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Abstract

We derive the Black-Scholes formula under weak assumptions. We also discuss how the assumptions fit real data, in particular how the fact that markets are not complete affects the portfolio value.

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1 Introduction

It is well-known that the option valuation formula of Black and Scholes is valid for assets whose prices follow the Black-Scholes model. It is also known that stock prices deviate significantly from this model. The formula works however fairly well in practice so there are reasons to believe that it can be derived without assuming the Black-Scholes model. We shall here give sufficient conditions for this. When doing this we try to avoid to assume more than necessary, so it is believed that these conditions are fairly weak. We also discuss how realistic these assumptions are.

Consider a portfolio which at time t consists of c_t EUR in cash and a number a_t of an asset whose price is S_t. The portfolio is rebalanced at time $\tau_0, \tau_1, \tau_2, ...$ by selling or buying assets. In the original derivation of the Black-Scholes formula (Black and Scholes (1973)) the authors assume that $\Delta\tau_k = \tau_k - \tau_{k-1}$, $k = 1, 2, ...$, are deterministic and the goal of the trading is to get the portfolio value $f(S_T)$ at a given time T, (with target functions of the form $f(s) = \max(0, s - K)$). The relative prices $S_{\tau_k}/S_{\tau_{k-1}}$ are then random.

We shall instead trade when the relative prices reaches fixed levels. In this case $\tau_1, \tau_2, ...$ are random and the goal of the trading is to get the portfolio value $f(S_{\tau_n})$ at time τ_n, where n is given in advance. This is done in Section 2.

In Section 3 we consider the limiting case when the portfolio is rebalanced continuously.

Real markets are rarely complete. In Section 4 we discuss how this fact affects the portfolio value.

So far the number of trades, n, has been deterministic. In Section 5 we shall instead consider a fixed clock time horizon, T. The number of trades will then be random, and hence also the portfolio value. We show that this randomness is negligible, provided we trade continuously.

We also show in this section that the following holds in a certain sense: The alternative to the Black-Scholes formula is not another formula, but no formula at all.

In the present paper we do not specify any particular model, but consider a class of models. Many specific alternative models to the Black-Scholes model have been proposed. We shall here only mention two: Eberlein and Keller (1995), and Barndorff-Nielsen (1998). These markets are however incomplete. An additional argument is therefore necessary in order to obtain an option pricing formula. Eberlein and Keller used the method of Gerber and Shiu (1996), and Bladt and Rydberg (1998) suggested another method and derived a formula assuming the above mentioned model of Barndorff-Nielsen. In contrast to our results these formulas do not coincide with the formula of Black and Scholes.

2 The Cox, Ross and Rubinstein model

By considering the present values of the asset and the portfolio we can and shall assume that the interest rate equals zero.

We shall choose to trade the asset at the levels of the binomial tree

$$
\begin{array}{ccccc}
0 & \tau_1 & \tau_2 & & \tau_n \\
 & & & & S_0u^n \\
 & & & \cdot & \\
 & & \cdot & & \\
 & & S_0u^2 & & \\
 & S_0u & & & \\
S_0 & & S_0ud & & \\
 & S_0d & & & \\
 & & S_0d^2 & & \\
 & & \cdot & & \\
 & & & \cdot & \\
 & & & & S_0d^n
\end{array}
$$

Here u and d are given positive numbers such that $d < 1 < u$. Therefore let $\tau_0 = 0$ and define τ_k for $k = 1, 2, ...$ as the first time after τ_{k-1} for which

$$S_{\tau_k}/S_{\tau_{k-1}} \geq u \text{ or } S_{\tau_k}/S_{\tau_{k-1}} \leq d. \tag{2.1}$$

The numbers $\tau_1, \tau_2, ...$ are thus not known in advance but determined by the movements of the price of the asset.

In order to be able to proceed we need an assumption:

Liquidity assumption *The asset can be traded in sufficiently large amounts at every level of the binomial tree.*

This assumption thus implies that we have equality in the inequalities (2.1).

Let $F_k(s)$ stand for the value of the portfolio at time τ_k when $S_{\tau_k} = s$. Then $F_n(s) = f(s)$, and it follows in the same way as in Cox, Ross and Rubinstein

(1979) that

$$F_{k-1}(s) = qF_k(su) + (1-q)F_k(sd), \tag{2.2}$$

for $k = n, n-1, ..., 0$. Here $q = \frac{1-d}{u-d}$. Hence

$$F_k(s) = M_f(n-k, s), \text{ where } M_f(r, s) = \sum_{i=0}^{r} \binom{r}{i} q^i (1-q)^{r-i} f(su^i d^{r-i}), \tag{2.3}$$

It also follows that the number of assets at time t equals $a_{k-1}(S_{\tau_{k-1}})$ for $\tau_{k-1} < t < \tau_k$, where

$$a_{k-1}(s) = \frac{F_k(su) - F_k(sd)}{s(u-d)}, \tag{2.4}$$

The amount of cash thus equals $C_{k-1}(S_{\tau_{k-1}})$ for $\tau_{k-1} < t < \tau_k$,, where

$$C_k(s) = F_k(s) - a_k(s)s. \tag{2.5}$$

In order to rebalance the portfolio at time τ_k we have to buy $a_k(S_{\tau_k}) - a_{k-1}(S_{\tau_{k-1}})$ assets. We thus need a second assumption:

Divisibility assumption *The asset can be divided and traded in sufficiently small parts.*

3 An asymptotic approximation

From now on we shall assume that $d = 1/u$ and write $u = e^\delta$. We shall approximate the portfolio value when $\delta \to 0$ and $n \to \infty$. Both the liquidity and the divisibility assumption will be violated when δ is small, but this will serve as an asymptotic approximation, and will be the link to the Black-Scholes formula.

Note that we have the following representation for the portfolio value at time τ_{n-r}

$$M_f(r, s) = Ef(su^{X_r} d^{r-X_r}) = Ef(se^{Y_r}), \tag{3.1}$$

Here X_r is a binomial random variable having parameters r and q, and $Y_r = \delta(2X_r - r)$. Note that this does not imply that we have made any assumptions about the stochastics of asset prices. This is just a mathematical identity.

We have

$$E[Y_r] = \delta r(2q-1) = -\frac{r\delta^2}{2}(1+O(\delta^2)),$$

$$\text{and} \tag{3.2}$$

$$\operatorname{Var}(Y_r) = \delta^2 4rq(1-q) = r\delta^2(1+O(\delta^2)).$$

The random variable Y_r therefore is asymptotically normally distributed with mean $-\frac{v}{2}$ and variance v as $\delta^2 r \to v$. Therefore one can expect that

$$M_f(r,s) \to I_f(v,s), \tag{3.3}$$

and that the number of assets at time t, $\tau_{n-r-1} < t < \tau_{n-r}$,

$$a_{n-r-1}(s) = \frac{M_f(r,se^{\delta}) - M_f(r,se^{-\delta})}{s(e^{\delta}-e^{-\delta})} \to \frac{d}{ds}I_f(v,s). \tag{3.4}$$

Here

$$I_f(v,s) = \int_{-\infty}^{\infty} f(se^{-\frac{v}{2}+\sqrt{v}z})\phi(z)dz \text{ where } \phi(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}. \tag{3.5}$$

The next theorem gives exact conditions under which these approximations hold.

Theorem 1 *Assume that f is continuous and that there are positive constants A and B such that*

$$|f(s)| \le A + Bs \tag{3.6}$$

for all $s > 0$. If $\delta \to 0$ and $r \to \infty$ in such a way that $\delta^2 r \to v > 0$, Then (3.3) and (3.4) hold. The convergence (3.3) is uniform in s on compact subsets of $(0,\infty)$.

See the appendix for a proof.

Note that the condition (3.6) is satisfied if f is the target function of a linear combination of calls and puts.

It follows from the proof that (3.3) holds also when $v = 0$, and it can be shown that the convergence (3.4) also holds in this case under the additional assumption that f is differentiable at s.

If one makes the substitution $y = \sqrt{v}z - \ln s$ in (3.5), differentiate with respect to s under the integral sign, and substitute back one gets the following expression for the number of assets

$$\frac{d}{ds}I_f(v,s) = -\frac{1}{s\sqrt{v}}\int_{-\infty}^{\infty} f(se^{-\frac{v}{2}+\sqrt{v}z})\phi'(z)dz. \tag{3.7}$$

4 Incomplete markets

Neither the liquidity nor the divisibility assumption will be met exactly in practice. The divisibility assumption will hold with good approximation for large stock portfolios. The liquidity assumption consists of two parts: 1 Sufficiently many asset can be traded without affecting the price. 2 Every level which is passed will be visited. That is, the market is complete.

We shall in this section discuss how incompleteness of the market will affect the portfolio value.

In order to discuss the behaviour of asset prices we shall as an example take the development of the stock price (last sale) of Ericsson (ERICY) on the NASDAQ Stock Market during one day (December 7, 1998) The price that was around \$ 28.5 moved with jumps. About 75% of the jumps were $\$\frac{1}{16}$ up or down and the remainder were mainly $\$\frac{1}{8}$ up or down. A few were a slightly bigger multiple of $\$\frac{1}{16}$.

We shall assume that asset prices move by a factor $e^{x\epsilon}$, where $x = \pm 1, \pm 2, ...$, and where ϵ is a small number. Also $\delta = h\epsilon$, where h is a positive integer. In the above example $\epsilon = 0.002$, and if $\delta = 4\%$, say, then $h = 20$.

Let s_k denote the asset price at time τ_k, and $\bar{s}_k$ the level in the binomial tree the price reached or passed at τ_k. Then $s_k = \bar{s}_k e^{y_k \epsilon}$, where $y_k = 0, \pm 1, \pm 2, ...$, and $y_k \geq 0$ if $\bar{s}_k = \bar{s}_{k-1}e^{\delta}$ and $y_k \leq 0$ if $\bar{s_k} = \bar{s}_{k-1}e^{-\delta}$.

The value of the portfolio will depend on how we rebalance. We shall here do this in the following way. Let $F_k(s)$, $a_k(s)$ and $C_k(s)$ be as in (2.3), (2.4), and (2.5).

If we were able to trade exactly on the levels of the binomial tree, then the portfolio value at τ_k would equal

$$F_k(\bar{s}_k) = a_{k-1}(\bar{s}_{k-1})\bar{s}_k + C_{k-1}(\bar{s}_{k-1}) = a_k(\bar{s}_k)\bar{s}_k + C_k(\bar{s}_k). \tag{4.1}$$

After the portfolio is rebalanced at τ_k the number of assets thus equals $a_k(\bar{s}_k)$ and the amount of cash equals $C_k(\bar{s}_k)$. This thus holds in this ideal case, but in general this portfolio will have different values after it is rebalanced than before. To avoid this we shall instead let the number of assets and the cash be proportional to the above numbers.

Thus let P_k stand for the portfolio value at τ_k, then $P_k = P_{k-1}\frac{U_k}{D_{k-1}}$, where

$$U_k = a_{k-1}(\bar{s}_{k-1})s_k + C_{k-1}(\bar{s}_{k-1}) \text{ and } D_{k-1} = a_{k-1}(\bar{s}_{k-1})s_{k-1} + C_{k-1}(\bar{s}_{k-1}).$$

Therefore

$$P_n = P_0\Pi_{k=1}^{n}\frac{U_k}{D_{k-1}} = U_n\Pi_{k=1}^{n-1}\frac{U_k}{D_k}.$$

We have

$$D_k = F_k(\bar{s}_k) + a_k(\bar{s}_k)(s_k - \bar{s}_k),$$

and it follows from (4.1) that

$$U_k = F_k(\bar{s}_k) + a_{k-1}(\bar{s}_{k-1})(s_k - \bar{s}_k).$$

In particular $U_n = f(\bar{s}_n) + O(\epsilon)$. The deviation of the portfolio value from the target thus mainly depends on the quotients

$$\frac{U_k}{D_k} = 1 + \frac{(a_{k-1}(\bar{s}_{k-1}) - a_k(\bar{s}_k))(s_k - \bar{s}_k)}{D_k} = 1 + \frac{\Delta_{n-k}(\bar{s}_k, y_k)}{D_k}, \qquad (4.2)$$

where

$$\Delta_{n-k}(s,y) = (a_{k-1}(se^{-\text{sgn}(y)\delta}) - a_k(s))s(e^{\epsilon y} - 1). \qquad (4.3)$$

Here we used that $\bar{s}_{k-1} = \bar{s}_k e^{-\text{sgn}(y_k)\delta}$ for $y_k \neq 0$.

Proposition 2 *Assume that f is continuous and has compact support. Then*

$$\Delta_r(s,y) = -s^2\frac{\partial^2}{\partial s^2}I_f(\delta^2 r, s)\delta\epsilon|y| + o(\epsilon^2), \qquad (4.4)$$

as $\epsilon \to 0$, $\delta \to 0$ and $r \to \infty$ in such a way that $\delta = h\epsilon$, and $\delta^2 r \to v > 0$,

The condition that f has compact support is convenient because it makes a rather complicated tail estimate unnecessary, but it can be replaced by the weaker (3.6).

See the appendix for a proof of the proposition.

Note that by Theorem 1

$$D_k = F_k(\bar{s}_k) + O(\epsilon) \approx I_f((n-k)\delta^2, \bar{s}_k),$$

and therefore the approximation

$$P_n \approx f(S_{\tau_n}) \exp[\frac{v}{h}\frac{1}{n}\sum_{k=0}^{n-1} J_f(\delta^2(n-k), S_{\tau_k})|y_k|] \tag{4.5}$$

holds under appropriate conditions on I_f. Here $v = \delta^2 n$, $h = \delta/\epsilon$, and

$$J_f(v,s) = -s^2\frac{\partial^2}{\partial s^2}I_f(v,s)/I_f(v,s). \tag{4.6}$$

In order to get a numerical value of the deviation we shall consider a trading strategy for which the function $J_f(v,s)$ does not depend on s.

Example Consider the trading strategy given by the target function $f(s) = Cs^\alpha$, where C is chosen so that the portfolio value at $t = 0$ equals 1. It follows from (2.3) that

$$F_k(s) = (s/S_0)^\alpha Q^{-k},$$

where

$$Q = \frac{e^{\alpha\delta} + e^{(1-\alpha)\delta}}{1+e^\delta} = 1 + O(\delta^2),$$

and hence from (2.4)

$$a_{k-1}(s) = \frac{\alpha_\delta}{s}F_k(s),$$

where

$$\alpha_\delta = \frac{e^{\alpha\delta} - e^{-\alpha\delta}}{e^\delta - e^{-\delta}} = \alpha + O(\delta^2).$$

This trading strategy thus holds the fixed proportion α_δ of the portfolio in the asset (and hence the proportion $1-\alpha_\delta$ in cash).

In this case

$$\frac{U_k}{D_k} = \frac{1 + \alpha_\delta \exp[\text{sgn}(y_k)\delta(1-\alpha)](e^{\epsilon y_k} - 1)}{1 + \alpha_\delta Q^{-1}(e^{\epsilon y_k} - 1)}.$$

This quotient is larger than 1 for $0 < \alpha < 1$ and $y_k \neq 0$. It equals 1 for $\alpha = 0$ and $\alpha = 1$, and it is less than 1 for all other alphas when $y_k \neq 0$. The imperfections of the market will thus be an advantage when $0 < \alpha < 1$, but a disadvantage when the portfolio has short positions in either the asset or cash.

We shall now give an asymptotic approximation of the portfolio value in this case when $\epsilon \to 0$, $\delta = h\epsilon$ and $n\delta^2 \to v$. We have

$$\ln \frac{U_k}{D_k} = \alpha(1-\alpha)\delta\epsilon|y_k| + O(\epsilon^3).$$

Neglecting terms of the order $O(\epsilon)$ we therefore get

$$P_n \approx F_n(\bar{s}_n) \exp[\frac{\alpha(1-\alpha)v}{h} \frac{1}{n} \sum_{k=1}^{n} |y_k|]. \tag{4.7}$$

We shall in Section 5 give conditions under which $v \approx \sigma^2 T$, where σ is the volatility of the asset price and T is the time horizon of the trading strategy.

In order to get a numerical value of the exponent in (4.7) we shall assume that the asset price behave as the stock Ericsson above. In this case 75% of the steps have the size ϵ and the remainder mainly 2ϵ. Let π denote the proportion of the y_k for which $|y_k| = 1$. The latter implies that the price moved a distance 2ϵ and hence $\pi < 0.25$.

This gives $\sum_1^n |y_k|/n = \pi < 0.25$. With $\alpha = 1/2$, $h = 20$, $\sigma = 0.5$ and $T = 1/4$ (three months) we get the upper bound 0.0002 for the exponent, which is negligible compared to $\epsilon = 0.002$. But this was a calm day. During more volatile conditions the error can be considerable.

5 Clock-time horizon

We have so far measured time by the number of levels in the binomial tree the asset price has visited. We shall now relate this number to the number of sunrises. To do this we shall make assumptions about the stochastics of asset prices.

We are looking for a trading strategy whose portfolio value equals $f(S_T)$ at time T. We therefore need to determine N so that $\tau_N \approx T$, or more precisely

$\tau_N \le T < \tau_{N+1}$. Therefore let $N_\delta(t,T)$ denote the number of τ_k in the interval $(t,T]$, and put $N_\delta(t) = N_\delta(0,t)$.

The value of the portfolio at time t thus equals the random variable

$$P_t^\delta = M_f(N_\delta(t,T), S_{\tau_{N_\delta(t)}}) + a_{N_\delta(t)}(S_{\tau_{N_\delta(t)}})(S_t - S_{\tau_{N_\delta(t)}}). \tag{5.1}$$

The quantities $S_{\tau_{N_\delta(t)}}$ and $N_\delta(t)$ are known at time t but $N_\delta(t,T)$ is still unknown (random).

We shall see that the second term to the right in (5.1) is negligible, and it follows from Theorem 1 that the first term is essentially a function of the number $\delta^2 N_\delta(t,T)$.

We shall in this section give conditions that implies that

$$\delta^2 N_\delta(t,T) \to \sigma^2(T-t), \tag{5.2}$$

in probability as $\delta \to 0$. Here σ is the volatility of the asset.

We have the following corollary to Theorem 1.

Corollary 3 *Assume that the liquidity and the divisibility assumptions hold. Assume further that f satisfies the conditions of Theorem 1. If*

$$\delta^2 N_\delta(t,T) \to v > 0 \tag{5.3}$$

in probability as $\delta \to 0$, then

$$Prob(|P_t^\delta - I_f(v,s)| > \eta \quad |S_t = s) \to 0 \tag{5.4}$$

for each $\eta > 0$.

The portfolio value can thus be approximated by the Black-Scholes formula if (5.2) holds.

Proof. We have $S_t e^{-\delta} < S_{\tau_{N_\delta(t)}} < S_t e^\delta$. It therefore follows from Theorem 1 that $M_f(N_\delta(t,T), S_{\tau_{N_\delta(t)}})$ tends to the desired limit. Also

$$|P_t^\delta - M_f(N_\delta(t,T), S_{\tau_{N_\delta(t)}})| = |a_{N_\delta(t)}(S_{\tau_{N_\delta(t)}})(S_t - S_{\tau_{N_\delta(t)}})| \le$$

$$|M_f(N_\delta(t,T)-1, S_{\tau_{N_\delta(t)}}e^\delta) - M_f(N_\delta(t,T)-1, S_{\tau_{N_\delta(t)}}e^{-\delta})|. \tag{5.5}$$

This is so because

$$e^{-\delta} - 1 < (S_t - S_{\tau_{N_\delta(t)}})/S_{\tau_{N_\delta(t)}} < e^{\delta} - 1$$

and

$$\frac{\max(|e^{-\delta} - 1|, e^{\delta} - 1)}{e^{\delta} - e^{-\delta}} = \frac{1}{1 + e^{-\delta}} < 1.$$

It follows from Theorem 1 that the difference (5.5) tends to zero as $\delta \to 0$.

We shall now make assumptions about the stochastic behaviour of the asset price. Let $S_t = S_0 e^{L_t}$. We shall assume that L_t is a jump process. Each jump is either ϵ or $-\epsilon$. This assumption which is part of the liquidity assumption will make the market complete. Thus if ξ_k denotes the k;th jump, and $M(t)$ the number of jumps in the time interval $(0, t]$. Then ξ_k can take the two values $\pm\epsilon$, and

$$L_t = U_{M(t)} \text{ where } U_m = \xi_1 + ... + \xi_m.$$

It is an empirical fact for stock prices that the signs of the daily returns are independent. We shall make the following assumption.

Independence assumption *The variables $\xi_1, \xi_2, ...$ are independent and identically distributed, and they are also independent of the process $M(t)$, $t \geq 0$.*

Let p denote the probability that $\xi_k = \epsilon$. It follows that

$$E(L_t) = E(\xi)E[M(t)] = \epsilon(2p - 1)E[M(t)], \tag{5.6}$$

and that

$$\begin{aligned} \text{Var}(L_t) &= \text{Var}(\xi)E[M(t)] + (E\xi)^2\text{Var}(M(t)) = \\ &\epsilon^2 4p(1-p)E[M(t)] + (\epsilon(2p-1))^2\text{Var}(M(t)). \end{aligned} \tag{5.7}$$

We shall add an assumption about the moments in order to be able to define drift and volatility.

Moment assumption

a) *The first two moments of $M(t)$ exist, and are continuous functions of t.*

b) *For each pair t_1, t_2 the first two moments of the increments $M(t_2 + s) - M(t_1 + s)$ are the same for all s.*

c) *The increments $M(t_2) - M(t_1)$, $M(t_3) - M(t_2)$ are uncorrelated for all $t_1 < t_2 < t_3$.*

We have

$$\mathrm{Cov}(L_{t_2} - L_{t_1}, L_{t_3} - Lt_2)) = (E\xi)^2 \mathrm{Cov}(M(t_2) - M(t_1), M(t_3) - M(t_2)),$$

and hence assumption c) implies that $L_{t_2} - L_{t_1}$, $L_{t_3} - L_{t_2}$ are uncorrelated. This is an empirical fact for stock prices although they are not independent; the sizes of the increments are positively correlated.

Lemma 4 *The Moment assumption implies*

$$E[M(t)] = tE[M(1)] \text{ and } \mathrm{Var}(M(t) = t\mathrm{Var}(M(1)). \tag{5.8}$$

Proof. Put $e(t) = E(M(t))$, $v(t) = \mathrm{Var}(M(t))$. Assumption b) implies that $e(t + s) = e(t) + e(s)$, and assumption b) and c) that $v(t + s) = v(t) + v(s)$. These identities imply together with assumption a) that $e(t) = e(1)t$ and $v(t) = v(1)t$.

It thus follows from (5.6), (5.7) and (5.8) that the expectation and variance of L_t thus have the form

$$E(L_t) = \nu t \text{ and } \mathrm{Var}(L_t) = \sigma^2 t, \tag{5.9}$$

with

$$\nu = \epsilon(2p - 1)E[M(1), \, \sigma^2 = \epsilon^2 4p(1 - p)E[M(1)] + (\epsilon(2p - 1))^2 \mathrm{Var}(M(1)). \tag{5.10}$$

We shall let $\epsilon \to 0$ in such a way that both the drift ν and the volatility σ is kept fixed. Both $p = p_\epsilon$ and the distribution of $M(t)$ will then depend on ϵ.

Example. Assume that the jumps occurs according to a Poisson process with intensity λ. This means that the time intervals between jumps are independent and exponentially distributed. In the example in the beginning of Section 4 about the development of the price of Ericsson the distribution of the times between jumps cannot be distinguished from the exponential distribution.

In this case $E[M(t)] = \text{Var}(M(t)) = \lambda t$, and hence we get the equations $\epsilon(2p-1)\lambda t = \nu t$, $\epsilon^2\lambda t = \sigma^2 t$. It follows that

$$\lambda = \sigma^2/\epsilon^2,\ p = \frac{1}{2}(1 + \frac{\nu\epsilon}{\sigma^2}),$$

and hence that

$$E[\epsilon^2 M(t)] = \sigma^2 t \text{ and } \text{Var}(\epsilon^2 M(t)) = \epsilon^2\sigma^2 t.$$

Thus in particular

$$\epsilon^2 M(t) \to \sigma^2 t \tag{5.11}$$

in probability, as $\epsilon \to 0$.

We now return to the number of levels visited in the binomial tree.

Theorem 5 *Let $\delta = \epsilon h$, where h is a fixed positive integer. Assume that the Independence assumption holds. If $M(t) \to \infty$ in probability, and $p_\epsilon \to \frac{1}{2}$, as $\epsilon \to 0$, then*

$$\frac{\delta^2 N_\delta(t)}{\epsilon^2 M(t)} \to 1,$$

in probability, and hence in particular the convergence (5.11) implies (5.2).

Proof. Let $\zeta_k = \xi_k/\epsilon$, and $V_n = \zeta_1 + ... + \zeta_n$. Then ζ_k takes the values ± 1, and $U_n = \epsilon V_n$. Define

$$H = \inf\{k \geq 0; |V_k| = h\}.$$

Then

$$H_1 + H_2 + ... + H_{N_\delta(t)} \leq M(t) < H_1 + H_2 + ... + H_{N_\delta(t)+1}, \tag{5.12}$$

where $H_1, H_2, ...$ are independent and have the same distribution as H above.

It follows that $N_\delta(t) \to \infty$, as $\epsilon \to 0$. Divide by $N_\delta(t)$ in (5.12). The extreme members tend to $E_0 H$, the expectation of H when $\epsilon = 0$, i.e. $p = 1/2$. Therefore also

$$\frac{M(t)}{N_\delta(t)} \to E_0 H.$$

The generating function of H is the sum of (4.11) and (4.12) in Ch. XIV of Feller Vol I (1967), with $a = 2h$ and $z = h$ and $p = 1/2$, and therefore equals

$$U(s) = \frac{2s^h}{(1+\sqrt{1-s^2})^h + (1-\sqrt{1-s^2})^h}.$$

A calculation shows that

$$\lim_{s\uparrow 1} \frac{d}{ds} \ln U(s) = h^2,$$

and hence $E_0 H = h^2$. This completes the proof of the theorem.

It is clear that (5.11) holds not only for the Poisson process but also for a wider class of processes, including some processes with dependent increments.

The next lemma gives sufficient conditions for (5.11).

Put

$$e_\epsilon = \epsilon^2 EM(1) = \text{ and } d_\epsilon = \epsilon^2 \sqrt{\mathrm{Var}(M(1))} = \sqrt{\mathrm{Var}(\epsilon^2 M(1))}.$$

In the example above we thus have $e_\epsilon = \sigma^2$ and $d_\epsilon = \epsilon\sigma$.

Lemma 6 *Assume that the Independence and Moment assumptions, and the identities (5.9) hold. If*

$$d_\epsilon / e_\epsilon \to 0 \text{ and } \epsilon / e_\epsilon \to 0 \tag{5.13}$$

as $\epsilon \to 0$, *then*

$$e_\epsilon \to \sigma^2 \text{ and } d_\epsilon \to 0, \tag{5.14}$$

and hence in particular (5.11) holds.

Proof. The equations to the left in (5.9) and (5.10) give

$$p = \frac{1}{2}(1 + \frac{\nu\epsilon}{e_\epsilon}), \tag{5.15}$$

and then the equations to the right in (5.9) and (5.10) give

$$\Big(1 - (\frac{\nu\epsilon}{e_\epsilon})^2\Big) e_\epsilon + (\frac{\nu d_\epsilon}{e_\epsilon})^2 = \sigma^2. \tag{5.16}$$

From which the Lemma follows.

Recall that the portfolio value (5.1) is essentially a function of the random variable $\delta^2 N_\delta(t,T)$, and hence according to Theorem 5 a function of $\epsilon^2 M(t,T)$, where $M(t,T)$ is the number of jumps of the process in the time interval $(t,T]$. In order to be able to predict this random variable it has to be almost constant, i.e. $\epsilon^2 M(t,T) \to v \geq 0$. In the generic case then $e_\epsilon(T-t) \to v$, and $d_\epsilon \to 0$. Thus if we exclude the degenerate case $v=0$ we have $v = \sigma^2(T-t)$, and hence the Black-Scholes formula holds. In this sense the alternative to the Black-Scholes formula is not another formula but no formula at all.

We shall now show that the models we have considered converge to the Black-Scholes model.

Proposition 7 *Assume that the assumptions of Lemma 6 and (5.13) hold. Then the finite dimensional distributions of L_t, $t \geq 0$, converge to those of $\nu t + \sigma W_t$, where W_t, $t \geq 0$, is a Wiener process.*

Therefore these models give the same result as the Black-Scholes model.

Proof. Let $t_0 < t_1 < ... < t_m$ and put

$$\psi(x_1, ..., x_m) = E[\Pi_{k=1}^m e^{ix_k(L_{t_k} - L_{t_{k-1}})}].$$

Then

$$\psi(x_1, ..., x_m) = EE[\Pi_{k=1}^m e^{ix_k(U_{M(t_k)} - U_{M(t_{k-1})})} | M(t_0), M(t_1), ..., M(t_m)] =$$

$$E[\Pi_{k=1}^m \beta(x_k)^{M(t_k) - M(t_{k-1})}],$$

where

$$\beta(x) = pe^{ix\epsilon} + (1-p)e^{-ix\epsilon} = 1 + \epsilon^2(ix\nu/e_\epsilon - x^2/2) + O(\epsilon^4) =$$
$$\exp(\epsilon^2(ix\nu/e_\epsilon - x^2/2) + O(\epsilon^4)).$$

Here we used (5.15).

It follows from (5.14) that

$$\epsilon^2(M(t_k) - M(t_{k-1})) \to \sigma^2(t_k - t_{k-1})$$

in probability, and hence also

$$\psi(x_1, ..., x_m) \to \Pi_{k=1}^m e^{(t_k - t_{k-1})(ix_k\nu - x_k^2\sigma^2/2)}$$

because $|\beta(x)| \leq 1$. The expression to the right is the characteristic function of the increments of the Black-Scholes model, and hence the proposition follows.

Appendix

Proof of the convergence (3.3).

We shall first show that (3.3) holds for a fixed s, and then show that the convergence is uniform on compacts.

We have $M_f(r, s) = \Sigma' + \Sigma''$, where

$$\Sigma' = E[f(se^{Y_r})I_K(Y_r)] \text{ and } \Sigma'' = E[f(se^{Y_r})(1 - I_K(Y_r))].$$

Here $I_K(x) = 1$ for $|x| \leq K$ and $=0$ otherwise.

The variables Y_r converge to Y_∞ in distribution as $r \to \infty$, where Y_∞ is normal with mean $-\frac{v}{2}$ and variance v. Therefore

$$\Sigma' \to E[f(se^{Y_\infty})I_K(Y_\infty)] \qquad (A.1)$$

for each $K > 0$.

The condition (3.6) implies

$$|E[f(se^{Y_\infty})(1 - I_K(Y_\infty))]| \leq E[(A + Bse^{Y_\infty})(1 - I_K(Y_\infty))], \qquad (A.2)$$

and this can be made arbitrarily small by choosing K large. The proof of (3.3) will thus be completed by showing that the same holds also for Σ''.

Let

$$b_p(r, k) = \binom{r}{k} p^k(1 - p)^{r-k},$$

for $i = 0, 1, ..., r$ and let $b_p(r, i) = 0$ otherwise. The identities $qu = 1 - q$ and $(1 - q)d = q$ give

$$b_q(r, i)u^i d^{r-i} = b_{1-q}(r, i) \qquad (A.3)$$

and hence by (3.6)

$$|\Sigma''| \leq A\Sigma_q'' + Bs\Sigma_{1-q}'', \qquad (A.4)$$

where

$$\Sigma''_q = \sum_{|\delta(2i-r)|>K} b_q(r,i).$$

It follows from Markov's inequality and (3.2) that both Σ''_q and Σ''_{1-q} are dominated by

$$\frac{\delta^2 r4q(1-q) + (\delta r(2q-1))^2}{K^2} = \frac{(v+\frac{v^2}{4})(1+O(\delta^2))}{K^2}, \tag{A.5}$$

where $v = r\delta^2$.

Therefore Σ'' can be made arbitrarily small by choosing K large.

The two tails (A.2) and (A.4) can obviously be made uniformly small on compacts. The convergence (A.1) is uniform on compacts because if $s \in S_0 = [a,b]$, where $0 < a < b$, then $se^{Y_r} \in S_1 = [ae^{-K}, be^K]$ for all $|Y_r| \le K$. The function $f(x)$ is uniformly continuous on S_1: for each $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x') - f(x'')| < \epsilon$ for all x' and x'' in S_1 satisfying $|x' - x''| < \delta$. Therefore

$$|f(s'e^{Y_r}) - f(s''e^{Y_r}| < \epsilon$$

for all $|Y_r| \le K$ and all s' and s'' in S_0 satisfying $|s' - s''| < \delta e^{-K}$. The compact set S_0 can be covered by finitely many intervals of lengths $< \delta e^{-K}$, and hence the convergence (A.1) is uniform in s.

Proof of the convergence (3.4).

The expression to the left in (3.4) equals

$$\frac{1}{s}\sum_{i=0}^{r+1} f(se^{\delta(2i-r-1)})d_q(r,i),$$

where

$$d_q(r,i) = \frac{b_q(r,i-1) - b_q(r,i)}{e^\delta - e^{-\delta}} \text{ for } i = 0, ..., r+1.$$

Lemma 8 *Let $h = \frac{1}{\sqrt{rq(1-q)}}$, and $x_i = h(i - rq)$. Then*

$$b_q(r,i) = h\phi(x_i)(1 + h\delta P_3(x_i) + h^2 P_4(x_i)) + O(h^4), \tag{A.6}$$

as $r \to \infty$, and $\delta \to 0$ in such a way that $r\delta^2 \to v > 0$. Here P_3 and P_4 are polynomials.

Proof. The expression to the left in (A.6) can be represented as a convolution; $b_q(r,i) = b_q^{r\star}(1,i)$, and hence it has an Edgeworth expansion

$$b_q(r,i) = h\phi(x_i)\Big(1+\frac{1}{\sqrt{r}}\frac{\mu_3}{6\sigma^3}H_3(x_i)+\frac{1}{r}\big(\frac{\mu_4-3\sigma^4}{24\sigma^4}H_4(x_i)+\frac{\mu_3^2}{72\sigma^6}H_3(x_i)\big)\Big)+O(h^4).$$

Here σ^2 is the variance, and μ_3 and μ_4 are the third and fourth central moment of a random variable that has the density $b_q(1,i)$. (See for example Theorem XVI.2.2 in Feller Vol. II (1966), where continuous densities are considered. The proof for discrete densities is similar but simpler.) In this case

$$\sigma^2 = q(1-q) = \frac{1}{4}(1+O(\delta^2)),\ \mu_3 = q(1-q)(1-2q) = q(1-q)\frac{\delta}{2}+O(\delta^3)$$

$$\mu_4 = q(1-q)(q^3+(1-q)^3) = q(1-q)\frac{1}{4}+O(\delta^2).$$

The lemma follows.

It follows from (A.6)) that

$$d_q(r,i) = \frac{h(\phi(x_i-h)-\phi(x_i))+O(h^3)}{e^{\delta}-e^{-\delta}} = \frac{-h^2\phi'(x_i)}{2\delta}+O(h^2). \qquad (A.7)$$

But $\frac{h}{2\delta} = \frac{1}{\sqrt{v}}+O(\delta^2)$, and $\delta(2i-r) = \sqrt{v}x_i - \frac{v}{2}+O(\delta)$, where $v = r\delta^2$. Therefore

$$\sum_{|\delta(2i-r)|\le K} f(se^{\delta(2i-r-1)})d_q(r,i) = -\frac{1}{\sqrt{v}}\sum_{|\delta(2i-r)|\le K} f(se^{\delta(2i-r-1)})\phi'(x_i)h+O(h)$$

$$\to -\frac{1}{\sqrt{v}}\int_{|\sqrt{v}x-\frac{v}{2}|\le K} f(se^{\sqrt{v}x-\frac{v}{2}})\phi'(x)dx.$$

It remains to show that the tails can be made arbitrarily small by choosing K large. We shall here only consider the right hand tail Σ_+/s, where

$$\Sigma_+ = \sum_{\delta(2i-r)>K} f(se^{\delta(2i-r-1)})d_q(r,i),$$

the other can treated analogously.

Let $t = \frac{1}{2}(r + \frac{K}{\delta})$. We have $d_q(r,i) \geq 0$ for $i \geq q(r+1)$, and $t > q(r+1)$ for $\delta < K$. Therefore

$$|\Sigma_+| \leq AS_0 + BsS_1,$$

where

$$S_m = \sum_{i>t} e^{m\delta(2i-r-1)} d_q(r,i) =$$

$$\frac{1}{e^{\delta} - e^{-\delta}} (e^{m\delta} \sum_{i>t-1} e^{m\delta(2i-r)} b_q(r,i) - e^{-m\delta} \sum_{i>t} e^{m\delta(2i-r)} b_q(r,i)).$$

Thus

$$S_0 = \frac{1}{e^{\delta} - e^{-\delta}} b_q(r,[t]),$$

and using (A.3)

$$S_1 = \frac{e^{\delta}}{e^{\delta} - e^{-\delta}} b_{1-q}(r,[t]) + \sum_{i>t} b_{1-q}(r,i). \qquad (A.8)$$

The second term to the right in (A.8) is dominated by (A.5), and it follows from Lemma 8 that the first term equals

$$\frac{h}{2\delta}\phi(h([t] - r(1-q))) + O(h^2/\delta) \to \frac{\phi(\frac{K-\frac{v}{2}}{\sqrt{v}})}{\sqrt{v}}.$$

In the same way we see that

$$S_0 \to \frac{\phi(\frac{K+\frac{v}{2}}{\sqrt{v}})}{\sqrt{v}}.$$

These expressions can thus be made arbitrarily small by choosing K large.

Proof of Proposition 2.

Let $z = \operatorname{sgn}(y)$. It follows from (2.4) and (2.3) that

$$a_{n-r-1}(se^{-z\delta}) - a_{n-r}(s) = \frac{1}{s} \sum_i \partial_q^z(r,i) f(se^{\delta(2i-r)}),$$

where

$$\partial_q^z(r,i) = \begin{cases} e^{\delta} d_q(r,i+1) - d_q(r-1,i) & \text{if } z = 1 \\ e^{-\delta} d_q(r,i) - d_q(r-1,i) & \text{if } z = -1. \end{cases} \tag{A.9}$$

It follows from (A.6) that

$$b_q(r,i) = h\phi(x_i)(1 + h^2 Q(x_i)) + O(h^4),$$

where

$$Q(x) = \sqrt{r\delta^2 q(1-q)}P_3(x) + P_4(x)$$

Using this we get a sharper version of (A.7)

$$d_q(r,i) = -\frac{h^2}{2\delta}\phi'(x_i) + O(h^3).$$

Let $h' = 1/\sqrt{(r-1)q(1-q)}$. Then $h' = h + O(h^3)$, and hence

$$d_q(r-1,i) = -\frac{h^2}{2\delta}\phi'(x_i + hq) + O(h^3).$$

Expansions of the expressions to the right in (A.9) in powers of δ and h now yields

$$\partial_q^z(r,i) = -z(\frac{h^3}{4\delta}\phi''(x_i) + \frac{h^2}{2}\phi'(x_i)) + O(h^3) = -z\frac{\delta}{v}(\phi''(x_i) + \sqrt{v}\phi'(x_i))h + O(h^3).$$

Therefore

$$\sum_{|\delta(2i-r)|\leq K} f(se^{\delta(2i-r)})\partial_q^z(r,i) =$$

$$-z\frac{\delta}{v}\sum_{|\delta(2i-r)|\leq K} f(se^{\delta(2i-r-1)})(\phi''(x_i) + \sqrt{v}\phi'(x_i))h + O(h^2) =$$

$$-z\frac{\delta}{v}\int_{|\sqrt{v}x - \frac{v}{2}|\leq K} f(se^{\sqrt{v}x - \frac{v}{2}})(\phi''(x) + \sqrt{v}\phi'(x))dx + o(h).$$

The identity

$$\frac{\partial^2 I_f(v,s)}{\partial s^2} = \frac{1}{s^2 v}\int f(se^{\sqrt{v}x-\frac{v}{2}})(\phi''(x)+\sqrt{v}\phi'(x))dx$$

follows in the same way as the identity (3.7).

This completes the proof of the proposition since we have assumed that f has compact support.

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