Mathematical Statistics
Stockholm University

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Research Report 2005:8

ISSN 1650-0377

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Mathematical Statistics
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Research Report **2005:8**,
http://www.math.su.se/matstat

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Ola Hammarlid and Anders Martin-Löf*

June 2005

Sammanfattning

In classical ruin theory the surplus can increase to infinity, which guarantees that the probability of ruin is strictly less than one. This assumption is substituted by a variable premium that depends on the level of the surplus. The variable premium is assumed to be linear such that the buffer is a generalized Ornstein-Uhlenbeck process. A consequence of this is that the probability of ruin is one. However, the variable premium is such that the expected time to ruin is large. Asymptotically the time to ruin with a certain scaling is shown to be exponentially distributed (and also for the the first passage time of a generalized Ornstein-Uhlenbeck process).

KEY WORDS: Insurance, time of ruin, variable premium,first passage time, generalized Ornstein-Uhlenbeck process

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1 Introduction

An insurance company covers expenses for specified unforseen events of the insured. These expenses are called claims. The collective of policy-holders pays a premium p to the insurance company for this commitment. Let the claims $\{Y_i\}_{i=1}^{\infty}$ be independent identically distributed with mean μ and distribution function $F(y)$. Further, the claim arrival process N_t is a Poisson process with intensity λ. The claim process is denoted by

$$S_t = \sum_{i=1}^{N_t} Y_i, \qquad \text{with} \qquad S_0 = 0,$$

that is, it is a compound Poisson process. The moment generating function of S_t is known to be

$$E\left[e^{\theta S_t}\right] = \exp\left(\lambda t \int_0^{\infty} (e^{\theta x} - 1) F(dx)\right) = \exp(\lambda t \gamma(\theta)),$$

where $\gamma(\theta) = \int_0^{\infty} (e^{\theta x} - 1) F(dx)$, see Grandell [11]. Therefore, the expected value of the claim process is $E[S_t] = \lambda \mu t$.

In classical risk theory the risk process is defined as

$$X_t = X_0 + pt - S_t,$$

where X_0 is the initial surplus.

Definition 1.1 *The insurance company is ruined when the risk process for the first time is equal to or below zero. The time of ruin is denoted by*

$$\tau = \inf_{t>0} \{t : X_t \leq 0\}.$$

The first estimate of the probability of ruin,

$$P(\tau < \infty) \approx C \exp(-RX_0), \tag{1}$$

where C is a constant and R is the root of the equation $\lambda\gamma(R) = pR$, is due to Cramér [5] and Lundberg [19]. Similar results using different techniques have been reported extendedly, see for example two dimensional renewal theory Höglund [16], ladder variables von Bahr [2], integral equation Segerdahl [22] and martingale techniques Grandell [11]. Further references to the insurance literature can be found in the textbooks by Grandell [11], Rolski *et al.* [20] or Assmusen [1].

As a modification Davidson [6], Djehiche [8] and Lundberg [19] suggested a variable premium depending on the risk process instead of a constant premium. However, the risk process was still such that $P(\tau < \infty) < 1$, for more references see Asmussen [1].

The fact that $P(\tau < \infty) < 1$ is an unrealistic feature since it depends on the fact that $X_t \to \infty$ as $t \to \infty$. Recently, Irbäck [17] studied the case where X_t is bounded, that is at a certain level no more premium were collected. In this case $P(\tau < \infty) = 1$, however the probability of ruin is still small in finite time if the barrier of the risk process is large.

We will introduce a feed-back in the premium rate, that is the premium is a decreasing function of X_t, $p(X_t)$. The premium function is linear and such that the probability of ruin is one. This is done in such manner that the expected time to ruin is very large. In our opinion this type of model is much more realistic, since no savings will, in real life, be untouched forever and grow to infinity.

We will show that the time to ruin, properly scaled, is asymptotically exponentially distributed when $\lambda \to \infty$. This result is of general interest since it states that the scaled first passage time of a generalized Ornstein-Uhlenbeck process to a distant barrier can be approximated by an exponential random variable. Hadjiev [13] derived the Laplace transform of the first passage time of the generalized Ornstein-Uhlenbeck process when coming from below. Results on the first passage time coming from above, like ruin, does not apply to his results.

2 The model and preparations

We consider the case when the variable premium is linear, that is $p(x) = \lambda a - bx$. The risk process is therefore on differential form written

$$dX_t = (\lambda a - bX_t)dt - dS_t.$$

There is an equilibrium value $\lambda \bar{x}$ of X_t, where the premium balances the average payments, that is $\bar{x} = (a - \mu)/b > 0$. We expect X_t to fluctuate around this equilibrium value and with time approach a steady state if not stopped by ruin. The risk process is analogous to the Ornstein-Uhlenbeck process and can similarly be expressed as

$$X_t = X_0 \mathrm{e}^{-bt} + \frac{\lambda a}{b}(1 - \mathrm{e}^{-bt}) - \int_0^t \mathrm{e}^{-b(t-u)} \mathrm{d}S_u. \tag{2}$$

Notice that X_t is bounded by $\lambda a/b$, under the assumption that $X_0 \leq \lambda a/b$. Further, let time go to infinity to derive the stationary limit risk process, that is

$$X_\infty = \frac{\lambda a}{b} - \int_0^\infty \mathrm{e}^{-bu} \mathrm{d}S_u. \tag{3}$$

We assume that the claim size distribution $F(y)$ has an exponentially decreasing tail so that $\gamma(\theta)$ is defined for $\theta < \bar{\theta}$ for some $\bar{\theta}$. Later we will have use for the cumulant generating function of the claim process and the cumulant generating function of its limit process.

Lemma 2.1 *The stochastic variable* $\int_0^\infty \mathrm{e}^{-bu}\mathrm{d}S_u$ *has a Lévy distribution with cumulant generating function* $\frac{\lambda}{b}g(\theta)$ *where,*

$$g(\theta) = \int_0^\infty \frac{\mathrm{e}^{\theta y} - 1}{y}\bar{F}(y)\mathrm{d}y \qquad \text{and} \qquad \bar{F}(y) = 1 - F(y).$$

Furthermore, $\int_0^t \mathrm{e}^{-b(t-u)}\mathrm{d}S_u$ *has a Lévy distribution defined by its cumulant generating function*

$$\frac{\lambda}{b}g(\theta, t) = \frac{\lambda}{b}\left(g(\theta) - g(\theta \mathrm{e}^{-bt})\right).$$

The Lévy density of this process is

$$\frac{\bar{F}(y) - \bar{F}(y\mathrm{e}^{bt})}{y}.$$

The distribution has finite mass bt and is hence a compound Poisson process.

Proof: The stochastic integral is the sum of independent contributions $\exp(-bu)\mathrm{d}S_u$. The cumulant generating functions of these contributions are given by,

$$\log\left(E\left[\exp\left(\theta \mathrm{e}^{-bu}\mathrm{d}S_u\right)\right]\right) = \lambda\gamma\left(\theta \mathrm{e}^{-bu}\right)\mathrm{d}u.$$

Integrating all these contributions together we get

$$\log\left(E\left[\exp\left(\theta\int_0^t \mathrm{e}^{-b(t-u)}\mathrm{d}S_u\right)\right]\right) = \lambda\int_0^t \gamma\left(\theta \mathrm{e}^{-bu}\right)\mathrm{d}u.$$

Let us continue by making the substitution $\mathrm{e}^{-bu} = v$, then

$$\begin{aligned}
\lambda\int_0^t \gamma\left(\theta \mathrm{e}^{-bu}\right)\mathrm{d}u &= \frac{\lambda}{b}\int_{\mathrm{e}^{-bt}}^1 \frac{\gamma(\theta v)}{v}\mathrm{d}v \\
&= \frac{\lambda}{b}\int_{\mathrm{e}^{-bt}}^1\int_0^\infty \frac{\mathrm{e}^{\theta v y} - 1}{v}F_Y(\mathrm{d}y)\mathrm{d}v \\
&= \frac{\lambda}{b}\int_{\mathrm{e}^{-bt}}^1\int_0^\infty\int_0^y \theta \mathrm{e}^{\theta v u}\mathrm{d}uF(\mathrm{d}y)\mathrm{d}v \\
&= \frac{\lambda}{b}\int_{\mathrm{e}^{-bt}}^1\int_0^\infty\int_u^\infty \theta \mathrm{e}^{\theta v u}F(\mathrm{d}y)\mathrm{d}u\mathrm{d}v \\
&= \frac{\lambda}{b}\int_0^\infty\int_{\mathrm{e}^{-bt}}^1 \theta \mathrm{e}^{\theta v u}\bar{F}(u)\mathrm{d}v\mathrm{d}u \\
&= \frac{\lambda}{b}\int_0^\infty \left(\mathrm{e}^{\theta u} - \mathrm{e}^{\theta u \mathrm{e}^{-bt}}\right)\frac{\bar{F}(u)}{u}\mathrm{d}u \\
&= \frac{\lambda}{b}\left(g(\theta) - g\left(\theta \mathrm{e}^{-bt}\right)\right).
\end{aligned}$$

Now the difference

$$g(\theta) - g\left(\theta \mathrm{e}^{-bt}\right) = \int_0^\infty \left(\mathrm{e}^{\theta x} - 1\right) \frac{\bar{F}(x) - \bar{F}(x\mathrm{e}^{bt})}{x} \mathrm{d}x$$

implies that the Lévy measure has the density $\left(\bar{F}(x) - \bar{F}(x\mathrm{e}^{bt})\right)/x$.

Left to show is that this Lévy measure has finite mass, that is

$$\begin{aligned}
\int_0^\infty \frac{\bar{F}(x) - \bar{F}(x\mathrm{e}^{bt})}{x} \mathrm{d}x &= \int_0^\infty \frac{1}{x} \int_x^{x\mathrm{e}^{bt}} \mathrm{d}F(y)\mathrm{d}x \\
&= \int_0^\infty \int_{y\mathrm{e}^{-bt}}^{y} \frac{\mathrm{d}x}{x} \mathrm{d}F(y) \\
&= \int_0^\infty bt\mathrm{d}F(y) \\
&= bt.
\end{aligned}$$

Trivially, we have that $\lim_{t\to\infty} g(\theta, t) = g(\theta)$. This implies weak convergence by the continuity theorem for transforms. Furthermore, if $\lambda/b \geq 2$, we have that the density $f(x)$ of $\int_0^\infty \mathrm{e}^{-bu}\mathrm{d}S_u$ is $[\lambda/b] - 1$ differentiable ($[\lambda/b]$ denotes the integer part of λ/b), see Sato [21].

The density $f(x,t)$ of $\int_0^t \mathrm{e}^{-bu}\mathrm{d}S_u$ is defined by

$$\int_0^\infty \mathrm{e}^{\theta x} f(x,t)\mathrm{d}x = \exp\left(\frac{\lambda}{b} g(\theta, t)\right).$$

However, the convergence of the cumulative distribution function does not directly imply convergence of $f(x,t)$ to the density $f(x)$. However, in Appendix A we show that in our setting the convergence actually holds true, that is $\lim_{t\to\infty} f(x,t) = f(x)$. This will turn out to be a crucial point in the proof of the main theorem.

We will also need a saddlepoint approximation of the density $f(x)$. The derivation of this is in Appendix B, more details on saddlepoint approximations can be found in the book by Jensen [18]. The Legendre-Fenchel transform of the cumulant generating function is called the rate function in the field of Large Deviations, see for example Bucklew [4] or Dembo and Zeitouni [7]. The rate function of the stochastic integral of the claim process is

$$h(x) = \sup_\theta \{\theta x - g(\theta)\}.$$

The saddlepoint approximation around the point $\lambda x/b$ is

$$f(\lambda x/b) = \sqrt{\frac{bh''(x)}{2\pi\lambda}} \exp\left(-\frac{\lambda}{b} h(x)\right) (1 + \mathcal{O}(1/\lambda)).$$

The risk process X_t is the difference of a deterministic function and the stochastic integral just discussed, see equation (2). Therefore, the density of X_t in $\lambda y/b$ is denoted by

$$f_X\left(\lambda x/b, \lambda y/b, t\right) = f\left(\frac{\lambda}{b}\left(x\mathrm{e}^{-bt} + a(1-\mathrm{e}^{-bt}) - y\right), t\right),$$

when starting in $X_0 = \lambda x/b$. When $t \to \infty$ we have that

$$f_X\left(\lambda x/b, \lambda y/b, \infty\right) = f\left(\frac{\lambda}{b}\left(a - y\right)\right). \tag{4}$$

Definition 2.2 *The risk process starting in x hits for the first time y (when moving upwards from below) at time*

$$T_\lambda(x, y) = \inf\{t > 0 : X_0 = x, X_t = y\}.$$

Every state below $\lambda a/b$ is possible to attain since X_t is ergodic and $P(T_\lambda(x, y) < \infty) = 1$. The successive reentries to level x form a renewal process. The sequence of independent identically distributed renewal times to level x are written $\{R^i_\lambda(x)\}_{i=1}^\infty$, which are distributed as $T_\lambda(x, x)$. The expected value of these renewal times are

$$E[R^i_\lambda(x)] = r_\lambda(x) = (p(x)f(x))^{-1}.$$

The last equality follows by an argument in the proof of the main theorem, see equation (8).

Definition 2.3 *The probability of a visit to y between two successive visits to x is written*

$$q_\lambda(x, y) = P\left(X_s = y, \text{for some } s, 0 \leq s \leq R^1_\lambda(x) | X_0 = x,\right).$$

Lemma 2.4

$$E\left[T_\lambda(x, y) + T_\lambda(y, x)\right] = \frac{1}{p(x)f(x)q_\lambda(x, y)} = \frac{1}{p(y)f(y)q_\lambda(y, x)}.$$

The idea of the proof of this theorem can be found in Harris [15]. However, the author dealt only with discrete Markov chains. *Proof:*Let K be the first cycle from x and back to x such that the state y is visited. The stopping time K is geometrically distributed, that is $E[K] = 1/q_\lambda(x,y)$. Recall that $E\left[R_\lambda^i(x)\right] = (p(x)f(x))^{-1}$, then the expected value can be derived, that is

$$E\left[T_\lambda(x,y) + T_\lambda(y,x)\right] = E\left[\sum_{i=1}^{K} R_\lambda^i(x)\right] = E\left[K\right] r_\lambda(x) = \frac{1}{q_\lambda(x,y)p(x)f(x)}.$$

The second equality is due to a simple martingale argument, see theorem 5.3 in Gut [12]. The lemma now follows by simply interchanging the order of x and y in the argument above.

Lemma 2.5 *The probability $q_\lambda(0, \lambda x/b)$ can be approximated by the probability of no ruin in a classical setting with a premium equal to λa starting in zero, that is*

$$\lim_{\lambda\to\infty} q_\lambda(0, \lambda x/b) = 1 - \mu/a.$$

Proof: First let us define two auxiliary coupled processes with identical claim process as X_t but with new drifts. The drift of the first process X'_t is,

$$p'(x) = \begin{cases} \lambda\left(a - b/\sqrt{\lambda}\right) & \text{when } x' \leq \sqrt{\lambda}, \\ p(x) & \text{otherwise.} \end{cases}$$

which implies that $p'(x) \leq p(x)$. The drift of the second coupled risk process X''_t is defined by,

$$p''(x) = \begin{cases} \lambda a & \text{when } x \geq 0, \\ p(x) & \text{otherwise.} \end{cases} \tag{5}$$

which gives that $p''(x) \geq p(x)$. All three processes start in $X_t = X'_t = X''_t = 0$, which in combinaiton with the definition of the premiums imply that,

$$X'_t \leq X_t \leq X''_t. \tag{6}$$

Let $q'_\lambda(0,y)$ denote the probability that X'_t has hit the level y on a cycle from 0 and back to 0 (in analogy with the original process X_t). The equivalent probability of the same event for X''_t is denoted $q''_\lambda(0,y)$.

The stochastic domination, Equation (6), implies a 'sandwich' inequality,

$$q'_\lambda(0,\infty) \leq q_\lambda(0, \lambda x/b) \leq q''_\lambda(0, \sqrt{\lambda}).$$

The limit of the first probability is just the probability of no ruin, that is

$$\lim_{\lambda\to\infty} q'_\lambda(0,\infty) = \lim_{\lambda\to\infty} 1 - \frac{\mu}{a - b/\sqrt{\lambda}} = 1 - \mu/a,$$

see one of the textbooks [1, 11, 20].

The process X_t'' is a classical ruin process on the positive real line, starting in 0. For a classical risk process the probability $q_\lambda''(0,\sqrt{\lambda})$ can be determined by a simple argument. The probability of no ruin is equal to the first passing level y and then have no ruin. Let τ_λ'' be the time of ruin for the process X_t'', then

$$1 - P(\tau_\lambda'' < \infty | X_0'' = 0) = (1 - P(\tau_\lambda'' < \infty | X_0'' = y)) q_\lambda''(0, y).$$

Put $y = \sqrt{\lambda}$, then by classical ruin theory we have that $P(\tau_\lambda' < \infty | X_0' = \sqrt{\lambda}) \sim C\mathrm{e}^{-\sqrt{\lambda}R}$, where R and C are some positive constants. This gives that

$$\lim_{\lambda\to\infty} q_\lambda''(0,\sqrt{\lambda}) = \lim_{\lambda\to\infty} \frac{1 - P(\tau'' < \infty | X_0'' = 0)}{1 - C\mathrm{e}^{-\sqrt{\lambda}R}} = 1 - \mu/a.$$

The conclusion now follows by the 'sandwich' inequality.

3 Main result - the asymptotic distribution of the normalized time to ruin

The time of ruin τ_λ is bounded by $T_\lambda(\lambda x/b, 0)$ We have indexed τ to stress the dependence of λ.

Theorem 3.1 *We have that*

$$\lim_{\lambda\to\infty} P\left(\frac{q_\lambda(\lambda x/b,0)T_\lambda(\lambda x/b,0)}{r_\lambda(\lambda x/b)} > t\right) = e^{-t}, t > 0.$$

Further, let $h(x) = \sup_\theta\{\theta x - g(\theta)\}$*, then we have that,*

$$\frac{r_\lambda(\lambda x/b)}{q_\lambda(\lambda x/b,0)} \sim \sqrt{\frac{2\pi\lambda}{(a-\mu)bh''(a)}} \exp\left(\frac{\lambda}{b}h(a)\right),$$

where $\sim$ *means that the quotient between the two sides converges to* 1.

*Proof:*In this proof the notation is shorted by $q = q_\lambda(\lambda x/b, 0)$, $r = r_\lambda(\lambda x/b)$ and $T = T_\lambda(\lambda x/b, 0)$, for simplicity. First, let us define the sum of recurrence times as $Z_m = \sum_{i=1}^m R_\lambda^i(\lambda x/b)$, where dependence of λ is suppressed. The probability of no visit to 0 during the m first excursions is

$$P(T > Z_m) = P\left(\frac{qT}{r} > \frac{qZ_m}{r}\right) = (1-q)^m.$$

Let $m = t/q$, then we have that

$$P\left(\frac{qT}{r} > \frac{tZ_m}{rm}\right) = (1-q)^{t/q}.$$

If we can prove the law of large numbers,

$$\lim_{m\to\infty} \frac{Z_m}{rm} \overset{P}{=} 1,$$

then

$$P\left(\frac{qT}{r} > t\right) \to e^{-t} \text{ when } q \to 0 \text{ and } m = t/q \to \infty.$$

Before we proceed, we will prove that the law of large numbers actually implies this asymptotic result. Fix $t_1 < t < t_2$ and $\varepsilon > 0$ small enough to satisfy

$$t_1(1-\varepsilon) < t_1(1+\varepsilon) < t < t_2(1-\varepsilon) < t_2(1+\varepsilon).$$

Such an ε does always exist and we write $I = (1-\varepsilon, 1+\varepsilon)$ and $m_i = t_i/q$, for $i = 1, 2$. Let us first study an upper bound constructed by the law of total probability, that is

$$\begin{aligned} P\left(\frac{qT}{r} > t\right) &\leq P\left(\frac{qT}{r} > t, \frac{Z_{m_1}}{rm_1} \in I\right) + P\left(\frac{Z_{m_1}}{rm_1} \notin I\right) \\ &\leq P\left(\frac{qT}{r} > \frac{Z_{m_1}}{rm_1}, \frac{Z_{m_1}}{rm_1} \in I\right) + P\left(\frac{Z_{m_1}}{rm_1} \notin I\right) \\ &\leq P\left(\frac{qT}{r} > \frac{Z_{m_1}}{rm_1}\right) + P\left(\frac{Z_{m_1}}{rm_1} \notin I\right). \end{aligned}$$

Then by the law of large numbers, which we prove later, we have that $\lim_{m_1\to\infty} P\left(\frac{Z_{m_1}}{rm_1} \notin I\right) = 0$ and $\lim_{m_1\to\infty} P\left(\frac{qT}{r} > \frac{tZ_{m_1}}{rm_1}\right) = \mathrm{e}^{-t_1}$.

The lower bound is constructed in a similar fashion,

$$\begin{aligned} P\left(\frac{qT}{r} > t\right) &\geq P\left(\frac{qT}{r} > t, \frac{Z_{m_2}}{rm_2} \in I\right) \\ &\geq P\left(\frac{qT}{r} > \frac{Z_{m_2}}{rm_2}, \frac{Z_{m_2}}{rm_2} \in I\right) \\ &\geq P\left(\frac{qT}{r} > \frac{Z_{m_2}}{rm_2}\right) - P\left(\frac{Z_{m_2}}{rm_2} \notin I\right) \end{aligned}$$

and $\lim_{m_1\to\infty} P\left(\frac{qT}{r} > \frac{tZ_{m_2}}{rm_2}\right) = \mathrm{e}^{-t_2}$. Therefore the convergence $\lim P\left(\frac{qT}{r} > t\right) = \mathrm{e}^{-t}$ follows by

$$\mathrm{e}^{-t_2} < \liminf P\left(\frac{qT}{r} > t\right) < \limsup P\left(\frac{qT}{r} > t\right) < \mathrm{e}^{-t_1}$$

and that t_1 and t_2 are arbitrary.

Left to show are two parts, the law of large numbers of $Z_m(rm)^{-1}$ and that $q \to 0$ as $\lambda \to \infty$. Fix $\delta > 0$ and define the interval $I(t) = [t, t+\delta]$. Now if $Z_m \in I(t)$ then the risk process is in the interval

$$\Delta(\lambda x/b) = [\lambda x/b, \lambda x/b + \delta p(\lambda x/b)] = [\lambda x/b, \lambda x/b + \delta\lambda(a-x)],$$

with a probability greater than $1 - \mathrm{e}^{-\lambda\delta}$. This leads to the renewal function

$$P(X_t \in \Delta(\lambda x/b)) = p(\lambda x/b) f_X(\lambda x/b, \lambda x/b, t)\delta + O(\delta^2) = \sum_{m=0}^{\infty} P(Z_m \in I(t)). \tag{7}$$

Let the Laplace transform of the renewal times be written

$$L(\theta, \lambda x/b) = E[\exp(-\theta R_\lambda^i(\lambda x/b))].$$

Then since $\{R_\lambda^i(\lambda x/b)\}_{i=1}^{\infty}$ are independent identically distributed we have that

$$E[\exp(-\theta Z_m)] = L(\theta, \lambda x/b)^m.$$

Hence, this and equation (7) imply

$$p(\lambda x/b)\theta \int_0^\infty e^{-\theta t} f_X(\lambda x/b, \lambda x/b, t)\mathrm{d}t \;=\; \theta \sum_{m=0}^{\infty} L(\theta, \lambda x/b)^m = \frac{\theta}{1 - L(\theta, \lambda x/b)}.$$

When we let $\theta \to 0$ then the only contribution in the integral on the left hand side is for $t = \infty$. Therefore let us substitute variables $v = t/\theta$ and use the bounded convergence theorem to see that,

$$\begin{aligned} p(\lambda x/b)\theta \int_0^\infty e^{-\theta t} f_X(\lambda x/b, \lambda x/b, t)\mathrm{d}t \;&=\; p(\lambda x/b) \int_0^\infty e^{-v} f_X(\lambda x/b, \lambda x/b, v/\theta)\mathrm{d}v \\ &\to\; p(\lambda x/b) f(\lambda(a-x)/b) \int_0^\infty e^{-v}\mathrm{d}v \\ &=\; p(\lambda x/b) f(\lambda(a-x)/b), \end{aligned}$$

when $\theta \to 0$. In this calculation we have used equation 4 and the convergence of the densities, see appendix A for a justification of this step. To conclude,

$$p(\lambda x/b) f(\lambda(a-x)/b) = \lim_{\theta \to 0} \frac{\theta}{1 - L(\theta, \lambda x/b)} = \frac{1}{L'(0, \lambda x/b)} = \frac{1}{r_\lambda(\lambda x/b)}. \tag{8}$$

We will now use this result to to show that $Z_m(rm)^{-1} \to 1$ in probability as $m = t/q \to \infty$. The idea is to show that the Laplace transform of $Z_m(rm)^{-1}$ converges to the Laplace transform of 1, hence

$$E\left[\exp\left(\theta \frac{Z_m}{rm}\right)\right] = L(\theta/rm, \lambda x/b)^m = (1 - (1 - L(\theta/rm, \lambda x/b)))^m .$$

We continue by Taylor expanding

$$1 - L(\theta/rm, \lambda x/b) = \frac{\theta L'(0, \lambda x/b)}{rm} + \mathcal{O}(m^{-2}) = \frac{\theta}{m} + \mathcal{O}(m^{-2}),$$

since $r = L'(0, \lambda x/b)$. This implies that

$$\lim_{m\to\infty} E\left[\exp\left(\theta\frac{Z_m}{rm}\right)\right] = \lim_{m\to\infty}\left(1 - \frac{\theta}{m} + \mathcal{O}(m^{-2})\right)^m = \exp(-\theta),$$

which completes the proof of the law of large numbers for $Z_m(rm)^{-1}$.

We are interested in the scale factor $r_\lambda(\lambda x/b)/q_\lambda(\lambda x/b, 0)$ and will use the relationship in lemma 2.4, to find an estimate. The expected renewal time at zero is $r_\lambda(0) = (\lambda a f(\lambda a))^{-1}$. The saddlepoint approximation of the density function gives us an estimate of the expected renewal time,

$$r_\lambda(0) = \sqrt{\frac{2\pi\lambda}{bh''(a)}}\exp\left(\frac{\lambda}{b}h(a)\right).$$

The probability $q_\lambda(0, \lambda x/b)$ can be approximated by the probability of no ruin in a classical setting with a premium equal to λa starting in zero. It is well known that this is equal to $1 - \mu/a$, see one of the textbooks [1, 11, 20]. In the limit $\lim_{\lambda\to\infty} q_\lambda(0, \lambda x/b) = 1 - \mu/a$. This implies that the quotient

$$\frac{r}{q} = \frac{r_\lambda(\lambda x/b)}{q_\lambda(0, \lambda x/b)} = \sqrt{\frac{2\pi\lambda}{(a-\mu)bh''(a)}}\exp\left(\frac{\lambda}{b}h(a)\right)$$

With this tool it is easy to check that $q \to 0$ as $\lambda \to \infty$, since

$$\begin{aligned} q_\lambda(\lambda x/b, 0) &= q_\lambda(0, \lambda x/b)\frac{r_\lambda(\lambda x/b)}{r_\lambda(0)} \\ &\leq C\frac{f(0)}{f(\lambda x)} \\ &\leq C\exp\left(\frac{\lambda}{b}(h(a) - h(a - bx))\right) \to 0, \text{ when } \lambda \to \infty, \end{aligned}$$

where $C > 0$ is a constant. The convergence to zero is true since the function $h(x)$ is minimal at μ, that is $a - bx = \mu$ and increasing for $\mu < x \leq a$.

Theorem 3.2 *We have that the time of ruin τ_λ is asymptotically distributed as*

$$\lim_{\lambda\to\infty} P\left(\frac{q_\lambda(\lambda x/b, 0)\tau_\lambda}{r_\lambda(\lambda x/b)} > t\right) = e^{-t}, t > 0.$$

Further, let $h(x) = \sup_\theta\{\theta x - g(\theta)\}$ *then we have that,*

$$\frac{r_\lambda(\lambda x/b)}{q_\lambda(\lambda x/b,0)} \sim \sqrt{\frac{2\pi\lambda}{(a-\mu)bh''(a)}}\exp\left(\frac{\lambda}{b}h(a)\right),$$

where $\sim$ *denotes that the quotient between the two sides converges to* 1.

*Proof:*Let us define the difference $T_\lambda(X_\tau,0) = T_\lambda(\lambda x/b,0) - \tau$. Conditioned on $X_\tau = x$ we have that $E\left[T_\lambda(x,0)|X_\tau = x\right] \leq |x|\lambda^{-1}(a-\mu)^{-1}$, since the premium is greater than λa, for $X_t < 0$. Therefore we have that

$$E\left[T_\lambda(X_\tau,0)\right] \leq \frac{E[|X_\tau|]}{\lambda(a-\mu)}.$$

Let us estimate the expected value of the asolute size of the overshoot. We define the event

$$\mathbf{A}_k = \left\{\sum_{i=0}^{k-1} R^i_\lambda(X_0) < T_\lambda(X_0,0) \leq \sum_{i=0}^{k} R^i_\lambda(X_0)\right\},$$

where $R^0_\lambda(X_0) = 0$. In each of these events the process X_t starts afresh in X_0. Therefore, we have that $P(X_\tau \leq x|\mathbf{A}_k) = P(X_\tau \leq x|\mathbf{A}_1)$. Now the Bayes' formula implies

$$\begin{aligned} P(X_\tau \leq x) &= \sum_{k=1}^{\infty} P(X_\tau \leq x|\mathbf{A}_k)P(\mathbf{A}_k) \\ &= P(X_\tau \leq x|\mathbf{A}_1)\sum_{k=1}^{\infty} P(\mathbf{A}_k) \\ &= P(X_\tau \leq x|\mathbf{A}_1). \end{aligned}$$

This conditioned probability can be rewritten by the law of total probability. This is done by splitting the sample space according to where the process was just before ruin, that is $X_t \in \mathrm{d}u$ and $t \in \mathrm{d}t$ hence,

$$P(X_\tau \leq x|\mathbf{A}_1) = \int_{t\geq 0}\int_{x\leq 0} \lambda P(X_t \in du, \tau > t, R^1_\lambda(X_0) > t|\mathbf{A}_1)\bar{F}(|x|+u)\mathrm{d}t\mathrm{d}u.$$

The observations $\bar{F}(|x|+u) \geq bF(|x|)$ and the formula for expected value of $R^i_\lambda(X_0)$ give us

$$\begin{aligned} &\int_{t\geq 0}\int_{x\leq 0} \lambda P(X_t \in du, \tau > t, R^1_\lambda(X_0) > t|\mathbf{A}_1)\bar{F}(|x|+u)\mathrm{d}t\mathrm{d}u \leq \\ &\leq \lambda\bar{F}(|x|)\int_{t\geq 0} P(R^1_\lambda(X_0) > t|\mathbf{A}_1)\mathrm{d}t \\ &= \lambda\bar{F}(|x|)r_\lambda(X_0). \end{aligned}$$

The expected value is therefore,

$$\begin{aligned} E[|X_\tau|] &\leq \int_{x\leq 0} P(X_\tau \leq x)\mathrm{d}x \\ &\leq \lambda r_\lambda(X_0)\int_{x\leq 0} \bar{F}(x)\mathrm{d}x \\ &\leq \lambda\mu r_\lambda(X_0). \end{aligned}$$

A simple application of the Markov inequailty gives

$$P\left(T_\lambda(X_\tau, 0) > \varepsilon\right) \leq \frac{E[|X_\tau|]}{\lambda(a-\mu)\varepsilon}.$$

Fix $\varepsilon > 0$. The inequality just above and the bound of $E[|X_\tau|]$ give that

$$P\left(\frac{q_\lambda(0, \lambda x/b)T_\lambda(X_\tau, 0)}{r_\lambda(0)} > \varepsilon\right) \leq \frac{q_\lambda(0, \lambda x/b)r_\lambda(X_0)\mu}{r_\lambda(0)(a-\mu)\varepsilon}.$$

Since $X_0 = \lambda x/b$ we know that

$$\lim_{\lambda\to\infty} \frac{q_\lambda(0, \lambda x/b)r_\lambda(\lambda x/b)}{r_\lambda(0)\varepsilon} \to 0,$$

exponentially, fast see the end of the proof of theorem 3.1. The conclusion follows since ε was arbitrary, that is

$$\lim_{\lambda\to\infty} \frac{q_\lambda(0, \lambda x/b)T_\lambda(X_\tau, 0)}{r_\lambda(0)} \overset{P}{=} 0.$$

A Convergence of the density

This section is dedicated to proving that the convergence of $g(\theta, t)$ to $g(\theta)$ when t goes to infinity also implies convergence of the density. Before proceeding any further we will set some notation. Let

$$V_t = \int_0^t \exp(-bu)\mathrm{d}S_u \text{ and } V_\infty = \int_0^\infty \exp(-bu)\mathrm{d}S_u.$$

The characteristic functions of theses processes are

$$\begin{aligned} E\left[\mathrm{e}^{i\theta V_t}\right] &= \exp\left(\frac{\lambda}{b}\left(g(i\theta) - g\left(i\theta\mathrm{e}^{-bt}\right)\right)\right) = \exp\left(\frac{\lambda}{b}g(i\theta, t)\right), \\ E\left[\mathrm{e}^{i\theta V_\infty}\right] &= \exp\left(\frac{\lambda}{b}g(i\theta)\right), \end{aligned}$$

where $g(i\theta, t) = g(i\theta) - g\left(i\theta\mathrm{e}^{-bt}\right)$.

Lemma A.1 *If* $1 < \beta < \lambda/b$*, then it holds true that*

$$\left|E\left[\mathrm{e}^{i\theta V_\infty}\right]\right| = \exp\left(\frac{\lambda}{b}\mathbf{Re}\{g(i\theta)\}\right) \le \min(C\theta^{-\beta}, 1),$$

where $C > 0$ *is some constant. Therefore, the characteristic function is in* $\mathcal{L}^1$ *and* V_∞ *has a density.*

*Proof:*We have that

$$\mathbf{Re}\{g(i\theta)\} = -\int_0^\infty \frac{1-\cos(\theta x)}{x}\bar{F}(x)\mathrm{d}x,$$

see Lemma 2.1. The probability $\bar{F}(x)$ is decreasing and bounded by one. Therefore, it is possible to choose $\gamma > 0$ such that $\lambda b^{-1}\bar{F}(\gamma) = \beta < \lambda b^{-1}$ and

$$\begin{aligned}
\frac{\lambda}{b}\mathbf{Re}\{g(i\theta)\} &\le -\beta\int_0^\gamma \frac{1-\cos(\theta x)}{x}\mathrm{d}x \\
&= \{\text{Variable substitution } y = \theta x,\} \\
&= -\beta\int_0^{\gamma\theta} \frac{1-\cos(y)}{y}\mathrm{d}y \\
&= \{\text{Use } \theta\gamma > 1,\} \\
&\le -\beta\int_1^{\gamma\theta} \frac{1-\cos(y)}{y}\mathrm{d}y \\
&= -\beta\log(\theta\gamma) + \beta\int_1^{\gamma\theta}\frac{\cos(y)}{y}\mathrm{d}y.
\end{aligned}$$

The last integral is bounded and therefore can we write $\exp(\mathbf{Re}\{g(i\theta)\}) \le \min\left(1, C\theta^{-\beta}\right)$, where $C > 0$ is some constant.

We have already established the weak convergence of the distribution of V_t to that of V_∞. In order to show the corresponding convergence of the densities, we argue as follows: The process V_t is compound Poisson distributed defined by

$$E[e^{i\theta Y_t}] = \exp\left(\lambda t\int_0^\infty \left(\mathrm{e}^{i\theta x} - 1\right) l(x,t)\mathrm{d}x\right),$$

with Lévy density

$$l(x,t) = \frac{\bar{F}(x) - \bar{F}(xe^{bt})}{btx}.$$

The total mass of $l(x,t)$ is easily checked to be equal to 1, by Lemma 2.1. Therefore, the density of V_t is

$$f(x,t) = e^{-\lambda t}\sum_{n=0}^\infty \frac{(\lambda t)^n l^{n*}(x,t)}{n!},$$

where $l^{n*}(x,t)$ is the n:th convolution of $l(x,t)$. The first two terms, $\mathrm{e}^{-\lambda t}$ and $\mathrm{e}^{-\lambda t}(\lambda t)l(x,t)$, converge to 0 as $t \to \infty$.

Therefore, we have to show that the sum from 2 to infinity converges to the density of V_∞. The densities can be written in terms of their characteristic functions by the Fourier inversion formula. Therefore, the convergence can be established if the characteristic function converges in $\mathcal{L}^1$.

The characteristic function of $l(x,t)$ is

$$\varphi(i\theta,t) = \int_0^\infty \mathrm{e}^{i\theta x} l(x,t)\mathrm{d}x,$$

and is related to the Lévy exponent by

$$\lambda t(\varphi(i\theta,t) - 1) = \lambda b^{-1} g(i\theta,t),$$

by the definition of the characteristic function of V_t. Let us put

$$\begin{aligned}
\Delta(i\theta,t) &= e^{-\lambda t} \sum_{n=2}^{\infty} \frac{(\lambda t)^n \varphi^n(i\theta,t)}{n!} - \exp\left(\lambda b^{-1} g(i\theta)\right) \\
&= \exp(\lambda b^{-1} g(i\theta,t)) - \mathrm{e}^{-\lambda t} - \lambda t \mathrm{e}^{-\lambda t}\varphi(i\theta,t) - \exp\left(\lambda b^{-1} g(i\theta)\right)
\end{aligned}$$

Therefore, we have

$$f(x,t) - \lambda t \mathrm{e}^{-\lambda t} l(x,t) - f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\theta x}\Delta(i\theta,t)\mathrm{d}\theta.$$

Hence, we need to show that

$$\int_0^\infty |\Delta(i\theta,t)|\mathrm{d}\theta \to 0, \text{ as } \to t \to \infty. \tag{9}$$

This also implies that $f(x,t)$ is bounded, since $\lambda \mathrm{e}^{-\lambda t} l(x,t)$ and $f(x)$ are bounded. Later on, the two following Lemmas will be useful.

Lemma A.2 *Let the characteristic function of the claims be denoted by*

$$\varphi(is) = \int_0^\infty \mathrm{e}^{isx}\mathrm{d}F(x).$$

Then characteristic function of V_t and V_∞ can be written as

$$\begin{aligned}
g(i\theta) &= \int_0^\theta \frac{\varphi(is) - 1}{s}\mathrm{d}s \\
g(i\theta,t) &= \int_{\theta \mathrm{e}^{-bt}}^\theta \frac{\varphi(is) - 1}{s}\mathrm{d}s.
\end{aligned}$$

*Proof:*The first order derivative of the cumulant generating function is by Lemma 2.1,

$$\begin{aligned}\frac{\mathrm{d}g(i\theta)}{\mathrm{d}\theta} &= \int_0^\infty i\mathrm{e}^{i\theta x}\bar{F}(x)\mathrm{d}x \\ &= \int_0^\infty \int_x^\infty i\mathrm{e}^{i\theta x}\mathrm{d}F(y)\mathrm{d}x \\ &= \int_0^\infty \int_0^y i\mathrm{e}^{i\theta x}\mathrm{d}x\mathrm{d}F(y) \\ &= \int_0^\infty \frac{e^{i\theta x}-1}{\theta}\mathrm{d}F(y) \\ &= \frac{\varphi(i\theta)-1}{\theta}.\end{aligned}$$

This implies that

$$\begin{aligned}g(i\theta) &= \int_0^\theta g'(is)\mathrm{d}s \\ &= \int_0^\theta \frac{\varphi(is)-1}{s}\mathrm{d}s.\end{aligned}$$

Furthermore, the cumulant generating function in finite time is

$$g(i\theta,t) = \int_{\theta\mathrm{e}^{-bt}}^\theta \frac{\varphi(is)-1}{s}\mathrm{d}s.$$

Lemma A.3 *We have that*

$$|g(i\theta)| \leq \mu|\theta|$$

*Proof:*Use Lemma 2.1 to substitute $g(i\theta)$, that is,

$$\begin{aligned}|g(i\theta)| &= \left|\int_0^\infty \frac{\mathrm{e}^{i\theta x}-1}{x}\bar{F}(x)\mathrm{d}x\right| \\ &\leq \int_0^\infty |\theta|\bar{F}(x)\mathrm{d}x = \mu\theta.\end{aligned}$$

Let us return to the integral of equation (9) and split the integration into three intervals

$$I_1 = [0,\mathrm{e}^{bt/2}), \quad I_2 = [\mathrm{e}^{bt/2},\mathrm{e}^{5bt/4}), \quad I_3 = [\mathrm{e}^{5bt/4},\infty).$$

It is on the first interval I_1 possible to bound the function

$$\begin{aligned}|\Delta(i\theta,t)| &\le \left|\exp\left(\lambda b^{-1}g(i\theta)\right)\right|\left|\exp\left(\lambda b^{-1}g(i\theta e^{-bt})\right)-1\right| + e^{-\lambda t}(1+\lambda t)\\ &= \{\text{Use Lemmas A.1 and A.3}\}\\ &\le \min(1, C\theta^{-\beta})\theta\mu\left(\exp(\lambda\mu e^{-bt/2})-1\right) + e^{-\lambda t}(1+\lambda t),\end{aligned}$$

where $C>0$ is some constant. Therefore if $\lambda/b>2$, then $\beta>2$ and then integral over the interval I_1 is integrable, by the derived bound. Hence the first integral converges to 0 as $t\to\infty$.

Let us now integrate over the interval I_2. According to Lemma A.1 is it possible to bound $|e^{\lambda g(i\theta,t)}| \le \min(1, C\theta^{-\beta})$, where $\beta<\lambda/b$. Furthermore,

$$\begin{aligned}\mathbf{Re}\{g(i\theta,t)\} &= -\int_{\theta e^{-bt}}^{\theta}\frac{1-\mathbf{Re}\{\varphi(is)\}}{s}\mathrm{d}s\\ &\le -\int_{e^{bt/4}}^{\theta}\frac{1-\mathbf{Re}\{\varphi(is)\}}{s}\mathrm{d}s,\end{aligned}$$

since $\theta\le e^{5bt/4}$. The characteristic function $\varphi(s)\to 0$ as $s\to\infty$ according to Riemann-Lebesgues lemma if $F'(x)$ is continuous. This gives that

$$\begin{aligned}\lambda b^{-1}\mathbf{Re}\{g(i\theta,t)\} &\le -\beta\int_{e^{bt/4}}^{\theta}\frac{\mathrm{d}s}{s}\\ &= -\beta\log(\theta e^{-bt/4}),\end{aligned}$$

if $\beta<\lambda b{-}1$. Then we have that

$$\left|\exp\left(\lambda b^{-1}g(i\theta,t)\right)\right| \le \theta^{-\beta}e^{\beta bt/4},$$

which is integrable if $\beta>1$. Thus, the integral over I_2 is,

$$\begin{aligned}\int_{e^{bt/2}}^{e^{5bt/4}}\left|\exp\left(\lambda b^{-1}g(i\theta,t)\right)\right|\mathrm{d}\theta &\le \int_{e^{bt/2}}^{\infty}\exp\left(\beta bt/4\right)\theta^{-\beta}\mathrm{d}\theta\\ &= \frac{\exp(-\beta bt/4+bt/2)}{\beta-1}\to 0,\end{aligned}$$

as $t\to\infty$, if $\beta>2$. The contribution from $e^{-\lambda t}(1+\lambda t)\to 0$ if $\lambda>5b/4$.

The characteristic function $\varphi(is)$ could have a discrete part (the remaining part has a continuous density) defined by $\sum_{k=1}^{\infty}e^{isy_k}p_k$, where p_k is the probability of a claim of size y_k ($\sum_k p_k \le 1$). Then, this allow us to bound $\int_{e^{bt/4}}^{\theta}\frac{\mathbf{Re}\{\varphi(is)\}}{s}\mathrm{d}s$. This is true since

$$\int_{e^{bt/4}}^{\theta}\frac{\cos(y_k)}{s}\mathrm{d}s = \left.\frac{\sin(y_k s)}{y_k s}\right|_{e^{bt/4}}^{\theta} + \int_{e^{bt/4}}^{\theta}\frac{\sin(y_k)}{y_k s^2}\mathrm{d}s,$$

is bounded uniformly in y_k as $\theta \to \infty$ (It is assumed that the claims y_k are bounded away from 0).

It only remains to estimate the integral over the last interval I_3. In $\Delta(i\theta, t)$ the contribution form $\exp(\lambda b^{-1} g(i\theta))$ converges to 0 as θ goes to infinity since it is integrable, Lemma A.1. The remaining part is

$$\mathrm{e}^{-\lambda t} \int_{\mathrm{e}^{5bt/4}}^{\infty} |\exp(\lambda t \varphi(i\theta, t)) - 1 - \lambda t \varphi(i\theta, t)| \, \mathrm{d}\theta.$$

Now, by Lemma A.2, we have that

$$\lambda t \varphi(i\theta, t) = \frac{\lambda}{b} \int_{\theta \mathrm{e}^{-bt}}^{\theta} \frac{\varphi(is)}{s} \mathrm{d}s.$$

This can be estimated when θ is large, even if $F(y)$ is a mixture of a discrete distribution and one with a smooth density.

Lemma A.4 *If $\varphi(is)$ is the characteristic function of a mixture of a discrete part and a continuous part, where $F''(y)$ is absolute integrable, then*

$$|\lambda t \varphi(i\theta, t)| \leq C \frac{\lambda \mathrm{e}^{bt}}{b|\theta|},$$

where $C > 0$ is some constant.

Proof: Take first one of the contributions from the discrete part, e^{isy_k}. By partial integration we get

$$\int_{\theta \mathrm{e}^{-bt}}^{\theta} \frac{\exp(isy_k)}{s} \mathrm{d}s = \left. \frac{\exp(isy_k)}{isy_k} \right|_{\theta \mathrm{e}^{-bt}}^{\theta} + \int_{\theta \mathrm{e}^{-bt}}^{\theta} \frac{\exp(isy_k)}{is^2 y_k} \mathrm{d}s,$$

which gives us the bound

$$\left| \int_{\theta \mathrm{e}^{-bt}}^{\theta} \frac{\exp(isy_k)}{s} \mathrm{d}s \right| \leq C \frac{\mathrm{e}^{bt}}{\theta y_k} \leq C \frac{\mathrm{e}^{bt}}{\theta},$$

since the y_k's are bonded away from zero.

For the continuous part we have analogously by partial integration

$$\begin{aligned} \int_0^{\infty} \exp(isy) F'(y) \mathrm{d}y &= \left. \frac{\exp(is)}{is} F'(y) \right|_0^{\infty} - \int_0^{\infty} \frac{\exp(isy)}{is} F''(y) \mathrm{d}y \\ &= -\frac{1}{is} \left(F'(0) + \int_0^{\infty} \exp(isy) F''(y) \mathrm{d}y \right), \end{aligned}$$

so

$$|\varphi(is)| \leq \frac{C}{|s|},$$

if $\int_0^\infty |F''(y)|dy < \infty$. Hence, also in the continuous part we have that,

$$\int_{\theta e^{-bt}}^{\theta} \frac{\varphi(is)}{s} \mathrm{d}s \leq \int_{\theta e^{-bt}}^{\theta} \frac{C}{s^2} \mathrm{d}s \leq \frac{C e^{bt}}{\theta}.$$

Since $\theta \geq e^{5bt/4}$ in I_3 we see that

$$|\lambda t \varphi(i\theta, t)| \leq \frac{\lambda}{b} C e^{-bt/4} \to 0$$

when $t \to \infty$. This implies that

$$e^{-\lambda t} |e^{\lambda t \varphi(i\theta,t)} - 1 - \lambda t \varphi(i\theta, t)| \leq C e^{-\lambda t} |\varphi(i\theta, t)|^2 \leq C \left(\frac{\lambda e^{bt}}{b\theta} \right)^2.$$

The integral of this can hence be bounded by

$$\int_0^\infty C \left(\frac{\lambda e^{bt}}{b\theta} \right)^2 \mathrm{d}\theta = C \left(\frac{\lambda}{b} \right)^2 \exp(3bt/4 - \lambda t) \to 0$$

if $\lambda > 3b/4$.

Hence, this implies the convergence of the density to the density of the limit process. The following lemma summarizes our findings in this appendix.

Lemma A.5 *If the continuous part of the distribution full fills $\int_0^\infty |F''(y)|dy < \infty$, and there is a $\varepsilon > 0$ such that $y_k \geq \varepsilon$ then*

$$\lim_{t \to \infty} f(x, t) = f(x),$$

and $f(x, t)$ is bounded.

B Saddle point approximation of the density

The distribution of the time of ruin is connected to the density of Y_∞. We need therefore an approximation of the density for values far from the mean. Saddle point approximation is the tool we will use.

In large deviation theory the exponential tilt of a measure, also called the Esscher transform, is defined as

$$f_\alpha(y) = f(y)\mathrm{e}^{\alpha y - g(\alpha)}.$$

Under the tilted measure the Lévy density is $\mathrm{e}^{\alpha y}\bar{F}(y)$ and the cumulant generating function $\lambda b^{-1} g_\alpha(\theta) = \lambda b^{-1}(g(\theta+\alpha) - g(\theta))$.

To carry this through we need a similar result as Lemma A.1

Lemma B.1 *If* $0 < \eta < 1$*, then it holds true that*

$$\left|\mathrm{e}^{g_\alpha(i\theta)}\right| = \exp\left(\frac{\lambda}{b}\mathbf{Re}\{g_\alpha(i\theta)\}\right) \leq \min(C\theta^{-\eta}, 1),$$

where $C > 0$ *is some constant. Therefore, the characteristic function is in* $\mathcal{L}^1$ *if* $\lambda\eta/b > 1$ *and* V_∞ *has a density.*

*Proof:*We have that

$$\mathbf{Re}\{g_\alpha(i\theta)\} = -\int_0^\infty \frac{1-\cos(\theta x)}{x}\mathrm{e}^{\alpha x}\bar{F}(x)\mathrm{d}x,$$

see Lemma 2.1. We have that $\mathrm{e}^{\alpha x}\bar{F}(x)/x \geq \bar{F}(x)/x$ for $\alpha > 0$. Therefore, trivially it is possible to choose $\gamma > 0$ such that $\bar{F}(\gamma) = \eta < 1$. When $\alpha \leq 0$ then it is possible to choose $\gamma > 0$ such that $\mathrm{e}^{\alpha x}\bar{F}(\gamma) = \eta < 1$, since then $\mathrm{e}^{\alpha}\bar{F}(x)$ is decreasing. Hence, for a suitable γ

$$\begin{aligned}
\mathbf{Re}\{g(i\theta)\} &\leq -\eta\int_0^\gamma \frac{1-\cos(\theta x)}{x}\mathrm{d}x \\
&= \{\text{Variable substitution } y = \theta x,\} \\
&= -\eta\int_0^{\gamma\theta} \frac{1-\cos(y)}{y}\mathrm{d}y \\
&= \{\text{Use } \theta\gamma > 1,\} \\
&\leq -\eta\int_1^{\gamma\theta} \frac{1-\cos(y)}{y}\mathrm{d}y \\
&= -\eta\log(\theta\gamma) + \eta\int_1^{\gamma\theta}\frac{\cos(y)}{y}\mathrm{d}y.
\end{aligned}$$

The last integral is bounded and therefore we can write $\exp(\mathbf{Re}\{g(i\theta)\}) \leq \min(1, C\theta^{-\eta})$, where $C > 0$ is some constant.

The characteristic function of the tilted density is

$$\exp(\lambda b^{-1} g_\alpha(i\theta)) = \exp(\lambda b^{-1}(g(i\theta+\alpha) - g(\alpha)).$$

Further the the expected value and variance under the tilted measure are given by

$$E_\alpha[V_\infty] = \frac{\lambda g'(\alpha)}{b} \text{ and } Var_\alpha(V_\infty) = \frac{\lambda g''(\alpha)}{b}.$$

If we choose α such that $g'(\alpha) = x$ then $f_\alpha(\lambda x/b)$ can be approximated by the Central Limit Theorem, when $\lambda/b \to \infty$. To be more concrete, the density is given by

$$f(\lambda x/b) \quad = \quad \exp\left(-\frac{\alpha\lambda x}{b} + \frac{\lambda}{b} g(\alpha)\right) f_\alpha(\lambda x/b).$$

The tilted density can be rewritten by the inversion formula

$$\sqrt{\frac{\lambda}{b}} f_\alpha(\lambda x/b) \quad = \quad \frac{1}{2\pi}\sqrt{\frac{\lambda}{b}} \int_{-\infty}^{\infty} \exp\left(-i\frac{\lambda x}{b} + \frac{\lambda}{b}(g(i\theta+\alpha) - g(\alpha)\right) \mathrm{d}\theta.$$

Taylor expanding the exponent and using that $x = g'(\alpha)$ we get

$$-i\frac{\lambda x}{b} + \frac{\lambda}{b}\left(g(i\theta+\alpha) - g(\alpha)\right) = -\frac{\lambda\theta^2}{2b}(g''(\alpha) + O((i\theta)^3)).$$

The remainder is of no consequence when $\lambda/b \to \infty$, since

$$\exp(g_\alpha(i\theta)) \in \mathcal{L}^1$$

see Theorem XV.5.2 in Feller II [10].

Therefore, after a variable substitution, we have that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{u^2}{2g''(\alpha)}} \mathrm{d}u = \frac{1}{\sqrt{2\pi g''(\alpha)}}.$$

Collecting the pieces in the argument above, we see that $h(x) = \alpha x - g(\alpha)$, where $x = g'(\alpha)$, which is the Legendre transform of $g(\alpha)$,

$$h(x) = \sup_\alpha (x\alpha - g(\alpha)).$$

The large deviation estimate is hence given by

$$f(\lambda x/b) \sim \frac{\exp\left(-\frac{\lambda}{b}h(x)\right)}{\sqrt{2\pi g''(\alpha)}}\sqrt{\frac{b}{\lambda}},$$

in the sense that the ratio of the two terms converges to 1 as $\lambda \to \infty$.

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