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Risk capital stress testing framework and the new capital adequacy rules

Håkan Andersson* and Andreas Lindell†

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Abstract

We propose a general framework for simulating the value of a financial portfolio over time. The main idea is to let experts specify the *long-term* behaviour of the economy, and use statistical models to generate the behaviour at intermediate time points. First, a set of expert scenarios specifying the values of the dominant risk factors at a few discrete time points is determined. Then, for each of these scenarios, a continuous-time model is specified that is consistent with the expert scenario.

As a working example we simulate the equity of a model bank mainly involved in retail business activities such as lending to households and small corporates, deposit services, brokerage, and different types of payment services. We specify a continuous-time model for the bank's daily revenues and costs given expert scenarios specifying the long-term development of the macroeconomy, client volumes, margins and fees. The ruin probability, i.e. the probability that the equity ever becomes negative, is investigated in some depth. Inspired by the work of Jokivuolle and Peura [7], we also show how the new capital adequacy rules (Basel II) can be included in our framework. The Basel accord specifies calculation rules for the eligible capital as well as for the amount of risk (referred to as 'Risk Weighted Assets') that the bank is exposed to. We perform a joint simulation of the capital and the risk weighted assets in order to investigate if the regulatory requirements are fulfilled in all the scenarios.

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1 Introduction

To ensure a stable global financial system, it is necessary that banks and other financial institutions hold substantial capital buffers to protect against large unexpected losses. Shareholders, investors and depositors need to be confident that the bank will not fall in distress whatever the future state of the economy. Stress testing of the bank capital is thus an important input to the capital adequacy process. In this work we present a framework for investigating whether the bank is solid enough to survive also under stressed economic conditions. The bank's result is affected by a large number of external factors, such as market variables, macroeconomic factors, variables governing the behaviour of the clients etc. A model of the joint time dynamics of all relevant variables would require the estimation of a huge number of parameters. Here we take the important standpoint that it is simply not feasible to perform this task. Instead, we assume that a number of predefined *master scenarios* form the basis of the model. Each master scenario specifies the long-term behaviour of variables that have a great impact on the bank's result—external variables as well as bank-specific quantities like client volumes and margins. Then, given a master scenario we run a continuous-time statistical model of the bank's daily revenues and costs. The set of master scenarios could contain historical scenarios as well as expert scenarios. Besides reducing the estimation problem, this approach has the advantage of being easily communicable to the management. Indeed, the model is intended to be an important tool for the management in the process of determining the required capital level, and it could also be used in the communication with regulators, rating agencies and the public. The disadvantage with the approach is of course that no probability measure can be assigned to the scenario set, hence a classical Value-at-Risk framework is not meaningful in this context.

The following related questions are of fundamental importance:

(1) *Solvency.* Given today's capital and given a master scenario, what is the probability that the capital will stay above zero between today and the specified risk horizon?

(2) *Capital estimation.* How large capital is needed today to ensure that the bank survives with high probability, no matter what master scenario in the scenario set we consider?

The first part of the paper will provide a quantitative framework for answering these questions.

The modelling framework is quite general and may in principle be applied to any financial institution. However, since a financial institution may be involved in a large variety of complex business activities, it is impossible to provide an exhaustive set of relevant examples in this text. Instead, for illustrative purposes we describe a simplified model that could be applied to a medium-sized retail bank with exclusively domestic business activities. We assume that our model bank focuses on the following types of fundamental retail business:

- Lending to households and small corporates;
- Deposit services;
- Asset fund management (equity funds);
- Different types of payment services.

In addition, the residual market risk evolving from these business activities is managed efficiently by a Treasury function. A retail bank is typically not very much involved in proprietary trading, hence we assume that no positions are taken for speculative purposes only. Within this context we will discuss the questions (1) and (2) above.

There is a large body of literature covering the market risk and credit risk of financial portfolios. For instance, the Value-at-Risk framework for market risk is discussed e.g. in Jorion [8] and in Dowd [4], and credit risk models are introduced in Bluhm et al [3]. Stress testing in the context of the Basel II framework is discussed in several BIS working papers, see e.g. [2]. However, we are not aware of any previous work that aims at describing a consistent mathematical methodology, taking a unified view on a bank's risk and capital adequacy assessment.

We turn to discuss the new capital adequacy framework that will have a great impact on the future capital strategies of the banks. The main purpose of the first Basel accord of 1988 was to provide a framework for the calculation of a *minimum capital requirement* for the banks. Banks with large volumes of risky assets were assigned large capital requirements. National supervisors all over the world soon adopted the Basel proposal and created legislation to be followed by the financial institutions in the respective countries. The original Basel accord only covered credit risk, but was complemented in the late nineties with a market risk amendment.

Over the years it has become clear that the very crude calculation rules for the credit risk capital requirement need to be replaced by a more risk sensitive framework. For instance, *all* corporate loans of a given size are considered equally risky under the first Basel accord, hence from a regulatory perspective the bank feels no incentive to steer its lending activities towards more creditworthy counterparties of the corporate segment. With such considerations in mind, the Basel committee decided to develop a new set of capital adequacy rules, henceforth referred to as *Basel II*. In many countries of the world, these rules will become legally binding in 2007. The new Basel accord is built on three so-called *Pillars*:

- Pillar 1: Minimum capital requirements for credit risk, market risk and operational risk. For each of these risk types, the bank needs to choose one out of several levels of sophistication. Selecting a more advanced calculation model requires additional investments from the bank's side, but tends to lead to lower capital requirements.

- Pillar 2: Supervisory review process. The bank needs to demonstrate that a sound internal capital adequacy process is in place, and that risks not covered under Pillar 1 are appropriately monitored and managed.
- Pillar 3: Market discipline. The bank is required to provide detailed information to the public on its risk profile, risk management and capital strategies.

We have no ambition to cover the entire Basel II framework in this work. The interested reader should consult e.g. [1] to get the full picture.

The final part of this document outlines a framework for the *joint* simulation of the bank's capital and the so-called *Risk Weighted Assets*, which we now define. Under Pillar 1 of Basel II, risk weights are calculated for all assets of the bank using supervisory functions that are designed so that risky assets are assigned large risk weights, and vice versa. The Risk Weighted Assets (RWA) for credit risk is simply the sum of these risk weights. In addition, RWA for market risk and operational risk are calculated using supervisory formulas. The bank's total RWA is the sum of these three components (credit risk, market risk, and operational risk). Finally, the *capital ratio* is defined as the ratio of the bank capital and the total RWA. The regulation requires the capital ratio of the financial institution to stay above a certain minimum capital ratio.

We argued above that it is crucial that the probability of getting a negative capital is virtually zero. However, it is well known that rating agencies, investors and depositors react negatively already when the capital approaches the minimum capital requirement from above. Hence, the bank management typically announces a slightly higher target capital ratio than the required minimum capital ratio for the business. Here we wish to answer the following questions:

(1') *Capital adequacy.* Given today's capital and given a master scenario, what is the probability that the capital ratio will stay above a target ratio specified by the management between today and the specified risk horizon?

(2') *Capital estimation.* How large capital is needed today in order to ensure that the capital ratio stays above the target ratio with high probability, no matter what master scenario in the scenario set we consider?

The methodology is inspired by the work of Jokivuolle and Peura [7]. The model can be used as a tool in the internal capital adequacy process. Furthermore, we believe that the results obtained using the model could serve as an important input to the Pillar 2 process.

The document is organized as follows. A detailed description of the modelling framework is given in Section 2. In Section 3 we discuss how to simulate the dominant parts of the profit and loss account of an idealized bank. Furthermore, ruin probabilities are simulated.

Section 4 outlines the Basel II regulatory framework together with the simulation model of the capital ratio. Then the result of a very small case study is reported. In Section 5, possible extensions are discussed. Conclusions are given in Section 6. An Appendix concludes the document.

2 Modelling framework

We are given a portfolio of financial contracts, and we wish to estimate how much the value of the portfolio may decrease from today, $t = 0$, to some specified risk horizon, $t = T$. Within the concept of Value-at-Risk, a risk calculation is typically performed as follows. Movements of relevant risk drivers are simulated using some statistical model. In each simulated scenario, the change in portfolio value is calculated. Finally, the desired left quantile is extracted from the simulated distribution of changes in portfolio value.

It is certainly feasible to adopt such an approach if a limited number of risk drivers are associated with the portfolio and if we only consider a short risk horizon, i.e. a few days. A model of the time dynamics of the risk drivers may be readily defined, and its parameters will be reasonably stable over time. This is usually the situation when the portfolio contains products traded on a liquid market. However, imagine that we pose the same question when the portfolio contains all the transactions of the entire bank and the risk horizon is a number of *years*. Then the set of risk drivers includes interest rates, currency rates, stock quotes, GDP, inflation, unemployment figures, client volumes, margins and fees etc, and it is neither practical nor desirable to set up a complicated stochastic model with a huge number of parameters where all of these variables are simulated over a long time period. Such a model runs the risk of becoming a 'black box' whose output is impossible to interpret clearly. Instead, we believe that the bank management appreciates a transparent model where specific expert scenarios, henceforth referred to as *master scenarios*, may be defined and the impact of these scenarios on the business may be clearly monitored.

Here follows a high-level description of the proposed model. The quantity of interest is henceforth simply referred to as the *value process*, the value of the bank portfolio being a prominent example. Assume that a set of risk drivers, called *risk factors*, of the value process has been identified. The model is built by performing the following steps:

- Define a small number of master scenarios for the *long-term* movement of the dominant risk factors, i.e. given their values at time $t = 0$, specify their values at time $t = T$, and possibly at a number of intermediate time points.
- Given such a master scenario, specify the joint time dynamics *between* $t = 0$ and $t = T$ of all risk factors. Possible dependencies should be captured by the model. Then, for each trajectory, calculate the value process at each time point (given the value at time $t = 0$).

- For each scenario, calculate summary statistics such as the expectation and appropriate quantiles of the value process.

Among the advantages of the approach we find that the model is easily communicated to the management. The main drawback of the approach is that no attempts are made to assign a probability measure on the set of master scenarios. Hence classical Value-at-Risk cannot be calculated.

We turn to give a more detailed description of the modelling framework.

2.1 Set of risk factors

We work in continuous time, measured in years. The risk factors may be of two types:

- Environmental variables;
- Bank-specific variables.

Additional noise variables are sometimes required in order to obtain a fully functional stochastic model[1], but these variables will not be regarded as true risk factors. Interest rates, stock prices, FX rates and traditional macro variables are examples of environmental variables, while typical bank-specific variables may be client volumes, margins and fees. Environmental variables and bank-specific variables are assumed to exist as random processes for all t, $0 \leq t \leq T$. Together with the noise variables, these objects define the sample space Ω. Also, a σ-field of events on Ω needs to be defined along with a filtration describing the information available at different time points. However, in order to keep the presentation on an elementary level we have decided not to be very rigorous regarding these matters. Denote the multivariate process of environmental variables and bank-specific variables by $\mathbf{S} = \{\mathbf{S}(t), 0 \leq t \leq T\}$.

2.2 Set of master scenarios and time dynamics

Recall that T denotes the risk horizon. Define intermediate time points $\tau_1, \tau_2, \ldots, \tau_K$, where $0 < \tau_1 < \tau_2 < \cdots < \tau_K = T$. A master scenario s is defined by putting restrictions on some (or all) of the risk factors at these intermediate time points. Such a restriction on $\mathbf{S}(\tau_k)$ is represented as the event A_k^s; of course, $A_k^s = \Omega$ means no restriction. In this way quite general master scenarios may be created. Moreover, we will soon have additional use of the concept of intermediate time points.

[1]For example, the calculation of credit losses requires simulation of the number of defaults.

Example. The calculation date is 30/9/2006, which corresponds to $t = 0$. The risk horizon is 31/12/2007, which corresponds to $t = T = 1\frac{1}{4}$. Here we assume that all quarters are of equal length, namely exactly 1/4 of a year. There are five intermediate time points, namely the final day of each quarter of the time period under consideration. The only risk factor is the short-term risk-free interest rate $r = \{r(t), 0 \leq t \leq T\}$. Today the interest rate is 2.5%, and the master scenario s specifies that the interest rate should be 3% by the end of 2006 and 5% by the end of 2007. Table 1 depicts the specification of the master scenario.

Date	τ_k	A_k^s
31/12/2006	1/4	$\{r(\tau_1) = 0.03\}$
31/3/2007	2/4	Ω
30/6/2007	3/4	Ω
30/9/2007	1	Ω
31/12/2007	1 1/4	$\{r(\tau_5) = 0.05\}$

Table 1. Specification of the master scenario of the example

There remains to specify, for each master scenario s, a continuous-time stochastic model of $\mathbf{S} = \{\mathbf{S}(t), 0 \leq t \leq T\}$ in which all the events A_k^s, $k = 1, \ldots, K$, are given probability one. We stress that each given master scenario s induces a probability measure $\mathbf{P}^s$ on the events on Ω in this way, and we do not require $\mathbf{P}^s$ for different s to be consistent in any way. (On the other hand, using entirely different approaches for different s is not liable to lead to a very credible framework.) For notational convenience, whenever a single master scenario is analyzed the superscript s will be suppressed.

2.3 Value process

The quantity of interest is represented as an n-dimensional function $\mathbf{X}(t)$ of the risk factors and time. In the simplest case, $\mathbf{X}(t)$ is just the value of the portfolio under consideration and is hence scalar, but we will see, notably in Section 4 below on capital adequacy, examples where it is natural to allow for multidimensional functions. Given a master scenario s and a realization $\mathbf{S}$ of the risk factor process, we calculate a realization of the value process, $\mathbf{X} = \{\mathbf{X}(t), 0 \leq t \leq T\}$.

2.4 Adjustments

Sometimes the value process is subject to adjustments causing large discrete jumps at certain time points, a typical example being the accumulated result of a certain business from which tax and dividends are drawn at the end of the fiscal year. We assume that such adjustments are made at the intermediate time points τ_k. Zero adjustments are of course allowed.

2.5 Approved events

Again fix a master scenario s. The components of the value process may be subject to internal and/or external requirements, such as lower safety bounds on the capital buffer. Here we represent such requirements as "approved events" G_k^s at the intermediate time points τ_k, $k = 1, \ldots, K$. The event $G_k^s = \Omega$ means that there is no special requirement at time τ_k. In order to achieve a general framework, we allow the events G_k^s to be different for different scenarios s; in most cases, however, the requirements would be specified regardless of the scenarios.

Example, cont'd. Assume that the value process only consists of the capital, $X = \{C(t), 0 \leq t \leq T\}$. It is required that the capital exceed 100 million EUR at each end-of-quarter; moreover, the capital at the end of each year should exceed 300 million EUR. The list of approved events is shown in Table 2.

Date	τ_k	G_k^s
31/12/2006	1/4	$\{C(\tau_1) \geq 300\}$
31/3/2007	2/4	$\{C(\tau_2) \geq 100\}$
30/6/2007	3/4	$\{C(\tau_3) \geq 100\}$
30/9/2007	1	$\{C(\tau_4) \geq 100\}$
31/12/2007	1 1/4	$\{C(\tau_5) \geq 300\}$

Table 2. Specification of the approved events of the example

2.6 Questions to be answered

If the value process fails to satisfy at least one of the approved events G_k^s, $1 \leq k \leq K$, then we say that *ruin* has occurred. With this terminology in mind, we now give a technical formulation of the fundamental questions stated in the introduction of this paper.

(1) Given a master scenario s, what is the probability that ruin is avoided, adjustments taken into account? In other words, how large is $\mathbf{P}^s\left(\bigcap_{k=1}^K G_k^s\right)$?

(2) Given a small number $\varepsilon > 0$, find the "optimal" start values of the value process (optimal in a sense that needs to be specified by the modeller) given the constraint that ruin is avoided with probability at least $1 - \varepsilon$ for each scenario, i.e. given the constraint that $\mathbf{P}^s\left(\bigcap_{k=1}^K G_k^s\right) \geq 1 - \varepsilon$ for each scenario s.

3 Solvency

As a first application of the modelling framework outlined in the previous section, we show how to simulate the equity (i.e., the difference between assets and liabilities) of an idealized bank over time. The equity at time $t = 0$ is given, and the revenues and costs are simulated from $t = 0$ to $t = T = 1$ given one single master scenario involving a broad domestic equity index and the short-term domestic risk-free interest rate. There are no true intermediate time points, i.e. $K = 1$. Furthermore, no special adjustments are modelled. The "approved event" is the event that the equity is nonnegative at time $t = T$. The simplified model proposed here could be applied to a medium-sized retail bank with exclusively domestic business activities. We assume that our model bank, named `MBANK`, focuses on the following types of fundamental retail business:

- Lending to households and small corporates;
- Deposit services;
- Asset fund management (equity funds);
- Different types of payment services.

In addition, the residual market risk evolving from these business activities is managed efficiently by a Treasury function. Finally, we assume that no positions are taken for speculative purposes only.

As mentioned above, the present model only consists of two environmental variables: A broad domestic equity index and the short-term domestic risk-free rate.

Equity index. Let $A = \{A(t), 0 \leq t \leq T\}$ denote the domestic equity index. Today we have $A(0) = A_0$, and the master scenario specifies that the index will have changed by $100\delta_A$ percent by time T, i.e. $A(T) = A_0 (1 + \delta_A)$. We model A as a geometric Brownian motion tied down at $t = T$. This is equivalent to modelling $\ln(A)$ as a Brownian bridge Y with $Y(0) = \ln A_0$ and $Y(T) = \ln A_0 + \ln(1 + \delta_A)$. In the Appendix, the distributional properties of this process are described and a simple method to simulate trajectories according to the specified distribution is outlined.

Short-term risk-free interest rate. Let $r = \{r(t), 0 \leq t \leq T\}$ denote the short-term domestic risk-free rate. Today we have $r(0) = r_0$, and the master scenario specifies that the rate will have changed by $100\delta_r$ percentage units by time T, i.e. $r(T) = r_0 + \delta_r$. We have chosen to model r as a Brownian motion tied down at $t = T$; again, see the Appendix for the necessary mathematical framework. Note that negative interest rates are not precluded; on the other hand, very few trajectories will entail negative rates if parameters have been set up properly. A large number of sophisticated short-term rate models have been suggested in the literature. It is however not necessary to appeal to any of these models in the present study, since the results that we are looking for are of a very crude nature indeed.

The environmental variables are presented in Table 3.

Variable	Name	Start value	Change	Type of change
Broad equity index	$A(t)$	A_0	δ_A	Relative
Short-term interest rate	$r(t)$	r_0	δ_r	Absolute

Table 3. List of the environmental variables used in the case study

For easy reference, all the relevant bank-specific variables are presented here, see Table 4. They will be further explained as we move along.

Variable	Name	Start value	Change	Type of change
Deposit volume	$V_D(t)$	$V_{D,0}$	δ_D	Relative
Household lending volume	$V_{LH}(t)$	$V_{LH,0}$	δ_{LH}	Relative
Corporate lending volume	$V_{LC}(t)$	$V_{LC,0}$	δ_{LC}	Relative
Equity position	$V_{EQ}(t)$	$V_{EQ,0}$	δ_{EQ}	Relative
Service volume	$V_{PS}(t)$	$V_{PS,0}$	δ_{PS}	Relative
Cost volume	$V_C(t)$	$V_{C,0}$	δ_C	Relative

Table 4. List of the bank-specific variables used in the case study

Let $\mathbf{S} = \{\mathbf{S}(t), 0 \leq t \leq T\}$ denote these eight quantities, i.e. environmental variables as well as bank-specific variables. The value process of Subsection 2.3,

$$X = \{C(t), 0 \leq t \leq T\},$$

is simply the bank equity. At time $t = 0$ the equity is equal to C_0, and the "approved event" as introduced in Subsection 2.5 can be written $G_1 = \{C(T) \geq 0\}$. We turn to describe simple models for the major business activities of `MBANK`. Note that, although models for the time dynamics of the risk factors are given explicitly here, the equations (1)–(6) specifying the revenues are presented in a form that indicates that much more general risk factor models could be invoked if desired.

3.1 Deposit margins

We assume that `MBANK` offers one single deposit account to its clients. At each time point, the revenue is given by the margin times the deposit volume. The margin taken by the bank follows the short-term rate—low interest rates imply low margins (indeed, the difference between the interest rate and the margin needs to stay positive) and vice versa. Hence we set up a suitable non-decreasing function $m_D(x)$ with $m_D(x) = 0$ for $x \leq 0$ and $m_D(x) \leq x$ for all positive x to model the deposit margin. The function that has been chosen for the present simulation is shown in Figure 1. Of course, in reality the bank updates the deposit margin in leaps, and there is a time lag between changes in the interest rate and margin updates. However, for mathematical convenience we assume a precise and immediate response.

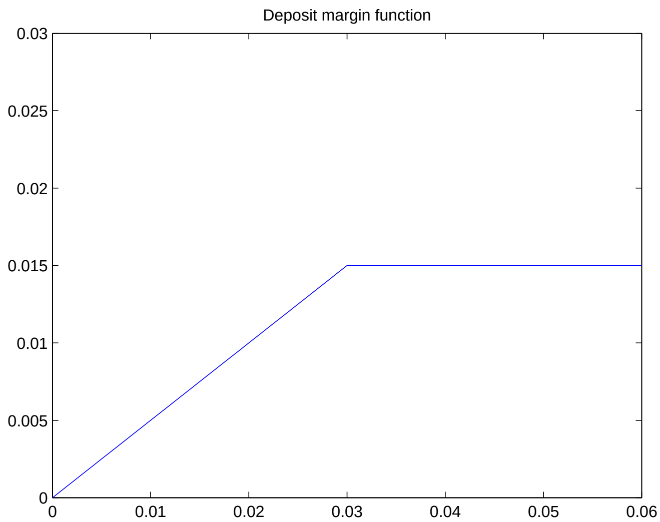

Figure 1. The deposit margin as a function of the interest rate

The deposit volume, $V_D = \{V_D(t),\, 0 \leq t \leq T\}$, is handled as follows. At time $t = 0$ the volume is equal to $V_D(0) = V_{D,0}$. Also, the volume at time $t = T$ is specified by the master scenario; $V_D(T) = V_{D,0}\,(1 + \delta_D)$. We assume a linear growth (or decline) between these time points, i.e. $V_D(t) = V_{D,0}\,(1 + \delta_D t/T)$. It follows that the accumulated revenue during the time interval $[0, t]$ is given by

$$P_D(t) = \int_0^t V_D(t') m_D\left(r(t')\right)\, dt', \tag{1}$$

where $r(t)$ refers to the short-term interest rate model defined above.

3.2 Asset fund commissions

We assume that `MBANK` offers one single domestic equity index fund to its clients. At each time point, the revenue is given by the total invested volume times the value of the index times the commission fee. The underlying equity index, $A = \{A(t),\, 0 \leq t \leq T\}$, has been discussed above. Commission fees are typically not updated very frequently. Hence we may assume here that the fee, m_{EQ}, is constant in time. The total asset volume invested by the clients is more difficult to model. It is clear that the interest among the public for equity investments increases as the stock index grows, implying substantial new investments. The opposite situation occurs during a falling trend. Here we make the simplifying assumption that the total position V_{EQ} evolves in time according to the master scenario, regardless of the equity index level. At time $t = 0$ the volume is equal to $V_{EQ}(0) = V_{EQ,0}$, and we assume that $V_{EQ}(t) = V_{EQ,0}\,(1 + \delta_{EQ} t/T)$, $0 \leq t \leq T$. The total asset volume at time t is of course given by $V_{EQ}(t)A(t)$.

It follows that the accumulated revenue during the time interval $[0, t]$ is given by

$$P_{EQ}(t) = \int_0^t V_{EQ}(t')A(t')m_{EQ}\,dt'. \tag{2}$$

3.3 Lending margins

Private persons and corporates may take loans at `MBANK`. The lending business is actually extremely hard to model accurately. At each time point a lending margin is specified for each maturity, and as time goes by, margins are accumulated in a very complicated manner. Here we need a pragmatic approach that captures the dynamics of the revenues of the lending business reasonably well.

We note that lending margins do *not* depend on the prevailing interest rate like deposit margins do, since there is no natural margin cap. Furthermore, at least in countries where loan prepayment risk is not an issue, there is no need to price fixed rate loans differently from floating rate loans. Hence we make the simplifying assumption that the lending margin, m_L, is *constant* in time and over scenarios. Turning to the lending volume, $V_L = \{V_L(t),\, 0 \leq t \leq T\}$, we assume a linear time development, i.e. $V_L(t) = V_{L,0}\,(1 + \delta_L t/T)$, where $V_{L,0}$ is the volume at time $t = 0$ and where the growth δ_L by time $t = T$ has been specified by the master scenario. It follows that the accumulated revenue during the time interval $[0, t]$ is given by

$$P_L(t) = \int_0^t V_L(t')m_L\,dt'. \tag{3}$$

In order to differentiate between household lending and corporate lending we will use subindices LH and LC, respectively, instead of just L.

3.4 Payment services

The revenue, $P_{PS} = \{P_{PS}(t),\, 0 \leq t \leq T\}$, from payment services (credit card services, telephone bank, Internet bank etc) is modelled in the simplest possible way. Historical revenues are analyzed, and based on this analysis a start value $V_{PS,0}$ and a growth rate δ_{PS} are specified. Assuming a linear trend, $V_{PS}(t) = V_{PS,0}\,(1 + \delta_{PS} t/T)$, we get that the accumulated revenue during the time interval $[0, t]$ is given by

$$P_{PS}(t) = \int_0^t V_{PS}(t')\,dt'. \tag{4}$$

3.5 Loan losses

The lending volume $V_L = \{V_L(t),\, 0 \le t \le T\}$ has been defined in Subsection 3.3 above. Inspired by the risk process concept in insurance mathematics, we specify the following continuous-time model for the credit losses in the lending portfolio (cf. the concept of *Cox processes*[2] in the insurance literature, e.g Grandell [6], and in Lando [10]). It is well known that in an economy with rising interest rates the clients will have difficulties fulfilling their payment obligations and as a consequence credit losses will start to build up. Hence, we model the time points of default as a Poisson process with a stochastic intensity λ_L that follows the short-term interest rate, i.e. $\lambda_L(t) = f_L\,(r(t))$. For the present simulation we have simply assumed that f_L is a linearly increasing function.

Turning to the size of the loss given default, we note that an exposure is in general secured by collateral. It is very complicated to model the recovery process properly, as this process may extend over a long time period and the amount actually recovered is very uncertain. In this simplified framework, we assume that the recovery process is contracted to a single time point; the time of default[3]. We also assume that the bank has estimated an average recovery rate based on historical recoveries. We remark in passing that having recovery processes that extend in time is analogous to having delays in claim settlement in the insurance context, and could preferably be handled using the same machinery, cf Mikosch [11].

The above reasoning leads to the following model. First assume that the volume V_L is constant in time. Let $X_{L,i}$ be independent and identically distributed nonnegative random variables representing net losses (i.e. exposure net recovery) on defaulted loans. Also, let $N_L = \{N_L(t),\, 0 \le t \le T\}$ be a Poisson process with stochastic intensity $\lambda_L(t) = f_L\,(r(t))$. The total loss up to time t is given by

$$\sum_{i=1}^{N_L(t)} X_{L,i}. \tag{5}$$

[2] A Cox process $N = \{N(t), 0 \le t < \infty\}$ is defined as follows. A nonnegative random process $\{\lambda(t), 0 \le t < \infty\}$ is specified. Conditioned on its realization, N is a non-homogeneous Poisson process with that realization as its intensity.

[3] Indeed, some institutions sell the defaulted contracts to third parties and receive a fraction of the face value up-front, hence the assumption is not entirely unreasonable.

The case with non-constant lending volume is handled by further modifying the intensity function of the Poisson process to be $\lambda_L(t) = f_L(r(t)) V_L(t)/V_L(0)$. Compare the concept of operational time scale in insurance mathematics, where an increasing consumer base is modelled by running the time according to a different clock.

As mentioned above, in order to differentiate between household lending and corporate lending we will use subindices LH and LC, respectively, instead of just L.

3.6 Costs

The cost side consists of e.g. staff costs, rent, equipment etc, and is not very volatile. Denote the total costs by $P_C = \{P_C(t), 0 \le t \le T\}$. We model the costs in the simplest possible way. Historical costs are analyzed, and based on this analysis a start value $V_{C,0}$ and a growth rate δ_C are specified. Assuming a linear trend, $V_C(t) = V_{C,0}(1 + \delta_C t/T)$, we get that the accumulated costs during the time interval $[0, t]$ are given by

$$P_C(t) = \int_0^t V_C(t')\, dt'. \tag{6}$$

3.7 Value process

Putting the pieces together, we have from Equations (1)–(6) that the equity at time t, i.e. the sum of the equity at time 0 and the accumulated result during the time interval $[0, t]$, is given by[4]

$$\begin{aligned}
C(t) &= C_0 \\
&+ \textstyle\int_0^t V_D(t') m_D(r(t'))\, dt' && \text{(deposits)} \\
&+ \textstyle\int_0^t V_{EQ}(t') A(t') m_{EQ}\, dt' && \text{(commissions)} \\
&+ \textstyle\int_0^t V_{LH}(t') m_{LH}\, dt' - \sum_{i=1}^{N_{LH}(t)} X_{LH,i} && \text{(household lending)} \\
&+ \textstyle\int_0^t V_{LC}(t') m_{LC}\, dt' - \sum_{i=1}^{N_{LC}(t)} X_{LC,i} && \text{(corporate lending)} \\
&+ \textstyle\int_0^t V_{PS}(t')\, dt' && \text{(payment services)} \\
&- \textstyle\int_0^t V_C(t')\, dt' && \text{(costs)}
\end{aligned}$$

We note that the costs and the revenues from payment services are deterministic. The commissions from equity funds follow the equity index and are independent of the other items. On the other hand, both deposits and lending are affected by the interest rates. Higher interest rate levels will lead to higher deposit revenues, but at the same time the loan losses will start to build up.

[4] The return on the equity is not taken into account here. It could conveniently be modelled by investing the equity using short money market instruments, implying essentially that the bank would receive the short-term rate. In reality, however, the management of the equity position is much more sophisticated.

We have performed a simulation of the equity, using the following numerical values. Regarding the development of the environmental variables, we assume that the short-term rate grows from 3% to 6% over the year, while the equity index falls by 20%. The model has been fed with realistic volatilities.

The initial equity is 100 million EUR. The bank has a total exposure of 1100 million EUR to households and 1000 million EUR to corporates. On the other hand, the total deposit volume is 2000 million EUR. These volumes are assumed to grow moderately with time (about 5-10% over the year), and the margins taken by the bank are about 1%. Next, the total fund asset volume is 1000 million EUR. The asset volume changes with the equity index, and the commission fee is 1.5%. Finally, the total revenue from payment services over the year is about 10 million EUR, and the total costs are about 60 million EUR.

We have used 100 time steps in the simulation. Figure 2 shows 100 trajectories of the value process. We note that most of the randomness can be attributed to the loan losses of the corporate segment. There are only a few defaults over the year, but each default leads to a loss of about 1 million EUR. On the other hand, there are many defaults on loans to households, but the loss on each of these defaulted loans is minor (about 20 000 EUR).

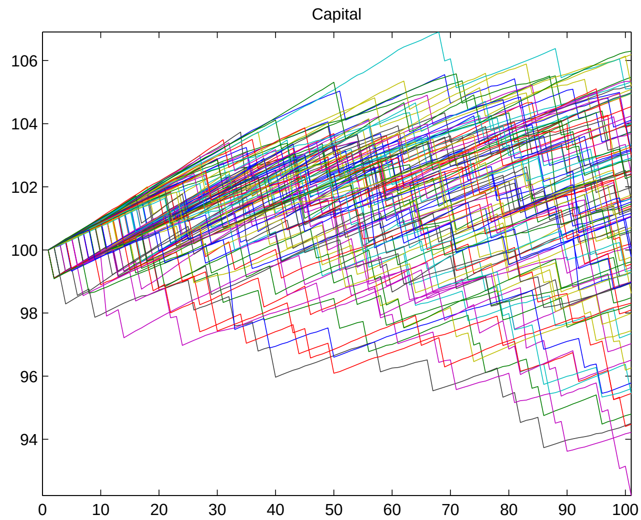

Figure 2. Trajectories of the value process (i.e. the capital) of the case study

Table 5 shows the constituents of the total result for three specific paths; one adverse path (Path 1), one normal path (Path 2), and one favourable path (Path 3). We note that some drivers of the result are very volatile while other drivers seem to be deterministic, or almost deterministic.

(million EUR)	Path 1	Path 2	Path 3
Deposit margins	30.76	30.73	29.79
Lending margins	16.03	16.03	16.03
Commissions	13.50	13.48	15.62
Payment services	10.00	10.00	10.00
Costs	-61.51	-61.51	-61.51
Loan losses	-16.56	-8.56	-4.60
Result	-7.78	0.17	5.33

Table 5. Result of the business activities of `MBANK`
(three paths corresponding to the same master scenario)

Finally, a simulation using 5000 realizations was performed, giving a lower 1%-quantile of 94.17 million EUR. Furthermore, the simulated probability of the approved event (nonnegative capital at the end of the time period) was 100%. It is virtually impossible to lose the entire start capital of 100 million EUR.

4 Capital adequacy

Our next task is to simulate the so-called *Tier 1 capital ratio* over time. Our model bank `MBANK` is required by the national supervisors to hold an adequate capital buffer against large potential losses. More precisely, the Tier 1 capital ratio is required to stay above 4%. The numerator of this ratio is called the Tier 1 capital and is obtained by making a number of adjustments of the equity[5]. In this work we simply assume that the equity and the Tier 1 capital are identical. The denominator of the Tier 1 capital ratio consists of the Risk Weighted Assets (RWA). Under the new capital adequacy rules, Basel II, risk weights are calculated for all assets of the bank using supervisory functions that are designed so that risky assets are assigned large risk weights, and vice versa. The Risk Weighted Assets for credit risk is simply the sum of these risk weights. In addition, RWA for market risk and operational risk are calculated using supervisory formulas. The bank's total RWA is the sum of these three components (credit risk, market risk and operational risk). For each of these risk types, the bank needs to choose one out of several levels of sophistication–selecting a more advanced calculation model requires additional investments from the bank's side, but tends to lead to lower RWA. In Subsection 4.2 below we present the model choices made by `MBANK`.

The simulation will use the same risk factors and the same master scenario as before (see Section 3). The risk horizon is still one year, and no special adjustments are assumed. This

[5]To obtain the Tier 1 capital, a number of supervisory deductions (involving e.g. goodwill and provisions) are applied to the equity. On the other hand, subordinated loans may in some countries be included in the Tier 1 capital to a certain extent.

time, however, the value process is bivariate: $\mathbf{X} = (C, R)$, where $C = \{C(t), 0 \leq t \leq T\}$ is the Tier 1 capital and $R = \{R(t), 0 \leq t \leq T\}$ is the RWA. The model of the Tier 1 capital is identical to the model of the equity as presented in Section 3. In Subsection 4.2, the RWA simulation model is depicted. As mentioned in the introduction of this paper, the bank management typically announces a slightly higher target capital ratio, α say, for the business than the minimum capital ratio of 4% specified by the regulators. This leads us to announce the following "approved event":

$$G_1 = \{C(T)/R(T) \geq \alpha\} = \{C(T) - \alpha R(T) \geq 0\}.$$

4.1 Equity

To reiterate, we know from Section 3 that the equity at time t, i.e. the sum of the equity at time 0 and the accumulated result during the time interval $[0, t]$, is given by

$$\begin{array}{rcll} C(t) & = & C_0 & \\ & + & \int_0^t V_D(t') m_D\left(r(t')\right)\, dt' & \text{(deposits)} \\ & + & \int_0^t V_{EQ}(t') A(t') m_{EQ}\, dt' & \text{(commissions)} \\ & + & \int_0^t V_{LH}(t') m_{LH}\, dt' - \sum_{i=1}^{N_{LH}(t)} X_{LH,i} & \text{(household lending)} \\ & + & \int_0^t V_{LC}(t') m_{LC}\, dt' - \sum_{i=1}^{N_{LC}(t)} X_{LC,i} & \text{(corporate lending)} \\ & + & \int_0^t V_{PS}(t')\, dt' & \text{(payment services)} \\ & - & \int_0^t V_C(t')\, dt' & \text{(costs)} \end{array} \qquad (7)$$

4.2 Risk weighted assets

We make the following definitions:

- $R_{CR}(t)$ = Risk Weighted Assets for credit risk at time t;
- $R_{MR}(t)$ = Risk Weighted Assets for market risk at time t;
- $R_{OR}(t)$ = Risk Weighted Assets for operational risk at time t.

Their sum is simply denoted by $R(t)$. We assume that MBANK has made the following choices regarding the calculation of the minimum capital requirement (i.e., Pillar 1 of Basel II):

- The *least* advanced method for credit risk (known as the Standardised Approach). More advanced Pillar 1 models require sophisticated methods to estimate the default probabilities for each counterparty, as well as the amount recovered given default for each deal. This is typically not yet implemented among the smaller banks.

- The *most* advanced method for market risk (known as the Internal Method). It has become best practice in the financial world to use Value-at-Risk to quantify the market risk of the trading book on a daily basis. With Value-at-Risk in place, the bank may use the most advanced of the available Pillar 1 calculation methods to quantify market risk.
- The *least* advanced method for operational risk (known as the Basic Indicator Approach). At the time of writing it is still not very common that a financial institution has developed advanced methods to quantify operational risk. Indeed, it is very difficult to estimate the probability and the severity of financial losses caused by e.g. IT system crashes, fraud, administration failures etc.

We turn to give a brief description of these three calculation models.

RWA for credit risk ("Standardised approach")

Under this approach, each loan is assigned a risk weight based on the type of loan. Table 6 shows some examples.

Loan type	Risk weight
Retail	75%
Mortgage	35%
Revolving credit	15%
Unrated corporate	100%
⋮	⋮

Table 6. Examples of risk weights according to the standardised approach

RWA for credit risk is then calculated as a weighted sum of the exposures to the various loan types, using these risk weights as weights. For the purpose of this example, we assume that all loans of the portfolio are classified as either Mortgage or Unrated corporate. It follows that $R_{CR}(t) = 0.35V_{LH}(t) + 1.00V_{LC}(t)$.

RWA for market risk ("Internal method")

Under this approach, R_{MR} is based on the bank's daily Value-at-Risk figures. The daily VaR covers equity risk in the trading book, interest rate risk in the trading book, and all currency risk. Since we have assumed that `MBANK` is involved in neither foreign activities nor proprietary trading, R_{MR} is typically just a small part of the total RWA. In this example we assume $R_{MR}(t)$ to be constant in time.

RWA for operational risk ("Basic indicator approach")

The philosophy behind this approach is that large revenues indicate vivid business activity, which in turn indicates large operational risk. Therefore, R_{OR} is based on the bank's gross income, namely a certain fraction of the average revenues of the three previous years.

In this example we have

$$R_{OR}(t) = 0.15 \times 12.5 \times \frac{P^{+}_{hist}(t) + P^{+}_{sim}(t)}{3}$$

where $P^{+}_{hist}(t)$ is the total revenue between the time points $t-3$ and 0, calculated using historical data, and where $P^{+}_{sim}(t)$ is the (simulated) total revenue between the time points 0 and t. The latter quantity is the sum of the right hand sides of Equations (1), (2), (3) and (4).

4.3 Capital ratio

We wish to estimate the probability that ruin is avoided in the given master scenario, i.e. the probability that $C(T) - \alpha R(T)$ is nonnegative. Figure 3 shows a simulation of 100 trajectories. In our example, R_{CR} grows deterministically with time (from 1385 million EUR to 1475 million EUR) while $R_{MR} = 100$ million EUR by assumption and R_{OR} depends on the simulated paths (R_{OR} is about 130 million EUR). The relative order of magnitude of these three quantities is quite typical for retail banks. In fact, RWA for credit risk is usually even more dominant than in this example. The trajectories of the capital process were shown in Figure 2 above. In Figure 3 (left) we see the corresponding paths of the risk weighted assets, $R(t)$. The picture to the right in Figure 3 shows the paths of the Tier 1 capital ratio.

Assume that the management has announced a capital ratio target of 5.5%, i.e. $\alpha = 0.055$. Then it is important to estimate the probability that the target is maintained given the master scenario. A simulation using 5000 realizations shows that this probability is 99.5%. Turning the discussion around, assume that the management is comfortable with the approved event having probability 95%. Then the same simulation shows that the initial capital may be decreased by 2.5 million EUR (from 100 million EUR to 97.5 million EUR), still keeping the bank on the safe side. Hence, in this simple setting we have managed to answer the fundamental questions stated in Subsection 2.6.

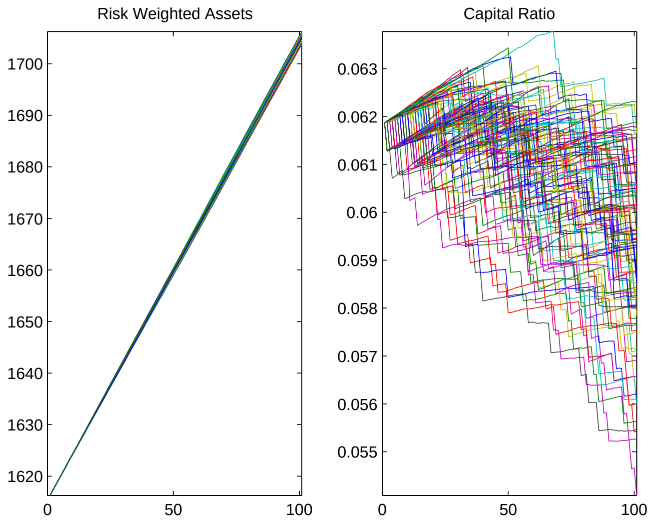

Figure 3. Trajectories of the total RWA (left)
and the Tier 1 capital ratio (right) of the case study

5 Possible extensions

5.1 More sophisticated bank business

We have deliberately chosen to study a very simple and idealized financial institution – the purpose has been to present the main ideas rather than to give detailed descriptions of all relevant models. The framework can however be extended in a number of directions. We proceed to give some examples.

A bank that is involved in FX trading is exposed to currency risk. The same is true for a bank having foreign branches. Currency effects are introduced into the model by including the relevant FX rates among the environmental variables. The master scenario specifies the long-term changes in the FX rates, and the continuous-time dynamics can be modelled as for the equity index in our example. Interesting dependencies between markets may be revealed by introducing such currency effects.

The lending volumes and the deposit volumes are typically not perfectly matched. The residual volume is actively managed by the Treasury function of the bank. The interest rate risk is managed by trading capital market instruments, and a certain degree of outright positions may also be allowed. Such activities could be modelled using the present framework. One or several relevant interest rate *curves* (i.e., not just the short-term rate) should be included in the risk factor set, and the master scenario should specify the long-term behaviour of a number of nodes on each curve. Then the interest rate portfolio should be simulated over time, using simple trading rules (for instance, as the instruments mature the trader reinvests the cash in order to maintain a specified target duration).

Proprietary trading introduces a volatile component that is hard to model. It is not practical to roll a trading portfolio forward for a number of years. One possible alternative solution would be to adopt a 'top-down' approach, where the volatility of the trading activity is inferred using time series of historical results. The trading portfolio might be simulated independently of all environmental variables.

Finally, a bank that is involved in lending to large corporates needs more sophisticated simulation models of loan losses. The creditworthyness of a large corporate is affected by internal factors rather than by macroeconomic variables, hence asset value models like the models developed by Moody's KMV corporation (see e.g. [3]) might be more appropriate for this segment.

5.2 The Internal Ratings Based framework

As mentioned above, when implementing Basel II Pillar 1, financial institutions need to make a choice between a number of available calculation models. Regarding the most important risk type, credit risk, the choice lies between the simpler standardised approach and the more sophisticated *internal ratings based* approaches. The common denominator of the different internal ratings based approaches is the requirement of a rating assessment of each individual loan[6]. In fact, for each loan an estimate of the probability of default within the next twelve months is required[7]. Our model bank `MBANK` has selected the standardised approach. However, larger financial institutions tend to adopt the more advanced framework, the reason being two-fold. First, the implementation of rating systems in the organization leads to a more sound credit granting process, which should eventually pay off in terms of a decrease in loan losses. Second, the Pillar 1 capital requirement tends to be lower under the more advanced approaches.

The framework outlined in Section 4 can be used without changes also in the case where the risk weighted assets for credit risk is calculated according to the internal ratings based approach. Indeed, using the model of loan losses, as described in Subsection 3.5, it is easy to extract default probabilities of all loans. Furthermore, exposures at default and losses given default may be extracted. Then, applying the appropriate supervisory risk weight formulas (see e.g. [1]) the quantity $R_{CR}(t)$ may be simulated over time.

Note that a new feature enters the RWA simulation, and thus the simulation of the Tier 1 ratio: Since the probability of default changes with changing macroeconomic conditions,

[6] The regulation specifies that individual default probabilities should be assigned on *counterparty* level for some segments and on *deal* level for other segments.

[7] Apart from default probabilities, estimates of losses given default and exposures at default are sometimes required, depending on the choice of method.

RWA for credit risk is liable to show substantial fluctuations over time[8]. This phenomenon is commonly referred to as the procyclicality effect, see e.g. Gordy and Howells [5]. Indeed, when entering a recession the bank will show weak results (which has an adverse effect on the capital) and *at the same time* RWA will increase. This will have a *double* negative effect on the Tier 1 capital ratio. Such effects are possible to capture using the present integrated simulation framework.

6 Conclusions

In this text we have shown how to perform a stress test of a financial portfolio where the risk horizon is very long, typically a number of years. Given such large time periods, in our opinion it is not feasible to set up a complicated stochastic model with a huge number of parameters to simulate all the relevant risk drivers over time. On the other hand, applying an instant shock to some risk factors and then fixing their modified values throughout the time period considered would be meaningless, since such a scenario would never occur in reality. The framework described in this paper in a sense interpolates between these two extreme approaches. A number of predefined master scenarios form the basis of the model. Each of these master scenarios specifies the long-term movement of the dominant risk factors, i.e. given the current values of the risk factors, the scenario specifies their values at the risk horizon, and possibly at a number of intermediate time points. Given such a master scenario, we set up a continuous-time stochastic model of the risk factors that is consistent with the scenario. The value of the financial portfolio is then simulated, and summary statistics such as the expectation and appropriate quantiles may be calculated.

We stress that it is sufficient to let the scenario set contain *very few* master scenarios. On the other hand, it is crucial that these scenarios are carefully designed. A consistent macroeconomic 'story' should lie behind each scenario. This approach will greatly facilitate the communication of the calculated results to the management.

In this work we have applied the stress testing framework in two different situations. First, we have investigated how the equity of an idealized retail bank develops over one year, given a simple master scenario specifying the behaviour of the short-term interest rate, the stock market, and client volumes for major business activities. Second, we have studied the development of the Tier 1 ratio (calculated according to the new capital adequacy rules, Basel II) for the same bank and under the same scenario. Using the model, it is possible to calculate the probability that the Tier 1 ratio stays above a predefined target ratio. In case the bank finds itself over-capitalized, the model can also be used to give an indication of the amount of capital that may be released. Of course, the output of our model does not alone give sufficient information to determine the appropriate capital level of the bank—strategic

[8]The regulation suggests that banks should use default probabilities that reflect 'normal' business conditions as input to the risk weight formulas. However, due to the lack of historical data, at present it is not easy to provide accurate estimates of such quantities.

considerations and rating goals, etc, should also be taken into account. Nevertheless, we believe that the model could serve as an important tool in the bank's capital adequacy assessment process. In addition, the model could provide important information to be used in the communication with both rating agencies and regulators (e.g., in the Pillar 2 process within Basel II).

Appendix: Brownian bridge

A (one-dimensional) *Brownian bridge* from a to b on the time interval $[0,T]$ is Brownian motion started at a and conditioned to arrive at b at time T. Here we give two different characterizations of the Brownian bridge; see e.g. [9] for a theoretical treatment.

Global characterization. Let $W = \{W(t), 0 \leq t < \infty\}$ be standard Brownian motion and define

$$B(t) = a + (b-a)\frac{t}{T} + \left(W(t) - \frac{t}{T}W(T)\right); \qquad 0 \leq t \leq T.$$

Then B is a Brownian bridge from a to b on $[0,T]$. This characterization is very useful for simulation purposes. To generate a path of the Brownian bridge, it is sufficient to simulate a Brownian trajectory on $[0,T]$ and then subtract $W(T)$ to the proper proportions from the simulated trajectory.

Local characterization. Let $W = \{W(t), 0 \leq t < \infty\}$ be standard Brownian motion and consider the linear stochastic differential equation

$$dB(t) = \frac{b - B(t)}{T-t}\, dt + dW(t); \qquad 0 \leq t < T,$$

with initial condition $B(0) = a$. The unique solution of this equation is given by the process

$$B(t) = \begin{cases} a + (b-a)\frac{t}{T} + (T-t)\int_0^t \frac{dW(s)}{T-s}; & 0 \leq t < T, \\ b; & t = T. \end{cases}$$

This is another representation of a Brownian bridge from a to b on $[0,T]$. The process has expectation function

$$m(t) = E(B(t)) = a + (b-a)\frac{t}{T}$$

and autocovariance function

$$\rho(s,t) = \mathrm{Cov}\left(B(s), B(t)\right) = \min(s,t) - \frac{st}{T}.$$

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