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Key words: Inflation, real bond, inflation-linked bond, hybrid derivative, option.

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EUROPEAN CALL OPTION WITH INFLATION LINKED STRIKE

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Abstract. A European call option with an inflation-linked strike is defined. The pricing formula is derived under the assumption that the quotient between the stock return and the price process of the inflation-linked bond is log-normally distributed. This is fulfilled if the real short rate is assumed to be one of the tree models, Vasicek, Ho-Lee or Hull-White, the inflation and the return processes are geometric Brownian motions. Also calculated are the first order derivatives that are used for hedging, also referred to as the "Greeks".

1. Introduction

In recent years structured products have become more and more popular. The most common construction is simple. It is made out of two parts, one zero coupon bond and a derivative with the same maturity. This construction assures capital safety, the invested money will always be returned. The difference between the value of the zero coupon bond and invested money is used to buy derivative. A common and simple derivative often used is the European call option written on a stock index.

This is an attractive investment opportunity if the stock market return out preforms the interest rate. However, what if the inflation is high. Then the return on the stock market might look good, but when compared to purchasing power the picture might be another. Therefore a structured product that incorporates a safety against inflation and still is exposed to some other risky asset like an equity index is attractive. Dodgson and Kainth [2] noted in their Conclusion section that "an increasingly popular retail product is a bond that pays the maximum of an equity index or a price index." However, they did not proceed and compute the price.

Key words and phrases. Inflation, real bond, inflation-linked bond, hybrid derivative, option.

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It could also be the other way around, with very low interest rates and low inflation expectation such that the nominal yield and the real yield are close. Then a speculation in that the inflation might raise unexpectedly is attractive.

Purchasing power is often measured in terms of Consumer Price Index (CPI). This is "tend to be aimed at being representative of the expiditure of a particular type of consumer" Dodgson and Kainth [2]. We denote the evolution of CPI through time t as $I(t)$.

The CPI is calculated and published by some national agency, with some delay. Therefore, the contracts are written in terms of a delayed index, where the contracted delay is ΔT. The delay is usually 3 month. We define the proportional change of CPI as an index factor

$$K(t,T) = \frac{I(T-\Delta T)}{I(t)}.$$

The payoff of a call option with an inflation-linked strike can be written

$$\max(R(0,T) - xK(t_{base},T),$$

where x is just a constant equivalent to moneyness of a regular European call and $R(0,T)$ is the return of the index over the time period $[0,T]$.

However, the CPI is not directly traded, but indirectly through inflation linked bonds. The redemption of an inflation linked bond at maturity T is proportional to the index change. Hence, there is a time of reference t_{base}, previous to the first day of the life of the contract, which defines a base index value $I(t_{base})$. This base value is then later used in the definition of the payoff of the index-linked bond. A zero coupon inflation-linked bond written at time zero and with a nominal equal to one pays $K(t_{base},T)$, at maturity. The value of this bond anytime in between t is written $P_{CPI}(t_{base},t,T)$. Hence, at maturity the payoff is

$$P_{CPI}(t_{base},T,T) = K(t_{base},T).$$

Furthermore, an inflation-linked bond is quoted in a real yield L, that is

$$P_{CPI}(t_{base},t,T) = K(t_{base},t)\exp(-L(T-t)).$$

The yield is actually a function of time, $L(t,T)$, but it is suppressed. However, we write this yield in continuous compounding, which is not the usual interest convention for real bonds. It is though equivalent and more convenient to work with mathematically.

Denote by Q the pricing martingale probability measure and write $\mathfrak{F}_t$ for the filtration, which is the information up to time t. We assume that the short rate $r(t)$ is stochastic. By the assumption of no arbitrage the price of a real zero coupon bond is,

$$\begin{aligned} P_{CPI}(t_{base}, t, T) &= E^Q \left[K(t_{base}, T) \exp\left(-\int_t^T r(s)\mathrm{d}s \right) \middle| \mathfrak{F}_t \right] \\ &= K(t_{base}, t) P_{CPI}(t - \Delta T, t, T) \\ &= K(t_{base}, t) \exp(-L(T - t)), \end{aligned} \tag{1.1}$$

where we have used that $K(t_{base}, t) \in \mathfrak{F}_t$ and that $K(t_{base}, T) = K(t_{base}, t)K(t-\Delta T, T)$.

Both Hughston and Jarrow & Yildirim made a currency analogy of the inflation to the currency between the nominal rate and the real rate [4, 5]. Hughston model was a one factor short rate model [4]. Jarrow & Yildirim used the Heath-Jarrow-Morton framework to price a call option on the inflation index. More advanced models have been suggested and used on derivatives of inflation risk, see Mercurio, [6], Mercurio & Moreni [7] and Hinnerich [3]. We will work with the most basic model, even if a generalization to a more general model is possible.

With the zero coupon real bonds it is possible to create purchasing power guaranteed structured products out of an inflation-linked bond plus any derivative. However, if the maximum of inflation and an equity index is the target then we have to define an option with a strike that depends on CPI. The strike of this option depends on the index factor $K(t_{base}, T)$. Let us denote the return of a equity index between time t and T by $R(t, T)$. The payoff of the option is given by

$$\max(R(0, T) - xK(t_{base}, T), 0) = \max(R(0, T) - xP_{CPI}(t_{base}, T, T), 0).$$

This simple rewriting of the payoff on the right hand side is important since it resolves the problem that $K(t, T)$ is not traded. We can therefore regard this option as an exchange option.

We will show that for the most common models of the real short interest rates the price of the option is given by the Black-Scholes formula. The real yield replaces the nominal yield and the volatility is computed form the parameters of the underlying processes.

2. Deriving the option price

We postpone to later the definition of the dynamics of the price processes. Let the option be written at time 0 and now we are at t, with maturity is still at T. The information at time t of relevance for the option price is the return $R(0, t)$ and the index factor $K(t_{base}, t)$. The CPI-striked option value at time t is denoted Π_t.

Theorem 2.1 (Black-Scholes formula)**.** *The price Π_{BS} of a European call option with strike x and maturity T at time t, where the underlying $R(0,t)$ follows a geometric Brownian motion with volatility σ and dividend yield q, is given by*

$$\begin{aligned}\Pi_{BS}(R(0,t),x,r,T-t,\sigma,q) &= R(0,t)N(d_1) - x\mathrm{e}^{-r(T-t)}N(d_2)\\ d_1 &= \frac{\log(R(0,t)/x) + (r-q+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\\ d_2 &= d_1 - \sigma\sqrt{T-t}.\end{aligned}$$

where $N(\cdot)$ is the cumulative standard normal distribution function and the interest rate r is here assumed to be constant.

Proof. See any textbook on pricing derivatives for example [1].

Theorem 2.2. *Let*

$$G(t) = \frac{R(0,t)}{P_{CPI}(t_{base},t,T)}$$

and assume that the dynamics of $R(0,t)$ and $P_{CPI}(t_{base},t,T)$ are such that $G(T)$ is log-normally distributed. Then

$$\Pi_t = \Pi_{BS}(R(0,t), xP_{CPI}(t_{base},t,T), 0, T-t, \hat{\sigma}, q),$$

where Π_{BS} denotes the Black-Scholes formula, q is the dividend yield and $\hat{\sigma}^2$ the average squared volatility over $[t,T]$ of $G(t)$ (and the interest rate is equal to zero).

We will later look at some models that will imply that $G(T)$ is log-normally distributed, see section 4.

Proof. If there should be no arbitrage, then there exist a probability measure Q^T such that the quotient

$$H_t = \frac{\Pi_t}{P_{CPI}(t_{base},t,T)},$$

is a Q^T-martingale, see Björk [1]. Especially, we have at maturity

$$H_T = \frac{\max(R(0,T) - xP_{CPI}(t_{base},T,T),0)}{P_{CPI}(t_{base},T,T)} = \max\left(G(T)-x,0\right). \tag{2.1}$$

Note that the right hand side of equation (2.1) is the payoff of an European call option.

Since H_s is a martingale we have that $H_t = \mathbb{E}^{Q^T}\left[H_T|\,\mathfrak{F}_t\right]$, which implies that

$$\begin{aligned}\Pi_t &= P_{CPI}(t_{base},t,T)\mathbb{E}^{Q^T}\left[\max\left(G(T)-x,0\right)|\,\mathfrak{F}_t\right]\\ &= P_{CPI}(t_{base},t,T)\Pi_{BS}(G(t),x,0,T-t,\hat{\sigma},q),\end{aligned}$$

where the volatility $\hat{\sigma}$ is the average volatility of $G(t)$ process. The last step is true since $G(t)$ is assumed to have log-normal distributed. It is the average volatility since it is only the standard deviation of the process at maturity that matters. The process can therefore in the formula be exchanged for a process with constant volatility. Furthermore,

$$\begin{aligned} \Pi_t &= P_{CPI}(t_{base}, t, T)\Pi_{BS}(G(t), x, 0, T-t, \hat{\sigma}, q) \\ &= \Pi_{BS}(G(t)P_{CPI}(t_{base}, t, T), xP_{CPI}(t_{base}, t, T), 0, T-t, \hat{\sigma}, q) \\ &= \Pi_{BS}(R(0,t), xP_{CPI}(t_{base}, t, T), 0, T-t, \hat{\sigma}, q), \end{aligned}$$

by the definition of $G(t)$. □

Our next goal is to define the dynamics of each component such that $G(t)$ is log-normally distributed and then explicitly compute $\hat{\sigma}$. The dynamics of $R(t,T)$, $K(t,T)$ and the short real rate $r_I(t)$ are defined by their Stochastic Differential Equations (SDE). The dynamics of the short interest rate $r(t)$ are not needed since the option price only depends on the zero coupon inflation linked bond. This can be compared with Jarrow and Yildirim [5]. The SDEs are under the measure Q,

$$\begin{aligned} \frac{dR(s,t)}{R(s,t)} &= (r(t)-q)dt + \sigma_R dW_R(t), \\ dr_I(t) &= a(t, r_I)dt + \sigma_r(t)dW_r(t), \\ \frac{dK(s,t)}{K(s,t)} &= \mu(s,t)dt + \sigma_K(s,t)dW_K(t), \end{aligned}$$

where $W_R(t)$, $W_r(t)$ and $W_K(t)$ are standard Brownian motions with correlation coefficients ρ_{Rr}, ρ_{RK} and ρ_{rK}.

Lemma 2.3. *Assume that the diffusion term of the real interest rate only depends on time, that is $\sigma_r(t)$ and*

$$a(t, r_I) = \alpha(t)r_I + \beta(t)$$

Then the model admits an affine term structure, that is

$$P_{CPI}(t-\Delta T, t, T) = \exp\left(A(t,T) - B(t,T)r_I(t)\right). \tag{2.2}$$

Proof. See Björk [1], page 258. □

Lemma 2.4. *Volatility of $P_{CPI}(t-\Delta T, t, T)$ is therefore*

$$\sigma_P(t,T) = -\sigma_r(t)B(t,T)$$

Proof. Use Itô calculus on equation (2.2). □

Lemma 2.5. *The process $G(t)$ satisfies the SDE under the Q^T measure,*

$$\begin{aligned}\frac{dG}{G} &= \sigma_R dW_R - \sigma_K(t_{base}, t)dW_K - 2\sigma_P(t,T)dW_r \\ &= \sigma_R dW_R - 2\sigma_K(t_{base}, t)dW_K + 2\sigma_r(t)B(t,T)dW_r,\end{aligned}$$

and the average volatility is given by

$$\begin{aligned}\hat{\sigma}^2 &= (T-t)^{-1}\int_t^T \Bigg(\sigma_R^2 + \sigma_K(t_{base}, s)^2 + \sigma_r(s)^2 B(s,T)^2 - \rho_{RK}\sigma_R\sigma_K(t_{base}, s) \\ &\quad + \rho_{Rr}\sigma_R\sigma_r(s)B(s,T) - \rho_{rK}\sigma_r(s)B(s,T)\sigma_K(t_{base}, s)\Bigg)\mathrm{d}s.\end{aligned}$$

Proof. The process can be rewritten as

$$G(t) = \frac{R(0,t)}{P_{CPI}(t_{base}, t, T)} = \frac{R(0,t)}{K(t_{base}, t)P_{CPI}(t-\Delta T, t, T)}.$$

Use Itô calculous on the right hand side of the equation and use the fact that $G(t)$ is a Q^T-martingale to conclude that the drift component is equal to zero. Furthermore, the diffusion are unchanged under the transformation from Q to Q^T. Integrate and divide by the interval of integration to get the time average. □

In section 4 we will assume that $K(t,T)$ is a geometric Brownian motion with constant volatility and compute $\hat{\sigma}$ for the three models Vasicek, Ho-Lee and Hull-White of the real short rate.

3. Hedging the option

In this section we derive the Greeks. The return $R(0,t)$ and the zero coupon real bond $P_{CPI}(t_{base}, t, T)$ will be denoted with the short notation R and P, when suited.

To hedge the change of the underlying $R(t,T)$, the usual delta hedge is used, that is

$$\Delta_{BS}(R, xP, 0, T-t, \hat{\sigma}, q) = \frac{\partial}{\partial R}\Pi_{BS}(R, xP, 0, T-t, \hat{\sigma}, q).$$

The equivalent hedge of the zero coupon inflation-linked bond is by the partial derivative of $P_{CPI}(t_{base}, t, T)$,

$$\begin{aligned}\frac{\partial}{\partial P}\Pi_{BS}(R, xP, 0, T-t, \hat{\sigma}, q) &= \frac{\partial}{\partial P}P\Pi_{BS}(R/P, x, 0, T-t, \hat{\sigma}, q) = \\ &= \Pi_{BS}(R/P, x, 0, T-t, \hat{\sigma}, q) - \frac{R}{P}\Delta_{BS}(R/P, x, 0, T-t, \hat{\sigma}, q),\end{aligned}$$

where we used the chain rule of derivation. Note that this partial derivative takes care of two risk factors at the same time. The risk from the index factor $K(t_{base}, t)$ and the risk of a change of the yield curve L.

We omit vega since it is just the same as in Black- Scholes model. Furthermore, the correspondence to rho is always equal to zero, since the interest rate component in the Black-Scholes formula is equal to zero.

However, when we calculate theta it is convenient to convert the pricing formula by

$$\begin{aligned}\Pi_{BS}(R, xP, 0, T-t, \hat{\sigma}, q) &= \Pi_{BS}(R(0,t), xK(t_{base}, t)P_{CPI}(t-\Delta T, t, T), 0, T-t, \hat{\sigma}, q) \\ &= \Pi_{BS}(R(0,t), xK(t_{base}, t), L, T-t, \hat{\sigma}, q),\end{aligned}$$

where we used that $P_{CPI}(t-\Delta T, t, T) = \exp(-L(T-t))$. Now, it is just to use the definition of theta on the right hand side, that is

$$\theta_{BS}(R, xK, L, T-t, \hat{\sigma}, q) = \frac{\partial}{\partial t}\Pi_{BS}(R, xK, L, T-t, \hat{\sigma}, q).$$

The "Greeks" are then used to hedge the instrument. However, note that in a real world situation the zero coupon inflation-linked bond does not exist and have to be created by duration matching by existing bonds (probably coupon bonds).

4. Computing the volatility and pricing

There are some interest rate models that we can compute an analytical expression for $\hat{\sigma}$. We will make this computation for the three most common basic models.

$$\begin{aligned}\text{Vasicek} \quad & \mathrm{d}r_I = (b - ar_I)\mathrm{d}t + \sigma_r \mathrm{d}W_r \\ \text{Ho-Lee} \quad & \mathrm{d}r_I = \theta(t)\mathrm{d}t + \sigma_r \mathrm{d}W_r \\ \text{Hull-White} \quad & \mathrm{d}r_I = (\theta(t) - ar_I)\mathrm{d}t + \sigma_r \mathrm{d}W_r.\end{aligned}$$

These models are affine, that is

$$P_{CPI}(t-\Delta T, t, T) = \exp(A(t,T) - B(t,T)r_I),$$

where $A(t,T)$ and $B(t,T)$ are real functions and

$$\begin{aligned}\text{Vasicek \& Hull-White:} \quad & B(t,T) = (1 - \exp(-a(T-t)))/a, \\ \text{Ho-Lee:} \quad & B(t,T) = T - t.\end{aligned}$$

Moreover, assume σ_K is a constant. We are interested in $B(t,T)$ since it only effects the average volatility. To compute the average volatility, we have to compute the integral of $B(t,T)$ and $B(t,T)^2$ since they are a part of the volatility of $P_{CPI}(t-\delta T,t,T)$, see lemma 2.4.

Lemma 4.1. *For the Ho-Lee model we have that the average volatility is given by,*

$$\hat{\sigma}^2 = \sigma_R^2 + \sigma_K^2 + \sigma_r^2 c_2 - 2\rho_{RK}\sigma_R\sigma_K + 2\rho_{Rr}\sigma_R\sigma_r c_1 - 2\rho_{rK}\sigma_K\sigma_r c_1,$$

where the constants are given by

$$c_1 = \frac{T-t}{2} \text{ and } c_2 = \frac{(T-t)^2}{3}.$$

Proof. First, the constants are the time averages derived by,

$$\begin{aligned} c_1 &= (T-t)^{-1}\int_t^T (T-s)\mathrm{d}s = \frac{T-t}{2} \\ c_2 &= (T-t)^{-1}\int_t^T (T-s)^2\mathrm{d}s = \frac{(T-t)^2}{3}. \end{aligned}$$

The pricing volatility is then for the Ho-Lee model

$$\begin{aligned} \hat{\sigma}^2 &= (T-t)^{-1}\Big(\sigma_R^2(T-t) + \sigma_K^2(T-t) + \sigma_r^2 c_2(T-t) - 2\rho_{RK}\sigma_R\sigma_K(T-t) \\ &\quad +2\rho_{Rr}\sigma_R\sigma_r c_1(T-t) - 2\rho_{rK}\sigma_K\sigma_r c_1(T-t)\Big) \\ &= \sigma_R^2 + \sigma_K^2 + \sigma_r^2 c_2 - 2\rho_{RK}\sigma_R\sigma_K + 2\rho_{Rr}\sigma_R\sigma_r c_1 - 2\rho_{rK}\sigma_K\sigma_r c_1 \end{aligned}$$

□

Lemma 4.2. *For models of Vasicek and Hull-White we have that*

$$\hat{\sigma}^2 = \sigma_R^2 + \sigma_K^2 + \sigma_r^2 b_2 - 2\rho_{RK}\sigma_R\sigma_K + 2\rho_{Rr}\sigma_R\sigma_r b_1 - 2\rho_{rK}\sigma_K\sigma_r b_1.$$

where the constants are given by

$$\begin{aligned} b_1 &= \frac{1}{a} - \frac{1-\exp(-a(T-t))}{a^2(T-t)}, \\ b_2 &= \frac{1}{a^2} - \frac{2(1-\exp(-a(T-t))}{a^3(T-t)} + \frac{(1-\exp(-2a(T-t))}{2a^3(T-t)} \end{aligned}$$

and this implies that the pricing volatility is given by

Proof. Integrate

$$\begin{aligned} ab_1 &= (T-t)^{-1}\int_t^T (1-\exp(-a(T-s)))\mathrm{d}s = 1-\frac{1-\exp(-a(T-t))}{a(T-t)}, \\ a^2b_2 &= (T-t)^{-1}\int_t^T (1-\exp(-a(T-s)))^2\mathrm{d}s \\ &= 1-\frac{2(1-\exp(-a(T-t))}{a(T-t)}+\frac{(1-\exp(-2a(T-t))}{2a(T-t)} \end{aligned}$$

and proceed as in the previous proof. □

In a practical situation all these parameters has to be fitted to existing volatility surfaces of options and yield curves of both real and nominal bonds. Best practice of this can for example be found in Jarrow and Yildirim [5]. The toughest part to estimate is the correlations, since that needs hybrid products. However, there is a practical side of this. If there is no contract that can be used to implicitly derive a parameter then the pricer of the option can more freely choose the parameter. This is due to the fact that there does not exist any other contract to use when trying to do an arbitrage.

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