\infty} \underline{\lim}_{\varepsilon\to 0} \sum_{n=0}^{N} e^{\beta n} f^{(\varepsilon)}(n) \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} e^{\beta n} f^{(0)}(n) = \phi^{(0)}(\beta) > 1.\end{aligned}$$

For any such β we can choose $\varepsilon_3 = \varepsilon_3(\beta)$ such that $\phi^{(\varepsilon)}(\beta) > 1$ for $\varepsilon \le \varepsilon_3$. Let $\varepsilon_0(\beta) = \{\varepsilon_1, \varepsilon_2, \varepsilon_3(\beta)\}$. Since $\phi^{(\varepsilon)}(0) = 1 - f^{(\varepsilon)} \le 1$, it follows that there exists a unique root $\rho^{(\varepsilon)} \in [0, \beta)$ to the characteristic equation (4) for

$\varepsilon \leq \varepsilon_0(\beta)$. To show that $\rho^{(\varepsilon)} \to \rho^{(0)}$, first note that since β can be chosen arbitrarily close from above to $\rho^{(0)}$ we have

$$\overline{\lim_{\varepsilon\to 0}} \rho^{(\varepsilon)} \leq \rho^{(0)}.$$

Using that β can be chosen such that $\rho^{(\varepsilon)} < \beta < \delta_c$ for ε small enough yields

$$\begin{aligned}
&\lim_{N\to\infty} \overline{\lim_{\varepsilon\to 0}} \sum_{n=N+1}^{\infty} e^{\rho^{(\varepsilon)}n} f^{(\varepsilon)}(n) \\
&\quad \leq \lim_{N\to\infty} \overline{\lim_{\varepsilon\to 0}} \sum_{n=N+1}^{\infty} e^{\beta n} f^{(\varepsilon)}(n) \\
&\quad \leq \lim_{N\to\infty} e^{-(\delta_c-\beta)(N+1)} \overline{\lim_{\varepsilon\to 0}} \phi^{(\varepsilon)}(\delta_c) = 0.
\end{aligned} \tag{9}$$

Assume that there exists a number $\gamma > 0$ and a subsequence $\varepsilon_k \to 0$ as $k \to \infty$ such that $\rho^{(\varepsilon_k)} \leq \rho^{(0)} - \gamma$ for all k. Then

$$\begin{aligned}
&\lim_{N\to\infty} \overline{\lim_{k\to\infty}} \sum_{n=0}^{N} e^{\rho^{(\varepsilon_k)}n} f^{(\varepsilon_k)}(n) \\
&\quad \leq \lim_{N\to\infty} \overline{\lim_{k\to\infty}} \sum_{n=0}^{N} e^{(\rho^{(0)}-\gamma)n} f^{(\varepsilon_k)}(n) \\
&\quad \leq \lim_{N\to\infty} \sum_{n=0}^{N} e^{(\rho^{(0)}-\gamma)n} f^{(0)}(n) = \phi^{(0)}(\rho^{(0)}-\gamma) < 1.
\end{aligned} \tag{10}$$

It follows from (9) and (10) that if such subsequence exists, then

$$\begin{aligned}
1 &= \overline{\lim_{k\to\infty}} \phi^{(\varepsilon_k)}(\rho^{(\varepsilon_k)}) \\
&= \lim_{N\to\infty} \overline{\lim_{k\to\infty}} \left(\sum_{n=0}^{N} e^{\rho^{(\varepsilon_k)}n} f^{(\varepsilon_k)}(n) + \sum_{n=N+1}^{\infty} e^{\rho^{(\varepsilon_k)}n} f^{(\varepsilon_k)}(n) \right) < 1.
\end{aligned}$$

This contradiction implies that $\underline{\lim}_{\varepsilon\to 0} \rho^{(\varepsilon)} \geq \rho^{(0)}$ and hence the first part of the theorem is proved.

For the proof of the second part of the theorem, set

$$\tilde{x}^{(\varepsilon)}(n) = e^{\rho^{(\varepsilon)}n} x^{(\varepsilon)}(n), \quad \tilde{q}^{(\varepsilon)}(n) = e^{\rho^{(\varepsilon)}n} q^{(\varepsilon)}(n), \quad \tilde{f}^{(\varepsilon)}(n) = e^{\rho^{(\varepsilon)}n} f^{(\varepsilon)}(n),$$

Multiplying both sides of the renewal equation (1) by $e^{\rho^{(\varepsilon)}n}$ yields

$$\tilde{x}^{(\varepsilon)}(n) = \tilde{q}^{(\varepsilon)}(n) + \sum_{k=0}^{n} \tilde{x}^{(\varepsilon)}(n-k) \tilde{f}^{(\varepsilon)}(k), \quad n = 0, 1, \ldots$$

Next step is to show that if $f^{(\varepsilon)}(n)$ and $q^{(\varepsilon)}(n)$ satisfy conditions **A**, **B** and **C**, then $\tilde{f}^{(\varepsilon)}(n)$ and $\tilde{q}^{(\varepsilon)}(n)$ satisfy conditions $\mathbf{A}'$ and $\mathbf{C}'$. It follows in a direct way that $\mathbf{A}'(\mathbf{a})$ holds for $\tilde{f}^{(\varepsilon)}(n)$. In particular, the distributions $\tilde{f}^{(\varepsilon)}(n)$ are proper since the definition of $\rho^{(\varepsilon)}$ as a root to the characteristic equation (4) implies that

$$\sum_{n=0}^{\infty} \tilde{f}^{(\varepsilon)}(n) = \sum_{n=0}^{\infty} e^{\rho^{(\varepsilon)} n} f^{(\varepsilon)}(n) = 1.$$

In order to prove that the expectations $\tilde{m}_1^{(\varepsilon)}$ of the distributions $\tilde{f}^{(\varepsilon)}(n)$ satisfy $\mathbf{A}'(\mathbf{b})$, first note that it follows from the first part of the proof that there exists $\gamma > 0$ such that $\rho^{(\varepsilon)} < \rho^{(0)} + \gamma < \delta_c$ for sufficiently small ε. Using this,

$$\begin{aligned} &\lim_{N\to\infty} \overline{\lim_{\varepsilon\to 0}} \sum_{n=N+1}^{\infty} n\tilde{f}^{(\varepsilon)}(n) \\ &\quad \leq \lim_{N\to\infty} \overline{\lim_{\varepsilon\to 0}} \sum_{n=N+1}^{\infty} n e^{(\rho^{(0)}+\gamma)n} f^{(\varepsilon)}(n) \\ &\quad \leq \lim_{N\to\infty} (N+1) e^{-(\delta_c - \rho^{(0)} - \gamma)(N+1)} \overline{\lim_{\varepsilon\to 0}} \phi^{(\varepsilon)}(\delta_c) = 0. \end{aligned} \tag{11}$$

Relation (11) and the fact that $\tilde{f}^{(\varepsilon)}(n)$ satisfies $\mathbf{A}'$ gives

$$\begin{aligned} \lim_{\varepsilon\to 0} \tilde{m}_1^{(\varepsilon)} &= \lim_{\varepsilon\to 0} \sum_{n=0}^{\infty} n\tilde{f}^{(\varepsilon)}(n) = \lim_{N\to\infty} \lim_{\varepsilon\to 0} \sum_{n=0}^{N} n\tilde{f}^{(\varepsilon)}(n) \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} n\tilde{f}^{(0)}(n) = \tilde{m}_1^{(0)}. \end{aligned}$$

Moreover, if **C** holds, then the functions $\tilde{q}^{(\varepsilon)}(n)$ satisfy condition $\mathbf{C}'$. The result now follows from Lemma 1. □

Proof of Theorem 2. Let $\Delta^{(\varepsilon)} = \rho^{(\varepsilon)} - \rho^{(0)}$. It follows from the Taylor expansion of the exponential function that for every $n = 0, 1, \dots,$

$$e^{\rho^{(\varepsilon)} n} = e^{\rho^{(0)} n} \left(\sum_{r=0}^{k} \frac{n^r (\Delta^{(\varepsilon)})^r}{r!} + \frac{n^{k+1} (\Delta^{(\varepsilon)})^{k+1}}{(k+1)!} e^{|\Delta^{(\varepsilon)}| n} \theta_{k+1}^{(\varepsilon)}(n) \right), \tag{12}$$

where $0 \leq \theta_{k+1}^{(\varepsilon)}(n) \leq 1$. Since $\rho^{(\varepsilon)} \to \rho^{(0)} < \delta_c$, there exist $\beta < \delta_c$ and $\varepsilon_1 = \varepsilon_1(\beta)$ such that

$$\rho^{(\varepsilon)} \leq \rho^{(0)} + |\Delta^{(\varepsilon)}| < \beta, \quad \varepsilon \leq \varepsilon_1.$$

It follows from the discussion in section 2 that for this β, there exists $\varepsilon_2 = \varepsilon_2(\beta)$ such that

$$\phi^{(\varepsilon)}(\rho^{(0)}, r) < \infty, \quad r = 0, 1, \ldots, \quad \varepsilon \leq \varepsilon_2.$$

Furthermore, the Cramér type condition **B** implies that

$$\varlimsup_{\varepsilon\to 0} \sum_{n=0}^{\infty} n^{k+1} e^{\beta n} f^{(\varepsilon)}(n) \leq \varlimsup_{\varepsilon\to 0} c_{k+1} \sum_{n=0}^{\infty} e^{\delta_c n} f^{(\varepsilon)}(n) < \infty,$$

where $c_{k+1} = \sup_{n\geq 0} n^{k+1} e^{-(\delta_c - \beta)n} < \infty$. From this it can be concluded that there exists $\varepsilon_3 = \varepsilon_3(\beta)$ such that

$$M_{k+1} = \frac{1}{(k+1)!} \sup_{\varepsilon\leq\varepsilon_3} \sum_{n=0}^{\infty} n^{k+1} e^{\beta n} f^{(\varepsilon)}(n) < \infty. \tag{13}$$

Set $\varepsilon_4 = \varepsilon_4(\beta) = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. By first multiplying both sides in (12) with $f^{(\varepsilon)}(n)$ and then summing both sides with respect to n, using the notation introduced in equation (5) and the fact that $\rho^{(\varepsilon)}$ is the solution to the characteristic function (4), we obtain for $\varepsilon \leq \varepsilon_4$,

$$1 = \sum_{r=0}^{k} \frac{(\Delta^{(\varepsilon)})^r}{r!} \phi^{(\varepsilon)}(\rho^{(0)}, r) + \frac{(\Delta^{(\varepsilon)})^{k+1}}{(k+1)!} \sum_{n=0}^{\infty} n^{k+1} e^{(\rho^{(0)}+|\Delta^{(\varepsilon)}|)n} \theta_{k+1}^{(\varepsilon)}(n) f^{(\varepsilon)}(n)$$

It follows from equation (13) that the last term on the right hand side can be written as $(\Delta^{(\varepsilon)})^{k+1} M_{k+1} \theta_{k+1}^{(\varepsilon)}$, where $M_{k+1} < \infty$ and $\theta_{k+1}^{(\varepsilon)} \in [0, 1]$. Hence, we have showed that for $\varepsilon \leq \varepsilon_4$,

$$\sum_{r=1}^{k} \frac{(\Delta^{(\varepsilon)})^r}{r!} \phi^{(\varepsilon)}(\rho^{(0)}, r) + (\Delta^{(\varepsilon)})^{k+1} M_{k+1} \theta_{k+1}^{(\varepsilon)} = 1 - \phi^{(\varepsilon)}(\rho^{(0)}, 0). \tag{14}$$

Putting $k = 1$ in equation (14) and dividing by ε gives

$$\frac{\Delta^{(\varepsilon)}}{\varepsilon} \left(\phi^{(\varepsilon)}(\rho^{(0)}, 1) + \Delta^{(\varepsilon)} M_2 \theta_2^{(\varepsilon)} \right) = \frac{1 - \phi^{(\varepsilon)}(\rho^{(0)}, 0)}{\varepsilon}, \quad 0 < \varepsilon \leq \varepsilon_4. \tag{15}$$

It follows from the perturbation condition $\mathbf{P^{(k)}}$ that $\phi^{(\varepsilon)}(\rho^{(0)}, 1) \to b_{0,1} \in (0, \infty)$ and $(1 - \phi^{(\varepsilon)}(\rho^{(0)}, 0))/\varepsilon \to b_{1,0} \in [0, \infty)$ as $\varepsilon \to 0$. From this and (15) it can be concluded that $\Delta^{(\varepsilon)}/\varepsilon \to -b_{1,0}/b_{0,1}$ as $\varepsilon \to 0$, which means that $\Delta^{(\varepsilon)}$ can be represented as

$$\Delta^{(\varepsilon)} = a_1 \varepsilon + \Delta_1^{(\varepsilon)}, \tag{16}$$

where $a_1 = -b_{1,0}/b_{0,1}$ and $\Delta_1^{(\varepsilon)}$ is of order $o(\varepsilon)$.

This proves part (**i**) in the case $k = 1$.

If the perturbation condition holds for some $k \geq 2$ we can continue and substitute (16) and $\mathbf{P^{(k)}}$ into (14) in the case when $k = 2$. This yields

$$\begin{aligned}(a_1\varepsilon + \Delta_1^{(\varepsilon)})(b_{0,1} + b_{1,1}\varepsilon + o(\varepsilon)) + \frac{1}{2}(a_1\varepsilon + \Delta_1^{(\varepsilon)})^2(b_{0,2} + o(1)) \\ + (a_1\varepsilon + \Delta_1^{(\varepsilon)})^3 M_3\theta_3^{(\varepsilon)} = -b_{1,0}\varepsilon - b_{2,0}\varepsilon^2 + o(\varepsilon^2).\end{aligned} \tag{17}$$

Dividing both sides of this equation by $b_{0,1}\varepsilon^2$, using the identity $a_1 = -b_{1,0}/b_{0,1}$ and letting ε tend to zero it can be concluded that

$$\frac{\Delta_1^{(\varepsilon)}}{\varepsilon^2} \to -\frac{1}{b_{0,1}}\left(b_{2,0} + b_{1,1}a_1 + \frac{1}{2}b_{0,2}a_1^2\right) \quad \text{as } \varepsilon \to 0.$$

Using this and equation (16), it is found that $\Delta^{(\varepsilon)}$ can be written in the form

$$\Delta^{(\varepsilon)} = a_1\varepsilon + a_2\varepsilon^2 + \Delta_2^{(\varepsilon)},$$

where

$$a_1 = -\frac{b_{1,0}}{b_{0,1}}, \quad a_2 = -\frac{1}{b_{0,1}}\left(b_{2,0} + b_{1,1}a_1 + \frac{1}{2}b_{0,2}a_1^2\right),$$

and the remainder term $\Delta_2^{(\varepsilon)}$ is order $o(\varepsilon^2)$.

This proves part (**i**) in the case $k = 2$.

Continuing in this way, the result is obtained for any positive integer k. However, once we know that the asymptotic expansion (6) exists, the coefficients $a_1, \ldots, a_k$ can be obtained in a simpler way. Consider the following formal equation.

$$\begin{aligned}&(b_{0,1} + b_{1,1}\varepsilon + \cdots)(a_1\varepsilon + a_2\varepsilon^2 + \cdots)/1! \\ &\quad + (b_{0,2} + b_{1,2}\varepsilon + \cdots)(a_1\varepsilon + a_2\varepsilon^2 + \cdots)^2/2! + \cdots \\ &\quad = -(b_{1,0}\varepsilon + b_{2,0}\varepsilon^2 + \cdots).\end{aligned} \tag{18}$$

By equating the coefficients of ε^n in the left and in the right-hand sides of (18) for $n = 1, \ldots, k$, the formula (7) for calculating the coefficients $a_1, \ldots, a_k$ is obtained. Let $\alpha_{q,m}$ denote the coefficient of ε^q in the expansion of $(a_1\varepsilon + a_2\varepsilon^2 + \cdots)^m/m!$. Using this notation, the left-hand side of equation (18) is

$$\begin{aligned}&(b_{0,1} + b_{1,1}\varepsilon + \cdots)(a_1\varepsilon + a_2\varepsilon^2 + \cdots) \\ &\quad + (b_{0,2} + b_{1,2}\varepsilon + \cdots)(\alpha_{2,2}\varepsilon^2 + \alpha_{3,2}\varepsilon^3 + \cdots) \\ &\quad + (b_{0,3} + b_{1,3}\varepsilon + \cdots)(\alpha_{3,3}\varepsilon^3 + \alpha_{4,3}\varepsilon^4 + \cdots) + \cdots\end{aligned} \tag{19}$$

The contribution from the first summand in (19) to the coefficient of ε^n is

$$b_{n-1,1}a_1 + b_{n-2,1}a_2 + \cdots + b_{0,1}a_n = b_{0,1}a_n + \sum_{q=1}^{n-1} b_{n-q,1}a_q. \tag{20}$$

If $2 \leq m \leq n$, then the contribution from the m^{th} summand in (19) to the coefficient of ε^n is

$$b_{n-m,m}\alpha_{m,m} + b_{n-m-1,m}\alpha_{m+1,m} + \cdots + b_{0,m}\alpha_{n,m} = \sum_{q=m}^{n} b_{n-q,m}\alpha_{q,m}. \tag{21}$$

No further contributions are given by the rest of the summands in (19). Using (19), (20) and (21) we see that equating the coefficients of ε^n in the left and in the right-hand sides of (18) yields $b_{0,1}a_1 = -b_{1,0}$ if $n = 1$ and

$$b_{0,1}a_n + \sum_{q=1}^{n-1} b_{n-q,1}a_q + \sum_{m=2}^{n}\sum_{q=m}^{n} b_{n-q,m}\alpha_{q,m} = -b_{n,0}, \quad \text{for } n = 2, \ldots, k.$$

To determine $\alpha_{q,m}$ it is sufficient to expand $(a_1\varepsilon + \cdots + a_{q-1}\varepsilon^{q-1})^m/m!$ and find the coefficient in front of ε^q. Using the multinomial theorem we obtain

$$\alpha_{q,m} = \sum_{n_1,\ldots,n_{q-1} \in D_{m,q}} \prod_{p=1}^{q-1} \frac{a_p^{n_p}}{n_p!},$$

where $D_{m,q}$ is the set of all non-negative integer solutions to the system (8).

This concludes the proof of part (**i**).

Part (**ii**) follows directly from the recurrence formula (7).

Finally, let us prove part (**iii**). Note that it follows from theorem 1 that

$$\frac{x^{(\varepsilon)}(n^{(\varepsilon)})}{\exp(-\rho^{(\varepsilon)}n^{(\varepsilon)})} \to \tilde{x}^{(0)}(\infty), \quad \text{as } \varepsilon \to 0. \tag{22}$$

If $\varepsilon^r n^{(\varepsilon)} \to \lambda_r \in [0, \infty)$ for some $1 \leq r \leq k$, it follows from part (**i**) that

$$\frac{\exp(-\rho^{(\varepsilon)}n^{(\varepsilon)})}{\exp(-(\rho^{(0)} + a_1\varepsilon + \cdots + a_{r-1}\varepsilon^{r-1})n^{(\varepsilon)})} \to e^{-\lambda_r a_r}, \quad \text{as } \varepsilon \to 0. \tag{23}$$

Part (**iii**) now follows from (22) and (23). □

4 Discrete Time Regenerative Processes

Let, for every $\varepsilon \geq 0$, $Z_n^{(\varepsilon)}, n = 0, 1, \ldots$, be a discrete time regenerative process with a phase space $\mathcal{X}$ with a σ-field of measurable subsets $\mathcal{B}_{\mathcal{X}}$ and regeneration times $0 < \tau_1^{(\varepsilon)} < \tau_2^{(\varepsilon)} < \cdots$, and let $\tau^{(\varepsilon)}$ be a regenerative stopping time, which regenerates together with the process $Z_n^{(\varepsilon)}$.

This means that the probability $P^{(\varepsilon)}(n, A) = \mathsf{P}\{Z_n^{(\varepsilon)} \in A, \tau^{(\varepsilon)} > n\}$ satisfies the following renewal equation,

$$P^{(\varepsilon)}(n, A) = q^{(\varepsilon)}(n, A) + \sum_{k=0}^{n} P^{(\varepsilon)}(n-k, A) f^{(\varepsilon)}(k), \ n = 0, 1, \ldots, \quad (24)$$

where

$$q^{(\varepsilon)}(n, A) = \mathsf{P}\{Z_n^{(\varepsilon)} \in A, \tau^{(\varepsilon)} \wedge \tau_1^{(\varepsilon)} > n\}, \ n = 0, 1, \ldots,$$

and

$$f^{(\varepsilon)}(n) = \mathsf{P}\{\tau_1^{(\varepsilon)} = n, \tau^{(\varepsilon)} > \tau_1^{(\varepsilon)}\}, \ n = 0, 1, \ldots.$$

A typical example of a regenerative stopping time is $\tau^{(\varepsilon)} = \min(n \geq 1 : Z_n^{(\varepsilon)} \in D^{(\varepsilon)})$, the first hitting time of the process $Z_n^{(\varepsilon)}$ into some set $D^{(\varepsilon)} \in \mathcal{B}_{\mathcal{X}}$.

The results presented in Theorems 1 and 2 can be reformulated for the perturbed renewal equation (24).

Note that $\rho^{(0)} > 0$ if and only if the stopping probability $f^{(0)} = \mathsf{P}\{\tau^{(0)} \leq \tau_1^{(0)}\} > 0$, i.e., the limiting distribution $f^{(0)}(k)$ is improper. In this case, Theorems 1 and 2 describe the so-called quasi-stationary asymptotics for the probabilities $P^{(\varepsilon)}(n, A)$.

Also, $\rho^{(0)} = 0$ if and only if the stopping probability $f^{(0)} = \mathsf{P}\{\tau^{(0)} \leq \tau_1^{(0)}\} = 0$, i.e., the limiting distribution $f^{(0)}(k)$ is proper. In this case, Theorems 1 and 2 describe the so-called pseudo-stationary asymptotics for the probabilities $P^{(\varepsilon)}(n, A)$.

One can find additional comments concerning quasi- and pseudo-stationary phenomena in the book Gyllenberg, Silvestrov (2008).

5 Queuing and Risk Applications

Suppose that a queuing system from the beginning contains u customers. The input flow of customers in the system is described by a sequence of i.i.d. random variables $V_n, n = 1, 2, \ldots$, taking values 1 and 0 with probabilities q and $1 - q$, respectively. If $V_n = 1$ then one customer go in the system at moment n. Such input flow is a discrete time analogue of a Poisson flow. The output flow is described by a sequence of i.i.d. random variables

$W_n, n = 1, 2, \dots$, taking values $0, 1, \dots$, with the corresponding probabilities $h(0), h(1), \dots$. If $W_n = k$ then k customers go out from the system at moment n. We also assume that the random sequences $V_n, n = 1, 2, \dots$, and $W_n, n = 1, 2, \dots$, are independent.

The total number of customers in the system at moment n is given by the following relation,

$$Z_{u,n} = u + \sum_{k=1}^{n} V_k - \sum_{k=1}^{n} W_k, \ n = 0, 1, \dots$$

Note that we formally admit the situation, where the random variables Z_n take negative values.

Usual assumptions are also that: **(a)** $q > 0$; **(b)** $\mathsf{P}\{W_1 > 1\} = 1 - h(0) - h(1) > 0$; **(c)** $\mathsf{E}W_1 = \sum_{x=0}^{\infty} x h(x) < \infty$.

It is possible and convenient to interpret the process $Z_{u,n}$ as a discrete time risk process. It can be done by defining a sequence of non-negative i.i.d. random variables $X_n = W_n - V_n + 1, n = 1, 2, \dots$, and representing the process $Z_{u,n}$ in the following form,

$$Z_{u,n} = u + n - \sum_{k=1}^{n} X_k, \ n = 0, 1, \dots$$

In this case, $Z_{u,n}$ may be interpreted as a capital of an insurance company (counted in units equivalent to expected premium per time unit) at moment n, and the random variable X_n as claims at moment n, counted in the same units.

An object of our interest is the following probability,

$$\psi(u) = \mathsf{P}\{\min_{n \geq 0} Z_{u,n} < 0\}, \ u = 0, 1, \dots, \tag{25}$$

which can be interpreted as a ruin probability for the infinite time horizon.

Let us denote,

$$p = \mathsf{P}\{X_1 > 0\}, \ \mu = \sum_{u=0}^{\infty} u g(u), \ g(u) = \mathsf{P}\{X_1 = u / X_1 > 0\}, \ u = 0, 1, \dots$$

It is not difficult to show that $\psi(u) = 1$ if $\mathsf{E}X_1 = \alpha = p\mu > 1$. Let, therefore, assume that **(d)** $\mathsf{E}X_1 = \alpha = p\mu \leq 1$.

In this case, the ruin probability $\psi(u)$ satisfy the following discrete time renewal equation,

$$\psi(u) = q(u) + \sum_{k=0}^{u} \psi(u-k) f(k), \ u = 0, 1, \dots, \tag{26}$$

where

$$G(u)=\sum_{k=0}^{u} g(k),\ f(u)=\alpha\frac{1-G(u)}{\mu},\ q(u)=\sum_{k=u+1}^{\infty} f(k),\ u=0,1,\dots$$

Just, in order to make the paper self-readable, let us shortly show the way of getting the renewal equation (26). It repeats the way of getting similar continuous time renewal equation for ruin probabilities, given, for example in Feller (1966) or Grandell (1991).

Let $\phi(u)=1-\psi(u)$. By conditioning on X_1 we obtain, for any $u=0,1,\dots,$

$$\begin{aligned}\phi(u)&=\sum_{x=0}^{u+1}\phi(u+1-x)\mathsf{P}\{X_1=x\}\\&=(1-p)\phi(u+1)+p\sum_{x=1}^{u+1}\phi(u+1-x)g(x).\end{aligned}$$

Rearranging this gives, for any $u=0,1,\dots,$

$$\phi(u+1)-\phi(u)=p\phi(u+1)-p\sum_{x=1}^{u+1}\phi(u+1-x)g(x).$$

Using this it follows that for any $u=1,2,\dots,$

$$\begin{aligned}\phi(u)-\phi(0)&=\sum_{t=0}^{u-1}(\phi(t+1)-\phi(t))\\&=p\sum_{t=1}^{u}\phi(t)-p\sum_{t=1}^{u}\sum_{x=1}^{t}\phi(t-x)g(x)\\&=-p\phi(0)+p\sum_{t=0}^{u}\phi(u-t)(1-G(t)).\end{aligned}$$

The left hand side is equal to the right hand side also for $u=0$, thus,

$$\phi(u)=(1-p)\phi(0)+p\sum_{t=0}^{u}\phi(u-t)(1-G(t)),\ u=0,1,\dots \tag{27}$$

It follows from the strong law of large numbers that $n^{-1}Z_{u,n}\to(1-\alpha)$ almost surely as $n\to\infty$. In the case $\alpha<1$ this implies that for almost every $\omega\in\Omega$, there exists a positive integer $N(\omega)$ such that $Z_{u,n}(\omega)>0$

for all $n \geq N(\omega)$. This yields that $\min_{n\geq 0} Z_{u,n}$ is almost surely finite and, therefore, we can conclude that $\phi(u) \to 1$ as $u \to \infty$. From this and (27) it follows by monotone convergence that $1 = (1-p)\phi(0) + \alpha$. Solving for $\phi(0)$, putting this into equation (27) and rearranging we obtain the renewal equation (26) for $\psi(u)$.

In the case, where the parameters of the above risk process depend on a perturbation parameter $\varepsilon \geq 0$, the results presented in Theorem 1 and 2 can be applied to the renewal equation (26) and yield exponential asymptotic expansions for the ruin probability $\psi^{(\varepsilon)}(u)$ which, in this case, also depends on the perturbation parameter ε.

It is worth to note that in the above model, $\rho^{(0)} > 0$ if and only if the limiting parameter $\alpha^{(0)} < 1$. In this case, the quasi-stationary asymptotics for $\psi^{(\varepsilon)}(u)$ given by Theorems 1 and 2 generalizes the classical Cramér-Lundberg approximation for these ruin probabilities to the more advanced form of an exponential asymptotic expansion. Also, $\rho^{(0)} = 0$ if and only if the limiting parameter $\alpha^{(0)} = 1$. In this case, the pseudo-stationary asymptotics for $\psi^{(\varepsilon)}(u)$ given in these theorems generalizes the so-called diffusion approximation for these ruin probabilities.

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Quasi-Stationary Distributions for Perturbed Discrete Time Regenerative Processes

Mikael Petersson*

Department of Mathematics
Stockholm University, Sweden

Abstract

Nonlinearly perturbed discrete time regenerative processes with regenerative stopping times are considered. We define the quasi-stationary distributions for such processes and present conditions for their convergence. Under some additional conditions, the quasi-stationary distributions can be expanded in an asymptotic power series with respect to the perturbation parameter. We give an explicit recurrence algorithm for calculating the coefficients of this asymptotic expansion. Applications to a perturbed alternating regenerative process with absorption are presented.

Key words: Regenerative process, renewal equation, nonlinear perturbation, quasi-stationary distribution, asymptotic expansion.

1 Introduction

Many stochastic systems has a random lifetime, the process is terminated due to some rare event. This means that the stationary distribution of such process will be degenerated. However, before the lifetime of the system goes to an end, one can often observe something that resembles a stationary distribution. It is often of interest to describe such behaviour, so-called quasi-stationary phenomena.

In this paper we study such phenomena for discrete time regenerative processes with regenerative stopping time. Roughly speaking, such a process

**E-mail address:* `mikpe@math.su.se`

$\xi(n)$, $n = 0, 1, \ldots$, regenerates at random times $\tau_1, \tau_2, \ldots$, and has random lifetime μ which regenerates jointly with the process.

In particular, such processes includes discrete time semi-Markov processes with absorption. For example, $\xi(n)$ can be a Markov chain, $\tau_1, \tau_2, \ldots$, the return times to some fixed state and μ the first hitting time of some fixed state.

As a special case, when $\mu = \infty$ almost surely, this class of processes includes regenerative processes without stopping time.

Under some conditions, it can be shown that for such processes there exists a probability distribution $\pi(A)$ such that

$$\mathsf{P}\{\xi(n) \in A/\mu > n\} \to \pi(A), \text{ as } n \to \infty.$$

We call this distribution the *quasi-stationary* distribution and use it to describe the quasi-stationary phenomena of the process. In the case $\mu = \infty$ almost surely, $\pi(A)$ is the usual stationary distribution.

Quasi-stationary distributions have been studied intensively since the 1960's. Some of the important early works were Vere-Jones (1962), Kingman (1963), Darroch, Seneta (1965, 1967) and Seneta, Vere-Jones (1966).

In this paper, we consider the case when $\xi(n)$ is perturbed and that the perturbation is described by a small parameter ε. Furthermore, it is assumed that some continuity conditions hold at $\varepsilon = 0$ for certain characteristics of the process $\xi^{(\varepsilon)}(n)$, regarded as a function of ε. This allows us to interpret $\xi^{(\varepsilon)}(n)$ as a perturbed version of the process $\xi^{(0)}(n)$.

We want the quasi-stationary distribution $\pi^{(\varepsilon)}(A)$ of the process $\xi^{(\varepsilon)}(n)$ to be an approximation of the quasi-stationary distribution $\pi^{(0)}(A)$ of the process $\xi^{(0)}(n)$, that is $\pi^{(\varepsilon)}(A) \to \pi^{(0)}(A)$ as $\varepsilon \to 0$.

We give conditions such that the quasi-stationary distribution can be expanded as

$$\pi^{(\varepsilon)}(A) = \pi^{(0)}(A) + f_1(A)\varepsilon + \cdots + f_k(A)\varepsilon^k + o(\varepsilon^k),$$

where the coefficients $f_1(A), \ldots, f_k(A)$ can be calculated from an explicit recurrence algorithm.

Theoretical results are illustrated with applications to an alternating regenerative process with absorption. Under perturbation conditions on distributions of sojourn times and absorption probabilities, we show explicitly how to build the asymptotic expansion of the quasi-stationary distribution for such a process.

The results in the present paper are based on the theory of the perturbed renewal equation in discrete time developed in Gyllenberg, Silvestrov (1994), Englund, Silvestrov (1997) and Petersson, Silvestrov (2012).

Corresponding results for perturbed regenerative processes in continuous time can be found in the book Gyllenberg, Silvestrov (2008) where one can also find an extended bibliography of works in the area. Some similar results for perturbed discrete time Markov chains are given in Silvestrov (2000).

Some works related to asymptotic expansions for perturbed Markov chains are Latouche (1988), Hassin, Haviv (1992), Khasminskii, Yin, Zhang (1996), Yin, Zhang (2003), Altman, Avrachenkov, Núñes-Queija (2004) and Yin, Nguyen (2009).

2 Quasi-Stationary Distributions for Regenerative Processes

For every $\varepsilon \geq 0$, let $\xi^{(\varepsilon)}(n)$ be a regenerative process in discrete time with a measurable phase space (X, Γ) and regeneration times $0 = \tau_0^{(\varepsilon)} < \tau_1^{(\varepsilon)} < \cdots$, and let $\mu^{(\varepsilon)}$ be a random variable defined on the same probability space $(\Omega, \mathcal{F}, \mathsf{P})$ as the process $\xi^{(\varepsilon)}(n)$ and the regeneration times $0 = \tau_0^{(\varepsilon)} < \tau_1^{(\varepsilon)} < \cdots$, and taking values in the set $\{0, 1, \dots, \infty\}$.

We call $\mu^{(\varepsilon)}$ a regenerative stopping time for the regenerative process $\xi^{(\varepsilon)}(n)$ if for any $A \in \Gamma$, the probabilities $P^{(\varepsilon)}(n, A) = \mathsf{P}\{\xi^{(\varepsilon)}(n) \in A, \mu^{(\varepsilon)} > n\}$ satisfies the renewal equation,

$$P^{(\varepsilon)}(n, A) = q^{(\varepsilon)}(n, A) + \sum_{k=0}^{n} P^{(\varepsilon)}(n-k, A) f^{(\varepsilon)}(k), \ n = 0, 1, \dots, \tag{1}$$

where

$$q^{(\varepsilon)}(n, A) = \mathsf{P}\{\xi^{(\varepsilon)}(n) \in A, \mu^{(\varepsilon)} > n, \tau_1^{(\varepsilon)} > n\}$$

and

$$f^{(\varepsilon)}(n) = \mathsf{P}\{\tau_1^{(\varepsilon)} = n, \mu^{(\varepsilon)} > \tau_1^{(\varepsilon)}\}.$$

Note that the defect $f^{(\varepsilon)}$ of the distribution $f^{(\varepsilon)}(n)$ is given by the stopping probability in one regeneration period for the process $\xi^{(\varepsilon)}(n)$, that is,

$$f^{(\varepsilon)} = 1 - \sum_{n=0}^{\infty} f^{(\varepsilon)}(n) = \mathsf{P}\{\mu^{(\varepsilon)} \leq \tau_1^{(\varepsilon)}\}.$$

We consider the case where the stopping probability in one regeneration period for the limiting process may be positive, i.e. $f^{(0)} \in [0, 1)$.

In the case $f^{(\varepsilon)} > 0$, the renewal equation (1) is improper. However, under some conditions, (1) can be transformed into a proper renewal equation. For this we define $\rho^{(\varepsilon)}$ as the solution to the characteristic equation

$$\sum_{n=0}^{\infty} e^{\rho n} f^{(\varepsilon)}(n) = 1. \tag{2}$$

The properties of $\rho^{(\varepsilon)}$ in the case where the limiting distribution is proper were studied in Englund, Silvestrov (1997). These results were extended to the more general case when the limiting distribution can be improper in Petersson, Silvestrov (2012).

Assume that the distributions $f^{(\varepsilon)}(n)$ satisfy the following conditions.

A: (**a**) $f^{(\varepsilon)}(n) \to f^{(0)}(n)$ as $\varepsilon \to 0$, $n = 0, 1, \ldots$, where the limiting distribution is non-periodic and not concentrated in zero.

(**b**) $f^{(\varepsilon)} \to f^{(0)} \in [0, 1)$ as $\varepsilon \to 0$.

B: There exists $\delta > 0$ such that

(**a**) $\overline{\lim}_{0 \leq \varepsilon \to 0} \sum_{n=0}^{\infty} e^{\delta n} f^{(\varepsilon)}(n) < \infty$.

(**b**) $\sum_{n=0}^{\infty} e^{\delta n} f^{(0)}(n) > 1$.

The following result from Petersson, Silvestrov (2012) gives some basic properties of $\rho^{(\varepsilon)}$ that will be used in what follows.

Lemma 1. *Assume that* **A** *and* **B** *hold. Then there exists a unique non-negative solution* $\rho^{(\varepsilon)}$ *of the characteristic equation* (2) *for* ε *small enough and* $\rho^{(\varepsilon)} \to \rho^{(0)} < \delta$ *as* $\varepsilon \to 0$.

For the rest of the paper, assume that **A** and **B** hold so that $\rho^{(\varepsilon)}$ is well defined for ε small enough. Also, to avoid repetition, we assume that ε always is small enough to satisfy the statements of Lemma 1. If both sides in (1) are multiplied by $e^{\rho^{(\varepsilon)} n}$, we see that the transformed probabilities $\tilde{P}(n, A) = e^{\rho^{(\varepsilon)} n} P(n, A)$ satisfy

$$\tilde{P}^{(\varepsilon)}(n, A) = \tilde{q}^{(\varepsilon)}(n, A) + \sum_{k=0}^{n} \tilde{P}^{(\varepsilon)}(n-k, A) \tilde{f}^{(\varepsilon)}(k), \ \ A \in \Gamma, \tag{3}$$

where

$$\tilde{q}^{(\varepsilon)}(n, A) = e^{\rho^{(\varepsilon)} n} q^{(\varepsilon)}(n, A), \ \ \tilde{f}^{(\varepsilon)}(n) = e^{\rho^{(\varepsilon)} n} f^{(\varepsilon)}(n).$$

It follows from the definition of $\rho^{(\varepsilon)}$, that (3) is a proper renewal equation. In order to apply the classical discrete time renewal theorem, the following condition is imposed on the tail probabilities of $\tau_1^{(\varepsilon)} \wedge \mu^{(\varepsilon)}$.

C: There exists $\gamma > 0$ such that

$$\overline{\lim_{\varepsilon\to 0}} \sum_{n=0}^{\infty} e^{(\rho^{(0)}+\gamma)n} q^{(\varepsilon)}(n, X) < \infty.$$

For any $\varepsilon \geq 0$, we define the *quasi-stationary distribution* of $\xi^{(\varepsilon)}(n)$ by

$$\pi^{(\varepsilon)}(A) = \frac{\sum_{n=0}^{\infty} e^{\rho^{(\varepsilon)}n} q^{(\varepsilon)}(n, A)}{\sum_{n=0}^{\infty} e^{\rho^{(\varepsilon)}n} q^{(\varepsilon)}(n, X)}, \ A \in \Gamma. \tag{4}$$

Under conditions **A**, **B** and **C** the quasi-stationary distribution is well defined for sufficiently small ε.

Let us also assume the following condition:

D: $X \in \Gamma_0 := \{A \in \Gamma : q^{(\varepsilon)}(n, A) \to q^{(0)}(n, A) \text{ as } \varepsilon \to 0, n = 0, 1, \ldots\}$.

Note that Γ_0 is an algebra but does not necessarily coincide with Γ.

The first part of the following result motivates why it is natural to call $\pi^{(\varepsilon)}(A)$ quasi-stationary distributions. The second part gives conditions for convergence of $\pi^{(\varepsilon)}(A)$ for sets $A \in \Gamma_0$.

Theorem 1. *Assume that* **A**, **B** *and* **C** *hold.*

(**i**) *Then for sufficiently small* ε,

$$\mathsf{P}\{\xi^{(\varepsilon)}(n) \in A/\mu^{(\varepsilon)} > n\} \to \pi^{(\varepsilon)}(A), \ \text{as } n \to \infty, \ A \in \Gamma.$$

(**ii**) *If, in addition, condition* **D** *holds, then*

$$\pi^{(\varepsilon)}(A) \to \pi^{(0)}(A), \ \text{as } \varepsilon \to 0, \ A \in \Gamma_0.$$

Proof. First note that if the limiting distribution $f^{(0)}(n)$ is non-periodic, then there exists a finite positive integer N such that

$$\gcd\{1 \leq n \leq N : f^{(0)}(n) > 0\} = 1.$$

It follows from condition **A** that the distributions $\tilde{f}^{(\varepsilon)}(n)$ are non-periodic for ε sufficiently small, say $\varepsilon \leq \varepsilon_1$. Let $\tilde{m}_1^{(\varepsilon)}$ denote the expectation of $\tilde{f}^{(\varepsilon)}(n)$. Since $\rho^{(0)} < \delta$ we can choose $\delta_0 > 0$ such that $\rho^{(0)} < \delta - \delta_0$. Let $C = \sup_{n\geq 0} ne^{-\delta_0 n}$. Since $\rho^{(\varepsilon)} \to \rho^{(0)}$ and condition **B** holds it follows that

$$\overline{\lim_{\varepsilon\to 0}} \, \tilde{m}_1^{(\varepsilon)} = \overline{\lim_{\varepsilon\to 0}} \sum_{n=0}^{\infty} ne^{\rho^{(\varepsilon)}n} f^{(\varepsilon)}(n) \leq \overline{\lim_{\varepsilon\to 0}} \sum_{n=0}^{\infty} ne^{(\delta-\delta_0)n} f^{(\varepsilon)}(n)$$
$$\leq C \, \overline{\lim_{\varepsilon\to 0}} \sum_{n=0}^{\infty} e^{\delta n} f^{(\varepsilon)}(n) < \infty.$$

It follows that $\tilde{m}_1^{(\varepsilon)}$ is finite for all ε small enough, say $\varepsilon \le \varepsilon_2$. Condition **C** implies that for any $A \in \Gamma$

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}} \sum_{n=0}^{\infty} \tilde{q}^{(\varepsilon)}(n,A) &= \overline{\lim_{\varepsilon\to 0}} \sum_{n=0}^{\infty} e^{\rho^{(\varepsilon)} n} \mathsf{P}\{\xi^{(\varepsilon)}(n) \in A, \mu^{(\varepsilon)} > n, \tau_1^{(\varepsilon)} > n\} \\ &\le \overline{\lim_{\varepsilon\to 0}} \sum_{n=0}^{\infty} e^{(\rho^{(0)}+\gamma)n} \mathsf{P}\{\tau_1^{(\varepsilon)} \wedge \mu^{(\varepsilon)} > n\} < \infty,\end{aligned}$$

so there exists $\varepsilon_3 > 0$ such that $\sum_{n=0}^{\infty} \tilde{q}^{(\varepsilon)}(n,A) < \infty$ for all $\varepsilon \le \varepsilon_3$.

Define $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. It follows from the classical discrete time renewal theorem that for any $\varepsilon \le \varepsilon_0$,

$$\tilde{P}^{(\varepsilon)}(n,A) \to \frac{1}{\tilde{m}_1^{(\varepsilon)}} \sum_{n=0}^{\infty} \tilde{q}^{(\varepsilon)}(n,A), \text{ as } n \to \infty, \ A \in \Gamma.$$

Part (**i**) follows from this since

$$\mathsf{P}\{\xi^{(\varepsilon)}(n) \in A / \mu^{(\varepsilon)} > n\} = \tilde{P}^{(\varepsilon)}(n,A) \,/\, \tilde{P}^{(\varepsilon)}(n,X).$$

Condition **C** and the definition of Γ_0 implies that

$$\lim_{\varepsilon\to 0} \sum_{n=0}^{\infty} e^{\rho^{(\varepsilon)} n} q^{(\varepsilon)}(n,A) = \sum_{n=0}^{\infty} e^{\rho^{(0)} n} q^{(0)}(n,A) < \infty, \ A \in \Gamma_0.$$

Part (**ii**) follows from this and condition **D**. □

3 Asymptotic Expansions of Quasi-Stationary Distributions

A problem with $\pi^{(\varepsilon)}(A)$ is that the expression defining it is rather complicated. Both numerator and denominator are represented as infinite sums and involves $\rho^{(\varepsilon)}$, which is only given as the solution to the nonlinear equation (2). However, under some perturbation conditions, $\pi^{(\varepsilon)}(A)$ can be expanded in an asymptotic power series with respect to ε.

In order to do this, we first need to expand $\rho^{(\varepsilon)}$. This can be done under some perturbation conditions on the following mixed power-exponential moments of the distributions $f^{(\varepsilon)}(n)$,

$$\phi^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} f^{(\varepsilon)}(n), \ \rho \ge 0, \ r = 0, 1, \dots$$

To expand the quasi-stationary distribution, some perturbation conditions on the following mixed power-exponential moment type functionals of $q^{(\varepsilon)}(n, A)$ are also needed,

$$\omega^{(\varepsilon)}(\rho, r, A) = \sum_{n=0}^{\infty} n^r e^{\rho n} q^{(\varepsilon)}(n, A),\ \rho \geq 0,\ r = 0, 1, \ldots,\ A \in \Gamma.$$

The perturbation conditions are the following:

$\mathbf{P_1^{(k)}}$: $\phi^{(\varepsilon)}(\rho^{(0)}, r) = \phi^{(0)}(\rho^{(0)}, r) + a_{1,r}\varepsilon + \cdots + a_{k-r,r}\varepsilon^{k-r} + o(\varepsilon^{k-r})$, for $r = 0, \ldots, k$, where $|a_{n,r}| < \infty$, $n = 1, \ldots, k-r$, $r = 0, \ldots, k$.

$\mathbf{P_2^{(k)}}$: $\omega^{(\varepsilon)}(\rho^{(0)}, r, A) = \omega^{(0)}(\rho^{(0)}, r, A) + b_{1,r}(A)\varepsilon + \cdots + b_{k-r,r}(A)\varepsilon^{k-r} + o(\varepsilon^{k-r})$, for $r = 0, \ldots, k$, where $A \in \Gamma_0$ and $|b_{n,r}(A)| < \infty$, $n = 1, \ldots, k-r$, $r = 0, \ldots, k$.

For convenience, we define $a_{0,r} = \phi^{(0)}(\rho^{(0)}, r)$ and $b_{0,r} = \omega^{(0)}(\rho^{(0)}, r, A)$ for $r = 0, \ldots, k$ and $A \in \Gamma_0$.

Now we are ready to give the expansion of $\pi^{(\varepsilon)}(A)$. The details are presented in the following theorem.

Theorem 2. *Suppose that* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{P_1^{(k)}}$ *hold.*

(**i**) *Then the root* $\rho^{(\varepsilon)}$ *of the characteristic equation* (2) *has the asymptotic expansion*

$$\rho^{(\varepsilon)} = \rho^{(0)} + c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k).$$

The coefficients $c_1, \ldots, c_k$ *are given by the recurrence formulas*

$$\begin{aligned} c_1 &= -a_{1,0}/a_{0,1}, \\ c_n &= -\frac{1}{a_{0,1}}\Bigg(a_{n,0} + \sum_{q=1}^{n-1} a_{n-q,1}c_q \\ &\qquad + \sum_{m=2}^{n}\sum_{q=m}^{n} a_{n-q,m} \cdot \sum_{n_1,\ldots,n_{q-1} \in D_{m,q}} \prod_{p=1}^{q-1} \frac{c_p^{n_p}}{n_p!}\Bigg),\ n = 2, \ldots, k, \end{aligned} \tag{5}$$

where $D_{m,q}$ *is the set of all nonnegative integer solutions to the system*

$$n_1 + \cdots + n_{q-1} = m,\ n_1 + \cdots + (q-1)n_{q-1} = q.$$

(ii) *If, in addition,* **C**, **D** *and* $\mathbf{P_2^{(k)}}$ *hold, then for any* $A \in \Gamma_0$ *the following asymptotic expansion holds,*

$$\pi^{(\varepsilon)}(A) = \pi^{(0)}(A) + f_1(A)\varepsilon + \cdots + f_k(A)\varepsilon^k + o(\varepsilon^k).$$

The coefficients $f_1(A), \ldots, f_n(A)$ *are given by*

$$f_n(A) = \frac{1}{d_0(X)} \left(d_n(A) - \sum_{q=0}^{n-1} d_{n-q}(X) f_q(A) \right), \tag{6}$$

where $d_0(A) = \omega^{(0)}(\rho^{(0)}, 0, A)$ *and* $f_0(A) = \pi^{(0)}(A)$*. The coefficients* $d_1(A), \ldots, d_k(A)$ *are given by*

$$\begin{aligned} d_1(A) &= b_{1,0}(A) + b_{0,1}(A)c_1, \\ d_n(A) &= b_{n,0}(A) + \sum_{q=1}^{n} b_{n-q,1}(A)c_q \\ &\quad + \sum_{m=2}^{n} \sum_{q=m}^{n} b_{n-q,m}(A) \cdot \sum_{n_1,\ldots,n_{q-1} \in D_{m,q}} \prod_{p=1}^{q-1} \frac{c_p^{n_p}}{n_p!}, \ n = 2, \ldots, k. \end{aligned} \tag{7}$$

Proof. For the proof of part (**i**), see Petersson, Silvestrov (2012). Here we give the proof of part (**ii**).

Let $\Delta^{(\varepsilon)} = \rho^{(\varepsilon)} - \rho^{(0)}$. Using the Taylor expansion of the exponential function, we obtain for any $n = 0, 1, \ldots,$

$$e^{\rho^{(\varepsilon)} n} = e^{\rho^{(0)} n} \left(\sum_{r=0}^{k} \frac{n^r (\Delta^{(\varepsilon)})^r}{r!} + \frac{n^{k+1} (\Delta^{(\varepsilon)})^{k+1}}{(k+1)!} e^{|\Delta^{(\varepsilon)}| n} \theta_{k+1}^{(\varepsilon)}(n) \right),$$

where $0 \leq \theta_{k+1}^{(\varepsilon)}(n) \leq 1$. Since $\rho^{(\varepsilon)} \to \rho^{(0)}$, there exists $\beta < \rho^{(0)} + \gamma$ and $\varepsilon_1 = \varepsilon_1(\beta)$ such that

$$\rho^{(0)} + |\Delta^{(\varepsilon)}| < \beta, \ \varepsilon \leq \varepsilon_1.$$

Let $\tilde{C}_r = \sup_{n \geq 0} n^r e^{(\rho^{(0)} + \gamma - \beta) n}$. From condition **C** it follows that there exists $\varepsilon_2 > 0$ and a constant C_r such that

$$\begin{aligned} \omega^{(\varepsilon)}(\beta, r, A) &= \sum_{n=0}^{\infty} n^r e^{\beta n} q^{(\varepsilon)}(n, A) \\ &\leq \tilde{C}_r \sum_{n=0}^{\infty} e^{(\rho^{(0)} + \gamma) n} \mathsf{P}\{\tau_1^{(\varepsilon)} \wedge \mu^{(\varepsilon)} > n\} \leq C_r, \ \varepsilon \leq \varepsilon_2. \end{aligned}$$

Define $\varepsilon_0 = \varepsilon_0(\beta) := \min\{\varepsilon_1(\beta), \varepsilon_2\}$. Substituting the Taylor expansion of $e^{\rho^{(\varepsilon)}n}$ into the definition of $\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A)$ yields

$$\begin{aligned}\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A) = \omega^{(\varepsilon)}(\rho^{(0)}, 0, A) + \omega^{(\varepsilon)}(\rho^{(0)}, 1, A)\Delta^{(\varepsilon)} + \cdots \\ + \omega^{(\varepsilon)}(\rho^{(0)}, k, A)(\Delta^{(\varepsilon)})^k/k! + r_{k+1}^{(\varepsilon)}(\Delta^{(\varepsilon)})^{k+1},\end{aligned} \tag{8}$$

where

$$r_{k+1}^{(\varepsilon)} = \frac{1}{(k+1)!}\sum_{n=0}^{\infty} n^{k+1} e^{(\rho^{(0)}+|\Delta^{(\varepsilon)}|)n}\theta_{k+1}^{(\varepsilon)}(n) q^{(\varepsilon)}(n, A).$$

If $\varepsilon \le \varepsilon_0$, the right hand side of (8) is finite and

$$r_{k+1}^{(\varepsilon)} \le \frac{1}{(k+1)!}\,\omega^{(\varepsilon)}(\beta, k+1, A) \le \frac{C_{k+1}}{(k+1)!}.$$

It follows that there exists a finite constant M_{k+1} and numbers $0 \le \theta_{k+1}^{(\varepsilon)} \le 1$ such that

$$r_{k+1}^{(\varepsilon)} = M_{k+1}\theta_{k+1}^{(\varepsilon)},\ \ \varepsilon \le \varepsilon_0. \tag{9}$$

Since **A**, **B** and $\mathbf{P_1^{(k)}}$ hold, it follows from part (**i**) that

$$\Delta^{(\varepsilon)} = c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k). \tag{10}$$

Substituting (9), (10) and condition $\mathbf{P_2^{(k)}}$ into the right hand side of (8) when $k = 0$ we see that $\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A) \to \omega^{(0)}(\rho^{(0)}, 0, A)$ as $\varepsilon \to 0$, which means that we have the representation

$$\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A) = \omega^{(0)}(\rho^{(0)}, 0, A) + \omega_0^{(\varepsilon)}(A), \tag{11}$$

where $\omega_0^{(\varepsilon)}(A) \to 0$ as $\varepsilon \to 0$.

Now assume that $k = 1$. If we substitute (9), (10), (11) and condition $\mathbf{P_2^{(k)}}$ into the right hand side of (8), divide by ε and let $\varepsilon \to 0$, it is found that

$$\frac{\omega_0^{(\varepsilon)}(A)}{\varepsilon} \to b_{1,0}(A) + b_{0,1}(A)c_1,\ \text{ as } \varepsilon \to 0. \tag{12}$$

Using (11) and (12) we obtain the asymptotic representation

$$\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A) = \omega^{(0)}(\rho^{(0)}, 0, A) + d_1(A)\varepsilon + \omega_1^{(\varepsilon)}(A),$$

where $d_1(A) = b_{1,0}(A) + b_{0,1}(A)c_1$ and $\omega_1^{(\varepsilon)}(A)$ is of order $o(\varepsilon)$.

If $k \ge 2$, we can continue in this way and build an asymptotic expansion of order k for $\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A)$. Once the existence of the expansion is proved, the coefficients can be found by collecting the coefficients of equal powers of ε in the expansion of the following expression,

$$\begin{aligned}
&(b_{0,0}(A) + \cdots + b_{k,0}(A)\varepsilon^k + o(\varepsilon^k)) \\
&\quad + (b_{0,1}(A) + \cdots + b_{k-1,1}(A)\varepsilon^{k-1} + o(\varepsilon^{k-1})) \\
&\quad \times (c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k)) + \cdots \\
&\quad + (b_{0,k}(A) + o(1))(c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k))^k/k! + o(\varepsilon^k).
\end{aligned}$$

This yields the expansion

$$\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A) = \omega^{(0)}(\rho^{(0)}, 0, A) + d_1(A)\varepsilon + \ldots + d_k(A)\varepsilon^k + o(\varepsilon^k), \tag{13}$$

where the coefficients $d_1(A), \ldots, d_k(A)$ are given according to (7).

The quasi-stationary distribution can be written as

$$\pi^{(\varepsilon)}(A) = \frac{\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A)}{\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, X)}, \ A \in \Gamma.$$

For sets $A \in \Gamma_0$, the numerator can be expanded as in equation (13). By condition **D**, we always have $X \in \Gamma_0$ so the denominator can also be expanded. Thus, for any $A \in \Gamma_0$,

$$\pi^{(\varepsilon)}(A) = \frac{\omega^{(0)}(\rho^{(0)}, 0, A) + d_1(A)\varepsilon + \ldots + d_k(A)\varepsilon^k + o(\varepsilon^k)}{\omega^{(0)}(\rho^{(0)}, 0, X) + d_1(X)\varepsilon + \ldots + d_k(X)\varepsilon^k + o(\varepsilon^k)}. \tag{14}$$

Using (14), we can build the expansion of $\pi^{(\varepsilon)}(A)$ similarly to how we built the expansion of $\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A)$. To do this, first note that with $k = 0$ in (14) it immediately follows that $\pi^{(\varepsilon)}(A) \to \pi^{(0)}(A)$, which means that we have the representation

$$\pi^{(\varepsilon)}(A) = \pi^{(0)}(A) + \pi_0^{(\varepsilon)}(A), \tag{15}$$

where $\pi_0^{(\varepsilon)}(A) \to 0$ as $\varepsilon \to 0$.

Now put $k = 1$ in (14). Since $\omega^{(0)}(\rho^{(0)}, 0, X) > 0$, it follows that the denominator of (14) is positive for ε small enough. Substituting (15) into the left hand side of (14), rearranging and using the identity $\pi^{(0)}(A)d_0(X) = d_0(A)$ gives the following for sufficiently small ε,

$$\pi_0^{(\varepsilon)}(A)d_0(X) + d_1(X)f_0(A) + o(\varepsilon) = d_1(A)\varepsilon + o(\varepsilon).$$

Dividing both sides by ε and letting $\varepsilon \to 0$, we conclude that

$$\frac{\pi_0^{(\varepsilon)}(A)}{\varepsilon} \to \frac{1}{d_0(X)} \left(d_1(A) - d_1(X)f_0(A)\right), \text{ as } \varepsilon \to 0.$$

Using this and (15), the following asymptotic representation is obtained,

$$\pi^{(\varepsilon)}(A) = \pi^{(0)}(A) + f_1(A)\varepsilon + \pi_1^{(\varepsilon)}(A),$$

where $f_1(A) = (d_1(A) - d_1(X)f_0(A))/d_0(X)$ and $\pi_1^{(\varepsilon)}(A)$ is of order $o(\varepsilon)$.

This proves part (**ii**) when $k = 1$.

If $k \geq 2$ we can continuing in this way and prove that the asymptotic expansion of $\pi^{(\varepsilon)}(A)$ exists. When we know that the expansion exists, the coefficients can be found in the following way. Consider the equation

$$\begin{aligned}&(f_0(A) + f_1(A)\varepsilon + \cdots + f_k(A)\varepsilon^k + o(\varepsilon^k))\\ &\quad \times (d_0(X) + d_1(X)\varepsilon + \cdots + d_k(X)\varepsilon^k + o(\varepsilon^k))\\ &\quad = (d_0(A) + d_1(A)\varepsilon + \cdots + d_k(A)\varepsilon^k + o(\varepsilon^k)).\end{aligned}$$

The coefficients $f_k(A)$ are obtained by equating the coefficients of ε^k in both sides of this equation. This yields

$$\pi^{(\varepsilon)}(A) = \pi^{(0)}(A) + f_1(A)\varepsilon + \cdots + f_k(A)\varepsilon^k + o(\varepsilon^k),$$

where $f_1(A), \ldots, f_k(A)$ are given according to the recurrent relation in equation (6). □

4 Applications

In this section, we consider a perturbed alternating regenerative process with absorption. We assume that the process $\eta^{(\varepsilon)}(n)$ starts in state 1 and stays there for a time with distribution $g_1^{(\varepsilon)}(n)$ before it jumps down to state 0. Then the process remains in state 0 for a time with distribution $g_0^{(\varepsilon)}(n)$. Now, with some small probability $p^{(\varepsilon)}$ the process is absorbed in state -1 or with probability $1 - p^{(\varepsilon)}$ the process starts over in state 1.

Such a process can be interpreted as the status of a machine which is successively repaired after break-downs. The states 0 and 1 then represents that the machine is broken or working. Respectively, $g_1^{(\varepsilon)}(n)$ is the distribution of the time between repair and failure and $g_0^{(\varepsilon)}(n)$ is the distribution of the time to locate the error after a break-down. The absorption probability $p^{(\varepsilon)}$ corresponds to a fatal error such that the machine can not be repaired.

We assume the following condition, preventing instant jumps.

E: $g_0^{(\varepsilon)}(0) = g_1^{(\varepsilon)}(0) = 0$ for all $\varepsilon \geq 0$.

Mathematically, this is described by a discrete time semi-Markov process.

Let $(\eta_k^{(\varepsilon)}, \kappa_k^{(\varepsilon)})$ be a Markov renewal chain with phase space $X \times \{1, 2, \ldots\}$, where $X = \{-1, 0, 1\}$, and with transition probabilities

$$q_{ij}^{(\varepsilon)}(n) = \mathsf{P}\{\eta_{k+1}^{(\varepsilon)} = j, \kappa_{k+1}^{(\varepsilon)} = n / \eta_k^{(\varepsilon)} = i\}, \ i, j \in X, \ n = 1, 2, \ldots,$$

given by

$$q_{ij}^{(\varepsilon)}(n) = \begin{cases} g_1^{(\varepsilon)}(n) & i = 1, j = 0, \\ (1 - p^{(\varepsilon)}) g_0^{(\varepsilon)}(n) & i = 0, j = 1, \\ p^{(\varepsilon)} g_0^{(\varepsilon)}(n) & i = 0, j = -1, \\ \chi(n = 1) & i = j = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\nu^{(\varepsilon)}(n) = \max\{k : \gamma^{(\varepsilon)}(k) \leq n\}$, where $\gamma^{(\varepsilon)}(0) = 0$ and $\gamma^{(\varepsilon)}(k) = \kappa_1^{(\varepsilon)} + \ldots + \kappa_k^{(\varepsilon)}$ for $k \geq 1$. The discrete time semi-Markov process corresponding to the Markov renewal chain $(\eta_k^{(\varepsilon)}, \kappa_k^{(\varepsilon)})$ is given by

$$\eta^{(\varepsilon)}(n) = \eta_{\nu^{(\varepsilon)}(n)}^{(\varepsilon)}, \ n = 0, 1, \ldots$$

Let $\nu_j^{(\varepsilon)} = \min\{k \geq 1 : \eta_k^{(\varepsilon)} = j\}$. Then the absorption time is given by $\mu^{(\varepsilon)} = \gamma^{(\varepsilon)}(\nu_{-1}^{(\varepsilon)})$ and the first regeneration time is given by $\tau_1^{(\varepsilon)} = \gamma^{(\varepsilon)}(\nu_1^{(\varepsilon)})$.

The process described above is illustrated in figure 1.

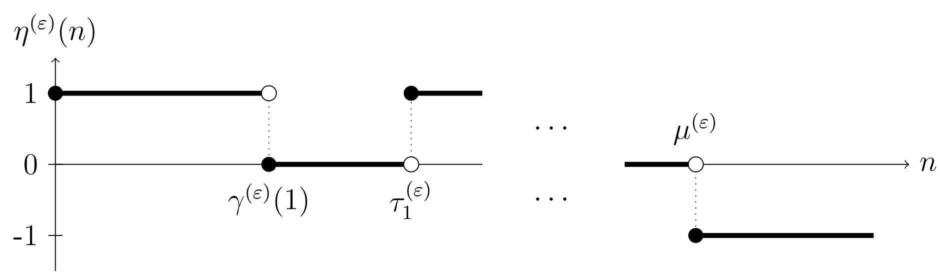

Figure 1: Realization of the process $\eta^{(\varepsilon)}(n)$.

In the definition of a regenerative process with regenerating stopping time it is assumed that the regeneration times are proper random variables. In the process defined above this is not the case. However, the transition probabilities from the absorbing state can be modified in such a way that the return times to state 1 are proper random variables, and that the probabilities $\mathsf{P}\{\eta^{(\varepsilon)}(n) = i, \mu^{(\varepsilon)} > n\}$ are the same for the modified process. We can then apply the results from Sections 2 and 3 to the modified process and interpret the result for the original process.

The weak continuity conditions at $\varepsilon = 0$ are formulated in terms of the local characteristics of the alternating regenerative process as follows.

F: **(a)** $g_i^{(\varepsilon)}(n) \to g_i^{(0)}(n)$ as $\varepsilon \to 0$, $n = 0, 1, \ldots$, $i = 0, 1$.
(b) $p^{(\varepsilon)} \to p^{(0)} \in [0, 1)$ as $\varepsilon \to 0$.

We also need the following non-periodicity condition.

G: At least one of the distributions $g_0^{(0)}(n)$ and $g_1^{(0)}(n)$ is non-periodic.

We introduce the following mixed power-exponential moment generating functions for distributions of sojourn times,

$$\psi_i^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} g_i^{(\varepsilon)}(n), \ \rho \geq 0, \ r = 0, 1, \ldots \ i = 0, 1. \tag{16}$$

Also, consider the following mixed power-exponential moment generating functions,

$$\phi^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} f^{(\varepsilon)}(n), \ \rho \geq 0, \ r = 0, 1, \ldots, \tag{17}$$

where

$$f^{(\varepsilon)}(n) = \mathsf{P}\{\tau_1^{(\varepsilon)} = n, \mu^{(\varepsilon)} > \tau_1^{(\varepsilon)}\}, \ n = 0, 1, \ldots$$

For the exponential moment generating functions, the following relation is obtained,

$$\begin{aligned} \phi^{(\varepsilon)}(\rho, 0) &= (1 - p^{(\varepsilon)}) \sum_{n=0}^{\infty} e^{\rho n} \mathsf{P}\{\kappa_1^{(\varepsilon)} + \kappa_2^{(\varepsilon)} = n\} \\ &= (1 - p^{(\varepsilon)}) \psi_0^{(\varepsilon)}(\rho, 0) \psi_1^{(\varepsilon)}(\rho, 0), \ \rho \geq 0. \end{aligned} \tag{18}$$

From this it follows that existence of (16) and (17) for ε small enough is guaranteed by the following Cramér type condition:

H: There exist $\delta > 0$ such that

(a) $\overline{\lim}_{0 \leq \varepsilon \to 0} \psi_i(\delta, 0) < \infty$, $i = 0, 1$.
(b) $(1 - p^{(0)}) \psi_0^{(0)}(\delta, 0) \psi_1^{(0)}(\delta, 0) > 1$.

We will also use the following mixed power-exponential moment generating functions,

$$\omega_i^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} q_i^{(\varepsilon)}(n), \ \rho \geq 0, \ r = 0, 1, \ldots, \ i = 0, 1, \tag{19}$$

where

$$q_i^{(\varepsilon)}(n) = \mathsf{P}\{\eta^{(\varepsilon)}(n) = i, \tau_1^{(\varepsilon)} \wedge \mu^{(\varepsilon)} > n\}, \ n = 0, 1, \ldots, \ i = 0, 1.$$

If condition $\mathbf{E} - \mathbf{H}$ hold, then condition $\mathbf{A} - \mathbf{D}$ hold, so the results in Section 2 can be applied. Lemma 1 implies that for ε small enough there exists a unique root $\rho^{(\varepsilon)}$ of the characteristic equation

$$\phi^{(\varepsilon)}(\rho, 0) = 1. \tag{20}$$

It is worth noticing that the solution to equation (20) satisfies $\rho^{(\varepsilon)} = 0$ if and only if $p^{(\varepsilon)} = 0$, and $\rho^{(\varepsilon)} > 0$ if and only if $p^{(\varepsilon)} > 0$.

It follows from Theorem 1 that that for ε sufficiently small,

$$\lim_{n \to \infty} \mathsf{P}\{\eta^{(\varepsilon)}(n) = j / \mu^{(\varepsilon)} > n\} = \pi_j^{(\varepsilon)}(\rho^{(\varepsilon)}), \ j = 0, 1,$$

where

$$\pi_j^{(\varepsilon)}(\rho^{(\varepsilon)}) = \frac{\omega_j^{(\varepsilon)}(\rho^{(\varepsilon)}, 0)}{\omega_0^{(\varepsilon)}(\rho^{(\varepsilon)}, 0) + \omega_1^{(\varepsilon)}(\rho^{(\varepsilon)}, 0)}, \ j = 0, 1. \tag{21}$$

If conditions $\mathbf{P_1^{(k)}}$ and $\mathbf{P_2^{(k)}}$ hold for the generating functions (17) and (19), it follows from Theorem 2 that we can build an asymptotic expansion for the quasi-stationary distribution (21). However, it is more convenient to use perturbation conditions for local characteristics of the process $\eta^{(\varepsilon)}(n)$. Therefore, we formulate perturbation conditions on the generating functions of the distributions of sojourn times and the absorption probabilities, and then show how these conditions are related to $\mathbf{P_1^{(k)}}$ and $\mathbf{P_2^{(k)}}$.

We assume the following:

$\mathbf{P_3^{(k)}}$: $p^{(\varepsilon)} = p^{(0)} + p[1]\varepsilon + \ldots + p[k]\varepsilon^k + o(\varepsilon^k)$, where $|p[n]| < \infty$, $n = 1, \ldots, k$.

$\mathbf{P_4^{(k)}}$: $\psi_i^{(\varepsilon)}(\rho^{(0)}, r) = \psi_i^{(0)}(\rho^{(0)}, r) + \psi_i[1, r]\varepsilon + \ldots + \psi_i[k-r, r]\varepsilon^{k-r} + o(\varepsilon^{k-r})$, for $r = 0, \ldots, k$, $i = 0, 1$, where $|\psi_i[n, r]| < \infty$, $n = 1, \ldots, k-r$, $r = 0, \ldots, k$, $i = 0, 1$.

Observe that for $n = 0, 1, \ldots,$

$$\mathsf{P}\{\eta^{(\varepsilon)}(n) = i, \tau_1^{(\varepsilon)} \wedge \mu^{(\varepsilon)} > n\} = \begin{cases} \mathsf{P}\{\kappa_1^{(\varepsilon)} \leq n, \kappa_1^{(\varepsilon)} + \kappa_2^{(\varepsilon)} > n\} & i = 0, \\ \mathsf{P}\{\kappa_1^{(\varepsilon)} > n\} & i = 1. \end{cases}$$

Using this relation, we obtain for $\rho \geq 0$,

$$\omega_i^{(\varepsilon)}(\rho, 0) = \begin{cases} \psi_1^{(\varepsilon)}(\rho, 0)\varphi_0^{(\varepsilon)}(\rho, 0) & i = 0, \\ \varphi_1^{(\varepsilon)}(\rho, 0) & i = 1, \end{cases} \tag{22}$$

where, for $i = 0, 1,$

$$\varphi_i^{(\varepsilon)}(\rho, 0) = \begin{cases} (\psi_i^{(\varepsilon)}(\rho, 0) - 1)/(e^\rho - 1) & \rho > 0, \\ \psi_i^{(\varepsilon)}(0, 1) & \rho = 0. \end{cases} \tag{23}$$

Under condition **H**, the derivative of any order of the function $\varphi_i^{(\varepsilon)}(\rho, 0)$ exists for $0 \leq \rho \leq \beta < \delta$ and sufficiently small ε. Denote the derivative of order r of this function by $\varphi_i^{(\varepsilon)}(\rho, r)$. It follows directly from (23) that

$$\psi_i^{(\varepsilon)}(\rho, 0) = \varphi_i^{(\varepsilon)}(\rho, 0)(e^\rho - 1) + 1, \ \rho \geq 0. \tag{24}$$

By differentiating equation (24) r times and rearranging, it follows that the derivative of order $r = 1, 2, \ldots,$ of the function $\varphi_i^{(\varepsilon)}(\rho, 0)$ is given by the recursive relation

$$\varphi_i^{(\varepsilon)}(\rho, r) = \begin{cases} (\psi_i^{(\varepsilon)}(\rho, r) - e^\rho \sum_{j=0}^{r-1} \binom{r}{j} \varphi_i^{(\varepsilon)}(\rho, j))/(e^\rho - 1) & \rho > 0, \\ (\psi_i^{(\varepsilon)}(0, r+1) - \sum_{j=0}^{r-1} \binom{r+1}{j} \varphi_i^{(\varepsilon)}(0, j))/(r+1) & \rho = 0. \end{cases}$$

In the following, suppose that condition $\mathbf{P_3^{(k)}}$ holds, together with condition $\mathbf{P_4^{(k)}}$ if $\rho^{(0)} > 0$, or together with condition $\mathbf{P_4^{(k+1)}}$ if $\rho^{(0)} = 0$. Then the following asymptotic expansion hold,

$$\varphi_i^{(\varepsilon)}(\rho^{(0)}, r) = \varphi_i^{(0)}(\rho^{(0)}, r) + \varphi_i[1, r]\varepsilon + \ldots + \varphi_i[k-r, r]\varepsilon^{k-r} + o(\varepsilon^{k-r}). \tag{25}$$

Denote $\varphi_i[0, r] = \varphi_i^{(0)}(\rho^{(0)}, r)$.

In the case $\rho^{(0)} > 0$, the coefficients for $i = 0, 1,$ are given by

$$\begin{aligned} &\varphi_i[n, r](e^{\rho^{(0)}} - 1) \\ &= \begin{cases} \psi_i[n, 0] - \delta(n, 0) & n = 0, \ldots, k, \ r = 0, \\ \psi_i[n, r] - e^{\rho^{(0)}} \sum_{j=0}^{r-1} \binom{r}{j} \varphi_i[n, j] & n = 0, \ldots, k-r, \ r = 1, \ldots, k. \end{cases} \end{aligned} \tag{26}$$

In the case $\rho^{(0)} = 0$, the coefficients for $i = 0, 1$, are given by

$$\varphi_i[n,r](r+1) = \begin{cases} \psi_i[n,1] & n = 0,\dots,k,\ r = 0, \\ \psi_i[n,r+1] - \sum_{j=0}^{r-1}\binom{r+1}{j}\varphi_i[n,j] & n = 0,...,k-r,\ r = 1,...,k. \end{cases} \tag{27}$$

Differentiating equation (18) and (22) r times with respect to ρ and evaluating at $\rho = \rho^{(0)}$ yields for any $r = 0, 1, \dots,$

$$\phi^{(\varepsilon)}(\rho^{(0)}, r) = (1 - p^{(\varepsilon)}) \sum_{j=0}^{r} \binom{r}{j} \psi_0^{(\varepsilon)}(\rho^{(0)}, j)\psi_1^{(\varepsilon)}(\rho^{(0)}, r-j), \tag{28}$$

$$\omega_0^{(\varepsilon)}(\rho^{(0)}, r) = \sum_{j=0}^{r} \binom{r}{j} \psi_1^{(\varepsilon)}(\rho^{(0)}, j)\varphi_0^{(\varepsilon)}(\rho^{(0)}, r-j), \tag{29}$$

$$\omega_1^{(\varepsilon)}(\rho^{(0)}, r) = \varphi_1^{(\varepsilon)}(\rho^{(0)}, r). \tag{30}$$

It follows from equations (25) $-$ (30) that conditions $\mathbf{P_1^{(k)}}$ and $\mathbf{P_2^{(k)}}$ are implied by conditions $\mathbf{P_3^{(k)}}$ and $\mathbf{P_4^{(k)}}$ in the case $\rho^{(0)} > 0$, and by conditions $\mathbf{P_3^{(k)}}$ and $\mathbf{P_4^{(k+1)}}$ in the case $\rho^{(0)} = 0$. We can find the relations between the coefficients by using arithmetic rules of asymptotic expansions.

The coefficients in condition $\mathbf{P_1^{(k)}}$ are for any $n = 0, \dots, k - r$ and $r = 0, \dots, k$ given by

$$\begin{aligned} a_{0,r} &= (1 - p^{(0)})h_{0,r},\ a_{n,r} = (1 - p^{(0)})h_{n,r} - \sum_{i=1}^{n} p[i]h_{n-i,r}, \\ h_{n,r} &= \sum_{i=0}^{n}\sum_{j=0}^{r} \binom{r}{j} \psi_0[i,j]\psi_1[n-i, r-j]. \end{aligned} \tag{31}$$

The coefficients in condition $\mathbf{P_2^{(k)}}$ are for any $n = 0, \dots, k - r$ and $r = 0, \dots, k$ given by

$$\begin{aligned} b_{n,r}(\{0\}) &= \sum_{i=0}^{n}\sum_{j=0}^{r} \binom{r}{j} \psi_1[i,j]\varphi_0[n-i, r-j], \\ b_{n,r}(\{1\}) &= \varphi_1[n,r],\ b_{n,r}(X) = b_{n,r}(\{0\}) + b_{n,r}(\{1\}). \end{aligned} \tag{32}$$

It follows from Theorem 2 that an asymptotic expansion of order k exists for the quasi-stationary distribution (21). We can build the expansion using equations (5), (6), (7), (26), (27), (31) and (32).

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