Mathematical Statistics
Stockholm University

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Research Report 2009:10

ISSN 1650-0377

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Keywords: American option; convergence of option rewards; binomial-trinomial tree approximation; optimal stopping; skeleton approximation; multivariate Markov price process.

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1 Introduction

An American option gives the holder the right to at any moment before some future time T, to exercise the option to receive a payoff determined by a function $g^{(\varepsilon)}(t, \vec{s})$. We study the model with payoff functions admitting power type upper bounds and multivariate exponential Markov price processes $\vec{S}^{(\varepsilon)}(t)$.

Optimal stopping problems for American type options have been also studied in Jacka (1991), Kim (1990), Peskir and Shiryaev (2006), for models with stochastic

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volatility by Zhang and Lim (2006), for American barrier options in Gau, Huang and Subrahmanyam (2000), and for generalized American knock out option in Lundgren (2007, 2010). Convergence of American option rewards have been studies in Amin and Khanna (1994), Coquet and Toldo (2007), Dupuis and Wang (2005), Jönsson (2001, 2005), Jönsson, Kukush and Silvestrov (2004, 2005), Kukush and Silvestrov (2000, 2001, 2004), Prigent (2003), Silvestrov, Galochkin and Sibirtsev (1999). For recent results concerning optimal stopping we refer to the works Dayanik and Karatzas (2003), Ekström, Lindberg, Tysk and Wanntorp (2009), Henderson and Hobson (2008).

We specially would like to mention the papers Silvestrov, Jönsson and Stenberg (2006, 2007, 2009) on convergence of optimal reward functionals for American type options for one-dimensional Markov price processes modulated by stochastic indices.

We consider the convergence of reward functionals for American type options in a multi-asset setting. In the first part of the paper, we generalize the results obtained in the papers mentioned above to the model of multivariate Markov price processes. Despite the main steps for analysis in multivariate case are similar with those in the univariate case, the multivariate aspects complicate the corresponding proofs and do require a separate consideration.

In the second part of the paper, we apply the general convergence results to the multivariate tree type approximation algorithms for American type options with general payoff functions. We present the corresponding approximations for multivariate geometric Brownian price processes and mean-reverse diffusion price processes.

The latter model was used in Schwartz (1997) for description of stochastic dynamics for energy prices. Similar model was also studied in Cortazar Gravet and Urzua (2008), where American options where studied using Monte Carlo simulation.

Further applications presented in the paper relate to the approximation tree type algorithms for the model of optimal reselling of European type options. We refer to the recent paper Lundgren, Silvestrov and Kukush (2008), where one can find additional details in analysis of the reselling problem.

The paper contains of 8 sections. In Section 2, we introduce the model of multivariate price processes and American type options with general payoff functions. In Section 3, we present skeleton type approximations of reward functionals for continuous time price processes by a similar functionals for simpler imbedded discrete time models. In Section 4, the results concerned conditions for convergence of reward functionals in discrete time models are given. Section 5 presents general results on convergence of reward functionals for American type options for continuous time price processes. In Sections 6, we illustrate our general convergence results by applying them to multivariate exponential price processes with independent increments. In Sections 7 and 8, we apply our general convergence results to approximation tree type algorithms for American type options for exponential diffusion type price processes mentioned above.

2 Reward functional for general American type options

In this section we introduce reward functionals for American type options with general payoff functions and formulate conditions which guarantee that the functionals are well defined, i.e take finite values. The corresponding inequalities are also used in following proofs of our convergence results.

For every $\varepsilon \geq 0$, let $\vec{Y}^{(\varepsilon)}(t) = (Y_1^{(\varepsilon)}(t), \ldots, Y_k^{(\varepsilon)}(t))$, $t \geq 0$ be a càdlàg Markov process with the phase space $\mathbb{R}^k$ and transition probabilities $P^{(\varepsilon)}(t, \vec{y}, t+s, A)$. We interpret $\vec{Y}^{(\varepsilon)}(t)$ as a vector log-price process.

Now, we define a vector price process $\vec{S}^{(\varepsilon)}(t) = (S_1^{(\varepsilon)}(t), \ldots, S_k^{(\varepsilon)}(t))$, $t \geq 0$ with the phase space $\mathbb{R}_+^k = \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$, by the relations

$$S_i^{(\varepsilon)}(t) = e^{Y_i^{(\varepsilon)}(t)}, i = 1, \ldots, k,\ t \geq 0. \tag{2.1}$$

Due to the one-to-one mapping and continuity properties of exponential function, $\vec{S}^{(\varepsilon)}(t)$ is also a càdlàg Markov process.

For every $\varepsilon \geq 0$, let $g^{(\varepsilon)}(t, \vec{s})$, $(t, \vec{s}) \in [0, \infty) \times \mathbb{R}_+^k$ be a payoff function. We assume $g^{(\varepsilon)}(t, \vec{s})$ to be a real-valued Borel measurable function. Note that we do not assume payoff functions to be non-negative.

Let $\mathcal{F}_t^{(\varepsilon)} = \sigma(\vec{Y}^{(\varepsilon)}(s), s \leq t)$, $t \geq 0$ be the natural filtration of σ-fields, associated with the vector log-price process $\vec{Y}^{(\varepsilon)}(t), t \geq 0$. It is useful to note that this filtration coincides with the natural filtration generated by the price process $\vec{S}^{(\varepsilon)}(t), t \geq 0$.

We consider Markov moments $\tau^{(\varepsilon)}$ with respect to the filtration $\mathcal{F}_t^{(\varepsilon)}, t \geq 0$. It means that $\tau^{(\varepsilon)}$ is a random variable which takes values in $[0, \infty]$ and with the property $\{\omega : \tau^{(\varepsilon)}(\omega) \leq t\} \in \mathcal{F}_t^{(\varepsilon)}, t \geq 0$.

Let $\mathcal{M}_{\max,T}^{(\varepsilon)}$ be the class of all Markov moments $\tau^{(\varepsilon)} \leq T$, where $T > 0$, and consider a class of Markov moments $\mathcal{M}_T^{(\varepsilon)} \subseteq \mathcal{M}_{\max,T}^{(\varepsilon)}$.

Below we impose conditions on price processes and payoff functions which guarantee that, for all ε small enough,

$$\sup_{\tau^{(\varepsilon)} \in \mathcal{M}_{\max,T}^{(\varepsilon)}} \mathsf{E}|g(\tau^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)})| < \infty. \tag{2.2}$$

The main object of our studies is the reward functional, that is, the maximal expected payoff over different classes of Markov moments, $\mathcal{M}_T^{(\varepsilon)}$,

$$\Phi(\mathcal{M}_T^{(\varepsilon)}) = \sup_{\tau^{(\varepsilon)} \in \mathcal{M}_T^{(\varepsilon)}} \mathsf{E} g^{(\varepsilon)}(\tau^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)})). \tag{2.3}$$

We are interested in conditions of convergence for reward functionals for different classes of stopping times. In particular, we formulate conditions implying the following convergence relation:

$$\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}) \to \Phi(\mathcal{M}^{(0)}_{\max,T}) \text{ as } \varepsilon \to 0. \tag{2.4}$$

The first condition assumes the absolute continuity of payoff functions and imposes power type upper bounds on their partial derivatives:

$\mathbf{A_1}$: There exists $\varepsilon_0 > 0$ such that for every $0 \leq \varepsilon \leq \varepsilon_0$:

- **(a)** function $g^{(\varepsilon)}(t, \vec{s})$ is absolutely continuous in t with respect to the Lebesgue measure on $[0, T]$ for every fixed $\vec{s} \in \mathbb{R}^k_+$ and in $\vec{s}$ with respect to the Lebesgue measure on $\mathbb{R}^k_+$ for every fixed $t \in [0, T]$;
- **(b)** for every $\vec{s} \in \mathbb{R}^k_+$, the partial derivative $|\frac{\partial g^{(\varepsilon)}(t,\vec{s})}{\partial t}| \leq K_1 + K_2 \sum_{j=1}^k s_j^{\gamma_0}$ for almost all $t \in [0, T]$ with respect to the Lebesgue measure on $[0, T]$, where $0 \leq K_1, K_2 < \infty$ and $\gamma_0 \geq 0$;
- **(c)** for every $t \in [0, T]$, the partial derivative $|\frac{\partial g^{(\varepsilon)}(t,\vec{s})}{\partial s_m}| \leq K_3 + K_4 \sum_{j=1}^k s_j^{\gamma_m}$ for almost all $\vec{s} \in \mathbb{R}^k_+$ with respect to the Lebesgue measure on $\mathbb{R}^k_+$, where $0 \leq K_3, K_4 < \infty$ and $\gamma_1, \ldots, \gamma_k \geq 0$, $m = 1, \ldots, k$.
- **(d)** for every $t \in [0, T]$ and for $\vec{s} \in \mathbb{R}^k_+$, the upper limit $\overline{\lim}_{\vec{s} \to \vec{0}} |g^{(\varepsilon)}(t, \vec{s})| \leq K_5$, where $0 \leq K_5 < \infty$.

Note that condition $\mathbf{A_1}$ implies that the function $g^{(\varepsilon)}(t, \vec{s})$ is continuous in $(t, \vec{s}) \in [0, T] \times \mathbb{R}^k_+$.

Denote $e^{\vec{y}} = (e^{y_1}, \ldots, e^{y_k})$. Then condition $\mathbf{A_1}$ can be re-written in the equivalent form in terms of the function $g^{(\varepsilon)}(t, e^{\vec{y}})$:

$\mathbf{A'_1}$: There exists $\varepsilon_0 > 0$ such that for every $0 \leq \varepsilon \leq \varepsilon_0$:

- **(a)** function $g^{(\varepsilon)}(t, e^{\vec{y}})$ is absolutely continuous in t with respect to the Lebesgue measure on $[0, T]$ for every fixed $\vec{y} \in \mathbb{R}^k$ and in $\vec{y}$ with respect to the Lebesgue measure on $\mathbb{R}^k$ for every fixed $t \in [0, T]$;
- **(b)** for every $\vec{y} \in \mathbb{R}^k$, the partial derivative in t is bounded as $\left|\frac{\partial g^{(\varepsilon)}(t,e^{\vec{y}})}{\partial t}\right| \leq K_1 + K_2 \sum_{j=1}^k e^{\gamma_0 y_j}$ for almost all $t \in [0, T]$ with respect to the Lebesgue measure on $[0, T]$, where $0 \leq K_1, K_2 < \infty$ and $\gamma_0 \geq 0$;
- **(c)** for every $t \in [0, T]$, the partial derivative in y_m is bounded as $\left|\frac{\partial g^{(\varepsilon)}(t,e^{\vec{y}})}{\partial y_m}\right| \leq (K_3 + K_4 \cdot \sum_{j=1}^k e^{\gamma_m y_j}) e^{y_m}$ for almost all $\vec{y} \in \mathbb{R}^k$ with respect to the Lebesgue measure on $\mathbb{R}^k$, where $0 \leq K_3, K_4 < \infty$ and $\gamma_1, \ldots, \gamma_k \geq 0$, $m = 1, \ldots, k$.

(d) for every $t \in [0, T]$, the upper limit $\overline{\lim}_{y_i \to -\infty, i=\overline{1,k}} |g^{(\varepsilon)}(t, e^{\vec{y}})| \leq K_5$, where $0 \leq K_5 < \infty$.

We use the notations $\mathsf{E}_{\vec{y},t}$ and $\mathsf{P}_{\vec{y},t}$ for expectation and probability calculated under condition $\vec{Y}^{(\varepsilon)}(t) = \vec{y}$.

For $\beta, c, T > 0, i = 1, \ldots, k$, define the exponential moment modulus of compactness for the càdlàg process $Y_i^{(\varepsilon)}(t)$, $t \geq 0$,

$$\Delta_\beta(Y_i^{(\varepsilon)}(\cdot), c, T) = \sup_{0 \leq t \leq t+u \leq t+c \leq T} \sup_{\vec{y} \in \mathbb{R}^k} \mathsf{E}_{\vec{y},t}(e^{\beta|Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t)|} - 1).$$

We use the following condition for exponential moment modulus of compactness for log-price processes:

$\mathbf{C_1}$: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \Delta_\beta(Y_i^{(\varepsilon)}(\cdot), c, T) = 0$, $i = 1, \ldots, k$ for some $\beta > \gamma$, where $\gamma = \max(\gamma_0, \gamma_1 + 1, \ldots, \gamma_k + 1)$ and $\gamma_0, \gamma_1, \ldots, \gamma_k$ are the parameters introduced in condition $\mathbf{A_1}$,

and also the following condition:

$\mathbf{C_2}$: $\overline{\lim}_{\varepsilon \to 0} \mathsf{E} e^{\beta|Y_i^{(\varepsilon)}(0)|} < \infty$, $i = 1, \ldots, k$, where β is the parameter introduced in condition $\mathbf{C_1}$.

The following lemmas gives upper bounds for moments of the maximum of price processes which are asymptotically uniform, with respect to perturbation parameter.

Lemma 2.1. *Let conditions $\mathbf{C_1}$ and $\mathbf{C_2}$ hold. Then, there exist $0 < \varepsilon_1 \leq \varepsilon_0$ and a constant $L_1 < \infty$ such that for every $\varepsilon \leq \varepsilon_1$, and $i = 1, \ldots, k$,*

$$\mathsf{E} \exp\{\beta \sup_{0 \leq u \leq T} |Y_i^{(\varepsilon)}(u)|\} \leq L_1. \tag{2.5}$$

Proof. The following equality holds for each $0 \leq t \leq T$ and $i = 1, \ldots, k$,

$${}_\beta S_i^{(\varepsilon)}(t) = \exp\{\beta \sup_{0 \leq u \leq t} |Y_i^{(\varepsilon)}(u)|\} = \sup_{0 \leq u \leq t} \exp\{\beta|Y_i^{(\varepsilon)}(u)|\}. \tag{2.6}$$

Note also that, by the definition, the random variable,

$${}_\beta S_i^{(\varepsilon)}(0) = \exp\{\beta|Y_i^{(\varepsilon)}(0)|\}. \tag{2.7}$$

Let us also introduce random variables,

$${}_\beta W_i^{(\varepsilon)}[t', t''] = \sup_{t' \leq t \leq t''} \exp\{\beta|Y_i^{(\varepsilon)}(t) - Y_i^{(\varepsilon)}(t')|\}, \ 0 \leq t' \leq t'' \leq T.$$

Define a partition $\tilde{\Pi}_m = \{0 = v_0^{(m)} < \ldots < v_m^{(m)} = T\}$ on the interval $[0, T]$ by points $v_n^{(m)} = nT/m, n = 0, \ldots, m$. Using equality (2.6) we can get the following inequalities for $n = 1, \ldots m$ and $i = 1, \ldots, k$,

$$\begin{aligned} {}_\beta S_i^{(\varepsilon)}(v_n^{(m)}) &\leq {}_\beta S_i^{(\varepsilon)}(v_{n-1}^{(m)}) + \sup_{v_{n-1}^{(m)} \leq u \leq v_n^{(m)}} \exp\{\beta|Y_i^{(\varepsilon)}(u)|\} \\ &\leq {}_\beta S_i^{(\varepsilon)}(v_{n-1}^{(m)}) + \exp\{\beta|Y_i^{(\varepsilon)}(v_{n-1}^{(m)})|\}_\beta W_i^{(\varepsilon)}[v_{n-1}^{(m)}, v_n^{(m)}] \\ &\leq {}_\beta S_i^{(\varepsilon)}(v_{n-1}^{(m)})({}_\beta W_i^{(\varepsilon)}[v_{n-1}^{(m)}, v_n^{(m)}] + 1). \end{aligned} \tag{2.8}$$

Condition $\mathbf{C_1}$ implies that for any constant $e^{-\beta} < L_5 < 1$ one can choose $c = c(L_5) > 0$ and then $\varepsilon_1 = \varepsilon_1(c) \leq \varepsilon_0$ such that for $\varepsilon \leq \varepsilon_1$, and $i = 1, \ldots, k$,

$$\frac{\Delta_\beta(Y_i^{(\varepsilon)}(\cdot), c, T) + 1}{e^\beta} \leq L_5. \tag{2.9}$$

Moreover, condition $\mathbf{C_2}$ implies that ε_1 can be chosen in such a way that, for some constant $L_6 = L_6(\varepsilon_1) < \infty$, the following inequality holds for $\varepsilon \leq \varepsilon_1$ and $i = 1, \ldots, k$,

$$\mathsf{E} \exp\{\beta|Y_i^{(\varepsilon)}(0)|\} \leq L_6. \tag{2.10}$$

We need to show that the following inequality holds for $\varepsilon \leq \varepsilon_1$ and $i = 1, \ldots, k$,

$$\sup_{0 \leq t' \leq t'' \leq t' + c \leq T} \sup_{\vec{y} \in \mathbb{R}^k} \mathsf{E}_{\vec{y}, t'} {}_\beta W_i^{(\varepsilon)}[t', t''] \leq L_7, \tag{2.11}$$

where

$$L_7 = \frac{e^\beta(e^\beta - 1)L_5}{1 - L_5} < \infty. \tag{2.12}$$

Using condition $\mathbf{C_2}$, relations (2.8), (2.10) – (2.12), and Markov property of the process $\vec{Y}^{(\varepsilon)}(t)$ we get, for $\varepsilon \leq \varepsilon_1$, $i = 1, \ldots, k$ and $m = [T/c] + 1$, and $n = 1, \ldots, m$,

$$\begin{aligned} \mathsf{E}\big({}_\beta S_i^{(\varepsilon)}(v_n^{(m)})\big) &\leq \mathsf{E}\big({}_\beta S_i^{(\varepsilon)}(v_{n-1}^{(m)})\mathsf{E}({}_\beta W_i^{(\varepsilon)}[v_{n-1}^{(m)}, v_n^{(m)}] + 1/\vec{Y}^{(\varepsilon)}(v_{n-1}^{(m)}))\big) \\ &\leq \mathsf{E}_\beta S_i^{(\varepsilon)}(v_{n-1}^{(m)})(L_7 + 1) \leq \cdots \leq \mathsf{E}_\beta S_i^{(\varepsilon)}(0)(L_7 + 1)^n \\ &\leq L_6(L_7 + 1)^n. \end{aligned}$$

Finally, for $\varepsilon \leq \varepsilon_1$ and $i = 1, \ldots, k$, the following inequality holds

$$\mathsf{E}_\beta S_i^{(\varepsilon)}(v_m^{(m)}) = \mathsf{E} \exp\{\beta \sup_{0 \leq u \leq T} |Y_i^{(\varepsilon)}(u)|\} \leq L_6(L_7 + 1)^m. \tag{2.13}$$

Relation (2.13) implies that inequality (2.5) holds, for $\varepsilon \leq \varepsilon_1$, with the constant,

$$L_1 = L_6(L_7 + 1)^m. \tag{2.14}$$

To show that relations (2.11) and (2.12) holds, we have from relation (2.9) that for every $\varepsilon \leq \varepsilon_1$ and $i = 1, \ldots, k$,

$$\sup_{0 \leq t' \leq t \leq t'' \leq t'+c \leq T} \sup_{\vec{y} \in \mathbb{R}^k} \mathsf{P}_{\vec{y},t}\{|Y_i^{(\varepsilon)}(t'') - Y_i^{(\varepsilon)}(t)| \geq 1\} \tag{2.15}$$

$$\leq \sup_{0 \leq t' \leq t \leq t'' \leq t'+c \leq T} \sup_{\vec{y} \in \mathbb{R}^k} \frac{\mathsf{E}_{\vec{y},t} \exp\{\beta |Y_i^{(\varepsilon)}(t'') - Y_i^{(\varepsilon)}(t)|\}}{e^\beta}$$
$$\leq \frac{\Delta_\beta(Y_i^{(\varepsilon)}(\cdot), c, T) + 1}{e^\beta} \leq L_5 < 1.$$

The process $Y_i^{(\varepsilon)}(t)$ is not a Markov process. Despite this, an analogue of Kolmogorov inequality can be obtained by slight modification of its standard proof for Markov processes, see, for example Gikhman and Skorokhod (1971). We formulate it in the form of a lemma.

Lemma 2.2. *Let $a, b > 0$ and for the process $\vec{Y}^{(\varepsilon)}(t)$ assume that the following condition hold* $\sup_{\vec{y} \in \mathbb{R}^k} \mathsf{P}_{\vec{y},t}\{|Y_i^{(\varepsilon)}(t'') - Y_i^{(\varepsilon)}(t)| \geq a\} \leq L < 1$, $t' \leq t \leq t''$. *Then, for any vector $\vec{y} \in \mathbb{R}^k$,*

$$\mathsf{P}_{\vec{y},t'}\{\sup_{t' \leq t \leq t''} |Y_i^{(\varepsilon)}(t) - Y_i^{(\varepsilon)}(t')| \geq a + b\} \tag{2.16}$$

$$\leq \frac{1}{1-L} \mathsf{P}_{\vec{y},t'}\{|Y_i^{(\varepsilon)}(t'') - Y_i^{(\varepsilon)}(t')| \geq b\}.$$

We refer to the report Silvestrov, Jönsson and Stenberg (2006) for the proof.

To shorten notations denote the random variables

$$W_i^+ = \sup_{t' \leq t \leq t''} |Y_i^{(\varepsilon)}(t) - Y_i^{(\varepsilon)}(t')|, \ W_i = |Y_i^{(\varepsilon)}(t'') - Y_i^{(\varepsilon)}(t')|.$$

.

Note that

$$e^{\beta W_i^+} = {}_\beta W_i^{(\varepsilon)}[t', t''].$$

Using (2.15) and Lemma 2.2, we get for every $\varepsilon \leq \varepsilon_1$, $i = 1, \ldots, k$, $0 \leq t' \leq t'' \leq t' + c \leq T$, $\vec{y} \in \mathbb{R}^k$, and $b > 0$,

$$\mathsf{P}_{\vec{y},t'}\{W_i^+ \geq 1 + b\} \leq \frac{1}{1-L_5} \mathsf{P}_{\vec{y},t'}\{W_i \geq b\}. \tag{2.17}$$

Relations (2.9) and (2.17) imply that for every $\varepsilon \leq \varepsilon_1$, $i = 1, \ldots, k$, $0 \leq t' \leq t'' \leq t' + c \leq T$ and $\vec{y} \in \mathbb{R}^k$,

$$
\begin{aligned}
\mathsf{E}_{\vec{y},t'} e^{\beta W_i^+} &= 1 + \beta \int_0^\infty e^{\beta b} \mathsf{P}_{\vec{y},t'}\{W_i^+ \geq b\} db \\
&\leq \ 1 + \beta \int_0^1 e^{\beta b} db + \beta \int_1^\infty e^{\beta b} \mathsf{P}_{\vec{y},t'}\{W_i^+ \geq b\} db \\
&= \ e^\beta + \beta \int_0^\infty e^{\beta(1+b)} \mathsf{P}_{\vec{y},t'}\{W_i^+ \geq 1 + b\} db \\
&\leq \ e^\beta + \frac{\beta e^\beta}{1 - L_5} \int_0^\infty e^{\beta b} \mathsf{P}_{\vec{y},t'}\{W_i \geq b\} db \\
&= \ e^\beta + \frac{\beta e^\beta}{1 - L_5} \frac{\mathsf{E}_{\vec{y},t'} e^{\beta W_i} - 1}{\beta} = \frac{e^\beta}{1 - L_5} (\mathsf{E}_{\vec{y},t'} e^{\beta W_i} - L_5) \\
&\leq \ \frac{e^\beta}{1 - L_5} (\Delta_\beta(Y_i^{(\varepsilon)}(\cdot), c, T) + 1 - L_5) \leq \frac{e^\beta (e^\beta - 1) L_5}{1 - L_5} = L_7.
\end{aligned}
\tag{2.18}
$$

Since inequality (2.18) holds for every $\varepsilon \leq \varepsilon_1$ and $0 \leq t' \leq t'' \leq t' + c \leq T$, $\vec{y} \in \mathbb{R}^k$, it imply relation (2.11). The proof is complete. $\square$

Lemma 2.3. *Let conditions* $\mathbf{A_1}$, $\mathbf{C_1}$, *and* $\mathbf{C_2}$ *hold. Then, there exists a constant* $L_2 < \infty$ *such that for every* $\varepsilon \leq \varepsilon_1$,

$$
\begin{aligned}
&\sup_{\tau^{(\varepsilon)} \in \mathcal{M}_{max,T}^{(\varepsilon)}} \mathsf{E}|g(\tau^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)})| \\
&\qquad \leq \mathsf{E}\big(\sup_{0 \leq u \leq T} |g^{(\varepsilon)}(u, \vec{S}^{(\varepsilon)}(u))|\big)^{\frac{\beta}{\gamma}} \leq L_2.
\end{aligned}
\tag{2.19}
$$

Proof. Consider the vectors $\vec{s}' = (s_1', \ldots, s_k'), \vec{s}'' = (s_1'', \ldots, s_k'') \in \mathbb{R}_+^k$ and construct the vectors $\vec{s}_i = (s_1', \ldots, s_i', s_{i+1}'', \ldots, s_k''), \vec{s}_i(v) = (s_1', \ldots, s_{i-1}', v, s_{i+1}'', \ldots, s_k'')$, $i = 1, \ldots, k$. By the definition, $\vec{s}_k = \vec{s}'$ and $\vec{s}_0 = \vec{s}''$. Define also $s_i^+ = s_i' \vee s_i'', s_i^- = s_i' \wedge s_i'', i = 1, \ldots, k$.

Using condition $\mathbf{A_1}$ we get the following estimates, for $\varepsilon \leq \varepsilon_0$ and $\vec{s}', \vec{s}'' \in \mathbb{R}_+^k$,

$$
|g^{(\varepsilon)}(u, \vec{s}') - g^{(\varepsilon)}(u, \vec{s}'')| \leq \sum_{i=1}^k |g^{(\varepsilon)}(u, \vec{s}_i) - g^{(\varepsilon)}(u, \vec{s}_{i-1})|
$$

$$
\begin{aligned}
&\le \sum_{i=1}^{k} \int_{s_i^-}^{s_i^+} |\frac{\partial g^{(\varepsilon)}(u, \vec{s}_i(v))}{\partial v}| dv \\
&\le \sum_{i=1}^{k} \int_{s_i^-}^{s_i^+} (K_3 + K_4 (\sum_{j=1}^{i-1} (s'_j)^{\gamma_i} + (v_i)^{\gamma_i} + \sum_{j=i+1}^{k} (s''_j)^{\gamma_i})) dv \\
&\le \sum_{i=1}^{k} (K_3 (s_i^+ - s_i^-) + K_4 (\sum_{j=1}^{k} (s_j^+)^{\gamma_i} (s_i^+ - s_i^-))) \\
&\le \sum_{i=1}^{k} (K_3 + K_4 \sum_{j=1}^{k} (s_j^+)^{\gamma_i})(s_i^+ - s_i^-).
\end{aligned}
\tag{2.20}
$$

By letting $\vec{s}' = \vec{s} = (s_1, \ldots, s_k) \in \mathbb{R}_+^k$ and $\vec{s}''_j \downarrow 0$ in (2.20) we get the following inequality, for $\varepsilon \le \varepsilon_0$ and $\vec{s} \in \mathbb{R}_+^k$,

$$
\begin{aligned}
|g^{(\varepsilon)}(u, \vec{s})| &\le K_5 + K_3 \sum_{i=1}^{k} s_i + K_4 \sum_{i=1}^{k} \sum_{j=1}^{k} (s_j)^{\gamma_i} s_i \\
&\le K_5 + K_3 \sum_{i=1}^{k} (1 \vee s_i)^{\gamma} + K_4 \sum_{i=1}^{k} \sum_{j=1}^{k} (1 \vee s_j \vee s_i)^{\gamma} \\
&\le K_5 + K_3 \sum_{i=1}^{k} (1 + (s_i)^{\gamma}) + K_4 \sum_{i=1}^{k} \sum_{j=1}^{k} (1 + (s_j)^{\gamma} + (s_i)^{\gamma}) \\
&= L_8 + L_9 \sum_{i=1}^{k} (s_i)^{\gamma},
\end{aligned}
\tag{2.21}
$$

where

$$
L_8 = K_5 + kK_3 + k^2 K_4, \quad L_9 = K_3 + 2kK_4.
$$

Let us denote for the moment

$$
S_i = \sup_{0 \le u \le T} S_i^{(\varepsilon)}(u), \ Y_i = \sup_{0 \le u \le T} |Y_i^{(\varepsilon)}(u)|, \ i = 1, \ldots, k.
$$

Using inequality (2.21) we get

$$
\sup_{0 \le u \le T} |g^{(\varepsilon)}(u, \vec{S}^{(\varepsilon)}(u))| \le L_8 + L_9 \sum_{i=1}^{k} S_i^{\gamma}. \tag{2.22}
$$

Using relation (2.6) from the proof of Lemma 2.1 and inequality (2.21) we get

$$
\begin{aligned}
&\big(\sup_{0\le u\le T}|g^{(\varepsilon)}(u,\vec{S}^{(\varepsilon)}(u))|\big)^{\frac{\beta}{\gamma}} \\
&\qquad \le (k+1)^{\frac{\beta}{\gamma}-1}(L_8^{\frac{\beta}{\gamma}} + L_9^{\frac{\beta}{\gamma}}\sum_{i=1}^{k}(1+S_i^{\beta}) \\
&\qquad = L_{10} + L_{11}\sum_{i=1}^{k} S_i^{\beta} \le L_{10} + L_{11}\sum_{i=1}^{k} e^{\beta Y_i},
\end{aligned}
\tag{2.23}
$$

where

$$
L_{10} = (k+1)^{\frac{\beta}{\gamma}-1}(L_8^{\frac{\beta}{\gamma}} + kL_9^{\frac{\beta}{\gamma}}), \quad L_{11} = (k+1)^{\frac{\beta}{\gamma}-1}L_9^{\frac{\beta}{\gamma}}.
$$

Thus, by taking expectation in (2.23) and using Lemma 2.1, we get for $\varepsilon \le \varepsilon_1$,

$$
\begin{aligned}
&\mathsf{E}\big(\sup_{0\le u\le T}|g^{(\varepsilon)}(u,\vec{S}^{(\varepsilon)}(u))|\big)^{\frac{\beta}{\gamma}} \\
&\qquad \le L_{10} + L_{11}\sum_{i=1}^{k}\mathsf{E}e^{\beta Y_i} \le L_{10} + kL_{11}L_1 = L_2 < \infty.
\end{aligned}
\tag{2.24}
$$

The proof is complete. □

Inequality (2.24) implies that for $\varepsilon \le \varepsilon_1$,

$$
\begin{aligned}
|\Phi(\mathcal{M}_{max,T}^{(\varepsilon)})| &\le \sup_{\tau^{(\varepsilon)}\in\mathcal{M}_{max,T}^{(\varepsilon)}} \mathsf{E}|g(\tau^{(\varepsilon)},\vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)}))| \\
&\le \mathsf{E}\sup_{0\le u\le T}|g^{(\varepsilon)}(u,\vec{S}^{(\varepsilon)}(u))| \le L_2^{\frac{\gamma}{\beta}} < \infty.
\end{aligned}
\tag{2.25}
$$

Thus, the reward functional $\Phi(\mathcal{M}_{max,T}^{(\varepsilon)})$ is well defined for $\varepsilon \le \varepsilon_1$.

3 Skeleton approximations for reward functionals

In this section we give explicit and uniform with respect to perturbation parameter estimates in so-called skeleton approximations where reward functionals for continuous time price processes are approximated by the corresponding reward functionals for discrete time priced processes. These approximations plays the key role in the proofs on the corresponding convergence results. They also have their own value. For example, such approximations can be used for justification of Monte Carlo type algorithms, where continuous type price processes should be approximated by the corresponding discrete time processes.

Let $\Pi = \{0 = t_0 < t_1 < \dots t_N = T\}$ be a partition on the interval $[0, T]$ and

$$d(\Pi) = \max_{1 \le i \le N} (t_i - t_{i-1}).$$

Consider the class $\hat{\mathcal{M}}^{(\varepsilon)}_{\Pi,T}$ of all Markov moments from $\mathcal{M}^{(\varepsilon)}_{\max,T}$, which only take the values $t_0, t_1, \dots t_N$, and the class $\mathcal{M}^{(\varepsilon)}_{\Pi,T}$ of all Markov moments $\tau^{(\varepsilon)}$ from $\hat{\mathcal{M}}^{(\varepsilon)}_{\Pi,T}$ such that the event $\{\omega : \tau^{(\varepsilon)}(\omega) = t_j\} \in \sigma(\vec{Y}^{(\varepsilon)}(t_0), \dots, \vec{Y}^{(\varepsilon)}(t_j))$ for $j = 0, \dots N$. By definition,

$$\mathcal{M}^{(\varepsilon)}_{\Pi,T} \subseteq \hat{\mathcal{M}}^{(\varepsilon)}_{\Pi,T} \subseteq \mathcal{M}^{(\varepsilon)}_{\max,T}. \tag{3.1}$$

Relations (3.1) imply that, under conditions of Lemma 2.2, for $\varepsilon \le \varepsilon_1$,

$$-\infty < \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T}) \le \Phi(\hat{\mathcal{M}}^{(\varepsilon)}_{\Pi,T}) \le \Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}) < \infty. \tag{3.2}$$

The reward functionals $\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T})$, $\Phi(\hat{\mathcal{M}}^{(\varepsilon)}_{\Pi,T})$, and $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T})$ correspond to American type option in continuous time, Bermudan type option in continuous time, and American type option in discrete time, respectively.

In the first two cases, the underlying price process is a continuous time Markov type price process, while in the third case the corresponding price process is a discrete time Markov type process.

The random variables $\vec{Y}^{(\varepsilon)}(t_0), \vec{Y}^{(\varepsilon)}(t_1), \dots, \vec{Y}^{(\varepsilon)}(t_N)$ are connected in a discrete time inhomogeneous Markov chain with the phase space $\mathbb{R}^k$, transition probabilities $P^{(\varepsilon)}(t_n, \vec{y}, t_{n+1}, A)$, and initial distribution $P^{(\varepsilon)}(A)$. Note that we have slightly modified the standard definition of a discrete time Markov chain by considering moments $t_0, \dots, t_N$ as the moments of jumps for the Markov chain $\vec{Y}^{(\varepsilon)}(t_n)$ instead of the moments $0, \dots, N$. This is done in order to synchronize the discrete and continuous time models. Thus, the optimization problem (2.3) for the class $\mathcal{M}_{\Pi,T}$ is really a problem of optimal expected reward for American type options in discrete time.

The following lemma establishes useful equality between reward functionals $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T})$ and $\Phi(\hat{\mathcal{M}}^{(\varepsilon)}_{\Pi,T})$.

Lemma 3.1. *Let conditions A_1, C_1 and C_2 hold. Then, for any partition $\Pi = \{0 = t_0 < t_1 < \dots < t_N = T\}$ on the interval $[0, T]$ and $\varepsilon \le \varepsilon_1$ where ε_1 is defined in (2.9) and (2.10),*

$$\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T}) = \Phi(\hat{\mathcal{M}}^{(\varepsilon)}_{\Pi,T}). \tag{3.3}$$

Proof. A similar result was given in Kukush and Silvestrov (2000, 2004) and we shortly present the modified version of the corresponding proof.

The optimization problem (2.3) for the class $\hat{\mathcal{M}}^{(\varepsilon)}_{\Pi,T}$ can be considered as a problem of optimal expected reward for American type options with discrete time. To see this let us add to the random variables $\vec{Y}_{t_n}$ additional components $\vec{Y}'^{(\varepsilon)}_n =$

$\{\vec{Y}^{(\varepsilon)}(t), t_{n-1} < t \leq t_n\}$ with the corresponding phase space $\mathbb{Y}'$ endowed by the corresponding σ-field. As $\vec{Y}_0'^{(\varepsilon)}$ we can take an arbitrary point in $\mathbb{Y}'$. Consider the extended Markov chain $\vec{Y}_n''^{(\varepsilon)} = (\vec{Y}^{(\varepsilon)}(t_n), \vec{Y}_n'^{(\varepsilon)})$ with the phase space $\mathbb{Y}'' = \mathbb{Y} \times \mathbb{Y}'$. As above, we slightly modify the standard definition and count moments $t_0, \ldots, t_N$ as moments of jumps for the this Markov chain instead of moments $0, \ldots, N$. This is done in order to synchronize the discrete and continuous time models.

Denote by $\tilde{\mathcal{M}}_{\Pi,T}^{(\varepsilon)}$ the class of all Markov moments $\tau^{(\varepsilon)} \leq t_N$ for the discrete time Markov chain $\vec{Y}_n''^{(\varepsilon)}$ and consider the reward functional,

$$\Phi(\tilde{\mathcal{M}}_{\Pi,T}^{(\varepsilon)}) = \sup_{\tau^{(\varepsilon)} \in \tilde{\mathcal{M}}_{\Pi,T}^{(\varepsilon)}} \mathsf{E} g^{(\varepsilon)}(\tau^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)})). \tag{3.4}$$

It is readily seen that the optimization problem (2.3) for the class $\hat{\mathcal{M}}_{\Pi,T}^{(\varepsilon)}$ is equivalent to the optimization problem (3.4), i.e.,

$$\Phi(\hat{\mathcal{M}}_{\Pi,T}^{(\varepsilon)}) = \Phi(\tilde{\mathcal{M}}_{\Pi,T}^{(\varepsilon)}). \tag{3.5}$$

It is well known, (See, for example Peskir and Shiryaev (2006)) that the optimal stopping moment $\tau^{(\varepsilon)}$ exists in any discrete time Markov model, and the optimal decision $\{\tau^{(\varepsilon)} = t_n\}$ depends only on the value $\vec{Y}_n''^{(\varepsilon)}$. Moreover the optimal Markov moment has a first hitting time structure, i.e., it is the first moment the process enters some set $\mathbb{D}$, that is $\tau^{(\varepsilon)} = \min(t_n : \vec{Y}_n'^{(\varepsilon)} \in \mathbb{D}''^{(\varepsilon)}_n)$, where $\mathbb{D}''^{(\varepsilon)}_n, n = 0, \ldots, N$ are some measurable subsets of the phase space $\mathbb{Y}''$. The optimal stopping domains are determined by the transition probabilities of the extended Markov chain $\vec{Y}_n''^{(\varepsilon)}$.

However, the extended Markov chain $\vec{Y}_n''^{(\varepsilon)}$ has transition probabilities depending only on values of the first component $\vec{Y}^{(\varepsilon)}(t_n)$. As was shown in Kukush and Silvestrov (2004), the optimal Markov moment has in this case the first hitting time structure of the form $\tau^{(\varepsilon)} = \min(t_n : \vec{Y}^{(\varepsilon)}(t_n) \in \mathbb{D}_n^{(\varepsilon)})$, where $\mathbb{D}_n^{(\varepsilon)}, n = 0, \ldots, N$ are some measurable subsets of the phase space of the first component $\mathbb{Y}$.

Therefore, for the optimal stopping moment $\tau^{(\varepsilon)}$ the decision $\{\tau^{(\varepsilon)} = t_n\}$ depends only on the value $\vec{Y}^{(\varepsilon)}(t_n)$, and $\tau^{(\varepsilon)} \in \mathcal{M}_{\Pi,T}^{(\varepsilon)}$. Hence,

$$\Phi(\mathcal{M}_{\Pi,T}) \geq \mathsf{E} g^{(\varepsilon)}(\tau^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)})) = \Phi(\hat{\mathcal{M}}_{\Pi,T}^{(\varepsilon)}). \tag{3.6}$$

Inequalities (3.2) and (3.6) imply the equality (3.3) and the proof is complete. □

The following theorem gives a skeleton approximation for the reward functionals $\Phi(\mathcal{M}_{\max,T}^{(\varepsilon)})$ which is asymptotically uniform with respect to the perturbation parameter ε.

Theorem 3.2. *Let conditions* $\mathbf{A_1}$, $\mathbf{C_1}$, *and* $\mathbf{C_2}$ *hold. Let also* $\varepsilon \leq \varepsilon_1$ *and* $d(\Pi) \leq c$, *where* ε_1 *and* c *are defined in relations* (2.9) and (2.10). *Then there exist constants*

$L_3, L_4 < \infty$ *such that the following skeleton approximation inequality holds, for* $\varepsilon \leq \varepsilon_1$,

$$\begin{aligned} &\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T}) \\ &\qquad \leq L_3 d(\Pi) + L_4 \sum_{i=1}^{k} \Delta_\beta(Y^{(\varepsilon)}_i(\cdot), d(\Pi), T)^{\frac{\beta-\gamma}{\beta}}. \end{aligned} \tag{3.7}$$

Proof. Let us assume that $\varepsilon \leq \varepsilon_1$ and $d(\Pi) \leq c$.

For any Markov moment $\tau^{(\varepsilon)} \in \mathcal{M}^{(\varepsilon)}_{\max,T}$ and a partition $\Pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ one can define the discretization of this moment,

$$\tau^{(\varepsilon)}[\Pi] = \begin{cases} t_k, & \text{if } \ t_{k-1} \leq \tau^{(\varepsilon)} < t_k, \ k = 1, \ldots N, \\ T, & \text{if } \ \tau^{(\varepsilon)} = T. \end{cases}$$

Let $\tau^{(\varepsilon)}_\delta$ be δ-optimal Markov moment in the class $\mathcal{M}^{(\varepsilon)}_{\max,T}$, i.e.,

$$\mathsf{E} g^{(\varepsilon)}(\tau^{(\varepsilon)}_\delta, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)}_\delta)) \geq \Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}) - \delta. \tag{3.8}$$

Such δ-optimal Markov moment always exists for any $\delta > 0$, by the definition of the reward functional $\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T})$.

By the definition, the Markov moment $\tau^{(\varepsilon)}_\delta[\Pi] \in \hat{\mathcal{M}}^{(\varepsilon)}_{\Pi,T}$. This fact and relation (3.3) given in Lemma 3.1 implies that

$$\begin{aligned} &\mathsf{E} g^{(\varepsilon)}(\tau^{(\varepsilon)}_\delta[\Pi], \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)}_\delta[\Pi])) \\ &\qquad \leq \Phi(\hat{\mathcal{M}}^{(\varepsilon)}_{\Pi,T}) = \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T}) \leq \Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}). \end{aligned} \tag{3.9}$$

By the definition,

$$\tau^{(\varepsilon)}_\delta \leq \tau^{(\varepsilon)}_\delta[\Pi] \leq \tau^{(\varepsilon)}_\delta + d(\Pi). \tag{3.10}$$

Now inequalities (3.8) and (3.9) imply the following skeleton approximation inequality,

$$\begin{aligned} 0 &\leq \Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T}) \\ &\leq \delta + \mathsf{E} g^{(\varepsilon)}(\tau^{(\varepsilon)}_\delta, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)}_\delta)) - \mathsf{E} g^{(\varepsilon)}(\tau^{(\varepsilon)}_\delta[\Pi], \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)}_\delta[\Pi])) \\ &\leq \delta + \mathsf{E} \left| g^{(\varepsilon)}(\tau^{(\varepsilon)}_\delta, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)}_\delta)) - g^{(\varepsilon)}(\tau^{(\varepsilon)}_\delta[\Pi], \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)}_\delta[\Pi])) \right|. \end{aligned} \tag{3.11}$$

Now, we need to improve relation (2.20). We use the same notations and let also $0 \leq u', u'' \leq T$ and $u^+ = u' \vee u'', u^- = u' \wedge u''$.

Using condition $\mathbf{A_1}$ we get the following estimates, for $\varepsilon \leq \varepsilon_0$ and $\vec{s}', \vec{s}'' \in \mathbb{R}_+^k$, $0 \leq u', u'' \leq T$,

$$
\begin{aligned}
|g^{(\varepsilon)}&(u', \vec{s}') - g^{(\varepsilon)}(u'', \vec{s}'')| \\
&\leq |g^{(\varepsilon)}(u', \vec{s}') - g^{(\varepsilon)}(u'', \vec{s}')| + |g^{(\varepsilon)}(u'', \vec{s}') - g^{(\varepsilon)}(u'', \vec{s}'')| \\
&\leq |g^{(\varepsilon)}(u', \vec{s}') - g^{(\varepsilon)}(u'', \vec{s}')| + \sum_{i=1}^{k} |g^{(\varepsilon)}(u'', \vec{s}_i) - g^{(\varepsilon)}(u'', \vec{s}_{i-1})| \\
&\leq \int_{u^-}^{u^+} |\frac{\partial g^{(\varepsilon)}(u, \vec{s}')}{\partial u}| du + \sum_{i=1}^{k} \int_{s_i^-}^{s_i^+} |\frac{\partial g^{(\varepsilon)}(u'', \vec{s}_i(v))}{\partial v}| dv \\
&\leq \int_{u^-}^{u^+} (K_1 + K_2 \sum_{j=1}^{k} (s'_j)^{\gamma_0}) du \\
&\quad + \sum_{i=1}^{k} \int_{s_i^-}^{s_i^+} (K_3 + K_4 (\sum_{j=1}^{i-1} (s'_j)^{\gamma_i} + (v_i)^{\gamma_i} + \sum_{j=i+1}^{k} (s''_j)^{\gamma_i})) dv \\
&\leq (K_1 + K_2 \sum_{j=1}^{k} (s_j^+)^{\gamma_0}) |u' - u''| \\
&\quad + \sum_{i=1}^{k} (K_3 + K_4 \sum_{j=1}^{k} (s_j^+)^{\gamma_i}) |s'_i - s''_i|.
\end{aligned}
\tag{3.12}
$$

In the case, where vectors $\vec{s}' = e^{\vec{y}'}, \vec{s}'' = e^{\vec{y}''}$, i.e., $s'_i = e^{y'_i}, s''_i = e^{y''_i}, i = 1, \ldots, k$, and, therefore, $s_i^{\pm} = e^{y_i^{\pm}}, i = 1, \ldots, k$, where $y_i^+ = y'_i \vee y''_i, y_i^- = y'_i \wedge y''_i, i = 1, \ldots, k$, inequality (3.12) can be transformed, using inequality $|e^{y_i^- - y_i^+} - 1| \leq |y_i^+ - y_i^-| = |y'_i - y''_i|$, to the following form,

$$
\begin{aligned}
|g^{(\varepsilon)}&(u', e^{\vec{y}'}) - g^{(\varepsilon)}(u'', e^{\vec{y}''})| \\
&\leq (K_1 + K_2 \sum_{j=1}^{k} e^{\gamma_0 y_j^+}) |u' - u''| \\
&\quad + \sum_{i=1}^{k} (K_3 + K_4 \sum_{j=1}^{k} e^{\gamma_i y_j^+} e^{y_i^+}) |y'_i - y''_i|.
\end{aligned}
\tag{3.13}
$$

To shorten notations denote the random variables

$$
\tau' = \tau_\delta^{(\varepsilon)}, \ \tau'' = \tau_\delta^{(\varepsilon)}[\Pi], \ Y'_i = Y_i^{(\varepsilon)}(\tau'), \ \ Y''_i = Y_i^{(\varepsilon)}(\tau''), \ i = 1, \ldots, k.
$$

and random vectors

$$
\vec{Y}' = (Y'_1, \ldots, Y'_k), \ \ \vec{Y}'' = (Y''_1, \ldots, Y''_k).
$$

Also define random variables $Y_i^+ = Y_i' \vee Y_i'', i = 1, \ldots, k$. Then,

$$|Y_i^+| \leq Y_i = \sup_{0 \leq u \leq T} |Y_i^{(\varepsilon)}(u)|, \ i = 1, \ldots, k. \tag{3.14}$$

Thus by substituting random variables introduced above in the inequality (3.13) and then using (3.14) we readily get that the following estimate holds under condition $\mathbf{A_1}$, for $\varepsilon \leq \varepsilon_0$,

$$\begin{aligned} &|g^{(\varepsilon)}(\tau', e^{\vec{Y}'}) - g^{(\varepsilon)}(\tau'', e^{\vec{Y}''})| \\ &\quad \leq (K_1 + K_2 \sum_{i=1}^{k} e^{\gamma_0 Y_i^+})(\tau'' - \tau') \\ &\qquad + \sum_{i=1}^{k} (K_3 + K_4 \sum_{j=1}^{k} e^{\gamma_i Y_j^+} e^{Y_i^+})|Y_i' - Y_i''| \\ &\quad \leq (K_1 + K_2 \sum_{i=1}^{k} e^{\gamma_0 Y_i})(\tau'' - \tau') \\ &\qquad + \sum_{i=1}^{k} (K_3 + K_4 \sum_{j=1}^{k} e^{\gamma_i Y_j} e^{Y_i})|Y_i' - Y_i''| \end{aligned} \tag{3.15}$$

Now, applying Hölder inequality (with parameters $p = \frac{\gamma}{\beta}$ and $q = \frac{\beta - \gamma}{\beta}$ and then with parameters $p = \frac{\gamma - 1}{\gamma}$ and $q = \frac{1}{\gamma}$) to the corresponding products of random variables on the right hand side in (3.15) and by using inequality (2.5) given in Lemma 2.1 and relation (3.10), we can write down the following estimate for the expectation on the right hand side in (3.11), for $\varepsilon \leq \varepsilon_1$,

$$\begin{aligned} &\mathsf{E}|g^{(\varepsilon)}(\tau_\delta^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau_\delta^{(\varepsilon)})) - g^{(\varepsilon)}(\tau_\delta^{(\varepsilon)}[\Pi], \vec{S}^{(\varepsilon)}(\tau_\delta^{(\varepsilon)}[\Pi]))| \\ &\quad = \mathsf{E}|g^{(\varepsilon)}(\tau', e^{\vec{Y}'}) - g^{(\varepsilon)}(\tau', e^{\vec{Y}''})| \\ &\quad \leq (K_1 + K_2 \sum_{i=1}^{k} \mathsf{E} e^{\gamma_0 Y_i}) d(\Pi) + K_3 \sum_{i=1}^{k} \mathsf{E}|Y_i' - Y_i''| \\ &\qquad + K_4 \sum_{i=1}^{k} \sum_{j=1}^{k} \mathsf{E} e^{\gamma_i Y_j} e^{Y_i} |Y_i' - Y_i''| \\ &\quad \leq (K_1 + K_2 \sum_{i=1}^{k} (\mathsf{E} e^{\frac{\gamma_0 \beta}{\gamma} Y_i})^{\frac{\gamma}{\beta}}) d(\Pi) + K_3 \sum_{i=1}^{k} (\mathsf{E}|Y_i' - Y_i''|^{\frac{\beta}{\beta - \gamma}})^{\frac{\beta - \gamma}{\beta}} \\ &\qquad + K_4 \sum_{i=1}^{k} \sum_{j=1}^{k} (\mathsf{E} e^{\frac{\gamma_i \beta}{\gamma} Y_j} e^{\frac{\beta}{\gamma} Y_i})^{\frac{\gamma}{\beta}} (\mathsf{E}|Y_i' - Y_i''|^{\frac{\beta}{\beta - \gamma}})^{\frac{\beta - \gamma}{\beta}} \end{aligned} \tag{3.16}$$

$$
\begin{aligned}
&\leq (K_1 + K_2 \sum_{i=1}^{k} (\mathsf{E} e^{\beta Y_i})^{\frac{\gamma}{\beta}}) d(\Pi) + K_3 \sum_{i=1}^{k} (\mathsf{E}|Y'_i - Y''_i|^{\frac{\beta}{\beta-\gamma}})^{\frac{\beta-\gamma}{\beta}} \\
&\quad + K_4 \sum_{i=1}^{k} \sum_{j=1}^{k} ((\mathsf{E} e^{\frac{\gamma_i \beta}{\gamma-1} Y_j})^{\frac{\gamma-1}{\gamma}} (\mathsf{E} e^{\beta Y_i})^{\frac{1}{\gamma}})^{\frac{\gamma}{\beta}} (\mathsf{E}|Y'_i - Y''_i|^{\frac{\beta}{\beta-\gamma}})^{\frac{\beta-\gamma}{\beta}} \\
&\leq (K_1 + kK_2 (L_1)^{\frac{\gamma}{\beta}}) d(\Pi) + K_3 \sum_{i=1}^{k} (\mathsf{E}|Y'_i - Y''_i|^{\frac{\beta}{\beta-\gamma}})^{\frac{\beta-\gamma}{\beta}} \\
&\quad + K_4 \sum_{i=1}^{k} \sum_{j=1}^{k} ((L_1)^{\frac{\gamma-1}{\gamma}} (L_1)^{\frac{1}{\gamma}})^{\frac{\gamma}{\beta}} (\mathsf{E}|Y'_i - Y''_i|^{\frac{\beta}{\beta-\gamma}})^{\frac{\beta-\gamma}{\beta}} \\
&\leq (K_1 + kK_2 (L_1)^{\frac{\gamma}{\beta}}) d(\Pi) \\
&\quad + (K_3 + kK_4 (L_1)^{\frac{\gamma}{\beta}}) \sum_{i=1}^{k} (\mathsf{E}|Y'_i - Y''_i|^{\frac{\beta}{\beta-\gamma}})^{\frac{\beta-\gamma}{\beta}}
\end{aligned}
$$

The last step in the proof is to show that, for $\varepsilon \leq \varepsilon_1$ and $i = 1, \ldots, k$,

$$
\mathsf{E}|Y'_i - Y''_i|^{\frac{\beta-\gamma}{\beta}} \leq L_{12} \Delta_\beta (Y_i^{(\varepsilon)}(\cdot), d(\Pi), T), \tag{3.17}
$$

where

$$
L_{12} = \sup_{y \geq 0} \frac{y^{\frac{\beta}{\beta-\gamma}}}{e^{\beta y} - 1} < \infty. \tag{3.18}
$$

Indeed, in this case, (3.16) take the form

$$
\begin{aligned}
&\mathsf{E}|g^{(\varepsilon)}(\tau_\delta^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau_\delta^{(\varepsilon)})) - g^{(\varepsilon)}(\tau_\delta^{(\varepsilon)}[\Pi], \vec{S}^{(\varepsilon)}(\tau_\delta^{(\varepsilon)}[\Pi]))| \\
&\qquad \leq L_3 d(\Pi) + L_4 \sum_{i=1}^{k} \Delta_\beta (Y_i^{(\varepsilon)}(\cdot), d(\Pi), T)^{\frac{\beta}{\beta-\gamma}},
\end{aligned} \tag{3.19}
$$

where

$$
L_3 = K_1 + kK_k (L_1)^{\frac{\gamma}{\beta}}, \ L_4 = (K_3 + kK_4 (L_1)^{\frac{\gamma}{\beta}})(L_{12})^{\frac{\beta}{\beta-\gamma}}. \tag{3.20}
$$

Note that the quantity on the right hand side in (3.19) does not depend on δ. Thus, we can substitute it in (3.11) and then pass δ to zero in this relation that will result in inequality (3.7) given in Theorem 3.2.

To get inequality (3.17), we employ the method from Silvestrov (1974) for estimation of moments for increments of stochastic processes stopped at Markov type stopping moments.

Introduce the function $f_\Pi(t) = t_{j+1} - t$ for $t_j \leq t < t_{j+1}, j = 0, \ldots, N-1$ and 0 for $t = t_N^{(N)}$. The function $f_\Pi(t)$ is continuous from the right on the interval $[0, T]$ and $0 \leq f_\Pi(t) \leq d(\Pi)$.

It follows from the definition of function $f_\Pi(t)$ that

$$\tau'' = \tau' + f_\Pi(\tau'),$$

and

$$|Y_i^{(\varepsilon)}(\tau_\delta^{(\varepsilon)}) - Y_i^{(\varepsilon)}(\tau_\delta^{(\varepsilon)}[\Pi])| = |Y_i^{(\varepsilon)}(\tau') - Y_i^{(\varepsilon)}(\tau' + f_\Pi(\tau'))|.$$

Use again partition $\tilde{\Pi}_m$ of interval $[0, T]$ by points $v_n^{(m)} = nT/m$, $n = 0, \dots, m$, and consider the random variables,

$$\tau'[\tilde{\Pi}_m] = \begin{cases} v_j^{(m)}, & \text{if } v_{j-1}^{(m)} \leq \tau' < v_j^{(m)},\ j = 1, \dots N, \\ T, & \text{if } \tau' = T. \end{cases}$$

Thus we have $\tau' \leq \tau'[\tilde{\Pi}_m] \leq \tau' + T/m$, and the random variables $\tau'[\tilde{\Pi}_m] \stackrel{a.s.}{\longrightarrow} \tau'$ as $m \to \infty$. Since the $\vec{Y}^{(\varepsilon)}(t)$ is a vector of càdlàg processes, we also get the following relation, for each $i = 1 \dots, k$

$$\begin{aligned} Q_{m,i}^{(\varepsilon)} &= |Y_i^{(\varepsilon)}(\tau'[\tilde{\Pi}_m]) - Y_i^{(\varepsilon)}(\tau'[\tilde{\Pi}_m] + f_\Pi(\tau'[\tilde{\Pi}_m]))|^{\frac{\beta}{\beta-\gamma}} \\ &\stackrel{a.s.}{\longrightarrow} Q_i^{(\varepsilon)} = |Y_i^{(\varepsilon)}(\tau') - Y_i^{(\varepsilon)}(\tau' + f_\Pi(\tau'))|^{\frac{\beta}{\beta-\gamma}} \text{ as } m \to \infty. \end{aligned} \tag{3.21}$$

Note also that $Q_{m,i}^{(\varepsilon)}$ are non-negative random variables and the following estimate holds for any $m = 1, \dots$ and $i = 1, \dots, k$,

$$\begin{aligned} Q_{m,i}^{(\varepsilon)} &\leq (|Y_i^{(\varepsilon)}(\tau'[\tilde{\Pi}_m])| + |Y_i^{(\varepsilon)}(\tau'[\tilde{\Pi}_m] + f_\Pi(\tau'[\tilde{\Pi}_m]))|)^{\frac{\beta}{\beta-\gamma}} \\ &\leq 2^{\frac{\beta}{\beta-\gamma}-1}(|Y_i^{(\varepsilon)}(\tau'[\tilde{\Pi}_m])|^{\frac{\beta}{\beta-\gamma}} + |Y_i^{(\varepsilon)}(\tau'[\tilde{\Pi}_m] + f_\Pi(\tau'[\tilde{\Pi}_m]))|^{\frac{\beta}{\beta-\gamma}}) \\ &\leq 2^{\frac{\beta}{\beta-\gamma}}(\sup_{0 \leq u \leq T} |Y_i^{(\varepsilon)}(u)|)^{\frac{\beta}{\beta-\gamma}} \\ &\leq 2^{\frac{\beta}{\beta-\gamma}} L_{12} \exp\{\beta \sup_{0 \leq u \leq T} |Y_i^{(\varepsilon)}(u)|\}. \end{aligned} \tag{3.22}$$

Inequality (2.5) given in Lemma 2.1, implies that the random variable on the right hand side in (3.22) has a finite expectation, and relation (3.21), we get by Lebesgue theorem that, for $\varepsilon \leq \varepsilon_1$ and $i = 1 \dots, k$,

$$\mathsf{E}Q_{m,i}^{(\varepsilon)} \to \mathsf{E}Q_i^{(\varepsilon)} \text{ as } m \to \infty. \tag{3.23}$$

Let us now find an estimate of $\mathsf{E}Q_m^{(\varepsilon)}$. To reduce the notation let us denote for the moment $Y'_{n+1,i} = Y_i^{(\varepsilon)}(v_{n+1}^{(m)})$ and $Y''_{n+1,i} = Y_i^{(\varepsilon)}(v_{n+1}^{(m)}) + f_\Pi(v_{n+1}^{(m)}))$. Since τ' is a Markov moment for the Markov process $\vec{Y}^{(\varepsilon)}(t)$, the random variables $\chi(v_n^{(m)} \leq \tau' < v_{n+1}^{(m)})$ and $|Y''_{n+1,i} - Y'_{n+1,i}|^{\frac{\beta}{\beta-\gamma}}$ are conditionally independent with respect to random vector

$\vec{Y}^{(\varepsilon)}(v^{(m)}_{n+1})$. Using this fact and inequality $f_\Pi(v^{(m)}_{n+1}) \leq d(\Pi)$, we get, for $\varepsilon \leq \varepsilon_1$, and $i = 1\ldots, k$,

$$\begin{aligned}
\mathsf{E}Q^{(\varepsilon)}_{m,i} &= \mathsf{E}|Y^{(\varepsilon)}_i(\tau'[\tilde{\Pi}_m]) - Y^{(\varepsilon)}_i(\tau'[\tilde{\Pi}_m] + f_\Pi(\tau'[\tilde{\Pi}_m]))|^{\frac{\beta}{\beta-\gamma}} \\
&= \sum_{n=0}^{m-1} \mathsf{E}|Y'_{n+1,i} - Y''_{n+1,i}|^{\frac{\beta}{\beta-\gamma}} \chi(v^{(m)}_n \leq \tau' < v^{(m)}_{n+1}) \\
&= \sum_{n=0}^{m-1} \mathsf{E}\{\chi(v^{(m)}_n \leq \tau' < v^{(m)}_{n+1})\mathsf{E}\{|Y'_{n+1,i} - Y''_{n+1,i}|^{\frac{\beta}{\beta-\gamma}} / \vec{Y}^{(\varepsilon)}(v^{(m)}_{n+1})\}\} \\
&\leq \sum_{n=0}^{m-1} \sup_{\vec{y}\in\mathbb{R}^k} \mathsf{E}_{\vec{y},v^{(m)}_{n+1}} |Y'_{n+1,i} - Y''_{n+1,i}|^{\frac{\beta}{\beta-\gamma}} \mathsf{P}\{v^{(m)}_n \leq \tau' < v^{(m)}_{n+1}\} \\
&\leq \sum_{n=0}^{m-1} L_{12} \sup_{\vec{y}\in\mathbb{R}^k} \mathsf{E}_{\vec{y},v^{(m)}_{n+1}} \exp\{\beta|Y'_{n+1,i} - Y''_{n+1,i}|\}\mathsf{P}\{v^{(m)}_n \leq \tau' < v^{(m)}_{n+1}\} \\
&\leq \sum_{n=0}^{m-1} L_{12}\Delta_\beta(Y^{(\varepsilon)}_i(\cdot), d(\Pi), T)\mathsf{P}\{v^{(m)}_n \leq \tau' < v^{(m)}_{n+1}\} \\
&\leq L_{12}\Delta_\beta(Y^{(\varepsilon)}_i(\cdot), d(\Pi), T).
\end{aligned} \tag{3.24}$$

Relations (3.23) and (3.24) imply that, for $\varepsilon \leq \varepsilon_1$ and $i = 1\ldots, k$,

$$\begin{aligned}
\mathsf{E}Q^{(\varepsilon)}_i &= \mathsf{E}|Y^{(\varepsilon)}_i(\tau') - Y^{(\varepsilon)}_i(\tau' + f_\Pi(\tau'))|^{\frac{\beta}{\beta-\gamma}} \\
&\leq L_{12}\Delta_\beta(Y^{(\varepsilon)}_i(\cdot), d(\Pi), T).
\end{aligned} \tag{3.25}$$

This inequality is equivalent to inequality (3.19). The proof is complete. □

4 Convergence of rewards for discrete time price processes

In this section we present convergence results for reward functionals for American type options for discrete time price processes. These results have their own value and are also essentially used in the proofs of the corresponding convergence results for continuous time price processes.

Let us now formulate conditions of convergence for discrete time reward functionals $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T})$ for a given partition $\Pi = \{0 = t_0 < t_1 \ldots < t_N = T\}$ on the interval $[0, T]$. In this case it is natural to use conditions based on the transition probabilities between the sequential moments of this partition and values of the payoff functions at the moments of this partition. Condition $\mathbf{A_1}$ is replaced by a simpler condition:

$\mathbf{A_2}$: There exists $\varepsilon_2 > 0$ such that, for every $0 \le \varepsilon \le \varepsilon_2$, function $|g^{(\varepsilon)}(t_n, \vec{s})| \le K_6 + K_7 \sum_{i=1}^k s_i^\gamma$, for $n = 0, \dots, N$ and $\vec{s} \in \mathbb{R}_+^k$ for some $\gamma \ge 1$ and constants $K_6, K_7 < \infty$.

Condition $\mathbf{A_2}$ can be re-written in terms of functions $g^{(\varepsilon)}(t, e^{\vec{y}})$:

$\mathbf{A_2'}$: There exists $\varepsilon_2 > 0$ such that, for every $0 \le \varepsilon \le \varepsilon_2$, function $|g^{(\varepsilon)}(t_n, e^{\vec{y}})| \le K_6 + K_7 \sum_{i=1}^k e^{\gamma y_i}$, for $n = 0, \dots, N$ and $\vec{y} \in \mathbb{R}^k$ for some $\gamma \ge 1$ and constants $K_6, K_7 < \infty$.

We also need an assumption about convergence of the payoff function.

In discrete time case, the derivatives of the payoff functions are not involved. In this case, the payoff functions can be discontinuous. This is compensated by a stronger assumption concerned the convergence of the payoff functions.

We require local uniform convergence for the payoff function on some sets, which later will be assumed to have the value 1 for the corresponding limit transition probabilities and the limit initial distribution:

$\mathbf{A_3}$: There exists a measurable set $\mathbb{S}'_{t_n} \subseteq \mathbb{R}_+^k$ for every $n = 0, \dots, N$, such that $g^{(\varepsilon)}(t_n, \vec{s}^{(\varepsilon)}) \to g^{(0)}(t_n, \vec{s})$ as $\varepsilon \to 0$, for any $\vec{s}^{(\varepsilon)} \to \vec{s} \in \mathbb{S}'_{t_n}$ and $n = 0, \dots, N$.

Condition $\mathbf{A_3}$ can be re-written in terms of functions $g^{(\varepsilon)}(t, e^{\vec{y}})$:

$\mathbf{A_3'}$: There exists measurable sets $\mathbb{Y}'_{t_n} \subseteq \mathbb{R}^k$, $n = 0, \dots, N$, such that the payoff function $g^{(\varepsilon)}(t_n, e^{\vec{y}^{(\varepsilon)}}) \to g^{(0)}(t_n, e^{\vec{y}})$ as $\varepsilon \to 0$ for any $\vec{y}^{(\varepsilon)} \to \vec{y} \in \mathbb{Y}'_{t_n}$ and $n = 0, \dots, N$.

The sets $\mathbb{S}'_{t_n}$ and $\mathbb{Y}'_{t_n}$ in conditions $\mathbf{A_3}$ and $\mathbf{A_3'}$ are connected by the following relation $\mathbb{S}'_{t_n} = \{\vec{s} = e^{\vec{y}} : \vec{y} \in \mathbb{Y}'_{t_n}\}, n = 0, \dots, N$.

Let us now formulate conditions assumed for the transition probabilities and initial distributions of the process $\vec{Y}^{(\varepsilon)}(t)$.

The first condition assumes weak convergence (denoted by the symbol $\Rightarrow$) of the transition probabilities that should be locally uniform with respect to initial states from some sets, and also that the corresponding limit measures are concentrated on these sets:

$\mathbf{B_1}$: There exist measurable sets $\mathbb{Y}''_{t_n} \subseteq \mathbb{R}^k$, $n = 0, \dots, N$ such that: **(a)** $P^{(\varepsilon)}(t_n, \vec{y}^{(\varepsilon)}, t_{n+1}, \cdot) \Rightarrow P^{(0)}(t_n, \vec{y}, t_{n+1}, \cdot)$ as $\varepsilon \to 0$, for any $\vec{y}^{(\varepsilon)} \to \vec{y} \in \mathbb{Y}''_{t_n}$ as $\varepsilon \to 0$ and $n = 0, \dots, N-1$; **(b)** $P^{(0)}(t_n, \vec{y}, t_{n+1}, \mathbb{Y}'_{t_{n+1}} \cap \mathbb{Y}''_{t_{n+1}}) = 1$ for every $\vec{y} \in \mathbb{Y}''_{t_n}$ and $n = 0, \dots, N-1$.

A typical example is when the sets $\bar{\mathbb{Y}}'_{t_n}$ and $\bar{\mathbb{Y}}''_{t_n}$ are at most finite or countable sets. Then the assumption that the measures $P^{(0)}(t_n, \vec{y}, t_{n+1}, A)$, $A \in \mathcal{B}_{\mathbb{R}^k}$ have no atom implies that condition $\mathbf{B_1}$ (**b**) holds.

The second condition assumes weak convergence of the initial distributions to some distribution that is assumed to be concentrated on the sets of convergence for the corresponding transition probabilities:

$\mathbf{B_2}$: **(a)** $P^{(\varepsilon)}(\cdot) \Rightarrow P^{(0)}(\cdot)$ as $\varepsilon \to 0$; **(b)** $P^{(0)}(\mathbb{Y}'_0 \cap \mathbb{Y}''_0) = 1$, where $\mathbb{Y}'_0$ and $\mathbb{Y}''_0$ are the sets introduced in conditions $\mathbf{A'_3}$ and $\mathbf{B_1}$.

A typical example is when the sets $\bar{\mathbb{Y}}'_{t_n}$ and $\bar{\mathbb{Y}}''_{t_n}$ are at most finite or countable sets. Then the assumption that the measures $P^{(0)}(A)$ have no atom implies that condition $\mathbf{B_2}$ (**b**) holds.

We also weaken condition $\mathbf{C_1}$ by replacing it by a simpler condition, which is implied by condition $\mathbf{C_1}$:

$\mathbf{C_3}$: $\overline{\lim}_{\varepsilon\to 0} \sup_{\vec{y}\in\mathbb{R}^k} \mathsf{E}_{\vec{y},t_n}(e^{\beta|Y_i^{(\varepsilon)}(t_{n+1})-Y_i^{(\varepsilon)}(t_n)|}-1) < \infty, n = 0, \ldots, N-1, i = 1, \ldots, k,$ for some $\beta > \gamma$, where γ is the parameter introduced in condition $\mathbf{A_2}$.

Condition $\mathbf{C_2}$ does not change and takes the following form:

$\mathbf{C_4}$: $\overline{\lim}_{\varepsilon\to 0} \mathsf{E} e^{\beta|Y_i^{(\varepsilon)}(t_0)|} < \infty, i = 1, \ldots, k$, where β is the parameter introduced in condition $\mathbf{C_3}$.

Condition $\mathbf{C_3}$ implies that there exists a constant $L_{13} < \infty$ and $\varepsilon_3 \leq \varepsilon_2$ such that for $n = 0, \ldots, N-1$, $\varepsilon \leq \varepsilon_3$, and $i = 1, \ldots k$,

$$\sup_{\vec{y}\in\mathbb{R}^k} \mathsf{E}_{\vec{y},t_n}(e^{\beta|Y_i^{(\varepsilon)}(t_{n+1})-Y_i^{(\varepsilon)}(t_n)|} - 1) \leq L_{13}. \tag{4.1}$$

Condition $\mathbf{C_4}$ implies that ε_3 can be chosen in such a way that, for some constant $L_{14} < \infty$, the following inequality holds for $\varepsilon \leq \varepsilon_3$ and $i = 1, \ldots, k$,

$$\mathsf{E} e^{\beta|Y_i^{(\varepsilon)}(0)|} \leq L_{14}. \tag{4.2}$$

A discrete analogue to inequality (2.25) can be written as

$$\begin{aligned} |\Phi(\mathcal{M}_{\Pi,T}^{(\varepsilon)})| &\leq \sup_{\tau^{(\varepsilon)}\in\mathcal{M}_{\Pi,T}^{(\varepsilon)}} \mathsf{E}|g(\tau^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)}))| \\ &\leq \mathsf{E} \max_{0\leq i\leq N} |g^{(\varepsilon)}(t_i, \vec{S}^{(\varepsilon)}(t_i))| \leq c_1^{\frac{\gamma}{\beta}} < \infty. \end{aligned} \tag{4.3}$$

The following theorem gives conditions of convergence for reward functionals $\Phi(\mathcal{M}_{\Pi,T}^{(\varepsilon)})$ for a given partition Π.

Theorem 4.1. *Let conditions* $\mathbf{A_2}$*,* $\mathbf{A_3}$*,* $\mathbf{B_1}$*,* $\mathbf{B_2}$*,* $\mathbf{C_3}$*, and* $\mathbf{C_4}$ *hold. Then, the following asymptotic relation holds for the partition* $\Pi = \{0 = t_0 < t_1 \cdots < t_N = T\}$ *on the interval* $[0, T]$*,*

$$\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T}) \to \Phi(\mathcal{M}^{(0)}_{\Pi,T}) \text{ as } \varepsilon \to 0. \tag{4.4}$$

Proof. Since $\varepsilon \geq 0$ in (4.4), we can assume that $\varepsilon \leq \varepsilon_2$. The reward functions are defined by the following recursive relations,

$$w^{(\varepsilon)}(t_N, \vec{y}) = g^{(\varepsilon)}(t_N, e^{\vec{y}}), \ \vec{y} \in \mathbb{R}^k,$$

and, for $n = 0, \dots, N-1$,

$$w^{(\varepsilon)}(t_n, \vec{y}) = \max(g^{(\varepsilon)}(t_n, e^{\vec{y}}), \ \mathsf{E}_{\vec{y},t_n} w^{(\varepsilon)}(t_{n+1}, \vec{Y}^{(\varepsilon)}(t_{n+1})), \ \vec{y} \in \mathbb{R}^k.$$

As follows from general results on optimal stopping for discrete time Markov processes, see Chow Robbins and Siegmund (1971), Shiryaev (1976), the reward functional,

$$\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T}) = \mathsf{E} w^{(\varepsilon)}(t_0, \vec{Y}^{(\varepsilon)}(0)). \tag{4.5}$$

Condition $\mathbf{A_2}$ directly implies that the following power type upper bound for the reward function $w^{(\varepsilon)}(t_N, \vec{y})$ holds, for $\varepsilon \leq \varepsilon_3$ and $\vec{y} \in \mathbb{R}^k$,

$$|w^{(\varepsilon)}(t_N, \vec{y})| \leq L_{1,N} + L_{2,N} \sum_{i=1}^{k} e^{\gamma |y_i|}, \tag{4.6}$$

where

$$L_{1,N} = K_6, \ L_{2,N} = K_7 < \infty. \tag{4.7}$$

Also, according conditions $\mathbf{A_3}$, for an arbitrary $\vec{y}^{(\varepsilon)} \to \vec{y}^{(0)}$ as $\varepsilon \to 0$, where $\vec{y}^{(0)} \in \tilde{\mathbb{Y}}'_{t_N} \cap \mathbb{Y}''_{t_N}$,

$$w^{(\varepsilon)}(t_N, \vec{y}^{(\varepsilon)}) \to w^{(0)}(t_N, \vec{y}^{(0)}) \text{ as } \varepsilon \to 0. \tag{4.8}$$

Let us prove that relations similar with (4.6), (4.7), and (4.8) hold for the reward functions $w^{(\varepsilon)}(t_{N-1}, \vec{y})$.

We get using relation (4.6), for $\varepsilon \leq \varepsilon_3$ and $\vec{y} \in \mathbb{R}^k$,

$$
\begin{aligned}
\mathsf{E}_{\vec{y},t_{N-1}} |g^{(\varepsilon)}(t_N, e^{\vec{Y}^{(\varepsilon)}(t_N)})| &\leq L_{1,N} + L_{2,N} \mathsf{E}_{\vec{y},t_{N-1}} \sum_{i=1}^{k} e^{\gamma|Y_i^{(\varepsilon)}(t_N)|} \\
&\leq L_{1,N} + L_{2,N} \mathsf{E}_{\vec{y},t_{N-1}} \sum_{i=1}^{k} e^{\gamma|y_i|} e^{\gamma|Y_i^{(\varepsilon)}(t_N) - y_i|} \\
&\leq L_{1,N} + L_{2,N} \sum_{i=1}^{k} e^{\gamma|y_i|} \mathsf{E}_{\vec{y},t_{N-1}} e^{\gamma|Y_i^{(\varepsilon)}(t_N) - Y_i^{(\varepsilon)}(t_{N-1})|} \\
&\leq L_{1,N} + L_{2,N} \sum_{i=1}^{k} e^{\gamma|y_i|} \mathsf{E}_{\vec{y},t_{N-1}} e^{\beta|Y_i^{(\varepsilon)}(t_N) - Y_i^{(\varepsilon)}(t_{N-1})|} \\
&\leq L_{1,N} + L_{2,N}(L_{13} + 1) \sum_{i=1}^{k} e^{\gamma|y_i|}.
\end{aligned}
\tag{4.9}
$$

Relation (4.9) implies that, for $\varepsilon \leq \varepsilon_3$ and $\vec{y} \in \mathbb{R}^k$,

$$
\begin{aligned}
|w^{(\varepsilon)}(t_{N-1}, \vec{y})| &\leq \max(|g^{(\varepsilon)}(t_{N-1}, e^{\vec{y}})|, \mathsf{E}_{\vec{y},t} |w^{(\varepsilon)}(t_N, \vec{Y}^{(\varepsilon)}(t_N))|) \\
&\leq K_6 + L_{1,N} + (K_7 + L_{2,N}(L_{13} + 1)) \sum_{i=1}^{k} e^{\gamma|y_i|} \\
&= L_{1,N-1} + L_{2,N-1} \sum_{i=1}^{k} e^{\gamma|y_i|},
\end{aligned}
\tag{4.10}
$$

where

$$
L_{1,N-1} = K_6 + L_{1,N}, \ \ L_{2,N-1} = K_7 + L_{2,N}(L_{13} + 1) < \infty. \tag{4.11}
$$

Let us introduce, for every $n = 0, \ldots, N-1$ and $\vec{y} \in \mathbb{R}^k$, a random vector $\vec{Y}_n^{(\varepsilon)}(\vec{y})$ such that $\mathsf{P}\{\vec{Y}_n^{(\varepsilon)}(\vec{y}) \in A\} = P^{(\varepsilon)}(t_n, \vec{y}, t_{n+1}, A)$, $A \in \mathcal{B}_{\mathbb{R}^k}$.

Let us prove that, for any $\vec{y}^{(\varepsilon)} \to \vec{y}^{(0)} \in \tilde{\mathbb{Y}}'_{t_{N-1}} \cap \mathbb{Y}''_{t_{N-1}}$ as $\varepsilon \to 0$, the following relation takes place,

$$
w^{(\varepsilon)}(t_N, \vec{Y}_{N-1}^{(\varepsilon)}(\vec{y}^{(\varepsilon)})) \Rightarrow w^{(0)}(t_N, \vec{Y}_{N-1}^{(0)}(\vec{y}^{(0)})) \text{ as } \varepsilon \to 0. \tag{4.12}
$$

Take an arbitrary sequence $\varepsilon_r \to \varepsilon_0 = 0$ as $r \to \infty$. From condition $\mathbf{B_1}$ we have for an arbitrary $\vec{y}^{(\varepsilon_r)} \to \vec{y}^{(\varepsilon_0)} \in \tilde{\mathbb{Y}}'_{t_{N-1}} \cap \mathbb{Y}''_{t_{N-1}}$ as $r \to \infty$,

$$
\vec{Y}_{N-1}^{(\varepsilon_r)}(\vec{y}^{(\varepsilon_r)}) \Rightarrow \vec{Y}_{N-1}^{(\varepsilon_0)}(\vec{y}^{(\varepsilon_0)}) \text{ as } r \to \infty, \tag{4.13}
$$

and

$$
\mathsf{P}\{\vec{Y}_{N-1}^{(\varepsilon_0)}(\vec{y}^{(\varepsilon_0)}) \in \tilde{\mathbb{Y}}'_{t_N} \cap \mathbb{Y}''_{t_N}\} = 1. \tag{4.14}
$$

Now, according to Skorokhod representation theorem, one can construct random variables
$\vec{Y}'^{(\varepsilon_r)}_{N-1}(\vec{y}^{(\varepsilon_r)})$, $r = 0, 1, \dots$ on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ such that, for every $r = 0, 1, \dots$, and $A \in \mathcal{B}_{\mathbb{R}^k}$,

$$\mathsf{P}\{\vec{Y}'^{(\varepsilon_r)}_{N-1}(\vec{y}^{(\varepsilon_r)}) \in A\} = \mathsf{P}\{\vec{Y}^{(\varepsilon_r)}_{N-1}(\vec{y}^{(\varepsilon_r)}) \in A\}, \tag{4.15}$$

and

$$\vec{Y}'^{(\varepsilon_r)}_{N-1}(\vec{y}^{(\varepsilon_r)}) \xrightarrow{a.s.} \vec{Y}'^{(\varepsilon_0)}_{N-1}(\vec{y}^{(\varepsilon_0)}) \ \text{ as } r \to \infty. \tag{4.16}$$

Denote

$$A_{N-1} = \{\omega \in \Omega : \vec{Y}'^{(\varepsilon_r)}_{N-1}(\vec{y}^{(\varepsilon_r)}, \omega) \to \vec{Y}'^{(\varepsilon_0)}_{N-1}(\vec{y}^{(\varepsilon_0)}, \omega) \text{ as } r \to \infty\}$$

and

$$B_{N-1} = \{\omega \in \Omega : \vec{Y}'^{(\varepsilon_0)}_{N-1}(\vec{y}^{(\varepsilon_0)}, \omega) \in \tilde{\mathbb{Y}}'_{t_N} \cap \mathbb{Y}''_{t_N}\}.$$

Relation (4.16) implies that $\mathsf{P}(A_{N-1}) = 1$. Also relations (4.14) and (4.15) imply that $\mathsf{P}(B_{N-1}) = 1$. Thus, $\mathsf{P}(A_{N-1} \cap B_{N-1}) = 1$.

By relation (4.8) and the definition of the sets A_{N-1} and B_{N-1}, the sequence $w^{(\varepsilon_r)}(t_N, \vec{Y}'^{(\varepsilon_r)}_{N-1}(\vec{y}^{(\varepsilon_r)}, \omega)) \to w^{(\varepsilon_0)}(t_N, \vec{Y}'^{(\varepsilon_0)}_{N-1}(\vec{y}^{(\varepsilon_0)}, \omega))$ as $r \to \infty$, for every $\omega \in A_{N-1} \cap B_{N-1}$, i.e., the random vectors

$$w^{(\varepsilon_r)}(t_N, \vec{Y}'^{(\varepsilon_r)}_{N-1}(\vec{y}^{(\varepsilon_r)})) \xrightarrow{a.s.} w^{(\varepsilon_0)}(t_N, \vec{Y}'^{(\varepsilon_0)}_{N-1}(\vec{y}^{(\varepsilon_0)})) \ \text{ as } r \to \infty. \tag{4.17}$$

Relation (4.15) implies that, for every $r = 0, 1, \dots$, and $A \in \mathcal{B}_{\mathbb{R}^k}$,

$$\mathsf{P}\{w^{(\varepsilon_r)}(t_N, \vec{Y}'^{(\varepsilon_r)}_{N-1}(\vec{y}^{(\varepsilon_r)})) \in A\} = \mathsf{P}\{w^{(\varepsilon_r)}(t_N, \vec{Y}^{(\varepsilon_r)}_{N-1}(\vec{y}^{(\varepsilon_r)})) \in A\}. \tag{4.18}$$

Relations (4.17) and (4.18) imply that the random variables

$$w^{(\varepsilon_r)}(t_N, \vec{Y}^{(\varepsilon_r)}_{N-1}(\vec{y}^{(\varepsilon_r)})) \Rightarrow w^{(\varepsilon_0)}(t_N, \vec{Y}^{(\varepsilon_0)}_{N-1}(\vec{y}^{(\varepsilon_0)})) \text{ as } r \to \infty. \tag{4.19}$$

Since an arbitrary choice of a sequence $\varepsilon_r \to \varepsilon_0$, relation (4.19) implies relation of weak convergence (4.12).

Using inequalities (4.1) and (4.10), and condition $\mathbf{C_3}$ we get for any sequence $\vec{y}^{(\varepsilon)} \to \vec{y}^{(0)} \in \tilde{\mathbb{Y}}'_{t_{N-1}} \cap \mathbb{Y}''_{t_{N-1}}$ as $\varepsilon \to 0$, and for $\varepsilon \le \varepsilon_3$,

$$\begin{aligned} \mathsf{E}|w^{(\varepsilon)}(t_N, \vec{Y}^{(\varepsilon)}_{N-1}(\vec{y}^{(\varepsilon)}))|^{\frac{\beta}{\gamma}} &= \mathsf{E}_{\vec{y}_\varepsilon, t_{N-1}}|w^{(\varepsilon)}(t_N, \vec{Y}^{(\varepsilon)}(t_N))|^{\frac{\beta}{\gamma}} \\ &\le \mathsf{E}_{\vec{y}^{(\varepsilon)},\, t_{N-1}}(L_{1,N} + L_{2,N}\sum_{i=1}^{k} e^{\gamma|Y_i^{(\varepsilon)}(t_N)|})^{\frac{\beta}{\gamma}} \\ &\le (k+1)^{\frac{\beta}{\gamma}-1}((L_{1,N})^{\frac{\beta}{\gamma}} \\ &\quad + (L_{2,N})^{\frac{\beta}{\gamma}}\mathsf{E}_{\vec{y}^{(\varepsilon)}, t_{N-1}}\sum_{i=1}^{k} e^{\beta|y_i^{(\varepsilon)}|}e^{\beta|Y_i^{(\varepsilon)}(t_N) - y_i^{(\varepsilon)}|}) \end{aligned} \tag{4.20}$$

$$\leq (k+1)^{\frac{\beta}{\gamma}-1}((L_{1,N})^{\frac{\beta}{\gamma}} + (L_{2,N})^{\frac{\beta}{\gamma}}(L_{13}+1)\sum_{i=1}^{k} e^{\beta|y_i^{(\varepsilon)}|})$$

and, therefore,

$$\overline{\lim_{\varepsilon\to 0}} \mathsf{E}|w^{(\varepsilon)}(t_N, \vec{Y}_{N-1}^{(\varepsilon)}(\vec{y}^{(\varepsilon)}))|^{\frac{\beta}{\gamma}} < \infty. \tag{4.21}$$

Relations (4.12) and (4.21) imply that for any sequence $\vec{y}_\varepsilon \to \vec{y}_0 \in \tilde{\mathbb{Y}}^k_{t_{N-1}} \cap \mathbb{Y}^k_{t_{N-1}}$ as $\varepsilon \to 0$,

$$\mathsf{E}_{\vec{y}_\varepsilon, t_{N-1}} w^{(\varepsilon)}(t_N, \vec{Y}^{(\varepsilon)}(t_N)) \to \mathsf{E}_{\vec{y}_0, t_{N-1}} w^{(0)}(t_N, \vec{Y}^{(0)}(t_N)) \text{ as } \varepsilon \to 0. \tag{4.22}$$

Relations (4.22), (2.1) and condition $\mathbf{A_3}$ imply that for any sequence $\vec{y}_\varepsilon \to \vec{y}_0 \in \tilde{\mathbb{Y}}^k_{t_{N-1}} \cap \mathbb{Y}^k_{t_{N-1}}$ as $\varepsilon \to 0$,

$$\begin{aligned} w^{(\varepsilon)}(t_{N-1}, \vec{y}^{(\varepsilon)}) &= g^{(\varepsilon)}(t_{N-1}, e^{\vec{y}^{(\varepsilon)}}) \vee \mathsf{E}_{\vec{y}^{(\varepsilon)}, t_{N-1}} w^{(\varepsilon)}(t_N, \vec{Y}^{(\varepsilon)}(t_N)) \\ &\to w^{(0)}(t_{N-1}, \vec{y}^{(0)}) \\ &= g^{(0)}(t_{N-1}, e^{\vec{y}^{(0)}}) \vee \mathsf{E}_{\vec{y}^{(0)}, t_{N-1}} w^{(0)}(t_N, \vec{Y}^{(0)}(t_N)) \text{ as } \varepsilon \to 0. \end{aligned} \tag{4.23}$$

Relations (4.10), (4.11), and (4.23) are analogues of relations (4.6), (4.7), and (4.8).

By repeating the recursive procedure described above, we finally get that, under conditions $\mathbf{A_2}$, $\mathbf{A_3}$, $\mathbf{B_1}$, and $\mathbf{C_3}$, for $\varepsilon \leq \varepsilon_3$, $n = 0, 1, \ldots, N$, and $\vec{y} \in \mathbb{R}^k$,

$$|w^{(\varepsilon)}(t_n, \vec{y})| \leq L_{1,n} + L_{2,n}\sum_{i=1}^{k} e^{\gamma|y_i|}, \tag{4.24}$$

where constants,

$$L_{1,n},\ L_{2,n} < \infty, \tag{4.25}$$

and that, for an arbitrary $y_n^{(\varepsilon)} \to y_n^{(0)}$ as $\varepsilon \to 0$, where $y_n^{(0)} \in \tilde{\mathbb{Y}}'_{t_n} \cap \mathbb{Y}''_{t_n}$, and for every $n = 0, 1, \ldots, N$,

$$w^{(\varepsilon)}(t_n, \vec{y}_n^{(\varepsilon)}) \to w^{(0)}(t_n, \vec{y}_n^{(0)}) \text{ as } \varepsilon \to 0. \tag{4.26}$$

Let us take an arbitrary sequence $\varepsilon_r \to \varepsilon_0 = 0$ as $r \to \infty$. By condition $\mathbf{B_2}$, the random vectors

$$\vec{Y}^{(\varepsilon_r)}(0) \Rightarrow \vec{Y}^{(\varepsilon_0)}(0) \text{ as } r \to \infty, \tag{4.27}$$

and

$$\mathsf{P}\{\vec{Y}^{(\varepsilon_0)}(0) \in \tilde{\mathbb{Y}}'_{t_0} \cap \mathbb{Y}''_{t_0}\} = 1. \tag{4.28}$$

According to Skorokhod representation theorem, one can construct random variables

$\vec{Y}'^{(\varepsilon_r)}(0), r = 0, 1, \ldots$ on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ such that for every $r = 0, 1, \ldots$, and $A \in \mathcal{B}_{\mathbb{R}^k}$,

$$\mathsf{P}\{\vec{Y}'^{(\varepsilon_r)}(0) \in A\} = \mathsf{P}\{\vec{Y}^{(\varepsilon_r)}(0) \in A\}, \tag{4.29}$$

and

$$\vec{Y}'^{(\varepsilon_r)}(0) \xrightarrow{a.s.} \vec{Y}'^{(\varepsilon_0)}(0) \ \text{ as } r \to \infty. \tag{4.30}$$

Let us denote

$$A = \{\omega \in \Omega : \vec{Y}'^{(\varepsilon_r)}(0, \omega) \to \vec{Y}'^{(\varepsilon_0)}(0, \omega) \text{ as } r \to \infty\}$$

and

$$B = \{\tilde{\omega} \in \Omega : \vec{Y}'^{(\varepsilon_0)}(0, \omega) \in \mathbb{Y}'_{t_0} \cap \mathbb{Y}''_{t_0}\}.$$

Relation (4.30) implies that $\mathsf{P}(A) = 1$. Relations (4.28) and (4.29) imply that $\mathsf{P}(B) = 1$. Thus, $\mathsf{P}(A \cap B) = 1$.

By relation (4.26) and the definition of sets A and B, the non-random sequence $w^{(\varepsilon_r)}(t_0, \vec{Y}'^{(\varepsilon_r)}(0, \omega)) \to w^{(\varepsilon_0)}(t_0, \vec{Y}'^{(\varepsilon_0)}(0, \omega))$ as $j \to \infty$, for every $\omega \in A \cap B$, i.e., the random vectors

$$w^{(\varepsilon_r)}(t_0, \vec{Y}'^{(\varepsilon_r)}(0)) \xrightarrow{a.s.} w^{(\varepsilon_0)}(t_0, \vec{Y}'^{(\varepsilon_0)}(0)) \ \text{ as } r \to \infty. \tag{4.31}$$

Relation (4.29) implies that for every $r = 0, 1, \ldots$, and $A \in \mathcal{B}_{\mathbb{R}^k}$,

$$\mathsf{P}\{w^{(\varepsilon_r)}(t_0, \vec{Y}'^{(\varepsilon_r)}(0)) \in A\} = \mathsf{P}\{w^{(\varepsilon_r)}(t_0, \vec{Y}^{(\varepsilon_r)}(0)) \in A\}, \tag{4.32}$$

Relations (4.31) and (4.32) imply that the random vectors

$$w^{(\varepsilon_r)}(t_N, \vec{Y}^{(\varepsilon_r)}(0)) \Rightarrow w^{(\varepsilon_0)}(t_N, \vec{Y}^{(\varepsilon_0)}(0)) \ \text{ as } r \to \infty. \tag{4.33}$$

Since an arbitrary choice of a sequence $\varepsilon_r \to \varepsilon_0$, relation (4.33) implies that

$$w^{(\varepsilon)}(t_0, \vec{S}^{(\varepsilon)}(0)) \Rightarrow w^{(0)}(t_0, \vec{S}^{(0)}(0)) \ \text{ as } \varepsilon \to 0. \tag{4.34}$$

Using inequalities (4.2) and (4.24), and condition $\mathbf{C_4}$ we get for $\varepsilon \leq \varepsilon_3$,

$$\begin{aligned} \mathsf{E}|w^{(\varepsilon)}(t_0, \vec{Y}^{(\varepsilon)}(0))|^{\frac{\beta}{\gamma}} &\leq \mathsf{E}(L_{1,0} + L_{2,0} \sum_{i=1}^{k} e^{\gamma|Y_i^{(\varepsilon)}(0)|})^{\frac{\beta}{\gamma}} \\ &\leq (k+1)^{\frac{\beta}{\gamma}-1}((L_{1,0})^{\frac{\beta}{\gamma}} + (L_{2,0})^{\frac{\beta}{\gamma}} \mathsf{E} \sum_{i=1}^{k} e^{\beta|Y_i^{(\varepsilon)}(0)|}) \\ &\leq (k+1)^{\frac{\beta}{\gamma}-1}((L_{1,0})^{\frac{\beta}{\gamma}} + (L_{2,0})^{\frac{\beta}{\gamma}} L_{14}), \end{aligned} \tag{4.35}$$

and, therefore,

$$\overline{\lim_{\varepsilon\to 0}}\mathsf{E}|w^{(\varepsilon)}(t_0,\vec{Y}^{(\varepsilon)}(0))|^{\frac{\beta}{\gamma}} < \infty. \tag{4.36}$$

Relations (4.34) and (4.36) imply that,

$$\mathsf{E}w^{(\varepsilon)}(t_0,\vec{Y}^{(\varepsilon)}(0)) \to \mathsf{E}w^{(0)}(t_0,\vec{Y}^{(\varepsilon)}(0)) \text{ as } \varepsilon \to 0. \tag{4.37}$$

Formula (4.5) and relation (4.37) imply relation (4.4) given in Theorem 4.1. □

5 Convergence of rewards for continuous time price processes

We are now ready to formulate and to prove our main convergence result for rewards of American type options for continuous time processes.

Now we formulate conditions of convergence for discrete time reward functionals $\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T})$ for continuous time model.

As was mentioned above, in the discrete time case, the payoff functions can be discontinuous. In the continuous time case, the derivatives of the payoff functions are involved in condition $\mathbf{A_1}$. The corresponding assumptions imply continuity of the payoff functions.

This give us possibility to weaken the assumption concerning the convergence of the payoff functions and just to require their pointwise convergence:

$\mathbf{A_4}$: $g^{(\varepsilon)}(t,\vec{s}) \to g^{(0)}(t,\vec{s})$ as $\varepsilon \to 0$, for every $(t,\vec{s}) \in [0,T]\times\mathbb{R}^k_+$.

Condition $\mathbf{A_4}$ can be re-written in terms of function $g^{(\varepsilon)}(t,e^{\vec{y}})$:

$\mathbf{A'_4}$: $g^{(\varepsilon)}(t,e^{\vec{y}}) \to g^{(0)}(t,e^{\vec{y}})$ as $\varepsilon \to 0$, for every $(t,\vec{y}) \in [0,T]\times\mathbb{R}^k$.

Remark 1. *See for example the paper Kukush and Silvestrov (2004) for an example where a payoff function are approximated by a perturbed discrete payoff function.*

Let us now formulate conditions assumed for the transition probabilities and the initial distributions of process $\vec{Y}^{(\varepsilon)}(t)$.

The first condition assumes weak convergence of the transition probabilities that should be locally uniform with respect to initial states from some sets, and also that the corresponding limit measures are concentrated on these sets:

$\mathbf{B_3}$: There exist measurable sets $\mathbb{Y}_t \subseteq \mathbb{R}^k$, $t \in [0,T]$ such that:

- **(a)** $P^{(\varepsilon)}(t,\vec{y}^{(\varepsilon)}, t+u,\cdot) \Rightarrow P^{(0)}(t,\vec{y},t+u,\cdot)$ as $\varepsilon \to 0$, for any $\vec{y}^{(\varepsilon)} \to \vec{y} \in \mathbb{Y}_t$ as $\varepsilon \to 0$ and $0 \le t < t+u \le T$;

(b) $P^{(0)}(t, \vec{y}, t+u, \mathbb{Y}_{t+u}) = 1$ for every $\vec{y} \in \mathbb{Y}_t$ and $0 \le t < t+u \le T$.

The second condition assumes weak convergence of the initial distributions to some distribution that is assumed to be concentrated on the sets of convergence for the corresponding transition probabilities:

$\mathbf{B_4}$: **(a)** $P^{(\varepsilon)}(\cdot) \Rightarrow P^{(0)}(\cdot)$ as $\varepsilon \to 0$;

(b) $P^{(0)}(\mathbb{Y}_0) = 1$, where $\mathbb{Y}_0$ is the set introduced in condition $\mathbf{B_3}$.

The following theorem presents our main convergence result. It gives conditions of convergence for reward functionals $\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T})$.

Theorem 5.1. *Let conditions* $\mathbf{A_1}$, $\mathbf{A_4}$, $\mathbf{B_3}$, $\mathbf{B_4}$, $\mathbf{C_1}$, *and* $\mathbf{C_2}$ *hold. Then*

$$\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}) \to \Phi(\mathcal{M}^{(0)}_{\max,T}) \text{ as } \varepsilon \to 0. \tag{5.1}$$

Proof. Let $\Pi_N = \{0 = t_0 < t_1 < \dots t_N = T\}$ be a sequence of partitions on the interval $[0, T]$. Since we study the case when $\varepsilon \to 0$, we can assume that $d(\Pi_n) \le c$ and $\varepsilon \le \varepsilon_1$ where c and ε_1 are defined in relations (2.9) and (2.10) this ensures that the corresponding reward functionals are finite.

The following lemmas play the key role in the proof.

Lemma 5.2. *Let conditions* $\mathbf{A_1}$, $\mathbf{C_1}$, *and* $\mathbf{C_2}$ *hold, the following relation holds for any sequence of partitions* Π_N *such that* $d(\Pi_N) \to 0$ *as* $N \to \infty$,

$$\lim_{N\to\infty} \overline{\lim_{\varepsilon\to 0}} (\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_N,T})) = 0. \tag{5.2}$$

Proof. This Lemma is a direct corollary of Theorem 3.2, which implies that, under conditions $\mathbf{A_1}$, $\mathbf{C_1}$, and $\mathbf{C_2}$, there exist constants $L_3, L_4 < \infty$ such that the following skeleton approximation inequality holds for $\varepsilon \le \varepsilon_1$ and N such that $d(\Pi_N) \le c$,

$$\begin{aligned} &\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_N,T}) \\ &\quad \le L_3 d(\Pi_N) + L_4 \Big(\sum_{i=1}^{k} \Delta_\beta(Y^{(\varepsilon)}_i(\cdot), d(\Pi_N), T)\Big)^{\frac{\beta-\gamma}{\beta}}. \end{aligned} \tag{5.3}$$

This estimate directly implies relation (5.2). □

Lemma 5.3. *Let conditions* $\mathbf{A_1}$, $\mathbf{A_4}$, $\mathbf{B_3}$, $\mathbf{B_4}$, $\mathbf{C_1}$, *and* $\mathbf{C_2}$ *hold. Then, conditions of Theorem 4.1 hold for any partition* $\Pi_N = \{0 = t_0 < t_1 \dots < t_N = T\}$ *on the interval* $[0, T]$ *and, therefore, the following asymptotic relation holds,*

$$\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_N,T}) \to \Phi(\mathcal{M}^{(0)}_{\Pi_N,T}) \text{ as } \varepsilon \to 0. \tag{5.4}$$

Proof. Take an arbitrary partition $\Pi_N = \{0 = t_0 < t_1 \cdots < t_N = T\}$ on the interval $[0, T]$.

Inequality (2.21) given in the proof of Lemma 2.3, implies that, for every $n = 0, \ldots, N$, $\vec{s} \in \mathbb{R}_+^k$, and $\varepsilon \leq \varepsilon_0$,

$$|g^{(\varepsilon)}(t_n, \vec{s})| \leq L_8 + L_9 \sum_{i=1}^{k} (s_i)^{\gamma}, \tag{5.5}$$

where $\gamma = \max(\gamma_0, \gamma_1 + 1, \ldots, \gamma_k + 1)$.

Thus condition $\mathbf{A_2}$ holds for partition Π_N with the constants $K_6 = L_8, K_7 = L_9$ and the parameter γ given above.

Also, inequality (2.20) given in the proof of Lemma 3, implies that for every $n = 0, \ldots, N$, $\vec{s}', \vec{s}'' \in \mathbb{R}_+^k$, and $\varepsilon \leq \varepsilon_0$,

$$|g^{(\varepsilon)}(t_n, \vec{s}') - g^{(\varepsilon)}(t_n, \vec{s}'')| \leq \sum_{i=1}^{k} (K_3 + K_4 \sum_{j=1}^{k} (s_j^+)^{\gamma_i})(s_i^+ - s_i^-). \tag{5.6}$$

and, therefore, for any $0 < u_i^- < u_i^+ < \infty, i = 1, \ldots, k$

$$\sup_{u_i^- \leq s_i', s_i'' < u_i^+, |s_i' - s_i''| \leq c, i=1,\ldots,k} |g^{(\varepsilon)}(t_n, \vec{s}') - g^{(\varepsilon)}(t_n, \vec{s}'')| \leq L_{15} c, \tag{5.7}$$

where

$$L_{15} = \sum_{i=1}^{k} (K_3 + K_4 \sum_{j=1}^{k} (u_j^+)^{\gamma_i}).$$

Condition $\mathbf{A_4}$ and inequalities (5.5) and (5.7) imply that for every $n = 0, \ldots, N$, and $0 < u_i^- < u_i^- < \infty, i = 1, \ldots, k$, the conditions of Ascoli-Arzelá theorem holds for payoff functions $g^{(\varepsilon)}(t_n, \vec{s})$, $u_i^- \leq s_i < u_i^+, i = 1, \ldots, k$, for every $n = 0, 1, \ldots, N$. Thus, these functions converge uniformly, i.e.,

$$\sup_{u_i^- \leq s_i < u_i^+, i=1,\ldots,k} |g^{(\varepsilon)}(t_n, \vec{s}) - g^{(0)}(t_n, \vec{s})| \to 0 \text{ as } \varepsilon \to 0. \tag{5.8}$$

Relation (5.8) implies that condition of locally uniform convergence $\mathbf{A_4}$ holds for partition Π_N with the corresponding sets $\mathbb{S}'_{t_n} = \mathbb{R}_+^k$, for $n = 0, 1, \ldots, N$.

It remain to show that condition $\mathbf{C_1}$ implies that condition $\mathbf{C_3}$ holds. Condition $\mathbf{C_1}$ implies that for any constant $L_{16} < \infty$ one can choose $c = c(L_{16}) > 0$ and then $\varepsilon_4 = \varepsilon_4(c) < \varepsilon_2$ such that for $\varepsilon \leq \varepsilon_4$ and $i = 1, \ldots, k$,

$$\Delta_\beta(Y_i^{(\varepsilon)}(\cdot), c, T) \leq L_{16}. \tag{5.9}$$

Take an arbitrary integer $0 \leq n \leq N$ and consider the uniform partition $t_n = u_0^{(m)} < \ldots < u_m^{(m)} = t_{n+1}$ on the interval $[t_n, t_{n+1}]$ by points $u_j^{(m)} = \frac{(t_{n+1}-t_n)j}{m}$. Relation (5.9) and the Markov property of the processes $\vec{Y}^{(\varepsilon)}(t)$ imply that for $\varepsilon \leq \varepsilon_4$, $m = [\frac{(t_{n+1}-t_n)}{c}] + 1$ (in this case $\frac{(t_{n+1}-t_n)}{m} \leq c$), $\vec{y} \in \mathbb{R}^k$, $j = 1, \ldots, m$ and $i = 1, \ldots, k$,

$$
\begin{aligned}
&\mathsf{E}_{\vec{y},t_n}(e^{\beta|Y_i^{(\varepsilon)}(u_j^{(m)})-Y_i^{(\varepsilon)}(u_0^{(m)})|} - 1) \\
&\quad \leq \mathsf{E}_{\vec{y},t_n} e^{\beta|Y_i^{(\varepsilon)}(u_{j-1}^{(m)})-Y_i^{(\varepsilon)}(u_0^{(m)})|} e^{\beta|Y_i^{(\varepsilon)}(u_j^{(m)})-Y_i^{(\varepsilon)}(u_{j-1}^{(m)})|} - 1 \\
&\quad = \mathsf{E}_{\vec{y},t_n}((e^{\beta|Y_i^{(\varepsilon)}(u_{j-1}^{(m)})-Y_i^{(\varepsilon)}(u_0^{(m)})|} - 1)e^{\beta|Y_i^{(\varepsilon)}(u_j^{(m)})-Y_i^{(\varepsilon)}(u_{j-1}^{(m)})|} \\
&\qquad + e^{\beta|Y_i^{(\varepsilon)}(u_j^{(m)})-Y_i^{(\varepsilon)}(u_{j-1}^{(m)})|} - 1) \\
&\quad = \mathsf{E}_{\vec{y},t_n}\{(e^{\beta|Y_i^{(\varepsilon)}(u_{j-1}^{(m)})-Y_i^{(\varepsilon)}(u_0^{(m)})|} - 1) \\
&\qquad \times \mathsf{E}\{(e^{\beta|Y_i^{(\varepsilon)}(u_j^{(m)})-Y_i^{(\varepsilon)}(u_{j-1}^{(m)})|} - 1 + 1)|\vec{Y}^{(\varepsilon)}(u_{j-1}^{(m)})\}\} \\
&\qquad + \mathsf{E}_{\vec{y},t_n}\{\mathsf{E}\{(e^{\beta|Y_i^{(\varepsilon)}(u_j^{(m)})-Y_i^{(\varepsilon)}(u_{j-1}^{(m)})|} - 1)|\vec{Y}^{(\varepsilon)}(u_{j-1}^{(m)})\}\} \\
&\quad \leq \mathsf{E}_{\vec{y},t_n}(e^{\beta|Y_i^{(\varepsilon)}(u_{j-1}^{(m)})-Y_i^{(\varepsilon)}(u_0^{(m)})|} - 1)(L_{16} + 1) + L_{16}
\end{aligned}
\tag{5.10}
$$

Finally, we get, for $\varepsilon \leq \varepsilon_4$, $\vec{y} \in \mathbb{R}^k$, and $i = 1, \ldots, k$,

$$
\begin{aligned}
&\mathsf{E}_{\vec{y},t_n}(e^{\beta|Y_i^{(\varepsilon)}(t_{n+1})-Y_i^{(\varepsilon)}(t_n)|} - 1) \\
&\quad = \mathsf{E}_{\vec{y},t_n}(e^{\beta|Y_i^{(\varepsilon)}(u_m^{(m)})-Y_i^{(\varepsilon)}(u_0^{(m)})|} - 1) \\
&\quad \leq ((L_{12} + 1)^m + L_{12} \sum_{j=0}^{m-1} (L_{12} + 1)^j) = (2(L_{12} + 1)^m - 1) < \infty.
\end{aligned}
\tag{5.11}
$$

Thus condition $\mathbf{C_3}$ holds with parameter β given in condition $\mathbf{A_1}$. Finally, condition $\mathbf{C_2}$ is equivalent to condition $\mathbf{C_4}$. The proof of Lemma 5.3 is complete. □

Lemmas 5.2 and 5.3 let us make the final fifth step in the proof of Theorem 5.1. We employ the following inequality that can be written down for any partition Π_N,

$$
\begin{aligned}
&|\Phi(\mathcal{M}_{\max,T}^{(\varepsilon)}) - \Phi(\mathcal{M}_{\max,T}^{(0)})| \leq |\Phi(\mathcal{M}_{\max,T}^{(\varepsilon)}) - \Phi(\mathcal{M}_{\Pi_N,T}^{(\varepsilon)})| \\
&\qquad + |\Phi(\mathcal{M}_{\max,T}^{(0)}) - \Phi(\mathcal{M}_{\Pi_N,T}^{(0)})| + |\Phi(\mathcal{M}_{\Pi_N,T}^{(\varepsilon)}) - \Phi(\mathcal{M}_{\Pi_N,T}^{(0)})|.
\end{aligned}
\tag{5.12}
$$

Using this inequality and relation (5.4) given in Lemma 5.3 we get for any partition Π_N and for $d(\Pi_N) < c$,

$$
\begin{aligned}
&\overline{\lim_{\varepsilon \to 0}} |\Phi(\mathcal{M}_{\max,T}^{(\varepsilon)}) - \Phi(\mathcal{M}_{\max,T}^{(0)})| \\
&\quad \leq \overline{\lim_{\varepsilon \to 0}} |\Phi(\mathcal{M}_{\max,T}^{(\varepsilon)}) - \Phi(\mathcal{M}_{\Pi_N,T}^{(\varepsilon)})| + |\Phi(\mathcal{M}_{\max,T}^{(0)}) - \Phi(\mathcal{M}_{\Pi_N,T}^{(0)})|.
\end{aligned}
\tag{5.13}
$$

Finally, relation (5.2) given in Lemma 5.2 implies (note that relation $\varepsilon \to 0$ admit also the case where $\varepsilon = 0$) that the expression on the right hand side in (5.13) can be forced to take a value less then any $\varepsilon > 0$ by choosing the partition Π_N with the diameter $d(\Pi_N)$ small enough.

This proves the asymptotic relation (5.1) and thus the proof of Theorem 5.1 is complete. $\square$

In conclusion of this section, let us formulate some useful sufficient conditions for an important condition of moment compactness $\mathbf{C_1}$.

Let us introduce the modulus of J-compactness, for $h, c > 0, i = 1, \ldots, k$,

$$\Delta(Y_i^{(\varepsilon)}(\cdot), h, c, T) = \sup_{0 \le t \le t+u \le t+c \le T} \sup_{\vec{y} \in \mathbb{R}^k} \mathsf{P}_{\vec{y},t}\{|Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t)| \ge h\}.$$

The following condition of J-compactness plays the key role in functional limit theorems for Markov type càdlàg processes:

$\mathbf{C_5}$: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \Delta(Y_i^{(\varepsilon)}(\cdot), h, c, T) = 0,\ h > 0, i = 1, \ldots, k.$

Introduce also the quantity, which represents the maximum of moment generating functions for increments of the log-price processes $Y_i^{(\varepsilon)}(t)$, $i = 1, \ldots, k$,

$$\Xi_\beta(Y_i^{(\varepsilon)}(\cdot), T) = \sup_{0 \le t \le t+u \le T} \sup_{\vec{y} \in \mathbb{R}^k} \mathsf{E}_{\vec{y},t} e^{\beta(Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t))},\ \beta \in \mathbb{R}_1.$$

The following condition formulated in terms of these moment generating functions can be effectively verified in many cases:

$\mathbf{C_6}$: $\overline{\lim}_{\varepsilon \to 0} \Xi_{\pm\beta'}(Y_i^{(\varepsilon)}(\cdot), T) < \infty, i = 1, \ldots, k$, for some $\beta' > \beta$, where β is the parameter introduced in condition $\mathbf{C_1}$.

Lemma 5.4. *Conditions* $\mathbf{C_5}$ *and* $\mathbf{C_6}$ *imply condition* $\mathbf{C_1}$.

Proof. Using Hölder inequality we get the following estimates, for every $0 \le t \le t + u \le T$, $\vec{y} \in \mathbb{R}_+^k$ and $i = 1, \ldots, k$,

$$\begin{aligned} & \mathsf{E}_{\vec{y},t} e^{\beta|Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t)|} - 1 \\ & \quad \le (e^{\beta h} - 1) + \mathsf{E}_{\vec{y},t} e^{\beta|Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t)|} \\ & \qquad \times \chi(|Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t)| \ge h) \\ & \quad \le (e^{\beta h} - 1) + (\mathsf{E}_{\vec{y},t} e^{\beta'|Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t)|})^{\frac{\beta}{\beta'}} \\ & \qquad \times \mathsf{P}_{\vec{y},t}\{|Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t)| \ge h\}. \end{aligned} \tag{5.14}$$

The following inequality, which connect the exponential moment modulus of compactness with the modulus of J-compactness, follows from (5.14),

$$\begin{aligned}\Delta_\beta(Y_i^{(\varepsilon)}(\cdot), c, T) &\leq (e^{\beta h} - 1) \\ &\quad + (\Delta_{\beta'}(Y_i^{(\varepsilon)}(\cdot), c, T) + 1)^{\frac{\beta}{\beta'}} \Delta(Y_i^{(\varepsilon)}(\cdot), h, c, T).\end{aligned} \tag{5.15}$$

Also, the following estimate takes place, for every $0 \leq t \leq t + u \leq T$, $\vec{y} \in \mathbb{R}_{k+}$, and $i = 1, \ldots, k$,

$$\begin{aligned}&\mathsf{E}_{\vec{y},t} e^{\beta'|Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t)|} \\ &\quad = \mathsf{E}_{\vec{y},t} e^{\beta'(Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t))} \chi(Y_i^{(\varepsilon)}(t+u) \geq Y_i^{(\varepsilon)}(t)) \\ &\quad\quad + \mathsf{E}_{\vec{y},t} e^{-\beta'(Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t))} \chi(Y_i^{(\varepsilon)}(t+u) < Y_i^{(\varepsilon)}(t)) \\ &\quad \leq \mathsf{E}_{\vec{y},t} e^{\beta'(Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t))} + \mathsf{E}_{\vec{y},t} e^{-\beta'(Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t))}.\end{aligned} \tag{5.16}$$

This estimate implies the following inequality,

$$\Delta_{\beta'}(Y_i^{(\varepsilon)}(\cdot), c, T) + 1 \leq \Xi_{\beta'}(Y_i^{(\varepsilon)}(\cdot), T) + \Xi_{-\beta'}(Y_i^{(\varepsilon)}(\cdot), T). \tag{5.17}$$

Relations (5.15) and (5.17) imply the statement of Lemma 5.4. □

6 Multivariate exponential price processes with independent increments

In this section we show general convergence results for the important case of exponential price processes with independent increments.

Let us consider the model where the log-price process $\vec{Y}^{(\varepsilon)}(t), t \geq 0$ is a càdlàg process with independent increments.

We also assume for simplicity that the initial state of process $\vec{Y}^{(\varepsilon)}(0) = \vec{y}^{(\varepsilon)} = (y_i^{(\varepsilon)}, i = 1, \ldots, k)$ is a constant.

The process $\vec{Y}^{(\varepsilon)}(t)$ is a càdlàg Markov process with transition probabilities which are connected with the distributions of increments for this process $P^{(\varepsilon)}(t, t+u, A)$ by the following relation,

$$\begin{aligned}P^{(\varepsilon)}(t, \vec{y}, t+u, A) &= P^{(\varepsilon)}(t, t+u, A - \vec{y}) \\ &= \mathsf{P}\{\vec{y} + \vec{Y}^{(\varepsilon)}(t+u) - \vec{Y}^{(\varepsilon)}(t) \in A\}.\end{aligned} \tag{6.1}$$

Let us assume the following standard condition of weak convergence for distributions of increments for log-price processes:

$\mathbf{D_1}$: $P^{(\varepsilon)}(t, t+u, \cdot) \Rightarrow P^{(0)}(t, t+u, \cdot)$ as $\varepsilon \to 0$, $0 \leq t \leq t+u \leq T$.

Representation (6.1) implies in an obvious way that condition $\mathbf{B_3}$ holds with the sets $\mathbb{Y}_t = \mathbb{R}^k, t \in [0, T]$, i.e., distributions of increments for the processes $Y_i^{(\varepsilon)}(t)$ locally uniformly weakly converge, if condition $\mathbf{D_1}$ holds.

Thus, in the case of processes with independent increments, the condition $\mathbf{B_3}$ with the sets $\mathbb{Y}_t = \mathbb{R}^k$ is, in fact, equivalent to the standard condition of weak convergence for such processes.

In this case the J-compactness modulus $\Delta(Y_i^{(\varepsilon)}(\cdot), h, c, T)$ takes the following form:

$$\Delta'(Y_i^{(\varepsilon)}(\cdot), h, c, T) = \sup_{0 \leq t \leq t+u \leq t+c \leq T} \mathsf{P}\{|Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t)| \geq h\}.$$

Thus, condition $\mathbf{C_5}$ is reduced to the standard J-compactness condition for the log-price processes with independent increments:

$\mathbf{D_2}$: $\overline{\lim}_{c \to 0} \lim_{\varepsilon \to 0} \Delta'(Y_i^{(\varepsilon)}(t), h, c, T) = 0$, $h > 0, i = 1, \ldots, k$.

Note that conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ imply J-convergence of processes $\vec{Y}^{(\varepsilon)}(t)$, $t \in [0, T]$ to process $\vec{Y}^{(0)}(t), t \in [0, T]$ as $\varepsilon \to 0$ and stochastic continuity of the limit process.

Also, the quantities $\Xi_\beta(Y_i^{(\varepsilon)}(\cdot), T), i = 1, \ldots, k$ take a simplified form,

$$\Xi'_\beta(Y_i^{(\varepsilon)}(\cdot), T) = \sup_{0 \leq t \leq t+u \leq T} \mathsf{E} e^{\beta(Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t))}, \ \beta \in \mathbb{R}_1.$$

Therefore, condition $\mathbf{C_6}$ takes the following form:

$\mathbf{D_3}$: $\overline{\lim}_{\varepsilon \to 0} \Xi'_{\pm\beta'}(Y_i^{(\varepsilon)}(\cdot), T) < \infty, i = 1, \ldots, k$, for some $\beta' > \beta$, where β is the parameter introduced in condition $\mathbf{C_1}$.

According to Lemma 5.4, conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ imply condition $\mathbf{C_1}$.

Condition $\mathbf{B_4}$ is reduced in this case to the following condition:

$\mathbf{D_4}$: $\lim_{\varepsilon \to 0} \vec{y}^{(\varepsilon)} = \vec{y}_0$.

Note that $\vec{y}_0$ can be any vector with real-valued components since the set $\mathbb{Y}_0 = \mathbb{R}^k$.

Note that, condition $\mathbf{D_4}$ implies also condition $\mathbf{C_4}$.

The following theorem summarizes the remarks above.

Theorem 6.1. *Let conditions* $\mathbf{A_1}$, $\mathbf{A_4}$, *and* $\mathbf{D_1}$ – $\mathbf{D_4}$ *hold for the exponential price processes with independents* $\vec{S}^{(\varepsilon)}(t)$. *Then*

$$\Phi(\mathcal{M}^{(\varepsilon)}_{\max, T}) \to \Phi(\mathcal{M}^{(0)}_{\max, T}) < \infty \text{ as } \varepsilon \to 0. \tag{6.2}$$

The skeleton approximations $\vec{Y}^{(\varepsilon)}(t) = \vec{Y}^{(0)}([t/\varepsilon])$, $t \geq 0$ for a stochastically continuous càdlàg log-price process $\vec{Y}^{(0)}(t)$, $t \geq 0$ with independent increments give a good example of the model introduced above.

In this case, conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ automatically hold.

Condition $\mathbf{D_3}$ is implied by the following condition:

$\mathbf{D_5}$: $\Xi'_{\pm\beta'}(Y_i^{(0)}(\cdot), T) < \infty, i = 1, \ldots, k$, for some $\beta' > \beta$, where β is the parameter introduced in condition $\mathbf{C_1}$.

Thus, if we let conditions $\mathbf{A_1}$, $\mathbf{A_4}$, and $\mathbf{D_5}$ hold, then the statement of Theorem 6.1 holds for the exponential price processes $\vec{S}^{(\varepsilon)}(t) = e^{\vec{Y}^{(0)}([t/\varepsilon])}, t \in [0, T]$.

Note that, in this case, Theorem 3.2 yields a stronger result in the form of explicit estimates for the accuracy of skeleton approximations for reward functions.

Note also that the optimal expected rewards for the skeleton price processes $\vec{S}^{(\varepsilon)}(t) = e^{\vec{Y}^{(0)}([t/\varepsilon])}$ can be estimated with the use of Monte Carlo simulation.

7 Binomial tree option reward approximations for multivariate Brownian motion

In order to illustrate the results presented above, let us consider the model where $k = 2$ and the bivariate geometric Brownian price process $\vec{S}^{(0)}(t) = e^{\vec{Y}^{(0)}(t)}, t \geq 0$, where the log-price process $\vec{Y}^{(0)}(t) = (Y_1^{(0)}(t), Y_2^{(0)}(t)), t \geq 0$ is a bivariate Brownian motion with components $Y_i^{(0)}(t) = y_i^{(0)} + \mu_i t + \sigma_i W_i(t), t \geq 0, i = 1, 2$, which are correlated, i.e., $\mathsf{E} W_1(t) W_2(t) = \rho t, t \geq 0$.

We approximate the process $\vec{Y}^{(0)}(t), t \geq 0$ with a bivariate binomial sum-process $\vec{Y}^{(\varepsilon)}(t) = (Y_1^{(\varepsilon)}(t), Y_2^{(\varepsilon)}(t)), t \geq 0$ with components $Y_i^{(\varepsilon)}(t) = y_i^{(0)} + \sum_{1 \leq k \leq [t/\varepsilon]} Y_{k,i}^{(\varepsilon)}, t \geq 0,\ i = 1, 2$.

Here $\vec{Y}_n^{(\varepsilon)} = (Y_{n,1}^{(\varepsilon)}, Y_{n,2}^{(\varepsilon)})$, $n = 1, 2, \ldots$ are, for every $\varepsilon > 0$, i.i.d. random vectors which have the following structure,

$$(Y_{n,1}^{(\varepsilon)}, Y_{n,2}^{(\varepsilon)}) = \begin{cases} (+u_1^{(\varepsilon)}, +u_2^{(\varepsilon)}) & & p_{++}^{(\varepsilon)}, \\ (+u_1^{(\varepsilon)}, -u_2^{(\varepsilon)}) & & p_{+-}^{(\varepsilon)}, \\ & \text{with prob.} & \\ (-u_1^{(\varepsilon)}, +u_2^{(\varepsilon)}) & & p_{-+}^{(\varepsilon)}, \\ (-u_1^{(\varepsilon)}, -u_2^{(\varepsilon)}) & & p_{--}^{(\varepsilon)}. \end{cases} \tag{7.1}$$

Since we assumed that the initial state $\vec{Y}^{(\varepsilon)}(0) = \vec{y}^{(0)} = (y_1^{(0)}, y_2^{(0)})$ is a constant which does not depend of ε, conditions $\mathbf{B_4}$ and $\mathbf{C_4}$ automatically hold.

In order to fit the bivariate binomial sum-processes defined above to the limit bivariate Brownian motion, we should fit expectations, variances, and covariance coefficients for summands $(Y_{n,1}^{(\varepsilon)}, Y_{n,2}^{(\varepsilon)})$ to the corresponding quantities for the increments of the bivariate Brownian motion $(\mu_1\varepsilon + \sigma_1(W_1((n+1)\varepsilon) - W_1(n\varepsilon)), \mu_2\varepsilon + \sigma_2(W_2((n+1)\varepsilon) - W_2(n\varepsilon)))$.

The following system of six equations with six unknowns should be solved:

$$\begin{cases} \mathsf{E}[Y_{1,1}^{(\varepsilon)}] & = u_1^{(\varepsilon)}(2(p_{++}^{(\varepsilon)} + p_{+-}^{(\varepsilon)}) - 1) & = \mu_1\varepsilon, \\ \mathsf{Var}[Y_{1,1}^{(\varepsilon)}] & = (u_1^{(\varepsilon)})^2 - (\mu_1\varepsilon)^2 & = \sigma_1^2\varepsilon, \\ \mathsf{E}[Y_{1,2}^{(\varepsilon)}] & = u_2^{(\varepsilon)}(2(p_{++}^{(\varepsilon)} + p_{-+}^{(\varepsilon)}) - 1) & = \mu_2\varepsilon, \\ \mathsf{Var}[Y_{1,2}^{(\varepsilon)}] & = (u_2^{(\varepsilon)})^2 - (\mu_2\varepsilon)^2 & = \sigma_2^2\varepsilon, \\ \mathsf{Cov}[Y_{1,1}^{(\varepsilon)}, Y_{1,2}^{(\varepsilon)}] & = \frac{u_1^{(\varepsilon)}u_2^{(\varepsilon)}(p_{++}^{(\varepsilon)} + p_{--}^{(\varepsilon)} - p_{-+}^{(\varepsilon)} - p_{+-}^{(\varepsilon)}) - \mu_1\mu_2\varepsilon^2}{\sigma_1\sigma_2\varepsilon} & = \rho, \\ & p_{++}^{(\varepsilon)} + p_{-+}^{(\varepsilon)} + p_{-+}^{(\varepsilon)} + p_{--}^{(\varepsilon)} & = 1. \end{cases} \tag{7.2}$$

This system has, for every $\varepsilon > 0$, the following unique solution,

$$\begin{cases} u_1^{(\varepsilon)} & = \sqrt{\varepsilon}\sqrt{\sigma_1^2 + \mu_1^2\varepsilon}, \\ u_2^{(\varepsilon)} & = \sqrt{\varepsilon}\sqrt{\sigma_2^2 + \mu_2^2\varepsilon}, \\ p_{++}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\sqrt{\varepsilon}\Big(\frac{\mu_1}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}} + \frac{\mu_2}{\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}\Big) + \frac{1}{4}\frac{\rho\sigma_1\sigma_2+\mu_1\mu_2\varepsilon}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}, \\ p_{+-}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\sqrt{\varepsilon}\Big(\frac{\mu_1}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}} - \frac{\mu_2}{\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}\Big) - \frac{1}{4}\frac{\rho\sigma_1\sigma_2+\mu_1\mu_2\varepsilon}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}, \\ p_{-+}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\sqrt{\varepsilon}\Big(- \frac{\mu_1}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}} + \frac{\mu_2}{\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}\Big) - \frac{1}{4}\frac{\rho\sigma_1\sigma_2+\mu_1\mu_2\varepsilon}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}, \\ p_{--}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\sqrt{\varepsilon}\Big(- \frac{\mu_1}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}} - \frac{\mu_2}{\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}\Big) + \frac{1}{4}\frac{\rho\sigma_1\sigma_2+\mu_1\mu_2\varepsilon}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}. \end{cases} \tag{7.3}$$

It is useful to note that the corresponding parameter have the following asymptotic expansions,

$$\begin{cases} u_1^{(\varepsilon)} & = \sqrt{\varepsilon}\sigma_1 + o(\varepsilon), \\ u_2^{(\varepsilon)} & = \sqrt{\varepsilon}\sigma_2 + o(\varepsilon), \\ p_{++}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\rho + \frac{1}{4}\Big(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\Big)\sqrt{\varepsilon} + \frac{1}{4}\Big(\frac{\mu_1\mu_2}{\sigma_1\sigma_2} - \frac{\rho\mu_1^2}{2\sigma_1^2} - \frac{\rho\mu_2^2}{2\sigma_2^2}\Big)\varepsilon + o(\varepsilon), \\ p_{+-}^{(\varepsilon)} & = \frac{1}{4} - \frac{1}{4}\rho + \frac{1}{4}\Big(\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\Big)\sqrt{\varepsilon} - \frac{1}{4}\Big(\frac{\mu_1\mu_2}{\sigma_1\sigma_2} - \frac{\rho\mu_1^2}{2\sigma_1^2} - \frac{\rho\mu_2^2}{2\sigma_2^2}\Big)\varepsilon + o(\varepsilon), \\ p_{-+}^{(\varepsilon)} & = \frac{1}{4} - \frac{1}{4}\rho - \frac{1}{4}\Big(\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\Big)\sqrt{\varepsilon} - \frac{1}{4}\Big(\frac{\mu_1\mu_2}{\sigma_1\sigma_2} - \frac{\rho\mu_1^2}{2\sigma_1^2} - \frac{\rho\mu_2^2}{2\sigma_2^2}\Big)\varepsilon + o(\varepsilon), \\ p_{--}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\rho - \frac{1}{4}\Big(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\Big)\sqrt{\varepsilon} + \frac{1}{4}\Big(\frac{\mu_1\mu_2}{\sigma_1\sigma_2} - \frac{\rho\mu_1^2}{2\sigma_1^2} - \frac{\rho\mu_2^2}{2\sigma_2^2}\Big)\varepsilon + o(\varepsilon). \end{cases} \tag{7.4}$$

Relation (7.4) guarantees that probabilities in (7.3) take values in interval $(0, 1)$ for ε small enough in the most interesting non-degenerate case, where $|\rho| < 1$. Note also that the sum of these probabilities is equal to 1, according to the last equation in (7.2).

The problem can be however reduced to more simple case where drift coefficients $\mu_1 = \mu_2 = 0$ and the initial state $\vec{y}^{(0)} = (0, 0)$.

Let us consider processes $\tilde{\vec{S}}^{(0)}(t) = e^{\tilde{\vec{Y}}^{(0)}(t)}, t \geq 0$, where the log-price process $\tilde{\vec{Y}}^{(0)}(t) = (\tilde{Y}_1^{(0)}(t), \tilde{Y}_2^{(0)}(t)), t \geq 0$ is a bivariate Brownian motion with components $\tilde{Y}_i^{(0)}(t) = \sigma_i W_i(t), t \geq 0, i = 1, 2$, which are correlated, i.e., $\mathsf{E} W_1(t) W_2(t) = \rho t, t \geq 0$. Obviously, the natural filtration $\mathcal{F}_t, t \geq 0$ is the same for processes $\vec{S}^{(0)}(t), t \geq 0$ and $\tilde{\vec{S}}^{(0)}(t), t \geq 0$.

Let $g^{(\varepsilon)}(t, \vec{s}) = g^{(\varepsilon)}(t, (s_1, s_2))$ be payoff functions that satisfy conditions $\mathbf{A_1}$ and $\mathbf{A_4}$.

Let us now consider the transformed payoff functions

$$\tilde{g}^{(\varepsilon)}(t, \vec{s}) = g^{(\varepsilon)}(t, (e^{y_1^{(0)} + \mu_1 t} s_1, e^{y_2^{(0)} + \mu_2 t} s_2)). \tag{7.5}$$

These functions also satisfy conditions $\mathbf{A_1}$ and $\mathbf{A_4}$ with some constants $K_i, i = 1, \ldots, 5$ and parameters $\gamma_0 = \gamma = \max(\gamma_0, \gamma_1 + 1, \gamma_2 + 1)$ and the same parameters γ_1 and γ_2.

It follows from the remarks above that the reward functional

$$\Phi(\mathcal{M}_T^{(0)}) = \sup_{\tau \in \mathcal{M}_T} \mathsf{E} g^{(0)}(\tau, \vec{S}^{(0)}(\tau)) = \sup_{\tau \in \mathcal{M}_T} \mathsf{E} \tilde{g}^{(0)}(\tau, \tilde{\vec{S}}^{(0)}(\tau)). \tag{7.6}$$

Now, we can approximate the bivariate Brownian processes $\tilde{\vec{Y}}^{(0)}(t)$ by the corresponding bivariate sum-processes $\tilde{\vec{Y}}^{(\varepsilon)}(t)$ as it was described above. In this case however the parameters μ_1 and μ_2 will take the value 0 in systems of equations (7.3) and (7.4). In this case, the solution to these systems will take the following simpler form,

$$\begin{cases} u_1^{(\varepsilon)} & = \sqrt{\varepsilon} \sigma_1, \\ u_2^{(\varepsilon)} & = \sqrt{\varepsilon} \sigma_2, \\ p_{++}^{(\varepsilon)} = p_{--}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4} \rho, \\ p_{+-}^{(\varepsilon)} = p_{-+}^{(\varepsilon)} & = \frac{1}{4} - \frac{1}{4} \rho. \end{cases} \tag{7.7}$$

The probabilities in (7.7) take non-negative values for any $|\rho| \leq 1$.

By applying convergence theorems for vector sum-processes with independent increments given for example in Skorokhod (1964) it is easy to check that the processes $\vec{Y}^{(\varepsilon)}(t), t \in [0, T]$ with parameters given in (7.3) weakly and moreover J-converge to process $\vec{Y}^{(0)}(t), t \in [0, T]$ as $\varepsilon \to 0$.

These statements remain true also if parameters of the approximating processes would be chosen equal to the corresponding sums of terms in the asymptotic expansions (7.4) with omitted terms $o(\varepsilon)$.

Also the processes $\tilde{\vec{Y}}^{(\varepsilon)}(t), t \in [0, T]$ with parameters given in (7.7) weakly J-converge to process $\tilde{\vec{Y}}^{(0)}(t), t \in [0, T]$ as $\varepsilon \to 0$.

Thus, conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ hold for processes $\vec{Y}^{(\varepsilon)}(t)$ and $\tilde{\vec{Y}}^{(\varepsilon)}(t)$.

Also, the moment generation functions $\mathsf{E}\exp\{\beta(Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t))\}$ exist for any $\beta \in \mathbb{R}$ and have an explicit form, namely, for $0 \leq t \leq t+u \leq T$, $i = 1, 2$,

$$\mathsf{E}\exp\{\beta(Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t))\}$$
$$= \begin{cases} (e^{\beta u_i^{(\varepsilon)}} p_i^{(\varepsilon)} + e^{-\beta u_i^{(\varepsilon)}} q_i^{(\varepsilon)})^{[(t+u)/\varepsilon]-[t/\varepsilon]}, & \text{if} \quad \varepsilon > 0, \\ \\ e^{\beta \mu_i u + \frac{\beta^2 \sigma_i^2 u}{2}}, & \text{if} \quad \varepsilon = 0, \end{cases} \tag{7.8}$$

where $p_1^{(\varepsilon)} = p_{++}^{(\varepsilon)} + p_{+-}^{(\varepsilon)}$, $q_1^{(\varepsilon)} = p_{-+}^{(\varepsilon)} + p_{--}^{(\varepsilon)}$ and $p_2^{(\varepsilon)} = p_{++}^{(\varepsilon)} + p_{-+}^{(\varepsilon)}$, $q_2^{(\varepsilon)} = p_{+-}^{(\varepsilon)} + p_{--}^{(\varepsilon)}$.

This makes it easy to check that condition $\mathbf{D_3}$ holds for processes $\vec{Y}^{(\varepsilon)}(t)$ for any $\beta' > \beta$.

Summarizing the remarks above, one can conclude that the conditions and, therefore, the statement of Theorem 6.1 holds, for the bivariate exponential price processes with independent increments $\vec{S}^{(\varepsilon)}(t) = \exp\{\vec{Y}^{(\varepsilon)}(t)\}, t \in [0, T]$, if conditions $\mathbf{A_1}$, $\mathbf{A_4}$ hold for the corresponding payoff functions, i.e.,

$$\Phi(\mathcal{M}_{\max,T}^{(\varepsilon)}) \to \Phi(\mathcal{M}_{\max,T}^{(0)}) \text{ as } \varepsilon \to 0. \tag{7.9}$$

Let assume for simplicity that $\varepsilon = T/N$ and consider the partition $\Pi_\varepsilon = \{t_0 = 0 < t_1 = \varepsilon < \cdots < t_{N-1} = (N-1)\varepsilon < t_N = T\}$ on the interval $[0, T]$.

In this case the Markov chain $(n, \vec{Y}^{(\varepsilon)}(n\varepsilon)), n = 0, 1, \ldots$ is a bivariate binomial tree model with the initial node $(0, \vec{y}^{(0)})$ and nodes of the form $(n, \vec{y}_{n,l_1,l_2})$, $l_1, l_2 = 0, 1, \ldots, n$, after $n \geq 1$ steps.

In the case of a non-reduced model with parameters of the approximating bivariate Bernoulli random vectors $\vec{Y}_n^{(\varepsilon)}$ defined by relations (7.1) and (7.3), the vector points $\vec{y}_{n,l_1,l_2}$, $l_1, l_2 = 0, 1, \ldots, n$, $n = 0, 1, \ldots, N$ should be defined by the formula

$$\vec{y}_{n,l_1,l_2} = \vec{y}^{(0)} + ((2l_1 - n)\sqrt{\varepsilon}\sqrt{\sigma_1^2 + \mu_1^2\varepsilon},\ (2l_2 - n)\sqrt{\varepsilon}\sqrt{\sigma_2^2 + \mu_2^2\varepsilon}), \tag{7.10}$$

while in the case of a reduced model with parameters of the approximating bivariate Bernoulli random vectors $(Y_{n,1}^{(\varepsilon)}, Y_{n,2}^{(\varepsilon)})$ defined by relations (7.1) and (7.7), the vector points $\vec{y}_{n,l_1,l_2}$, $l_1, l_2 = 0, 1, \ldots, n$, $n = 0, 1, \ldots, N$ should be defined by the simpler formula

$$\vec{y}_{n,l_1,l_2} = ((2l_1 - n)\sqrt{\varepsilon}\sigma_1,\ (2l_2 - n)\sqrt{\varepsilon}\sigma_2). \tag{7.11}$$

The corresponding tree has $(n+1)^2$ nodes after n steps. The number of nodes is a quadratic function of n.

The standard backward procedure can be applied in order to find the optimal expected reward at moment 0 for the discrete time exponential bivariate binomial price process $\vec{S}^{(\varepsilon)}(t_n) = e^{\vec{Y}^{(\varepsilon)}(t_n)}, t_n = n\varepsilon, n = 0, 1, \ldots, N$. This optimal expected reward coincides, in this case, with the reward functional $\Phi(\mathcal{M}_{\Pi_\varepsilon,T}^{(\varepsilon)})$ for the continuous time bivariate exponential price processes $\vec{S}^{(\varepsilon)}(t) = e^{\vec{Y}^{(\varepsilon)}(t)}, t \in [0, T]$.

To estimate the difference $\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T})$ we can use Theorem 1. In this case, $d(\Pi_\varepsilon) = \varepsilon$ and for $\beta \in \mathbb{R}$ and $i = 1, 2$,

$$\Delta_\beta(Y^{(\varepsilon)}_i(\cdot), \varepsilon, T) = \mathsf{E} e^{\beta |Y^{(\varepsilon)}_{1,i}|} - 1 \leq e^{\beta u^{(\varepsilon)}_i} - 1. \tag{7.12}$$

Theorem 3.2 yields in this case the following relation,

$$\begin{aligned} &\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T}) \\ &\qquad \leq L_3 \varepsilon + L_4((e^{\beta u^{(\varepsilon)}_1} - 1)^{\frac{\beta-\gamma}{\beta}} + (e^{\beta u^{(\varepsilon)}_2} - 1)^{\frac{\beta-\gamma}{\beta}}) \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{7.13}$$

As was pointed out in Section 3, the reward functional $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T})$ is the option optimal expected reward for American type options in discrete time that correspond to the partition to the discrete time Markov log-price process $\vec{Y}^{(\varepsilon)}(t_n), t_n = n\varepsilon, n = 0, 1, \dots, N$ with parameter $\varepsilon = T/N$. Introduce the corresponding reward functions,

$$w^{(\varepsilon)}(t_n, \vec{y}_{n,l_1,l_2}) = \sup_{\tau \in \mathcal{M}_{\Pi_\varepsilon,T},\, t_n \leq \tau \leq t_N = T} \mathsf{E}_{t_n, \vec{y}_{n,l_1,l_2}} g^{(\varepsilon)}(\tau, e^{\vec{Y}^{(\varepsilon)}(\tau)}),$$

where vector points $\vec{y}_{n,l_1,l_2}$, $l_1, l_2 = 0, \dots, n, n = 0, \dots, N$ were defined in (7.10), and, in particular, $\vec{y}_{0,0,0} = \vec{y}^{(0)}$.

Then, from relation (4.5),

$$\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T}) = w^{(\varepsilon)}(0, \vec{y}^{(0)}). \tag{7.14}$$

The reward functions $w^{(\varepsilon)}(t_n, \vec{y}_{n,l_1,l_2})$ can be found using the following backward recurrence relations, for $n = N-1, \dots, 0$,

$$\begin{aligned} w^{(\varepsilon)}(t_n, \vec{y}_{n,l_1,l_2}) &= g^{(\varepsilon)}(t_n, e^{\vec{y}_{n,l_1,l_2}}) \vee \big(w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1+1,l_2+1}) p^{(\varepsilon)}_{++} \\ &+ w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1+1,l_2}) p^{(\varepsilon)}_{+-} + w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1,l_2+1}) p^{(\varepsilon)}_{-+} \\ &+ w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1,l_2}) p^{(\varepsilon)}_{--}\big), \\ &l_1, l_2 = 0, \dots, n, \end{aligned} \tag{7.15}$$

with the boundary conditions

$$w^{(\varepsilon)}(t_N, \vec{y}_{N,l_1,l_2}) = g^{(\varepsilon)}(t_N, e^{\vec{y}_{N,l_1,l_2}}),\ l_1, l_2 = 0, \dots, N. \tag{7.16}$$

It is also useful to note that the modification of the approximation algorithm based on the reduced model described in (7.7) and (7.10) can be built and utilized in a similar way.

The corresponding approximation result for the bivariate binomial tree algorithm describe above can be summarized in the following theorem.

Theorem 7.1. *Let conditions* $\mathbf{A_1}$, $\mathbf{A_4}$ *hold. Then*

$$w^{(\varepsilon)}(0, \vec{y}^{(0)}) = \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon, T}) \to \Phi(\mathcal{M}^{(0)}_{\max, T}) \text{ as } \varepsilon \to 0. \tag{7.17}$$

As an other example of typical a payoff function, is a linear combination of payoff functions for a portfolio of options $g(t, \vec{s}) = e^{-rt}(a_1[s_1 - K_1]^+ + a_2[s_2 - K_2]^+)$ can be mentioned.

Let us also mention the model of exchange of assets with payoff function $g(t, \vec{s}) = e^{-rt}(s_1 - s_2)$. Note that this is an example of payoff function which is not nonnegative. The optimal stopping strategies for this model were recently studied in Mishura and Shevchenko (2009).

In both cases, the payoff functions are continuous, do not depend on perturbation parameter ε, and have obviously a polynomial rate of growth. Thus, conditions $\mathbf{A_1}$ and $\mathbf{A_4}$ automatically hold.

Therefore, according to Theorem 7.1 the optimal expected reward functions for the described above bivariate binomial exponential model converge to the corresponding optimal expected reward functionals for the corresponding bivariate geometric Brownian motion.

It is useful to note that the results concerning bivariate binomial models admit an obvious generalization to the case of multivariate binomial models.

Let us give some numerical example for the model of exchange of assets with payoff function $g(t, \vec{s}) = e^{-rt}(s_1 - s_2)$. We consider the case when the holder of the option has the right to change asset 1 for asset 2. The option has maturity in 6 months. Asset 1 has initial price 10 with a drift estimated to be 0.02 and volatility are estimated to 0.1 per year. Asset 2 has initial price 9.5 with a drift estimated to be 0.08 and volatility are estimated to 0.35 per year. The correlation between the to assets are assumed to be $\rho = 0.3$. The risk free interest rate are assumed to be $r = 0.04$ for the time period of the contract.

Study show that tree size $N = 100$ is enough, the expected reward for a tree with size $N = 100$ is 0.0850, this should be compared with the expected reward for a tree with size $N = 150$ which is 0.0858. The calculation time for this tree size is 5.11 seconds on an 1.73 GHz Intel® Pentium-M processor, 1GB internal memory using Matlab®.

Figure 1 illustrate the reward for exchanging Asset 1 for Asset 2 when parameters of the model are as above except the volatility vary on the interval $[0.05, 1]$ for the two different assets. It is worth to note that for some combinations of volatility the reward will be negative and thus exchange is not profitable. This question does however require an additional investigation.

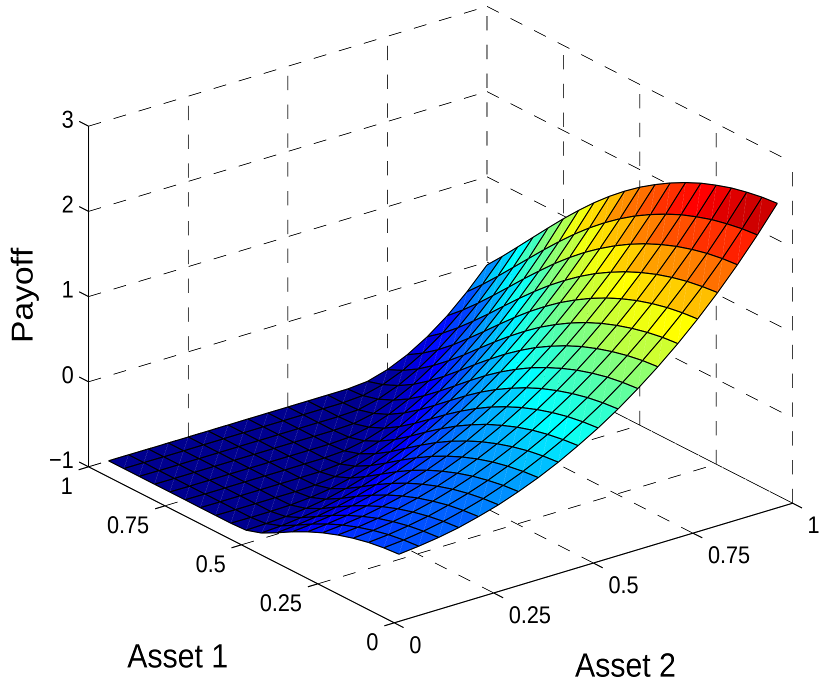

Figure 1: Rewards for exchange of Asset 1 for Asset 2, with volatilities $0.05 \leq \sigma_1 \leq 1$ and $0.05 \leq \sigma_2 \leq 1$.

8 Binomial tree approximations for mean reverse price processes

Let us consider the model of price process introduced by Schwartz (1997) for modeling energy prices. It has the following form

$$d \ln S(t) = -\alpha(\ln S(t) - \ln S(0))dt + \nu dW(t), \ t \geq 0, \tag{8.1}$$

where $\alpha, \nu > 0$, $W(t)$ is a standard Brownian motion, and the initial state $S(0) = s_0 > 0$ is a constant.

We consider further the case when we want to price an American call option, that is where a payoff function of the form $g(t,s) = e^{-rt}[s-K]^+$, where r, $K > 0$. It is easy to check that the payoff function satisfies conditions $\mathbf{A_1}$ and $\mathbf{A_4}$ with some constants $K_i, i = 1, \ldots, 5$ and parameters $\gamma_0 = 1$ and $\gamma_1 = 0$.

Thus our object of interest is in this case the reward functional

$$\Phi(\mathcal{M}^{(0)}_{\max,T}) = \sup_{0 \leq \tau \leq T} \mathsf{E} e^{-r\tau}[S(\tau) - K]^+. \tag{8.2}$$

The stochastic differential equation (8.1) has the following solution

$$S(t) = s_0\, e^{\nu e^{-\alpha t} \int_0^t e^{\alpha s} dW(s)}, \ t \geq 0. \tag{8.3}$$

The process $S(t)$ is a diffusion process, which however can be represented as a non-random transformation of a simpler exponential Gaussian process with independent increments

$$S^{(0)}(t) = e^{\nu e^{-\alpha T} \int_0^t e^{\alpha s} dW(s)}, \ t \geq 0. \tag{8.4}$$

It follows from (8.3) and (8.4) that

$$S(t) = s_0\, (S^{(0)}(t))^{e^{\alpha(T-t)}}, \ t \geq 0. \tag{8.5}$$

The processes $S(t), t \geq 0$ and $S^{(0)}(t), t \geq 0$ have the same natural filtration $\mathcal{F}_t, t \geq 0$ and therefore the class $\Phi(\mathcal{M}^{(0)}_{\max,T})$ of all Markov moments $0 \leq \tau \leq T$ is also the same for these processes.

Let us now consider the transformed payoff function

$$\tilde{g}^{(\varepsilon)}(t, s) = e^{-rt}[s_0 s^{e^{\alpha(T-t)}} - K]^+. \tag{8.6}$$

This function also satisfies conditions $\mathbf{A_1}$ and $\mathbf{A_4}$ with some constants $K_i', i = 1, \ldots, 5$ and parameters $\gamma_0 = e^{\alpha T} + 1$, $\gamma_1 = e^{\alpha T} - 1$.

It follows from the remarks above that the reward functional

$$\begin{aligned}\Phi(\mathcal{M}^{(0)}_{\max,T}) &= \sup_{0 \leq \tau \leq T} \mathsf{E} e^{-r\tau}[S(\tau) - K]^+ \\ &= \sup_{0 \leq \tau \leq T} \mathsf{E} e^{-r\tau}[s_0 S^{(0)}(\tau)^{e^{\alpha(T-\tau)}} - K]^+.\end{aligned} \tag{8.7}$$

The process $S^{(0)}(t) = e^{Y^{(0)}(t)}, t \geq 0$, where $Y^{(0)}(t) = \nu e^{-\alpha T} \int_0^t e^{\alpha s} dW(s)$, $t \geq 0$, is a Gaussian process with independent increments.

We approximate the process $Y^{(0)}(t), t \geq 0$ with a trinomial sum-process $Y^{(\varepsilon)}(t) = \sum_{1 \leq k \leq [t/\varepsilon]} Y^{(\varepsilon)}_n, t \geq 0$, where $Y^{(\varepsilon)}_n$, $n = 1, 2, \ldots$ are, for every $\varepsilon > 0$, independent random variables that have the following structure

$$Y^{(\varepsilon)}_n = \begin{cases} u^{(\varepsilon)} & & p^{(\varepsilon)}_{n,+}, \\ 0 & \text{with prob.} & p^{(\varepsilon)}_{n,\cdot}, \\ -u^{(\varepsilon)} & & p^{(\varepsilon)}_{n,-}. \end{cases} \tag{8.8}$$

Let us assume that $\varepsilon = T/N$.

In order to fit the trinomial sum-processes $Y^{(\varepsilon)}(t)$ to the limit process $Y^{(0)}(t)$, we should fit expectation and variance for the random variables $Y^{(\varepsilon)}_n$ and increment of the limit process $Y^{(0)}(n\varepsilon) - Y^{(0)}((n-1)\varepsilon)$ for $n = 1, \ldots, N$.

Simple calculations show that

$$\mathsf{E}(Y^{(0)}(n\varepsilon) - Y^{(0)}((n-1)\varepsilon)) = \mathsf{E}\nu e^{-\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} \nu e^{\alpha s} dW(s) = 0 \tag{8.9}$$

and

$$\begin{aligned}
\sigma^2_{n,\varepsilon} &= \mathsf{Var}(Y^{(0)}(n\varepsilon) - Y^{(0)}((n-1)\varepsilon)) \\
&= \mathsf{Var}\, \nu e^{-\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} e^{\alpha s} dW(s) \\
&= \nu^2 e^{-2\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} e^{2\alpha s} ds = \nu^2 e^{-2\alpha T} \frac{e^{2\alpha n\varepsilon} - e^{2\alpha(n-1)\varepsilon}}{2\alpha} \\
&= \nu^2 e^{-2\alpha T} e^{2\alpha n\varepsilon} \frac{1 - e^{-2\alpha\varepsilon}}{2\alpha}.
\end{aligned}$$

The following system of $3N$ equations and $3N+1$ unknowns should be solved,

$$\begin{cases}
\mathsf{E}Y^{(\varepsilon)}_n & = u^{(\varepsilon)}(p^{(\varepsilon)}_{n,+} - p^{(\varepsilon)}_{n,-}) & = 0, \\
\mathsf{Var}Y^{(\varepsilon)}_n & = (u^{(\varepsilon)})^2(p^{(\varepsilon)}_{n,+} + p^{(\varepsilon)}_{n,-}) & = \sigma^2_{n,\varepsilon}, \\
 & \quad p^{(\varepsilon)}_{n,+} + p^{(\varepsilon)}_{n,-} + p^{(\varepsilon)}_{n,\cdot} & = 1, \\
n = 1, \dots, N. & &
\end{cases} \tag{8.10}$$

The system above has the solution of the following form:

$$\begin{cases}
u^{(\varepsilon)} & = \kappa\sqrt{\varepsilon}, \\
p^{(\varepsilon)}_{n,+} = p^{(\varepsilon)}_{n,-} & = \frac{\sigma^2_{n,\varepsilon}}{2\kappa^2\varepsilon}, \\
p^{(\varepsilon)}_{n,\cdot} & = 1 - \frac{\sigma^2_{n,\varepsilon}}{\kappa^2\varepsilon}, \\
n = 1, \dots, N. &
\end{cases} \tag{8.11}$$

Let us consider the probability $p^{(\varepsilon)}_{n,+} = \frac{\sigma^2_{n,\varepsilon}}{2\kappa^2\varepsilon}$. It is easy to show that $\nu^2 e^{-2\alpha T} \leq \frac{\sigma^2_{n,\varepsilon}}{\varepsilon} \leq \nu^2 e^{-2\alpha T} e^{2\alpha T} = \nu^2$ for $n = 1, \dots N$. Thus, it is possible to choose κ large enough such that the values of probabilities $0 \leq \frac{\nu^2 e^{-2\alpha T}}{2u^2} \leq p^{(\varepsilon)}_{n,+} \leq \frac{\nu^2}{2u^2} \leq \frac{1}{2}$ for $n = 1, \dots, N$. In fact one can take any $\kappa \geq \nu$.

The defining relation (7.1) implies that for any $\delta > 0$ if ε is small enough, namely, if $\kappa\sqrt{\varepsilon} \leq \delta$, then

$$\sum_{n \leq [T/\varepsilon]} \mathsf{P}\{|Y^{(\varepsilon)}_n| > \delta\} = 0. \tag{8.12}$$

Also, by the definition of processes $Y^{(\varepsilon)}(t)$, for $\varepsilon \geq 0$ and $0 \leq t \leq T$,

$$\mathsf{E}Y^{(\varepsilon)}(t) = 0, \tag{8.13}$$

and, for every $0 \le t \le T$,

$$
\begin{aligned}
\mathsf{Var} Y^{(\varepsilon)}(t) &= \nu^2 e^{-2\alpha T} \int_0^{[t/\varepsilon]\varepsilon} e^{2\alpha s} ds = \nu^2 e^{-2\alpha T} \frac{e^{2\alpha[t/\varepsilon]\varepsilon} - 1}{2\alpha} \\
&\to \mathsf{Var} Y^{(0)}(t) = \nu^2 e^{-2\alpha T} \frac{e^{2\alpha t} - 1}{2\alpha} \text{ as } 0 < \varepsilon \to 0,
\end{aligned}
\tag{8.14}
$$

Since the functions in (8.14) are monotone, and the corresponding limit function is continuous, this convergence is also uniform in interval $[0, T]$ and, therefore, conditions of Ascoli-Arzelá theorem, in particular, condition of compactness in uniform topology holds as $\varepsilon \to 0$.

The remarks above imply that conditions of convergence theorems for vector sum-processes with independent increments, given, for example, in Skorokhod (1964), hold for processes $\vec{Y}^{(\varepsilon)}(t), t \in [0, T]$.

Thus, the processes $Y^{(\varepsilon)}(t), t \in [0, T]$ weakly and, moreover, J-converge to the process $Y^{(0)}(t), t \in [0, T]$ as $\varepsilon \to 0$.

Therefore, conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ hold for step-sum processes with independent increments $Y^{(\varepsilon)}(t)$.

Also, the moment generation function $\mathsf{E} \exp\{\beta(Y^{(\varepsilon)}(t+s) - Y^{(\varepsilon)}(t))\}$ exists for any $\beta \in \mathbb{R}$ and has an explicit form, namely, for $0 \le t \le t + s \le T$,

$$
\begin{aligned}
&\mathsf{E} \exp\{\beta(Y^{(\varepsilon)}(t+s) - Y^{(\varepsilon)}(t))\} \\
&\quad = \begin{cases} \prod_{[t/\varepsilon]+1}^{[(t+s)/\varepsilon]} (e^{\beta\kappa\sqrt{\varepsilon}} p_{n,+}^{(\varepsilon)} + e^{-\beta\kappa\sqrt{\varepsilon}} p_{n,-}^{(\varepsilon)} + p_{n,\cdot}^{(\varepsilon)}) & \text{if} \quad \varepsilon > 0, \\ e^{\frac{1}{2}\beta^2\nu^2 e^{-2\alpha T} \int_t^{t+s} e^{2\alpha v} dv} & \text{if} \quad \varepsilon = 0. \end{cases}
\end{aligned}
\tag{8.15}
$$

Using formula (8.15) it is possible to check that condition $\mathbf{D_3}$ holds for any $\beta' > \beta$. Indeed, since $\nu \le \kappa$ and $\frac{\sigma_{n,\varepsilon}^2}{2\kappa^2\varepsilon} \le \frac{1}{2}$ for $n = 1, \ldots, N$, we get,

$$
\begin{aligned}
\Xi'_{\pm\beta'}(Y^{(\varepsilon)}(\cdot), T) &= \sup_{0 \le t \le t+u \le T} \mathsf{E} e^{\pm\beta'(Y^{(\varepsilon)}(t+u) - Y^{(\varepsilon)}(t))} \\
&\le \prod_{n=1}^{[T/\varepsilon]} (1 + \frac{\sigma_{n,\varepsilon}^2}{2\kappa^2\varepsilon} (e^{\beta'\kappa\sqrt{\varepsilon}} + e^{-\beta'\kappa\sqrt{\varepsilon}} - 2)) \\
&\le (1 + \frac{1}{2}(e^{\beta'\kappa\sqrt{\varepsilon}} + e^{-\beta'\kappa\sqrt{\varepsilon}} - 2))^{T/\varepsilon} \to e^{\frac{\beta'^2\kappa^2 T}{2}} < \infty \text{ as } \varepsilon \to 0.
\end{aligned}
\tag{8.16}
$$

Therefore, all conditions of Theorem 6.1 hold for log-price processes $Y^{(\varepsilon)}(t)$ and, therefore,

$$
\Phi(\mathcal{M}_{\max,T}^{(\varepsilon)}) \to \Phi(\mathcal{M}_{\max,T}^{(0)}) \text{ as } \varepsilon \to 0. \tag{8.17}
$$

Let us now consider the partition $\Pi_\varepsilon = \{t_0 = 0 < t_1 = \varepsilon < \cdots < t_{N-1} = (N-1)\varepsilon < t_N = T\}$ on the interval $[0, T]$.

In this case, the Markov chain $(n, Y^{(\varepsilon)}(n\varepsilon))$, $n = 0, 1, \dots$ is a trinomial tree model with the initial node $(0, 0)$ and nodes of the form $(n, y_{n,l})$, $l = 0, \pm 1, \dots, \pm n$, after $n \geq 1$ steps.

In the case of model with parameters of the approximating trinomial random variables $Y_n^{(\varepsilon)}$ defined by relations (8.8) and (8.11), the points $y_{n,l}, l = 0, \pm 1, \dots, \pm n,\ n = 0, 1, \dots, N$ should be defined by the formula

$$y_{n,l} = l\sqrt{\varepsilon}\kappa. \tag{8.18}$$

The corresponding tree has $2n + 1$ nodes after n steps. The number of nodes is a linear function of n.

The standard backward procedure can be applied in order to find the optimal expected reward at moment 0 for the discrete time exponential trinomial price process $S^{(\varepsilon)}(t_n) = e^{Y^{(\varepsilon)}(t_n)}, t_n = n\varepsilon, n = 0, 1, \dots, N$. This optimal expected reward coincides, in this case, with the reward functional $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T})$ for the continuous time exponential price processes $S^{(\varepsilon)}(t) = e^{Y^{(\varepsilon)}(t)}, t \in [0, T]$.

To estimate the difference $\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T})$ we can use Theorem 3.2. In this case, $d(\Pi_\varepsilon) = \varepsilon$ and and for $\beta \in \mathbb{R}$,

$$\Delta_\beta(Y^{(\varepsilon)}(\cdot), \varepsilon, T) = \max_{1 \leq n \leq N} (\mathsf{E} e^{\beta |Y_n^{(\varepsilon)}|} - 1) \leq e^{\beta\kappa\sqrt{\varepsilon}} - 1. \tag{8.19}$$

Theorem 3.2 yields in this case the following relation,

$$\begin{aligned}\Phi(\mathcal{M}^{(\varepsilon)}_{\max,T}) &- \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T}) \\ &\leq L_3\varepsilon + L_4(e^{\beta\kappa\sqrt{\varepsilon}} - 1)^{\frac{\beta-\gamma}{\beta}} \to 0 \text{ as } \varepsilon \to 0.\end{aligned} \tag{8.20}$$

As was pointed out in Section 3, the reward functional $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T})$ is the option optimal expected reward for American type options in discrete time that correspond to the partition to the discrete time Markov log-price process $Y^{(\varepsilon)}(t_n),\ t_n = n\varepsilon,\ n = 0, 1, \dots, N$ with parameter $\varepsilon = T/N$. Let introduce the corresponding reward functions,

$$w^{(\varepsilon)}(t_n, y_{n,l}) = \sup_{\tau \in \mathcal{M}_{\Pi_\varepsilon,T},\, t_n \leq \tau \leq t_N = T} \mathsf{E}_{t_n, y_{n,l}} e^{-r\tau} [s_0 e^{Y^{(\varepsilon)}(\tau) e^{\alpha(T-\tau)}} - K]^+,$$

where $y_l = l\sqrt{\varepsilon}u,\ l = 0, \pm 1, \dots, \pm n,\ n = 0, \dots, N$.

Then, (see for example relation (4.5)),

$$\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T}) = w^{(\varepsilon)}(0, 0). \tag{8.21}$$

The reward functions $w^{(\varepsilon)}(t_n, y_l)$ can be found using the following backward recurrence relations, for $n = N-1, \ldots, 0$,

$$
\begin{aligned}
w^{(\varepsilon)}(t_n, y_{n,l}) = e^{-rt_n}[s_0 e^{y_{n,l}e^{\alpha(T-t_n)}} - K]^+ \vee \big(& w^{(\varepsilon)}(t_{n+1}, y_{n+1,l+1}) p^{(\varepsilon)}_{n,+} \\
& + w^{(\varepsilon)}(t_{n+1}, y_{n+1,l}) p^{(\varepsilon)}_{n,\cdot} + w^{(\varepsilon)}(t_{n+1}, y_{n+1,l-1}) p^{(\varepsilon)}_{n,-}\big), \\
& l = 0, \pm 1, \ldots, \pm n,
\end{aligned}
\tag{8.22}
$$

with the boundary conditions

$$
w^{(\varepsilon)}(t_N, y_{N,l}) = e^{-rt_N}[s_0 e^{y_{N,l}} - K]^+, \; l = 0, \pm 1, \ldots, \pm N. \tag{8.23}
$$

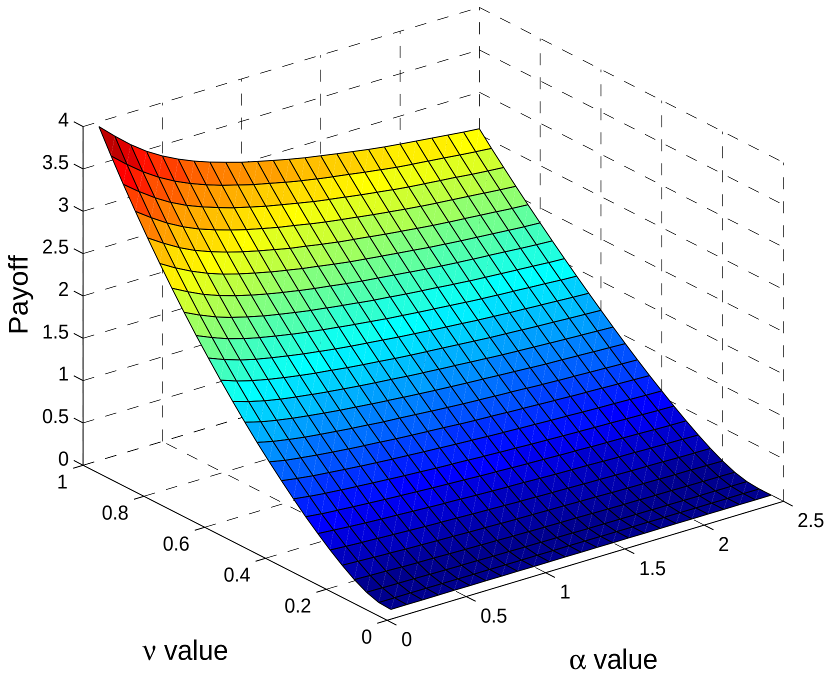

Figure 2: Reward of an American option a commodity assumed to follow the Schwartz model, having parameters $0.1 \leq \alpha \leq 2.5$ and $0.04 \leq \nu \leq 1$.

The corresponding convergence result of the Schwartz model can be summarized in the following theorem.

Theorem 8.1. *Let the price process corresponds to Schwartz model and consider an American option that have the standard payoff function. Then*

$$
w^{(\varepsilon)}(0,0) = \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon, T}) \to \Phi(\mathcal{M}^{(0)}_{\max,T}) \text{ as } \varepsilon \to 0. \tag{8.24}
$$

It is not out of picture to note that the standard payoff function $g(t,s) = e^{-rt}[s-K]^+$ can be replaced by any payoff function depending on parameter ε and satisfying conditions $\mathbf{A_1}$ and $\mathbf{A_4}$ in the described above approximation tree algorithm as well as in Theorem 8.1.

Let us now consider a numerical example when a standard American call option written on a commodity that are assumed to follow the Schwartz model. The commodity are currently traded at 10 and has an estimated volatility of 0.25 and mean reverting coefficient of $\alpha = 1$. The option has a strike price $K = 11$ and maturity $T = 0.5$ years. Finally, the risk free interest rate are $r = 0.04$.

Study show that tree size $N = 50$ is enough, the expected reward for a tree with size $N = 50$ is 0.2802, this should be compared with the expected reward for a tree with size $N = 100$ which is 0.2863. The calculation time for this tree size is 0.963 seconds on an 1.73 GHz Intel® Pentium-M processor, 1GB internal memory using Matlab®.

For low values of the ν parameter the option reward reaches its minimum, and when the ν value are high and the α value are low the option reward reaches its maximum.

9 Reselling of European options

We consider the geometric Brownian motion as a price process given by the stochastic differential equation

$$d \ln S(t) = \mu dt + \sigma dW_1(t),\ t \geq 0, \tag{9.1}$$

where $\mu \in \mathbb{R}, \sigma > 0$; $W_1(t)$ is a standard Brownian motion, and the initial state $S(0) = s_0 > 0$ is a constant.

It is also assumed that the continuously compounded interest model with a riskless interest rate $r > 0$ is used.

In this case, the price (at moment t and under condition that $S(t) = S$) for a European option, with the strike price $K > 0$ and maturity $T > 0$, is given by the Black–Scholes formula,

$$C(t, S, \sigma) = SN(d_t) - Ke^{-r(T-t)} N(d_t - \sigma\sqrt{T-t}), \tag{9.2}$$

where

$$d_t = \frac{\ln(S/K) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2},\ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy.$$

It is well known that the market price of European option deviates from the theoretical price. One of the explanations is that an implied volatility $\sigma(t)$ is used in formula (9.2) instead of σ.

We use a model given by the mean reverting Ornstein-Uhlenbeck process for implied volatility,

$$d(\ln \sigma(t) - \ln \sigma) = -\alpha(\ln \sigma(t) - \ln \sigma)dt + \nu dW_2(t),\ t \geq 0, \tag{9.3}$$

where $\alpha, \nu > 0$, $W_2(t)$ is also a standard Brownian motion, and the boundary condition is $\sigma(0) = \sigma$.

Finally, we assume that the process $\vec{W}(t) = (W_1(t), W_2(t))$ is the bivariate Brownian motion with correlated components, i.e.,

$$\mathsf{E}W_1(t)W_2(t) = \rho t,\ t \geq 0, \tag{9.4}$$

where $\rho \in [-1, 1]$. Note that the process $(S(t), \sigma(t))$ is a diffusion process.

The use of the market price $C(t, S(t), \sigma(t))$ actualises the problem for reselling of European option. In this case it is assumed that an owner of the option can resell the option at some stopping time from the class $\mathcal{M}_T$ which includes all stopping times $0 \leq \tau \leq T$ that are Markov moments with respect to the filtration $\mathcal{F}_t = \sigma((S(s), \sigma(s)), s \leq t), t \geq 0$ generated by the vector process $(S(t), \sigma(t))$. It is worth to note that the process $\sigma(t)$ is indirectly observable as an implied volatility corresponding to the observable market price of an option.

It should also be noted that the problem are considered under the assumption that the option are already bought, thus we are only interested into finding optimal expected reward for reselling the option.

The object of our studies is the reward functional

$$\Phi(\mathcal{M}_T) = \sup_{\tau \in \mathcal{M}_T} \mathsf{E}e^{-r\tau}C(\tau, S(\tau), \sigma(\tau)). \tag{9.5}$$

Thus, the problem of reselling the European option is imbedded in the problem of optimal execution of American type option with the payoff function $e^{-rt}C(t, S, \sigma)$ for the two-dimensional process $(S(t), \sigma(t))$.

In this model, there exists the unique solution to the system of stochastic differential equations (9.1) and (9.3) supplemented by the correlation relation (9.4). It is a diffusion process given by the following explicit formulas,

$$\begin{cases} S(t) & = S(0)e^{\mu t + \sigma W_1(t)},\ t \geq 0 \\ \sigma(t) & = \sigma e^{\nu e^{-\alpha t} \int_0^t e^{\alpha s} dW_2(s)},\ t \geq 0, \end{cases} \tag{9.6}$$

where $\vec{W}(t) = (W_1(t), W_2(t)), t \geq 0$ is the bivariate Brownian motion defined in (9.1), (9.3), and (9.4).

Therefore, our object is the reward functional $\Phi(\mathcal{M}_T)$ for American type option with the payoff function $e^{-rt}C(t, S, \sigma)$ for this bivariate diffusion process $(S(t), \sigma(t))$.

The problem can be however reduced to the simpler case of a bivariate process with independent increments.

Let us consider processes

$$S_1^{(0)}(t) = e^{\sigma W_1(t)}, \ t \geq 0, \ S_2^{(0)}(t) = e^{\nu e^{-\alpha T} \int_0^t e^{\alpha s} dW_2(s)}, \ t \geq 0. \tag{9.7}$$

By the definition, $S(t) = s_0 e^{\mu t} S_1^{(0)}(t), t \geq 0$ and $\sigma(t) = \sigma(S_2^{(0)}(t))^{e^{\alpha(T-t)}}, \ t \geq 0$, i.e., the process $(S(t), \sigma(t))$ is a non-random transformation of the process $(S_1^{(0)}(t), S_2^{(0)}(t))$ given by the above formulas.

The vector process $\vec{S}^{(0)}(t) = (S_1^{(0)}(t), S_2^{(0)}(t)), t \geq 0$ is a bivariate continuous non-homogeneous exponential Gaussian process with independent increments.

In some sense, this process is simpler than the process $(S(t), \sigma(t))$. It is more suitable for construction of the corresponding tree approximations.

The filtration $\mathcal{F}_t = \sigma((S(s), \sigma(s)), s \leq t), \ t \geq 0$ generated by the vector process $(S(t), \sigma(t))$ coincides with the filtration $\mathcal{F}_t = \sigma((S_1^{(0)}(s), S_2^{(0)}(s)), \ s \leq t), t \geq 0$ generated by the bivariate process $\vec{S}^{(0)}(t)$. Thus, the class $\mathcal{M}_T$, which includes all stopping times $0 \leq \tau \leq T$ that are Markov moments with respect to the filtration $\mathcal{F}_t, t \geq 0$, does not depend on which bivariate process is taken as a generator of this filtration, i.e., $\mathcal{M}_T = \mathcal{M}_{max,T}^{(0)}$.

Let us now define a payoff function,

$$g(t, \vec{s}) = e^{-rt} C(t, s_0 e^{\mu t} s_1, \sigma s_2^{e^{\alpha(T-t)}}). \tag{9.8}$$

Note that its derivatives have not more than polynomial rates of growth. More precisely, condition $\mathbf{A_1}$ holds for this function with some constants $K_i, i = 1, \ldots, 5$ and the parameters $\gamma_0 = 2 + e^{2\alpha T}, \gamma_1 = 0$, and $\gamma_2 = e^{2\alpha T}$, and, therefore, $\gamma = 2 + e^{2\alpha T}$.

It follows from the remarks above that the reward functional,

$$\begin{aligned} \Phi(\mathcal{M}_T) &= \sup_{\tau \in \mathcal{M}_T} \mathsf{E} e^{-r\tau} C(\tau, S(\tau), \sigma(\tau)) \\ &= \sup_{\tau \in \mathcal{M}_{max,T}^{(0)}} \mathsf{E} g(\tau, \vec{S}^{(0)}(\tau)). \end{aligned} \tag{9.9}$$

Therefore, the reward functional $\Phi(\mathcal{M}_T) = \Phi(\mathcal{M}_{max,T}^{(0)})$ is the optimal expected reward for American type option with the payoff function $g(t, \vec{s})$ for this bivariate exponential process with independent increments $\vec{S}^{(0)}(t)$.

Consider the corresponding bivariate log-price process $\vec{Y}^{(0)}(t) = (Y_1^{(0)}(t), Y_2^{(0)}(t))$, $t \geq 0$ with the components

$$Y_1^{(0)}(t) = \sigma W_1(t), \ t \geq 0, \ Y_2^{(0)}(t) = \nu e^{-\alpha T} \int_0^t e^{\alpha s} dW_2(s), \ t \geq 0. \tag{9.10}$$

We approximate the process $\vec{Y}^{(0)}(t), t \geq 0$ by a bivariate binomial-trinomial sum-process $\vec{Y}^{(\varepsilon)}(t) = (Y_1^{(\varepsilon)}(t), Y_2^{(\varepsilon)}(t)), t \geq 0$ with components

$$Y_i^{(\varepsilon)}(t) = \sum_{1 \leq n \leq [t/\varepsilon]} Y_{n,i}^{(\varepsilon)}, \ t \geq 0, \ i = 1, 2. \tag{9.11}$$

Here, $\vec{Y}_n^{(\varepsilon)} = (Y_{n,1}^{(\varepsilon)}, Y_{n,2}^{(\varepsilon)})$, $n = 1, 2, \ldots$ are, for every $\varepsilon > 0$, independent random vectors which have the following structure,

$$(Y_{n,1}^{(\varepsilon)}, Y_{n,2}^{(\varepsilon)}) = \begin{cases} (+u_{n,1}^{(\varepsilon)}, +u_{n,2}^{(\varepsilon)}) & & p_{n,++}^{(\varepsilon)} \\ (+u_{n,1}^{(\varepsilon)}, 0) & & p_{n,+\cdot}^{(\varepsilon)} \\ (+u_{n,1}^{(\varepsilon)}, -u_{n,2}^{(\varepsilon)}) & & p_{n,+-}^{(\varepsilon)} \\ & \text{with prob.} & \\ (-u_{n,1}^{(\varepsilon)}, +u_{n,2}^{(\varepsilon)}) & & p_{n,-+}^{(\varepsilon)} \\ (-u_{n,1}^{(\varepsilon)}, 0) & & p_{n,-\cdot}^{(\varepsilon)} \\ (-u_{n,1}^{(\varepsilon)}, -u_{n,2}^{(\varepsilon)}) & & p_{n,--}^{(\varepsilon)} \end{cases} \tag{9.12}$$

Respectively, the process $\vec{S}^{(0)}(t), t \geq 0$ is approximated by a bivariate exponential binomial-trinomial process $\vec{S}^{(\varepsilon)}(t) = e^{\vec{Y}^{(\varepsilon)}(t)}, t \geq 0$.

Let assume for simplicity that $\varepsilon = T/N$.

We shall try to fit the bivariate binomial-trinomial sum-process $\vec{Y}^{(\varepsilon)}(t)$ to the bivariate process $\vec{Y}^{(0)}(t)$ by fitting expectations, variances, and covariance between components for random vectors $\vec{Y}_n^{(\varepsilon)}$ to the corresponding quantities for the increments $\vec{Y}^{(0)}(n\varepsilon) - \vec{Y}^{(0)}((n-1)\varepsilon)$, for every $n = 1, \ldots, N$.

The corresponding quantities are given by the following formulas

$$\mathsf{E}\,\sigma(W_1(n\varepsilon) - W_1((n-1)\varepsilon)) = 0, \ \mathsf{E}\,\nu e^{-\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} e^{\alpha s} dW_2(s) = 0. \tag{9.13}$$

$$\begin{aligned} \sigma^2\varepsilon &= \mathsf{Var}\,(\sigma(W_1(n\varepsilon) - W_1((n-1)\varepsilon))), \\ \sigma_{n,\varepsilon}^2 &= \mathsf{Var}\,\nu e^{-\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} e^{\alpha s} dW_2(s) \\ &= \nu^2 e^{-2\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} e^{2\alpha s} ds = \nu^2 e^{-2\alpha T} e^{2\alpha n\varepsilon} \frac{1 - e^{-2\alpha\varepsilon}}{2\alpha}, \end{aligned} \tag{9.14}$$

and

$$\begin{aligned} \varrho_{n,\varepsilon} &= \mathsf{E}\,\sigma(W_1(n\varepsilon) - W_1((n-1)\varepsilon)) \cdot \nu e^{-\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} e^{\alpha s} dW_2(s) \\ &= \rho\sigma\nu e^{-\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} e^{\alpha s} ds = \sigma\rho\nu e^{-\alpha T} e^{\alpha n\varepsilon} \frac{1 - e^{-\alpha\varepsilon}}{\alpha}. \end{aligned} \tag{9.15}$$

The following system of $6N$ equations with $8N$ unknowns should be solved,

$$\begin{cases} \mathsf{E}[Y^{(\varepsilon)}_{n,1}] & = u^{(\varepsilon)}_{n,1}(2(p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,+-} + p^{(\varepsilon)}_{n,+\cdot}) - 1) & = 0, \\ \mathsf{Var}[Y^{(\varepsilon)}_{n,1}] & = (u^{(\varepsilon)}_{n,1})^2 & = \sigma^2\varepsilon, \\ \mathsf{E}[Y^{(\varepsilon)}_{n,2}] & = u^{(\varepsilon)}_{n,2}(p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,-+} - p^{(\varepsilon)}_{n,+-} - p^{(\varepsilon)}_{n,--}) & = 0, \\ \mathsf{Var}[Y^{(\varepsilon)}_{n,2}] & = (u^{(\varepsilon)}_{n,2})^2(p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,-+} + p^{(\varepsilon)}_{n,+-} + p^{(\varepsilon)}_{n,--}) & = \sigma^2_{n,\varepsilon}, \\ \mathsf{E}Y^{(\varepsilon)}_{n,1}Y^{(\varepsilon)}_{n,2} & = u^{(\varepsilon)}_{n,1}u^{(\varepsilon)}_{n,2}(p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,--} - p^{(\varepsilon)}_{n,-+} - p^{(\varepsilon)}_{+-}) & = \varrho_{n,\varepsilon}, \\ & \quad p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,-+} + p^{(\varepsilon)}_{n,+-} + p^{(\varepsilon)}_{n,--} + p^{(\varepsilon)}_{n,+\cdot} + p^{(\varepsilon)}_{n,-\cdot} & = 1, \\ n = 1, \ldots, N. \end{cases} \tag{9.16}$$

This system have the solution of the following form:

$$\begin{cases} u^{(\varepsilon)}_{n,1} & = \sigma\sqrt{\varepsilon}, \\ u^{(\varepsilon)}_{n,2} & = \kappa\sqrt{\varepsilon}, \\ p^{(\varepsilon)}_{n,++} = p^{(\varepsilon)}_{n,--} & = \frac{\sigma^2_{n,\varepsilon}}{4\kappa^2\varepsilon} + \frac{\varrho_{n,\varepsilon}}{4\sigma\kappa\varepsilon} \\ & = \frac{\nu^2 e^{-2\alpha T}}{4\kappa^2} e^{2\alpha n\varepsilon} \frac{1-e^{-2\alpha\varepsilon}}{2\alpha\varepsilon} + \frac{\rho\nu e^{-\alpha T}}{4\kappa} e^{\alpha n\varepsilon} \frac{1-e^{-\alpha\varepsilon}}{\alpha\varepsilon}, \\ p^{(\varepsilon)}_{n,+-} = p^{(\varepsilon)}_{n,-+} & = \frac{\sigma^2_{n,\varepsilon}}{4\kappa^2\varepsilon} - \frac{\varrho_{n,\varepsilon}}{4\sigma\kappa\varepsilon} \\ & = \frac{\nu^2 e^{-2\alpha T}}{4\kappa^2} e^{2\alpha n\varepsilon} \frac{1-e^{-2\alpha\varepsilon}}{2\alpha\varepsilon} - \frac{\rho\nu e^{-\alpha T}}{4\kappa} e^{\alpha n\varepsilon} \frac{1-e^{-\alpha\varepsilon}}{\alpha\varepsilon}, \\ p^{(\varepsilon)}_{n,+\cdot} = p^{(\varepsilon)}_{n,-\cdot} & = \frac{1}{2} - \frac{\sigma^2_{n,\varepsilon}}{2\kappa^2\varepsilon} \\ & = \frac{1}{2} - \frac{\nu^2 e^{-2\alpha T}}{2\kappa^2} e^{2\alpha n\varepsilon} \frac{1-e^{-2\alpha\varepsilon}}{2\alpha\varepsilon}, \\ n = 1, \ldots, N. \end{cases} \tag{9.17}$$

The probability components of this solution should also satisfy the obvious inequalities

$$0 \le p^{(\varepsilon)}_{n,\pm+},\ p^{(\varepsilon)}_{n,\pm-},\ p^{(\varepsilon)}_{n,\pm\cdot} \le 1,\ n = 1, \ldots, N. \tag{9.18}$$

Let us restrict consideration by the case, where the following condition of weak correlation for noise terms in the reselling models holds:

$\mathbf{E_1}$: $|\rho| < e^{-\alpha T}$.

This condition implies inequalities (9.18) to hold if to choose

$$\nu \le \kappa \le \nu|\rho|^{-1}e^{-\alpha T}. \tag{9.19}$$

As above, let assume that $\varepsilon = T/N$ and consider the partition $\Pi_\varepsilon = \{t_0 = 0 < t_1 = \varepsilon < \cdots < t_{N-1} = (N-1)\varepsilon < t_N = T\}$ on the interval $[0, T]$.

In this case the Markov chain $(n, \vec{Y}^{(\varepsilon)}(n\varepsilon)), n = 0, 1, \ldots$ is a bivariate binomial-trinomial tree model with the initial node $(0, (0, 0))$ and nodes of the form $(n, \vec{y}_{n,l_1,l_2})$, $l_1 = 0, 1, \ldots, n$, $l_2 = 0, \pm 1, \ldots, \pm n$, after n steps.

In the case of model with parameters of the approximating trinomial random vectors $\vec{Y}_n^{(\varepsilon)}$ defined by relations (9.12) and (9.17), the vector points $\vec{y}_{n,l_1,l_2}, l_1 = 0, 1, \ldots, n,\ l_2 = 0, \pm 1, \ldots, \pm n,\ n = 0, 1, \ldots, N$ should be defined by the formula

$$\vec{y}_{n,l_1,l_2} = (2l_1 - n)\sigma\sqrt{\varepsilon},\ l_2\kappa\sqrt{\varepsilon}) \tag{9.20}$$

The corresponding tree has $(n+1)(2n+1)$ nodes after n steps. The number of nodes is a quadratic function of n.

The standard backward procedure can be applied in order to find the optimal expected reward at moment 0 for the discrete time exponential trinomial price process $\vec{S}^{(\varepsilon)}(t_n) = e^{\vec{Y}^{(\varepsilon)}(t_n)},\ t_n = n\varepsilon,\ n = 0, 1, \ldots, N$. This optimal expected reward coincides, in this case, with the reward functional $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon, T})$ for the continuous time exponential price processes $\vec{S}^{(\varepsilon)}(t) = e^{\vec{Y}^{(\varepsilon)}(t)}, t \in [0, T]$.

The reward functional $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon, T})$ is the optimal expected reward for American type option in discrete time that corresponds to the discrete time Markov log-price process $\vec{Y}^{(\varepsilon)}(t_n), t_n = n\varepsilon, n = 0, 1, \ldots, N$ with parameter $\varepsilon = T/N$ and the payoff function $g(t, e^{\vec{y}}) = e^{-rt}C(t, s_0 e^{\mu t} e^{y_1}, \sigma e^{y_2 e^{\alpha(T-t)}}), t \in [0, t],\ \vec{y} = (y_1, y_2) \in \mathbb{R}^2$ defined according to relation (9.8).

Let us introduce the corresponding reward functions,

$$w^{(\varepsilon)}(t_n, \vec{y}_{n,l_1,l_2}) = \sup_{\tau \in \mathcal{M}_{\Pi_\varepsilon, T},\, t_n \le \tau \le t_N = T} \mathsf{E}_{t_n, \vec{y}_{n,l_1,l_2}} g(\tau, e^{\vec{Y}^{(\varepsilon)}(\tau)}),$$

where the vector points $\vec{y}_{n,l_1,l_2},\ l_1 = 0, 1, \ldots, n, l_2 = 0, \pm 1, \ldots, \pm n, n = 0, \ldots, N$ are defined as above, and, in particular, $\vec{y}_{0,0,0} = (0, (0, 0))$.

Then, by the definition,

$$\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon, T}) = w^{(\varepsilon)}(0, (0, 0)). \tag{9.21}$$

The reward functions $w^{(\varepsilon)}(t_n, \vec{y}_{n,l_1,l_2})$ can be found using the following recurrence relations, for $n = 0, 1, \ldots, N-1$,

$$\begin{aligned} w^{(\varepsilon)}(t_n, \vec{y}_{n,l_1,l_2}) &= g(t_n, e^{\vec{y}_{n,l_1,l_2}}) \vee \big(w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1+1,l_2+1})p^{(\varepsilon)}_{n,++} \\ &\quad + w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1+1,l_2})p^{(\varepsilon)}_{n,+\cdot} + w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1+1,l_2-1})p^{(\varepsilon)}_{n,+-} \\ &\quad + w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1,l_2+1})p^{(\varepsilon)}_{n,-+} + w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1,l_2})p^{(\varepsilon)}_{n,-\cdot} \\ &\quad + w^{(\varepsilon)}(t_{n+1}, \vec{y}_{n+1,l_1,l_2-1})p^{(\varepsilon)}_{n,--}\big), \\ &\quad l_1 = 0, 1, \ldots, n,\ l_2 = 0, \pm 1, \ldots, \pm n, \end{aligned} \tag{9.22}$$

with the boundary conditions,

$$w^{(\varepsilon)}(t_N, \vec{y}_{N,l_1,l_2}) = g(t_N, e^{\vec{y}_{N,l_1,l_2}}),\ l_1 = 0, 1, \ldots, N,\ l_2 = 0, \pm 1, \ldots, \pm N. \tag{9.23}$$

The following theorem, which proof is analogous to the proof of Theorem 8.1 given in Section 8, presents the corresponding approximation result for the bivariate binomial-trinomial tree algorithm describe above.

Theorem 9.1. *Let condition* $\mathbf{E_1}$ *holds. Then the optimal reselling rewards,*

$$w^{(\varepsilon)}(0,(0,0)) = \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T}) \to \Phi(\mathcal{M}^{(0)}_{max,T}) = \Phi(\mathcal{M}_T) \text{ as } \varepsilon \to 0. \tag{9.24}$$

We refer to the recent paper Lundgren and Silvestrov (2010), where one can find the detailed presentation of the proof of Theorem 9.1, comments concerned modifications of the algorithm described above for the models, where condition $\mathbf{E_1}$ does not holds, as well as the corresponding numerical examples.

References

Amin, K., Khanna, A. (1994). Convergence of American option values from discrete- to continuous- time financial models, *Math. Finance*, **4**, no. 4, 289–304.

Chow, Y.S., Robbins, H. Siegmund, D. (1971). *The theory of optimal stopping,* Houghton Mifflin Comp.

Coquet, F., Toldo, S. (2007). Convergence of values in optimal stopping and convergence in optimal stopping times, *Electr. J. Probab.*, **12**, no. 8, 207–228.

Cortazar, G., Gravet, M., Urzua, J. (2008). The valuation of multidimensional American real options using the LSM simulation method, *Comp. Oper. Res.*, **35**, 113–129.

Dayanik, S., Karatzas, I. (2003). On the optimal stopping problem for one-dimensional diffusions, *Stoch. Proc. Appl.*, **107**, 173–212.

Dupuis, P., Wang, H. (2005). On the convergence from discrete to continuous time in an optimal stopping problem, *Ann. Appl. Probab.*, **15**, 1339–1366.

Ekström, E., Lindberg, C., Tysk, J., Wanntorp, H. (2009). Optimal liquidation of a call spread, Preprint Uppsala University, Sweden.

Gau, B., J.Z. Huang, Subrahmanyam, M. (2000). The valuation of American barrier options using the decomposition technique, *J. Econom. Dynam. Control,* **24**, 1783–1827.

Gikhman, I.I., Skorokhod, A. V. (1971). *The Theory of Stochastic Processes, vol I*, Nauka, Moscow, (English translation: Springer, New York, (1974)).

Henderson, V., Hobson, D. (2008). An explicit solution for an optimal stopping/optimal control problem which models an asset sale *Ann. Appl. Probab.*, **18**, 1681–1705.

Jacka, S.D. (1991). Optimal stopping and the American put, *Math. Finance,* **1**, 1–14.

Jönsson, H. (2001). Monte Carlo studies of American type options with discrete time, *Theory Stoch. Process.*, **7(23)**, no. 1-2, 163–188.

Jönsson, H. (2005). Optimal Stopping Domains and Reward Functions for Discrete Time American Type Options, PhD Thesis No. 22, Mälardalen University.

Jönsson, H., Kukush, A.G., Silvestrov, D.S. (2004). Threshold structure of optimal stopping strategies for American type options. I, *Theor. Ĭmovirn. Mat. Stat.,* **71**, 113–123 (English translation in *Theory Probab. Math. Statist.*, **71**, 93–103).

Jönsson, H., Kukush, A.G., Silvestrov, D.S. (2005). Threshold structure of optimal stopping strategies for American type options. II, *Theor. Ĭmovirn. Mat. Stat.,* **72**, 42–53 (English translation in *Theory Probab. Math. Statist.,* **72**, 47–58).

Kim, I. (1990). The analytic valuation of American options, *Rev. Financ. Studies,* **3**, 547-572.

Kukush, A.G., Silvestrov, D.S. (2000). Structure of optimal stopping strategies for American type options, In: Uryasev, S. (ed.) Probabilistic Constrained Optimization: Methodology and Applications, Kluwer, 173–185.

Kukush, A.G., Silvestrov, D.S. (2001). Skeleton approximations of optimal stopping strategies for American type options with continuous time, *Theory Stoch. Process.,* **7(23)**, no. 1-2 215–230.

Kukush, A.G., Silvestrov, D.S. (2004). Optimal pricing for American type options with discrete time, *Theory Stoch. Process.*, **10(26)**, no. 1-2, 72–96.

Longstaff, F.A., Schwartz, E.S. (2001). Valuing American options by simulation: A simple least-squares approach, *Rev. Financ. Studies*, **14**, no. 1, 113–147.

Lundgren R. (2007). Structure of optimal stopping domains for American options with knock out domains, *Theory Stoch. Process.,* **13(29)**, no. 4, 98–129.

Lundgren R. (2010). Simulation studies of stopping domains for American knockout options, *J. Stat. Plan. Infer.*, (*to appear*).

Lundgren, R., Silvestrov, D., Kukush, A. (2008). Reselling of options and convergence of option rewards *J. Numer. Appl. Math.* **1(96)**, 149–172.

Lundgren, R., Silvestrov, D. (2010). Optimal stopping and reselling of European options, *Stochastics*, (*to appear*).

Mishura, Yu., Shevchenko, G. (2009). The optimal time to exchange one asset for another on finite interval, In: Delbaen, F., Rasonyi, M., Stricker, C. *Optimality and Risk - Modern Trends in Mathematical Finance.* Springer, 197–210.

Peskir, G, Shiryaev, A. (2006). *Optimal Stopping and Free-Boundary Problems*, Birkhäuser, Basel.

Prigent, J.L. (2003). *Weak Convergence of Financial Markets*, Springer, New York.

Schwartz, E.S. (1997). The Stochastic Behaviour of Commodity Prices: Implications for Valuation and Hedgeing, *J. Finance.,* **52** No 3, 923–973.

Shiryaev, A.N. (1976). *Sequential Analysis. Optimal Stopping Rules*, Nauka, Moscow, (English edition: *Optimal Stopping Rules*, Springer, New York, 1978).

Silvestrov, D.S., (1974). *Limit Theorems for Composite Random Functions.* Vysshaya Shkola and Izdatel'stvo Kievskogo Iniversiteta, Kiev.

Silvestrov, D.S., Galochkin, V.G., Sibirtsev, V.G. (1999). Algorithms and Programs for optimal Monte Carlo pricing of American type options, *Theory Stoch. Process.,* **5(21)**, no. 1-2, 175–187.

Silvestrov, D.S., Jönsson, H., Stenberg F., (2006). Convergence of option rewards for Markov type price processes controlled by stochastic indices. 1, Research Report 2006-1, Department of Mathematics and Physics, Mälardalen University.

Silvestrov, D., Jönsson, H., Stenberg, F. (2007). Convergence of option rewards for Markov type price processes, *Theory Stoch. Process.*, **13(29)**, no. 4, 174–185.

Silvestrov, D., Jönsson, H., Stenberg, F. (2009). Convergence of option rewards for Markov type price processes modulated by stochastic indices I, II, *Theor. Ĭmovirn. Mat. Stat.,* **79** (2008), 149–165, **80** (2009) (English translation in *Theory Probab. Math. Statist.*, **79**, **80**).

Skorokhod, A.V. (1964). *Random Processes with Independent Increments*, Probability Theory and Mathematical Statistics, Nauka, Moscow, (English edition: Nat. Lending Library for Sci. and Tech., Boston Spa, 1971).

Zhang, Z., Lim, K-G., (2006). A non-lattice pricing model of American options under stochastic volatility, *J. Futures Markets*, **26**, no. 5, 417–448.