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When is a convex barrier passed?

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When is a convex barrier passed?

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Abstract

This licentiate thesis contains two articles.

1. **Tools to estimate the first passage time to a convex barrier**
2. **A large deviation estimate of the first passage time to a convex barrier**

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Tools to estimate the first passage time to a convex barrier

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Abstract

The first passage time of a random walk to a barrier is of great importance in many areas, such as insurance, finance and sequential analysis. The barrier used in these areas is usually constant or slowly varying. Here, the barrier $cb(n/c)$ is convex, where c is a scale parameter and n is time. It is shown by large deviation techniques that the limit distribution of the first passage time decays exponentially in c. The exponential part is multiplied by a slowly changing function which is computed by means of a tilt of measure. Under the tilted measure, which changes the drift, it is proved that: The limit distribution of the overshoot is distributed as an overshoot over a linear barrier. Properly normalized the stopping time is shown to be asymptotically normally distributed. The overshoot and the asymptotic normal part are asymptotically independent. The combination of these three building blocks gives the slowly changing constant.

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1 Introduction

A sum S_n, that starts in zero, of independent, identically distributed random variables makes the unlikely event that it hits a distant upper convex barrier $cb(n/c)$. What is the asymptotic distribution of the stopping time, when the scale parameter c goes to infinity?

Early on in insurance, Cramér [3] and Lundberg [11] studied the probability of ever hitting a constant barrier, a prospect which meant ruin. The probability of ruin (also before a finite time) has been analyzed by the use of many different techniques; two dimensional renewal theory Höglund [10], ladder variables von Bahr [1], integral equation Segerdahl [13] and martingale techniques Grandell [6]. More recently ruin probability has become an interest in finance, when handling credit risk in a loan or bond portfolio Dembo, Deuschel and Duffie [5].

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We will use large deviation techniques which loosely state that $P(S_n/n \approx x) \approx \exp(-nI(x))$ for $x > \mu$, where μ is the mean and $I(x)$ is called the rate function. Hitting the barrier at the stopping time N implies a drift $S_N/N \approx (c/N)b(N/c)$ and a rate $I[(c/N)b(N/c)]$. Therefore we expect an estimate of

$$P(N/c \approx s) \approx K(c)\exp(-csI(b(s)/s)) = K(c)\exp(-cR(s)),$$

where $R(s) = sI(b(s)/s)$ and $K(c)$ is slowly changing compared to the exponential part. When the scale parameter is large, the time T, defined by $R(T) = \inf_s R(s)$, will dominate, such that

$$P(N/c < \infty) \approx \sum_s K(c)\exp(-cR(s)) \approx K(c)\exp(-cR(T))).$$

Furthermore, for $t \geq T$, by the same type of argument, we have $P(N/c < t) \approx K(c)\exp(-cR(T))$. When $t < T$ then the dominating time is t and $P(N/c < t) \approx K(c)\exp[-cR(t)]$.

Martin-Löf [12] studied $t \geq T$, using Wald's identity and large deviation techniques. We will use a more probabilistic approach and study in more depth the properties in both $t \geq T$ and $t < T$.

The Esscher transform tilts the true distribution P to a distribution P_θ, so that the drift changes. The drift is changed to the tilted drift $b(T)/T$, by a appropriate choice of θ. The variance of the increments, σ^2, is unchanged by the tilt of measure. Then we use Anscombe's theorem, an idea borrowed from Gut [7], to prove,

$$\lim_{c\to\infty} c^{-1/2}(N - cT) \stackrel{P_\theta}{=} Y \sim N(0, a^2),$$

where $a^2 = \sigma^2 T^3 (b(T) - b'(T)T)^{-2}$. Furthermore, the overshoot is proved to be asymptotically independent of Y and converges in distribution to an overshoot over a linear barrier. In sequential analysis you find similar results of asymptotic normality, distribution of the overshoot and asymptotic independence for perturbed random walks, or for a random walk passing a slowly changing barrier, Gut [7] and Siegmund [14].

We want to estimate $P(N \leq cT + c^{1/2}y)$, and therefore introduce the indicator function $\mathbf{1}(N \leq cT + c^{1/2}y)$ and write $Z = S_N - cb(N/c)$ for the overshoot. After expressing the probability as an expectation of the indicator function, tilting the distribution, Taylor expanding the barrier and use the asymptotic independence we get,

$$e^{cR(T)}P(N \leq cT + c^{1/2}y) = E_\theta\left[e^{-\theta Z}\right] E_\theta\left[\mathbf{1}(N \leq cT + c^{1/2}y)e^{-\theta\frac{b''(T)}{2c}(N-cT)^2}\right].$$

The quadratic part $0.5c^{-1}b''(T)(N-cT)^2$ is incorporated by a change of variance of the limiting normal random variable Y, to $\eta^{-2} = a^{-2} + b''(T)\theta$. Combining this, the asymptotic normality and the convergence in distribution to a linear overshoot gives that, for some real y

$$\lim_{c\to\infty} e^{cR(T)}P(N \leq cT + \sqrt{c}y) = \frac{\eta}{a}E_\theta[e^{-\theta Z_l}]\Phi(y/\eta),$$

where Z_l is the overshoot over the linear barrier $cb(T) + b'(T)(n - cT)$.

In the finite time horizon case, when $t < T$ asymptotic normality, weak convergence of the distribution of the overshoot to the distribution of the overshoot over a linear barrier and asymptotic independence are also proved under the tilted distribution. The tilt of the distribution is however not the same as before, the drift is now changed to $b(t)/t$, by another choice of θ.

The slowly changing function is in the finite time horizon case not that easy to compute. It is only made likely that

$$K(c) \approx \frac{E_\theta\left[e^{-\theta Z_l}\right]}{\left(1 - e^{R'(t)}\right)\sqrt{2\pi c a}},$$

where $a^2 = \sigma^2 t^3 (b(t) - b'(t)t)^{-2}$. Thus

$$\lim_{c\to\infty} c^{1/2} e^{cR(t)} P\left(N \le ct\right) = \frac{E_\theta\left[e^{-\theta Z_l}\right]}{\left(1 - e^{R'(t)}\right)\sqrt{2\pi a}}.$$

This conjecture is proven right in Hammarlid [8]. It is probably possible to make a unified formulation of $K(c)$ for the two cases $t < T$ and $t \ge T$ as Höglund did in the case of a sum of independent random variables [9].

The outline of the article is as follows. Section 2 gives a brief introduction to large deviations and the properties of the rate function. The properties are then used to derive the equations determining the dominating point of the barrier and the exponential rate of decrease and its properties. In Section 3 the tilting of a distribution and its implications for the rate function and convergence of the normalized stopping time are presented. All the results regarding asymptotic normality, independence and the distribution of the overshoot are proved in Section 4. A formal proof of the main theorem is given in Section 5.

2 Large deviations and a first passage time estimate

We will start to introduce the large deviation tools needed. Let $S_n = \sum_{i=1}^n X_i$, where X_i are independent identically distributed with $E[X_i] = \mu$. It is also assumed that the cumulant function $g(\theta) = \log(E[e^{\theta X_i}])$ exists for θ in some open set. Then, for $b > \mu$ and $\theta > 0$,

$$P\left(\frac{S_n}{n} > b\right) = \int_{s>nb} dP_{S_n}(s) \le e^{-bn\theta} \int_{s>nb} e^{\theta s} dP_{S_n}(s). \tag{1}$$

If the domain of integration, of the integral on the right hand side, is extended to the real line, then $P(S_n > nb) \le \exp[-n(b\theta - g(\theta))]$. To make the best possible approximation we minimize this over θ and define a rate function $I(x) = \sup_\theta (x\theta - g(\theta))$ to get

$$P(S_n > nb) \le \exp\left(-n \sup_\theta (b\theta - g(\theta))\right) = \exp(-nI(b)). \tag{2}$$

This bound is called the Chernoff bound. There is also a lower bound such that for any convex set B,

$$\lim_{n\to\infty} n^{-1} \log\left(P\left(\frac{S_n}{n} \in B\right)\right) = -\inf_{x\in B} I(x). \tag{3}$$

This result is usually stated as two separate limit theorems, one upper bound for closed sets, and one lower bound for open sets. It is only when working with nice sets, such as intervals and convex sets, that it is guaranteed that the upper bound equals the lower bound. We will work with nice sets. The rate function has some well known properties:

$$\begin{array}{lllllll} I(x) & \geq & 0, \forall x & \text{and} & I(\mu) & = & 0, \\ I(x) & = & \theta x - g(\theta) & \text{where} & g'(\theta) & = & x, \\ I'(x) & = & \theta & \text{and} & \theta' & = & 1/g''(\theta). \end{array} \quad (4)$$

All of these properties follow from manipulations of the definition of rate function, see for example Bucklew [2] or Dembo and Zeitouni [4].

We will now continue by specifying the problem mathematically and see how large deviations fit in. The distant barrier that the process tries to pass is $cb(n/c)$, where c is a scale parameter tending to infinity. The barrier is convex, three times continuously differentiable and satisfying $b(0) > 0$.

Definition 1 *The time when the process S_n for the first time passes or hits $cb(n/c)$ is denoted by $N = \inf\{n : S_n \geq cb(n/c)\}$.*

There is a counterpart to the rate function of a sum, a Cramér-Lundberg type of constant, for the stopping time.

Lemma 1 *Let $\mu = E[X_i]$ and assume that $\mu s < b(s)$ for all s. Then the function $R(s) = sI(b(s)/s)$ is convex and it attains its minimum at T, which is determined by*

$$g(\theta) = b'(T)\theta \qquad \text{and} \qquad g'(\theta) = b(T)/T. \quad (5)$$

The function $R(s) = \theta b(s) - g(\theta)s$, where $g'(\theta) = b(s)/s$ so that

$$\lim_{c\to\infty} c^{-1} \log\left(P(N \leq ct)\right) = \begin{cases} -R(t) & \text{for } t < T \\ -R(T) & \text{for } t \geq T. \end{cases} \quad (6)$$

Remark: In the case $t \geq T$ it is possible to write $R(T) = \theta(b(T) - Tb'(T)) = T(\theta g'(\theta) - g(\theta))$.

Remark: When c is large we write that $P(N \leq ct) \approx K(c)\exp(-cR(t))$, where $K(c)$ is some slowly changing function in comparison to the exponential part.

Proof. The proof is carried out in two steps. First we show that $c^{-1}\log[P(N \leq ct)]$ has an upper and a lower bound that in the limit are equal. Then the first and second order derivatives of $R(s)$ are computed, where $R''(s) \geq 0$ and the optimal time T is the unique solution to $R'(s) = 0$.

The probability we want to estimate is

$$P(N \leq ct) = P\left(\cup_{n=1}^{[ct]}\{S_n \geq cb(n/c)\}\right),$$

where $[\cdot]$ denotes the integer part. This probability has an upper bound, derived from the Chernoff's bound (2), since $(c/n)b(n/c) > \mu$ for all n,

$$P\left(S_n \geq cb(n/c)\right) = P\left(S_n/n \geq (c/n)b(n/c)\right) \leq \exp\left(-c(n/c)I[(c/n)b(n/c)]\right).$$

By Boole's inequality,

$$P\left(\cup_{n=1}^{[ct]}\{S_n \geq cb(n/c)\}\right) \leq \sum_{n=1}^{[ct]} P\left(S_n \geq cb(n/c)\right) \leq \sum_{n=1}^{[ct]} e^{-c(n/c)I[(c/n)b(n/c)]}.$$

Only the largest probability in the sum on the right will contribute to the total probability. Hence if $s = n/c$ then,

$$P\left(\cup_{n=1}^{[ct]} S_n \geq cb(n/c)\right) \leq ct \exp\left(-c \inf_{0\leq s\leq t} sI(b(s)/s)\right). \tag{7}$$

Let us turn to the lower bound of $P(N \leq ct)$. It is always true that for any $n \leq ct$,

$$P\left(\frac{S_n}{n} \geq \frac{c}{n}b(n/c)\right) \leq P\left(\cup_{n=1}^{[ct]}\{S_n \geq cb(n/c)\}\right). \tag{8}$$

Then the lower bound assures that

$$-sI(b(s)/s) \leq \lim_{c\to\infty} c^{-1}\log\left(P\left(\frac{S_n}{n} \geq \frac{c}{n}b(n/c)\right)\right), \qquad s = n/c. \tag{9}$$

The right choice of s is to chose the time that maximizes the probability on the left hand side of (9), which is the dominating point in (7). Take the logarithm on both sides of (7) and divide by c, then in the limit in combination with the lower bound (9) we have that,

$$\lim_{c\to\infty} c^{-1}\log\left(P\left(N \leq ct\right)\right) = -\inf_{0\leq s\leq t} sI(b(s)/s) = -\inf_{0\leq s\leq t} R(s).$$

The second step of the proof is to compute the first and second order derivative of $R(s)$ in order to find the unique optimal point. The first order derivative is $R'(s) = I(b(s)/s) + I'(b(s)/s)[b'(s)s - b(s)]/s$, which we simplified by the properties of the rate function (4) to

$$R'(s) = \theta b'(s) - g(\theta). \tag{10}$$

Without restrictions on s the minimum of $R(s)$ is attained when $R'(s) = 0$, and hence by (10) we have that $b'(T)\theta = g(\theta)$, where $g'(\theta) = b(T)/T$.

The second order derivative of $R(s)$ is by (10), $R''(s) = \frac{d}{ds}(\theta b'(s) - g(\theta))$. Use the chain rule where $\theta' = 1/g''(\theta)$ and exchange $g'(\theta) = b(s)/s$ to get

$$R''(s) = \frac{(b'(s)s - b(s))^2}{g''(\theta)s^3} + \theta b''(s). \tag{11}$$

The barrier and the cumulant function are convex and hence $b''(s) \geq 0$ for all s and $g''(\theta) \geq 0$ for all θ in the definition set. We know that $g'(\theta)$ is non-decreasing since $g''(\theta) \geq 0$ and that $g'(0) = \mu$. This implies that the θ that solves $g'(\theta) = b(s)/s$ must be greater than zero when $b(s)/s > \mu$ and less than zero when $b(s)/s < \mu$. Therefore, since $b(s)/s > \mu$ for all s and $s > 0$ have that $R''(s) \geq 0$.

If $t < T$ then the optimal solution is not feasible, and the solution has to be on the boundary. The function $R(s)$ is decreasing in $s \leq T$ so that $\inf_{s\leq t}(R(s)) = R(t)$. □

Remark: Note if $s = T$ in equation (11), then by the remark under Lemma 1,

$$R''(T) = \frac{R(T)^2}{g''(\theta)\theta^2 T^3} + \theta b''(T). \qquad (12)$$

Every point $s > T^*$ is on the 'shadow side' where T^* solves $b'(s) = b(s)/s$ and $T^* = \infty$ is allowed. We call it the shadow side because every point on that side cannot be reached by a straight line from zero without passing the barrier along the way. The barrier is convex and therefore

$$b(s)/s > b'(s), \qquad \text{for every } s \leq T^*. \qquad (13)$$

When the barrier is crossed by the drift line the function $R(s)$ is not convex anymore.

Lemma 2 *Assume that the convex barrier $b(s)$ is three times continuously differentiable, $b(0) > 0$ and that there is a T that solves $\mu T = b(T)$. Then $R(T) = R'(T) = 0$ and the function $R(s)$ is not convex. More precisely, $R''(s) \geq 0$ for all $s \leq T$ and $R''(T^*) \leq 0$.*

Proof. The minimum of $I(\mu) = 0$, by the properties of the rate function (4) and the assumption $\mu T = b(T)$. Therefore $R(T) = TI(b(T)/T) = 0$ and $\theta = 0$ solves $g'(\theta) = b(T)/T$ which by (10) gives $R'(T) = 0$. When $s \leq T$ then $b(s)/s \geq \mu$, which implies that $\theta \geq 0$ and therefore $R''(s) \geq 0$, see the discussion under equation (11). On the other hand $\theta < 0$ when $g'(\theta) = b(T^*)/T^* < \mu$ and $R''(T^*) = \theta b''(T^*) \leq 0$. Hence the function $R(s)$ is not convex. □

Remark: What is the relationship of the rate function between two points $u < T^* < v$ that are the crossing times of a line from origin to the barrier $b(s)$? The relation is linear, because the slope is the same, that is $b(v)/v = b(u)/u$ and

$$vI(b(v)/v) = uI\left(\frac{b(u)}{u}\right) + (v-u)I\left(\frac{b(u)}{u}\right).$$

If $b(s)/s \to \eta$ when $s \to \infty$ and $\inf\{s|\eta s = b(s)/s\}$ then asymptotically the function $R(s)$ will grow linearly $sI(b(s)/s) = uI(\eta) + (s-u)I(\eta)$. By this, one realizes that $T \leq T^*$.

3 Tilted distribution

One important tool is the tilted distribution, sometimes called an exponential change of measure or the Esscher transform, see for example Bucklew [2] and Martin-Löf [12]. For every θ in the open definition set, a tilting of a distribution $F(x)$ is defined as, $dF_\theta(x) = \exp(\theta x - g(\theta))dF(x)$. The tilted expectation is denoted $E_\theta[\cdot]$ and the tilted cumulant function $g_\theta(\gamma) = E_\theta[\exp(\gamma X)]$. It can easily be shown that $g_\theta(\gamma) = g(\gamma + \theta) - g(\theta)$. The expected value and variance under the tilted measure is therefore derived by the first and second order derivative of the tilted cumulant at zero,

$$E_\theta[X] = g'(\theta) \qquad \text{and} \qquad Var_\theta(X) = g''(\theta). \qquad (14)$$

The tilted rate function is $I_\theta(x) = \gamma x - g_\theta(\gamma)$, where γ solves $g'_\theta(\gamma) = x$ and $I_\theta(g'(\theta)) = 0$ since $g'(\theta) = E_\theta[X]$. Also $I'_\theta(g'(\theta)) = 0$, since $I'_\theta(x) = \gamma$ according to the properties of the rate function (4) and that $\gamma = 0$ solves $g'_\theta(\gamma) = g'(\theta) = E_\theta[X] = \mu_\theta$.

Lemma 3 *Fix $\varepsilon > 0$ and let θ solve $g'(\theta) = b(t)/t$. The function $R_\theta(s) = sI_\theta(b(s)/s)$, is not generally convex, but $R_\theta(t) = 0$, $R'_\theta(t) = 0$ and $R''_\theta(t) \geq 0$. Furthermore, there is a constant $d > 0$ such that, $R_\theta(t \pm \varepsilon) \leq d\varepsilon^2$. Especially*

$$\begin{array}{lcl} P_\theta(S_n \geq cb(n/c), \text{ for some } n \leq c(t-\varepsilon)) & \leq & cte^{-cd\varepsilon^2}, \\ P_\theta(S_n < cb(n/c), \text{ for all } n \leq c(t+\varepsilon)) & \leq & e^{-cd\varepsilon^2}. \end{array} \tag{15}$$

Remark: The most important case is when $t = T$.

Proof. The tilted drift is $\mu_\theta = g'(\theta) = b(t)/t$, which crosses the barrier at time t. The minimum $R_\theta(t) = R'_\theta(t) = 0$ and that $R_\theta(t)$ is not generally convex but $R''_\theta(t) \geq 0$, follows by Lemma 2. Therefore, the Taylor expansion of $R_\theta(t \pm \varepsilon) = R''_\theta(\xi)\varepsilon^2/2$, where $|\xi - t| \leq \varepsilon$. Chose therefore $d \leq R''_\theta(s)/2$, for all $|s - t| \leq \varepsilon$.

By the same large deviation estimates as before and Boole's inequality,

$$\begin{array}{lcl} P_\theta(S_n \geq cb(n/c), \text{ for some } n \leq c(t-\varepsilon)) & \leq & ct\exp\left(-c\inf_{0\leq s\leq t-\varepsilon} sI_\theta(b(s)/s)\right), \\ P_\theta(S_n < cb(n/c), \text{ for all } n \leq c(t+\varepsilon)) & \leq & \exp\left(-c\inf_{s\geq t+\varepsilon} sI_\theta(b(s)/s)\right). \end{array}$$

The definition of $R_\theta(s) = sI_\theta(b(s)/s)$ and the Taylor expansion give equation (15). □

Lemma 4 *Fix $\varepsilon > 0$ and let $\gamma > 0$ and $\alpha > 0$ such that $\alpha < 2\gamma$. Then under the tilted distribution there is a $d > 0$ so that,*

$$P_\theta\left(\left|\frac{(N-ct)^\alpha}{c^\gamma}\right| > \varepsilon\right) \leq 2ct\exp\left(-c^{2\gamma/\alpha-1}d\varepsilon^{2/\alpha}\right),$$

where $g'(\theta) = b(t)/t$. Also, $c^{-\gamma}(N-ct)^\alpha \xrightarrow{a.s} 0$, as $c \to \infty$.

Proof. The set $\{\omega : |N/c - t| > \varepsilon\}$ is the union of the two disjoint sets $A' = \{\omega : S_n \geq cb(n/c), \text{ for some } n \leq c(t-\varepsilon)\}$ and $A'' = \{\omega : S_n < cb(n/c), \text{ for all } n \leq c(t+\varepsilon)\}$, hence $P_\theta(|N/c - t| > \varepsilon) = P_\theta(A') + P_\theta(A'')$. Therefore, by the large deviation estimates of these probabilities in Lemma 3, we have,

$$P_\theta(|N/c - t| > \varepsilon) \leq 2cte^{-cd\varepsilon^2}.$$

Take $\varepsilon > 0$, then by the last equation,

$$P_\theta\left(\left|\frac{(N-ct)^\alpha}{c^\gamma}\right| > \varepsilon\right) = P_\theta\left(\left|\frac{N-ct}{c}\right| > c^{\gamma/\alpha-1}\varepsilon^{1/\alpha}\right) \leq 2ct\exp\left(-c^{2\gamma/\alpha-1}d\varepsilon^{2/\alpha}\right).$$

If $2\gamma > \alpha$ then $ct\exp\left(-d\varepsilon^{2/\alpha}c^{2\gamma/\alpha-1}\right) \to 0$ as c goes to infinity. Choose $\varepsilon = c^{-\eta}$ where $\eta > 0$ fulfills $\gamma - \eta > \alpha/2$, then

$$\sum_{c=1}^{\infty} P_\theta\left(\left|\frac{(N-ct)^\alpha}{c^\gamma}\right| > \frac{1}{c^\eta}\right) \leq 2\sum_{c=1}^{\infty} ct\exp\left(-c^{2(\gamma-\eta)/\alpha-1}d\right) < \infty.$$

Borel-Cantelli lemma states that the convergence is almost sure. □

Corollary 1 *Fix ε. There is a constant $d > 0$ such that under the tilted distribution $P_\theta(|N - cT| > c\varepsilon) \leq 2cte^{-cd\varepsilon^2}$, where $g'(\theta) = b(T)/T$ and $N/c \xrightarrow{a.s} T$, as $c \to \infty$.*

Proof. This is a direct consequence of Lemma 4 where $t = T$, $\gamma = 1$ and $\alpha = 1$. □

We will need that the overshoot, defined as the difference between the stopped random walk and the barrier, converges to zero when scaled by $c^{-\alpha}$. The overshoot is always, however, less than or equal to the last increment.

Lemma 5 *Let $\alpha > 0$ then under the tilted distribution*

$$\frac{X_N}{c^\alpha} \xrightarrow{a.s} 0, \text{ as } c \to \infty.$$

Proof. Take an $\varepsilon > 0$ and a $\gamma > 0$ such that $g_\theta(\gamma) < \infty$ and $g_\theta(-\gamma) < \infty$. Split the probability $P_\theta(|X_n| > \varepsilon n^\alpha) = P_\theta(X_n > \varepsilon n^\alpha) + P_\theta(X_n < -\varepsilon n^\alpha)$ and use on each of these probabilities the technique deriving the Chernoff bound, equation (2), to get

$$P_\theta(|X_n| > \varepsilon n^\alpha) \leq \left(e^{g_\theta(\gamma)} + e^{g_\theta(-\gamma)}\right) e^{-n^\alpha \gamma \varepsilon}.$$

Let $\varepsilon = n^{-\alpha/2}$ and sum for all n

$$\sum_{n=1}^{\infty} P_\theta(|X_n| > n^{\alpha/2}) \leq \left(e^{g_\theta(\gamma)} + e^{g_\theta(-\gamma)}\right) \sum_{n=1}^{\infty} e^{-n^{\alpha/2}\gamma} < \infty.$$

The sum is convergent and therefore by the Borel-Cantelli lemma is $n^{-\alpha} X_n \xrightarrow{a.s} 0$ when n tends to infinity. When c goes to infinity then $N^\alpha \to \infty$ and therefore $N^{-\alpha} X_N \xrightarrow{a.s} 0$. We have $\lim_{c\to\infty} c^{-1}N = t$ by Lemma 4 and hence $\lim_{c\to\infty} c^{-\alpha} N^\alpha N^{-\alpha} X_N \stackrel{a.s}{=} 0$. □

4 Normality, overshoot and independence

4.1 Asymptotic normality

Gut [7] uses Anscombe's theorem as key factor in proving asymptotic normality for stopping times of random walks hitting a linear or slowly varying boundary. We will use the same technique.

Theorem 1 (Anscombe's Theorem) *Let $\{X_i, i \geq 1\}$ be a sequence of independent, identically distributed random variables with mean 0 and variance σ^2 and let $\{S_n, n \geq 1\}$ denote their partial sums. Further, assume that $N/c \xrightarrow{P} T$, as $c \to \infty$. Then*

$$\begin{aligned} \lim_{c\to\infty} P\left(S_N \leq y\sigma\sqrt{N}\right) &= \Phi(y) \\ \lim_{c\to\infty} P\left(S_N \leq y\sigma\sqrt{cT}\right) &= \Phi(y). \end{aligned}$$

The formulation of the theorem is changed to this context. For a proof see Gut [7].

Theorem 2 *Assume that* $Var_\theta(X_i) = \sigma^2 = g''(\theta) < \infty$ *and* $g'(\theta) = b(t)/t$, *then for* $t \leq T^*$

$$\lim_{c\to\infty} P_\theta\left(\frac{b(t)-b'(t)t}{t}(N-ct) \leq y\sigma\sqrt{ct}\right) \quad = \quad \Phi(y).$$

In the special case $g'(\theta) = b(T)/T$, *we have* $b(T) - b'(T)T = \theta^{-1}R(T)$ *and*

$$\lim_{c\to\infty} P_\theta\left(\frac{R(T)}{\theta T}(N-cT) \leq y\sigma\sqrt{cT}\right) \quad = \quad \Phi(y).$$

Proof. Start with Anscombe's theorem,

$$\lim_{c\to\infty} P_\theta\left(S_N - g'(\theta)N \leq y\sigma\sqrt{ct}\right) = \Phi(y).$$

The stopped process is by Definition 1 greater than or equal to the barrier. Furthermore the overshoot $Z \leq X_N$, so that

$$\frac{cb(N/c)-g'(\theta)N}{\sigma\sqrt{ct}} \leq \frac{S_N - g'(\theta)N}{\sigma\sqrt{ct}} \leq \frac{cb(N/c)-g'(\theta)N}{\sigma\sqrt{ct}} + \frac{X_N}{\sigma\sqrt{ct}}. \tag{16}$$

The last term on the right hand side converges to zero in probability by Lemma 5. Taylor expand $cb(N/c) = cb(t) + b'(t)(N-ct) + \frac{b''(\xi)}{2c}(N-ct)^2$, where ξ is between t and N/c, and plug the expansion in (16) and use that $g'(\theta) = b(t)/t$, which gives

$$\begin{aligned}
\frac{cb(N/c)-g'(\theta)N}{\sigma\sqrt{ct}} &= \frac{cb(t) + b'(t)(N-ct) + \frac{b''(\xi)}{2c}(N-ct)^2 - g'(\theta)N}{\sigma\sqrt{ct}} \\
&= \frac{cb(t) + b'(t)(N-ct) + \frac{b''(\xi)}{2c}(N-ct)^2}{\sigma\sqrt{ct}} \\
&\quad -\frac{g'(\theta)(N-ct)+cb(t)}{\sigma\sqrt{ct}} \\
&= -\frac{(b(t)-b'(t)t)(N-ct)}{\sigma t\sqrt{ct}} + \frac{b''(\xi)(N-ct)^2}{2c^{3/2}t^{1/2}\sigma}.
\end{aligned}$$

When $s \leq T^*$ then $b''(s)$ is bounded and $c^{-3/2}(N-ct)^2 \stackrel{P_\theta}{\to} 0$, as c goes to infinity by Lemma 4. Hence the quadratic part converges to zero. The theorem follows by the symmetry of the normal distribution. In the special case $g'(\theta) = b(T)/T$ then by the remark in connection to Lemma 1, imply that,

$$\frac{(b(T)-b'(T)T)(N-cT)}{\sigma T\sqrt{cT}} = \frac{R(T)(N-cT)}{\theta T\sigma\sqrt{cT}}.$$

□

Note that for arbitrary $0 < \eta \neq 1/2$ it is easy to show that $c^{-\eta}(N-ct)$ and the overshoot are asymptotically independent.

4.2 Overshoot

In this section we show that the overshoot is distributed as an overshoot over a linear barrier. In our case the linear barrier of interest is

$$cl(n/c) = cb(t) + b'(t)(n - ct). \tag{17}$$

Assume that X_i are non-arithmetic and let

$$\begin{aligned} N_+ &= \inf\{n : n \geq 1, S_n - nb'(t) > 0\} \\ N_- &= \inf\{n : n \geq 1, S_n - nb'(t) < 0\} \\ M &= \inf\{m : S_m \geq cl(m/c)\}. \end{aligned}$$

Define the linear overshoot as

$$Z_l = S_M - cl(M/c). \tag{18}$$

Lemma 6 *Assume that* $0 < E_\theta[X_i] = b(t)/t < \infty$*, for* $t \leq T^*$*. Then*

$$\begin{aligned} \lim_{c\to\infty} P_\theta(Z_l > z) &= \frac{1}{E_\theta[S_{N_+} - b'(t)N_+]} \int_{(z,\infty)} P_\theta(S_{N_+} - b'(t)N_+ > x)dx. \\ &= Q_\theta(z) \end{aligned}$$

where $g'(\theta) = b(t)/t$*.*

Proof. It is well known, that when $b'(t) = 0$ then

$$Q_\theta(z) = \lim_{c\to\infty} P_\theta(S_M - cb(t) > z) = \frac{1}{E_\theta[S_{N_+}]} \int_{(z,\infty)} P_\theta(S_{N_+} > x)dx, \tag{19}$$

see Siegmund [14]. In our case the overshoot is over a linear barrier $cb(t) - b'(t)(n - ct)$ and the drift has to be changed to $g'(\theta) - b'(t)$ to fit (19). The drift is still positive toward the barrier due to the fact that t is not on the shadow side (13) and $g'(\theta) = b(t)/t \geq b'(t)$. The barrier the process has to pass is $c(b(t) - b'(t)t)$, which in the case $t = T$ is equal to $cR(T)/\theta$. □

Lemma 7 *Assume that* $0 < E_\theta[X_i] = g'(\theta)$*. Then for* $\theta > 0$ *and* $t \leq T^*$

$$\begin{aligned} \lim_{c\to\infty} E_\theta\left[e^{-\theta Z_l}\right] &= \frac{P_\theta(N_- = \infty)}{(b(t)/t - b'(t))\theta} \left(1 - E\left[\exp\left(\theta(S_N - S_{N_+}) - g(\theta)N + \theta b'(t)N_+\right)\right]\right) \\ &= 1 + O(\theta), \end{aligned}$$

where $g'(\theta) = b(t)/t$*.*

Proof. We start by using the distribution of the overshoot Lemma 6 and partially integrate to get

$$\begin{aligned} \lim_{c\to\infty} E_\theta\left[e^{-\theta Z_l}\right] &= \frac{1}{E_\theta[S_{N_+} - b'(t)N_+]} \int_{(0,\infty)} e^{-\theta x} P_\theta(S_{N_+} - b'(t)N_+ > x)dx \\ &= \frac{1 - E_\theta\left[\exp\left(-\theta(S_{N_+} - b'(t)N_+)\right)\right]}{\theta E_\theta[S_{N_+} - b'(t)N_+]}. \end{aligned} \tag{20}$$

Corollary 8.39 in [14] provides us with $E_\theta[N_+] = 1/P_\theta(N_- = \infty)$. This and Wald's identity, $g'(\theta) = b(t)/t$, imply that

$$E_\theta[S_{N_+} - b'(t)N_+] = (g'(\theta) - b'(t))E_\theta[N_+] = \frac{b(t)/t - b'(t)}{P_\theta(N_- = \infty)}. \tag{21}$$

Now a change back to the original measure gives,

$$E_\theta\left[\exp\left(-\theta(S_{N_+} - b'(t)N_+)\right)\right] = E\left[\exp\left(\theta(S_N - S_{N_+}) - g(\theta)N + \theta b'(t)N_+\right)\right]. \tag{22}$$

For θ close to zero we can approximate

$$\begin{aligned} 1 - E_\theta\left[\exp\left(-\theta(S_{N_+} - b'(t)N_+)\right)\right] &\approx \\ \approx E_\theta[N_+]\theta(b(t)/t - b'(t)) &- \frac{\theta^2}{2}E_\theta[(S_{N_+} - b'(t)N_+)^2] \\ = \frac{\theta(b(t)/t - b'(t))}{P_\theta(N_- = \infty)} &+ O(\theta^2). \end{aligned}$$

By this and (21) plugged into (20) we see that $\lim_{c\to\infty} E_\theta\left[e^{-\theta Z_l}\right] = 1 + O(\theta)$. □

We now turn to the overshoot $Z = S_N - cb(N/c)$. The aim is to show that the overshoot converges in distribution to the overshoot over the linear barrier $cl(n/c)$. This could be proved by a transformation to a perturbed random walk and use of the results in Siegmund [14] and then transforming back. The intuition and simplicity of the idea is then however lost.

Define the auxiliary stopping time, for any $t \leq T^*$

$$N_\alpha = \inf\{n : S_n > (c - c^\alpha)b(t) + b'(t)(n - (c - c^\alpha)t)\} \tag{23}$$

and denote by $\Delta N = N - N_\alpha$.

The choice of the constant $\alpha > 0$ will later be such that the distance left to the true barrier can be neglected in limit on the scale c, but far enough for the overshoot to gain the asymptotic properties.

The overshoot over the auxiliary barrier is called Z_α. The probability that

$$\begin{aligned} P_\theta(N = N_\alpha) &\leq P_\theta(Z_\alpha > c^\alpha(b(t) - b'(t)t)) \\ &\leq \exp\left(-c^\alpha\gamma(b(t) - b'(t)t) + g_\theta^{Z_\alpha}(\gamma)\right), \end{aligned} \tag{24}$$

where $\gamma > 0$ is such that $g_\theta^{Z_\alpha}(\gamma) = \log\left(E_\theta[\exp(\gamma Z)]\right) < \infty$. The proof that such a γ exists can be found in Hammarlid [8]. Furthermore, $b(t) - b'(t)t > 0$ for all $t \leq T^*$, which follows by equation (13).

Lemma 8 *Assume that* $0 < \alpha < 1$. *Then under the tilted distribution for arbitrary* $0 < \delta < b(t) - b'(t)t$*, there is a time* $\hat{t} \leq T^*$ *and a constant* $d > 0$ *such that*

$$P_\theta\left(|\Delta N - c^\alpha\hat{t}| > c^\alpha\delta | N_\alpha = n_\alpha, Z_\alpha = z\right) \leq 3c^\alpha\hat{t}e^{-c^\alpha d\delta^2},$$

where $g'(\theta) = b(t)/t$ *and* $c^{-\alpha}\Delta N \xrightarrow{a.s} t$ *as* $c \to \infty$.

Proof. The idea of the proof is that when the auxiliary barrier is passed then the remaining distance is of order c^α and therefore also the remaining time.

The set $\{\omega : |\Delta N - c^\alpha \hat{t}| > c^\alpha \delta\} = A' \cup A''$ where

$$\begin{array}{rcl} A' & = & \{S_n \geq cb(n/c), \text{ for some } n \leq N_\alpha + c^\alpha(\hat{t} - \delta)\}, \\ A'' & = & \{S_n < cb(n/c), \text{ for all } n \leq N_\alpha + c^\alpha(\hat{t} + \delta)\}. \end{array}$$

Write $\Delta n = n - N_\alpha$ when $n > N_\alpha$ and denote by $\Delta S_n = S_n - S_{N_\alpha}$. The partial sum can in this notation be written as

$$S_n = \Delta S_n + S_{N_\alpha} = \Delta S_n + (c - c^\alpha)b(t) + b'(t)(N_\alpha - (c - c^\alpha)t) + Z_\alpha.$$

Hence if $c^\alpha h(\Delta n/c^\alpha) = cb(n/c) - S_{N_\alpha}$ then,

$$\begin{array}{rcl} A' & = & \{\Delta S_n \geq c^\alpha h(\Delta n/c^\alpha), \text{ for some } \Delta n \leq c^\alpha(\hat{t} - \delta)\}, \\ A'' & = & \{\Delta S_n \geq c^\alpha h(\Delta n/c^\alpha), \text{ for all } \Delta n \leq c^\alpha(\hat{t} + \delta)\}. \end{array}$$

Expand $c^\alpha h(\Delta n/c^\alpha)$ around t,

$$c^\alpha h(\Delta n/c^\alpha) = c^\alpha b(t) + b'(t)(\Delta n - c^\alpha t) + \frac{b''(t)}{2c}(n - ct)^2 - Z_\alpha.$$

The most probable time of hitting $c^\alpha h(\Delta n/c^\alpha)$ is when the barrier is crossed by the drift line $g'(\theta)\Delta n$. The most probable time $\hat{t} = \Delta n/c^\alpha$ is the solution to

$$g'(\theta)\frac{\Delta n}{c^\alpha} = b(t) + b'(t)\left(\frac{\Delta n}{c^\alpha} - t\right) + \frac{b''(t)}{2c^{1+\alpha}}(n - ct)^2 - \frac{Z_\alpha}{c^\alpha}.$$

The overshoot part converges almost surely to zero by combining $Z_\alpha \leq X_{N_\alpha}$ and Lemma 5. The quadratic term converges to zero since $N_\alpha \leq n \leq N$ and $c^{-(1+\alpha)}(n - ct)^2 \leq c^{-(1+\alpha)}((N_\alpha - ct)^2 + (N - ct)^2)$, which by Lemma 4 converges almost surely to zero. The equation

$$g'(\theta)\hat{t} = b(t) + b'(t)(\hat{t} - t) + O(1/c^\alpha) \tag{25}$$

does always have a solution when c is large enough, because $t \leq T^*$ and therefore $g'(\theta) = b(t)/t > b'(t)$ and $b(0) > 0$, see the implication of not being on the shadow side, equation (13). Take a $d \leq \gamma$ where $g_\theta^{Z_\alpha}(\gamma) < \infty$ and let c be such that $c^\alpha t \geq \exp\left(g_\theta^{Z_\alpha}(\gamma)\right)$. Then by Lemma 3 and the law of total probability in combination with equation (24) imply that

$$P_\theta\left(|\Delta N - c^\alpha \hat{t}| > c^\alpha \delta | N_\alpha, Z_\alpha\right) \leq P_\theta\left(A' \cup A''\right) + P_\theta\left(N = N_\alpha\right) \leq 3c^\alpha \hat{t} e^{-c^\alpha d \delta^2}.$$

We use the equation determining the dominating point (5) to find the solution $\hat{t} = t + O(1/c^\alpha)$ to (25). Now if we put $\delta = c^{-\alpha/4}$ and sum over c,

$$\sum_{c=1}^{\infty} P_\theta\left(|\Delta N - c^\alpha t| > \frac{c^\alpha}{c^{\alpha/4}} \middle| N_\alpha = n_\alpha, Z_\alpha = z_\alpha\right) \leq 3\sum_{c=1}^{\infty} c^\alpha \hat{t} e^{-c^{\alpha/2} d} < \infty.$$

Borel-Cantelli lemma assures the almost sure convergence. □

Lemma 9 *Let $0 < \alpha < 1/4$ and $g'(\theta) = b(t)/t$, where $t \leq T^*$. Then under the tilted distribution*

$$\lim_{c\to\infty} c^{-1}\left((N-ct)^2 - (N_\alpha - (c-c^\alpha)t)^2\right) \overset{a.s}{=} 0.$$

Proof. The trick of the proof is to split the sample space into two sets, the likely event A and its complement A^*. The probability of the unlikely event converges to zero, and on the likely event the distance between the two quadratic terms converges to zero.

Denote $\hat{c} = c - c^\alpha$ and take arbitrary $\varepsilon > 0$, $\eta > 0$ and $0 < \delta < 1$ to define the set

$$A = \{\omega : |N_\alpha - \hat{c}t| < \hat{c}\varepsilon\} \cap \left\{|\Delta N - c^\alpha \hat{t}| < c^\alpha \delta\right\} \cap \{|N - ct| < c\varepsilon\}. \qquad (26)$$

On the set A we have by the conjugate rule that,

$$\begin{aligned}\left|(N-ct)^2 - (N_\alpha - \hat{c}t)^2\right| &= |N - N_\alpha - c^\alpha t|\,|N - ct + N_\alpha - \hat{c}t| \\ &\leq c^\alpha \delta\varepsilon(c+\hat{c}) \leq 2c^{\alpha+1}\varepsilon.\end{aligned}$$

We chose $\varepsilon = c^{-2\alpha}/2$ then

$$c^{-1}\left|(N-ct)^2 - (N_\alpha - \hat{c}t)^2\right| \leq c^{-\alpha} \leq \eta \quad \text{when} \quad c > \eta^{-1/\alpha}. \qquad (27)$$

Hence $P_\theta\left(\left|(N-ct)^2 - (N_\alpha - \hat{c}t)^2\right| > c\eta, A\right) = 0$. By this, the law of total probability, Lemma 4 and Lemma 8 we have for $c \geq \eta^{-1/\alpha}$,

$$\begin{aligned}&P_\theta\left(\left|(N-ct)^2 - (N_\alpha - \hat{c}t)^2\right| > c\eta\right) \leq \\ &\leq P_\theta(A^*) \\ &\leq 3c^\alpha \hat{t} e^{-c^\alpha d\delta^2} + 2\hat{c}te^{-\hat{c}c^{-4\alpha}d} + 2cte^{-c^{1-4\alpha}d} \\ &\leq 4ct\left(e^{-c^\alpha d\delta^2} + e^{-\hat{c}c^{-4\alpha}d}\right).\end{aligned}$$

In the last inequality we used $\hat{c} \leq c$ and assumed that $\hat{t} \leq t$. There could as well be the other way around, $\hat{t} \geq t$, which causes no practical change.

Let $\eta = c^{-1/4}$. Then since $\alpha < 1/4$ we have $c^{1/4\alpha} \leq c$, which imply a convergent sum,

$$\sum_{c=1}^{\infty} P_\theta\left(c^{-1}\left|(N-ct)^2 - (N_\alpha - \hat{c}t)^2\right| > 1/c^{1/4}\right) \leq 4\sum_{c=1}^{\infty} ct\left(e^{-c^\alpha d\delta^2} + e^{-\hat{c}c^{-4\alpha}d}\right).$$

The convergence is therefore almost sure by Borel-Cantelli lemma.

□

The idea now is to look at a stopping time of the process to a barrier that is just before the true barrier. This barrier does not have any contribution to the curvature part after N_α. This auxiliary stopping time is defined as

$$N_l = \inf\{n : S_n > cl(n/c) + b''(t)(N_\alpha - (c-c^\alpha)t)^2/2c - b''(t)/2c^\alpha\}$$

and corresponding auxiliary overshoot

$$\tilde{Z}_l = S_{N_l} - cl(N_l/c) - b''(t)(N_\alpha - (c-c^\alpha)t)^2/2c + b''(t)/2c^\alpha,$$

where $t \leq T^*$. The extra term $b''(t)/2c^\alpha$ is added to make sure that this auxiliary barrier is in front of the true barrier on a certain set and therefore $N_l \leq N$.

Lemma 10 *Under the tilted distribution where $g'(\theta) = b(t)/t$,*

$$N - N_l \xrightarrow{P_\theta} 0, \qquad \text{as } c \to \infty.$$

Proof. Denote as before $\hat{c} = (c - c^\alpha)$. On the set A defined in (26) with $\varepsilon = c^{-2\alpha}/2$ and $\alpha < 1/4$ we have for $N_\alpha \le n \le N$ that

$$\frac{b''(t)}{2c}\left((n-ct)^2 - (N_\alpha - \hat{c}t)^2\right) + \frac{b''(t)}{2c^\alpha} \ge \frac{b''(t)}{2c^\alpha} - \frac{b''(t)}{2c^\alpha} = 0,$$

by the same reasoning that lead to equation (27), hence $N_l \le N$. Let us now study when the stopping times are not equal on A. That is the set

$$\begin{aligned}
\{\omega : N > N_l\} &= \left\{\omega : cl(N_l/c) + \frac{b''(t)}{2c}(N_\alpha - \hat{c}t)^2 - \frac{b''(t)}{2c^\alpha} \le S_{N_l} < cb(N_l/c)\right\} \\
&= \left\{\omega : 0 \le \tilde{Z}_l < \frac{b''(t)}{2c}\left((N_l - ct)^2 - (N_\alpha - \hat{c}t)^2\right) + \frac{b''(t)}{2c^\alpha}\right\} \\
&\subseteq \left\{\omega : 0 \le \tilde{Z}_l < \frac{b''(t)}{c^\alpha}\right\}.
\end{aligned}$$

We have by the law of total probability, equation (24) and $\lim_{c\to\infty} P_\theta(A^*) = 0$ that,

$$\begin{aligned}
\lim_{c\to\infty} P_\theta(N \ne N_l) &\le \lim_{c\to\infty} P_\theta(N > N_l, A) + P_\theta(A^*) + P(N = N_\alpha) \\
&\le \lim_{c\to\infty} P_\theta(0 \le Z_l < c^{-\alpha} b''(t)) = \lim_{c\to\infty} F^c_{Z_l}(c^{-\alpha} b''(t)),
\end{aligned}$$

where $F^c_{Z_l}(z)$ is the distribution function of the overshoot for a fixed c. The limit distribution $F_{Z_l}(z)$ is continuous, Lemma 6. Therefore, for fixed $\nu > 0$, there is a c such that $c^{-\alpha} b''(t) < \nu$ and $F^c_{Z_l}(c^{-\alpha} b''(t)) \le F^c_{Z_l}(\nu)$. The limit $\lim_{c\to\infty} F^c_{Z_l}(c^{-\alpha} b''(t)) \le F_{Z_l}(\nu) \le \nu$, but ν is arbitrary. □

Theorem 3 *The distribution of the overshoot under the tilted distribution, where $g'(\theta) = b(t)/t$ and $t \le T^*$ satisfy,*

$$\lim_{c\to\infty} P_\theta(Z \le z) = \lim_{c\to\infty} P_\theta(S_M - cb(t) - b'(t)(M - ct) \le z) = Q_\theta(z).$$

$Q_\theta(z)$ is the asymptotic distribution of the overshoot given in Lemma 6 and M is the stopping time to the linear barrier.

Proof. The idea of the proof is to condition on when and where the process crosses the auxiliary barrier $cl(n/c)$. This barrier is close enough to capture the curvature of the real barrier at the stopping time, distant to keep the asymptotic properties of the overshoot. The real barrier is then exchanged to a linear barrier translated by the curvature at the stopping time of the auxiliary barrier.

In the Taylor expansion of the barrier the remainder, the third order term, is ignored because, according to Lemma 4 it converges to zero. Thus, $cb(N/c) \overset{P_\theta}{=} cb(t) + b'(t)(N - ct) + c^{-1} b''(t)(N - ct)^2$, when the scale parameter c goes to infinity. The partial sum hits the auxiliary barrier before the original barrier at N_α. The difference in curvature is vanishing almost surely, so that $c^{-1}(N - ct)^2$ can be exchanged by $c^{-1}(N_\alpha - (c - c^\alpha)t)^2$ according to Lemma 9 and the difference between N and N_l equals zero almost surely as $c \to \infty$ by Lemma 8. We have

$$S_N - cb(N/c) \overset{P_\theta}{=} S_{N_l} - cb(t) - b'(t)(N_l - ct) - \frac{b''(t)}{2c}(N_\alpha - ct)^2 + \frac{b''(t)}{2c^\alpha},$$

when c goes to infinity. The quadratic term on the right hand side is asymptotically a χ^2-distributed random variable. Therefore we have

$$S_{N_l} - cb(t) - b'(t)(N_l - ct) - \frac{b''(t)}{2c}(N_\alpha - ct)^2 \overset{d}{=} S_M - cb(t) - b'(t)(M - ct),$$

when c goes to infinity. □

4.3 Asymptotic independence

Theorem 4 *Under the tilted distribution, when $g'(\theta) = b(t)/t$ and $t \leq T^*$, then the overshoot Z and $Y = a^{-1}c^{-1/2}(N - ct)$ are asymptotically independent, where the variance $a^2 = g''(\theta)t^3(b(t) - b'(t)t)^{-2}$, so that*

$$\lim_{c\to\infty} P_\theta(Z \leq z, Y \leq y) = Q_\theta(z)\Phi(y).$$

When $t = T$ then $a^2 = g''(\theta)\theta^2 T^3/R(T)^2$.

Proof. Take $0 < \alpha < 1/2$. When we condition on N_α and S_{N_α} it is possible to write

$$\begin{aligned} Z &= \Delta S_N - c^\alpha h(\Delta N/c^\alpha), \\ Y &= a^{-1}c^{-1/2}(N - ct). \end{aligned}$$

Write $Y_\alpha = a^{-1}c^{-1/2}(N_\alpha - ct)$ and define

$$A = \{\omega : |Y_\alpha - Y| < \varepsilon\} = \{\omega : |\Delta N - c^\alpha t| < c^{1/2}\varepsilon\}.$$

Note that $\lim_{c\to\infty} P_\theta(A^*) = 0$ by Lemma 8. By the law of total probability

$$P_\theta(Z \leq z, Y \leq y) \leq P_\theta(Z \leq z, Y_\alpha \leq y + \varepsilon, A) + P(A^*).$$

We use Bayes' theorem and that Y_α and $Z = \Delta S_N - c^\alpha h(\Delta N/c^\alpha)$ are conditionally independent by construction to get an upper bound,

$$P_\theta(Z \leq z, Y \leq y, A) \leq \int_{A, Y_\alpha \leq y+\varepsilon} P_\theta(Z \leq z | N_\alpha, S_{N_\alpha}) dP_\theta(N_\alpha, S_{N_\alpha}).$$

Therefore by Theorem 2, Theorem 3, $\lim_{c\to\infty} P(A) = 1$ and the Bounded convergence theorem we have

$$\lim_{c\to\infty} P_\theta(Z \leq z, Y \leq y) \leq Q_\theta(z)\Phi(y + \varepsilon).$$

A lower bound $Q_\theta(z)\Phi(y - \varepsilon) \leq \lim_{c\to\infty} P_\theta(Z \leq z, Y \leq y)$ is found in a similar way. Thus

$$Q_\theta(z)\Phi(y - \varepsilon) \leq \lim_{c\to\infty} P_\theta(Z \leq z, Y \leq y) \leq Q_\theta(z)\Phi(y + \varepsilon),$$

but ε is arbitrary and $\lim_{c\to\infty} P_\theta(Z \leq z, Y \leq y) = Q_\theta(z)\Phi(y)$. □

5 Main result

We will start by a heuristic large deviation argument, in the spirit of the introduction. The expansion of $R(s) = R(T) + 0.5R''(T)(s-T)^2$ since $R'(T) = 0$ and

$$P(N \approx cT + c^{1/2}y) \approx \text{constant} \cdot \exp\left(-cR(T) - R''(T)y^2/2\right),$$

where $R''(T) = R(T)^2/g''(\theta)^2\theta^2T^3 + \theta b''(T)$ according to (12). The quadratic part is almost the density of a normal distribution and therefore one would expect that

$$P(N \le cT + c^{1/2}y) \approx \text{constant} \cdot e^{-cR(T)}\Phi\left(\sqrt{R''(T)}y\right).$$

Theorem 5 *Assume that the barrier $b(s)$ is convex, three times continuously differentiable, satisfies $b(0) > 0$ and that $\mu s < b(s)$ for all s. The rate of the stopping time N is $R(T) = \theta b'(T) - g(\theta)T$, where the parameters solve,*

$$g'(\theta) = \frac{b(T)}{T} \qquad \text{and} \qquad g(\theta) = \theta b'(T).$$

Then the asymptotic distribution of the first passage time,

$$\lim_{c\to\infty} e^{cR(T)}P\left(N \le cT + \sqrt{c}y\right) = a^{-1}\eta E_\theta\left[e^{-\theta Z_l}\right]\Phi(y/\eta),$$

where $a^2 = g''(\theta)^2\theta^2T^3/R(T)^2$, the variance $\eta^{-2} = a^{-2} + b''(T)\theta = R''(T)$ and Z_l is the overshoot over the barrier $cb(T) + b'(T)(n - cT)$.

Proof. Denote the indicator function $\mathbf{1} = \mathbf{1}(N \le cT + \sqrt{c}y)$, then

$$e^{cR(T)}P\left(N \le cT + \sqrt{c}y\right) = E\left[\mathbf{1}e^{cR(T)}\right].$$

Tilt the distribution by adding and subtracting θS_N and $g(\theta)N$ in the exponent, $E[\mathbf{1}\exp(cR(T))] = E_\theta[\mathbf{1}\exp(cR(T) + g(\theta)N - \theta S_N)]$, where $g'(\theta) = b(T)/T$. Expand S_N to order three around cT,

$$S_N = cb(T) + b'(T)(N - cT) + b''(T)\frac{(N-cT)^2}{2c} + b'''(\xi)\frac{(N-cT)^3}{3!c^2} + Z. \quad (28)$$

We will skip the rest term since it converges to zero by Lemma 4, since ξ is between t and N/c and that $b'''(s)$ is bounded for all $s \le T^*$.

The rate $R(T) = \theta b(T) - g(\theta)T$ and the first derivative $R'(T) = \theta b'(T) - g(\theta) = 0$. Therefore,

$$E_\theta\left[\mathbf{1}\exp\left(cR(T) + g(\theta)N - \theta S_N\right)\right] = E_\theta\left[\mathbf{1}\exp\left(-\theta\frac{b''(T)}{2c}(N-cT)^2 - \theta Z\right)\right]. \quad (29)$$

Write $Y = c^{-1/2}(N - cT)$, which by Theorem 2 converges in probability to a normal random variable Y, with variance $a^2 = R(T)^{-2}g''(\theta)^2\theta^2T^3$. Theorem 4 gives that Y and the overshoot Z are asymptotically independent. Furthermore the overshoot converges to the overshoot over a linear barrier Z_l by Theorem 3. We have that by the bounded convergence Theorem,

$$\lim_{c\to\infty} E_\theta\left[\mathbf{1}\exp\left(-\theta b''(T)\frac{(N-cT)^2}{2c} - \theta Z\right)\right] = E_\theta\left[e^{-\theta Z_l}\right]E_\theta\left[\mathbf{1}\exp\left(-\theta b''(T)\frac{Y^2}{2}\right)\right]. \quad (30)$$

To handle the squared normal distributed random variable, we will change variance,

$$E_\theta\left[\mathbf{1}\exp\left(-\theta b''(T)\frac{Y^2}{2}\right)\right]=\frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{y}\exp\left(-\theta\frac{b''(T)u^2}{2}-\frac{u^2}{2a^2}\right)du.$$

The goal is to be able to write the probability in terms of a standard normal distribution. As a start towards this goal let η^2 be the variance of a new normal distribution, where

$$\eta^{-2}=\theta b''(T)+a^{-2}=\theta b''(T)+\frac{R(T)^2}{g''(\theta)^2\theta^2T^3}=R''(T)$$

according to (12). Then after a change of variables,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{y}\exp\left(-\theta\frac{b''(T)u^2}{2}-\frac{u^2}{2a^2}\right)du &= \frac{\eta}{a}\int_{-\infty}^{y/\eta}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right)dv\\ &= \frac{\eta}{a}\Phi(y/\eta).\end{aligned}$$

□

When $t<T$ then $R'(t)\neq 0$ and the equivalent to equation (29) then becomes

$$e^{cR(t)}P(N\le ct)=E_\theta\left[\mathbf{1}\exp\left(-R'(t)(N-ct)-\theta\frac{b''(t)}{2c}(N-ct)^2-\theta Z\right)\right],$$

where now $g'(\theta)=b(t)/t$. The overshoot still factors out by asymptotic independence, but the rest of the slowly changing function is not easily handled. Heuristically, if we write this expectation as a sum and exchange the probability by its Fourier inversion of the characteristic function, then since the central part converges to the characteristic function of a normally distributed random variable, we can after some calculations believe that

$$\lim_{c\to\infty}c^{1/2}e^{cR(t)}P\left(N\le ct\right)=\frac{E_\theta\left[e^{-\theta Z_l}\right]}{\left(1-e^{R'(t)}\right)\sqrt{2\pi}a},\qquad a^2=g''(\theta)t^3(b(t)-b'(t)t)^{-2}.$$

This conjecture is proved to be right in Hammarlid [8] by use of a sequence of auxiliary stopping times.

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A large deviation estimate of the first passage time to a convex barrier

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Abstract

The asymptotic distribution of the first passage time of a random walk to a scaled convex barrier decays exponentially in the scale parameter multiplied by a slowly changing function. The exponential rate of decay is connected to the most probable time of passing the barrier. The rate and the slowly changing function alters for stopping times strictly before the most probable time. Here the slowly changing function is shown to be a constant multiplied by the square root of the scale parameter. This is proved using large deviation techniques and the characteristic function. A small but difficult technicality in the computations is to show that the integral of the non central parts of the characteristic function multiplied by the square root of the scale parameter vanishes. The convergence of this integral is proved with the help of some auxiliary stopping times who's sum equals the true stopping time and that are conditionally independent.

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1 Introduction

A sum $S_n = \sum_{i=1}^{n} X_i$ that starts at zero and where the increments are independent identically distributed, makes the unlikely event that it hits a distant upper convex barrier $cb(n/c)$. The barrier is convex, three times continuously differentiable, $b(0) > 0$ and c is a scale parameter.

Definition 1.1 *The time when the process S_n, for the first time, passes or hits $cb(n/c)$ is denoted by $N = \inf\{n : S_n \geq cb(n/c)\}$.*

What is the asymptotic distribution of the stopping time, when the scale parameter c goes to infinity?

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Assume that the cumulant generating function $g(\theta) = \log\left(E\left[e^{\theta X}\right]\right)$ exists for all θ in some open set. In the field of large deviations the rate function $I(x) = \sup_\theta(x\theta - g(\theta))$ is central and for any convex set B,

$$\lim_{n\to\infty} \frac{1}{n} \log\left(P\left(\frac{S_n}{n} \in B\right)\right) = -\inf_{x\in B} I(x), \qquad (1)$$

see Bucklew [2] or Dembo and Zeitouni [4].

There is also a rate function for the asymptotic distribution of the stopping time to a convex barrier. Martin-Löf used large deviation techniques and Wald's identity to prove the exponential decay when $t \geq T$. Hammarlid [5] used a more probabilistic approach to compute the rate function for the two cases $t < T$ and $t \geq T$ and proved the following lemma.

Lemma 1.2 *Let $\mu = E[X_i]$ and assume that $\mu s < b(s)$ for all s. Then the function $R(s) = sI(b(s)/s)$ is convex and it attains its minimum at T, which is determined by*

$$g(\theta) = b'(T)\theta \qquad \text{and} \qquad g'(\theta) = b(T)/T. \qquad (2)$$

The function $R(s) = \theta b(s) - g(\theta)s$, where $g'(\theta) = b(s)/s$ and

$$\lim_{c\to\infty} c^{-1} \log\left(P(N \leq ct)\right) = \begin{cases} -R(t) & \text{for } t < T \\ -R(T) & \text{for } t \geq T. \end{cases} \qquad (3)$$

Furthermore, under a tilted measure P_θ, which changes the drift to $b(T)/T$, it is proved in Hammarlid [5] that: The limit distribution of the overshoot $Z = S_N - cb(N/c)$ is distributed as an overshoot over a linear barrier, $Z_l = S_N - cb(T) - b'(t)(N - cT)$, when c goes to infinity. Properly normalized the stopping time is asymptotically normally distributed. The overshoot and the asymptotic normal part are asymptotically independent. These results were then used to prove that, when $g'(\theta) = b(T)/T$, then

$$\lim_{c\to\infty} e^{cR(T)} P\left(N \leq cT + \sqrt{c}y\right) = a^{-1}\eta E_\theta\left[e^{-\theta Z_l}\right]\Phi(y/\eta),$$

where $\eta^{-2} = a^{-2} + b''(T)\theta$ and $a^2 = g''(\theta)^2\theta^2 T^3/R(T)$.

We write when c is large that $P(N \leq ct) \approx K(c)\exp(-cR(t))$, where $K(c)$ is some slowly changing function in comparison to the exponential part.

This article is dedicated to show that $K(c) \approx \text{constant} \cdot c^{-1/2}$ in the case $t < T$. That is when $g'(\theta) = b(t)/t$ and $Z_l = S_N - cb(t) - b'(t)(N - ct)$ then

$$\lim_{c\to\infty} c^{1/2} e^{cR(t)} P\left(N \leq ct\right) = \frac{E_\theta\left[e^{-\theta Z_l}\right]}{\left(1 - e^{R'(t)}\right)\sqrt{2\pi}a},$$

where $a^2 = \sigma^2 t^3 (b(t)/t - b'(t))^{-2}$ and $R'(t) = \theta b'(t) - g(\theta) < 0$.

The proof of the this result, that is found in Section 2, involves the inversion formula of the characteristic function. The essential observation of the proof is that the central part of the characteristic function multiplied by $c^{1/2}$ converges to the characteristic function of a normally distributed random variable and that the remaining part vanishes, when c goes to infinity. The convergence to zero of the noncentral part is proved by the help of some auxiliary stopping times that are conditionally independent. These auxiliary stopping times have the properties one would expect, compared to the properties of N. The lemmas describing these properties are therefore put in Appendix A.

2 Main result

It is useful to tilt the distribution of S_N such that the drift changes to $b(t)/t$, which means that the most likely time to hit the barrier under the tilted measure is t. Tilting a distribution $F(x)$ is defined by $dF_\theta(x) = \exp(\theta x - g(\theta))dF(x)$, for every θ in the open set of definition, see Bucklew [2] or Martin-Löf [8]. The tilted expectation is denoted $E_\theta[\cdot]$ and the tilted cumulant function $g_\theta(\gamma) = E_\theta[\exp(\gamma X)]$. It can easily be shown that $g_\theta(\gamma) = g(\gamma+\theta) - g(\theta)$ and therefore, from the first and second order derivative of the tilted cumulant function at zero,

$$E_\theta[X] = g'(\theta) \qquad \text{and} \qquad Var_\theta(X) = g''(\theta). \tag{4}$$

The tilted rate function is $I_\theta(b(t)/t) = \gamma b(t)/t - g_\theta(\gamma)$, where γ solves $g'_\theta(\gamma) = b(t)/t$ and $I_\theta(g'(\theta)) = I_\theta(b(t)/t) = I'_\theta(b(t)/t) = 0$. The following lemma and corollary were shown in Hammarlid [5].

Lemma 2.1 *Fix $\varepsilon > 0$ and let $\gamma > 0$ and $\alpha > 0$ such that $\alpha < 2\gamma$. Then under the tilted distribution there is a $d > 0$ so that,*

$$P_\theta\left(\left|\frac{(N-ct)^\alpha}{c^\gamma}\right| > \varepsilon\right) \le 2ct\exp\left(-d\varepsilon^{2/\alpha}c^{2\gamma/\alpha-1}\right),$$

where $g'(\theta) = b(t)/t$. Also, $c^{-\gamma}(N-ct)^\alpha \xrightarrow{a.s} 0, \text{ as } c\to\infty$.

Corollary 2.2 *Fix $\varepsilon > 0$. Then under the tilted distribution there is a constant $d > 0$ such that, $P_\theta(|N-ct| > c\varepsilon) \le 2cte^{-cd\varepsilon^2}$, where $g'(\theta) = b(t)/t$ and $N/c \xrightarrow{a.s} t, \text{ as } c\to\infty$.*

We will split the stopping time N into some convenient auxiliary stopping times.

Definition 2.3 *Let $0 < \alpha < 2/3$ and $N_1 = \inf\left\{n : S_n \ge c^{\alpha/2}b(t) + b'(t)(n - c^{\alpha/2}t)\right\}$. Denote the auxiliary stopping times and the overshoots by*

$$N_k = \inf\left\{n > N_{k-1} : S_n \ge kc^{\alpha/2}b(t) + b'(t)(n - kc^{\alpha/2}t) + \sum_{j=1}^{k-1} Z_j\right\}, \tag{5}$$

$$Z_j = S_{N_j} - kc^{\alpha/2}b(t) - b'(t)(N_j - kc^{\alpha/2}t) - \sum_{i=1}^{j-1} Z_i, \tag{6}$$

where $Z_1 = S_{N_1} - c^{\alpha/2}b(t) - b'(t)(N_1 - c^{\alpha/2}t)$.

We are first and foremost interested in $N_{[c^{\alpha/2}]}$, where $[c^{\alpha/2}]$ is the integer part of $c^{\alpha/2}$. Call the difference $\Delta n_j = n_j - N_{j-1}$ and the partial sum $\Delta S_{n_j} = S_{n_j} - S_{N_{j-1}}$, then

$$N_{[c^{\alpha/2}]} = \sum_{j=1}^{[c^{\alpha/2}]} \Delta N_j, \qquad \text{where} \qquad \Delta N_j = N_j - N_{j-1},$$

$$\Delta N_j = \inf\left\{\Delta n_j : \Delta S_{n_j} \ge c^{\alpha/2}b(t) + b'(t)(\Delta n_j - c^{\alpha/2}t)\right\}.$$

One important observation is that ΔN_j are independent by construction. The sequence of overshoots can also be rewritten,

$$Z_j = \Delta S_{N_j} - c^{\alpha/2} b(t) - b'(t)(\Delta N_j - c^{\alpha/2} t). \tag{7}$$

In Appendix A it is shown that $N_{[c^{\alpha/2}]} \sim c^{\alpha} t$, Lemma A.2 and that the probability of $N_{[c^{\alpha/2}]} > N$ is decreasing exponentially to zero in c, Lemma A.3.

The idea of the next auxiliary stopping time, $\tilde{N}$, is that it is equal to to N in the limit. The difference between the two, is that $\tilde{N}$ is not dependent of $N_{[c^{\alpha/2}]}$ from the curvature part of the Taylor expansion of the barrier.

Definition 2.4 *Fix $\nu > 0$ that fulfills $0 < 1 - 3\alpha/2 - 2\nu$ and define a function $h(c) = 3b''(t)c^{-\nu/2}/2$. The first time the process passes or hits the barrier*

$$c\tilde{b}(n/c) = cb(t) + b'(t)(n - ct) + \frac{b''(t)}{2c}\left(n - N_{[c^{\alpha/2}]} - (c - c^{\alpha})\, t\right)^2 - h(c)$$

is denoted by $\tilde{N} = \inf\left\{n : S_n \geq c\tilde{b}(n/c)\right\}$ *and the overshoot is written* $Z_{\tilde{N}} = S_{\tilde{N}} - c\tilde{b}(\tilde{N}/c)$.

The function $h(c)$ guarantees that the probability of $\tilde{N} \leq N$ converges to one. How this works will be evident later in the proof of Lemma A.6. The probability of $N_{[c^{\alpha/2}]} > \tilde{N}$ converges to to zero when c goes to infinity by Lemma A.4.

The main theorem is proved by the help of conditional independence, where the unconditioned dependence is through the overshoots. To see how this dependence comes about put $\hat{c} = c - c^{\alpha}$ and let $\Delta S_n = S_n - S_{N_{[c^{\alpha/2}]}}$ for $n \geq N_{[c^{\alpha/2}]}$. Then

$$\begin{aligned}\tilde{N} - N_{[c^{\alpha/2}]} \quad = \quad & \inf\Bigg\{m = n - N_{[c^{\alpha/2}]} : \Delta S_n \geq \hat{c}b(t) + b'(t)(m - \hat{c}t) \\ & + \frac{b''(t)}{2c}(m - \hat{c}t)^2 - \sum_{j=1}^{[c^{\alpha/2}]} Z_j - h(c)\Bigg\},\end{aligned} \tag{8}$$

which follows from the Definition 2.3 and Definition 2.4.

The distribution of the overshoot Z under the tilted measure converges in distribution to the distribution $Q_\theta(z)$ of the overshoot $Z_l = S_N - cb(t) - b'(t)(N - ct)$. The overshoot is also for $\eta > 0$ asymptotically independent of $c^{-\eta}(N - ct)$, see Hammarlid [5]. A consequence of this result is that $c^{-\eta}(\Delta N_j - c^{\alpha/2}t)$ and Z_j are asymptotically independent, which implies that $N_{[c^{\alpha/2}]}$ and $\sum_{j=1}^{[c^{\alpha}/2]} Z_j$ are asymptotically independent.

We can now split the stopping time

$$N = N - \tilde{N} + \tilde{N} - N_{[c^{\alpha/2}]} + N_{[c^{\alpha/2}]},$$

where the different parts are conditionally independent by the following lemma.

Lemma 2.5 *The differences of the stopping times $N - \tilde{N}$, $\tilde{N} - N_{[c^{\alpha/2}]}$ and $N_{[c^{\alpha/2}]}$ are conditionally independent, conditioned on $\mathbf{Z} \cap \mathbf{A}$, where $\mathbf{Z} = \{Z_0, \ldots, Z_{[c^{\alpha/2}]}, Z_{\tilde{N}}\}$ and $\mathbf{A} = \{N_{[c^{\alpha/2}]} < \tilde{N} < N\}$, so that*

$$\begin{aligned}E_\theta\left[e^{i\zeta N}\middle|\, \mathbf{Z} \cap \mathbf{A}\right] \quad = \quad & E_\theta\left[e^{i\zeta(N - \tilde{N})}\middle|\, \mathbf{Z} \cap \mathbf{A}\right] E_\theta\left[e^{i\zeta(\tilde{N} - N_{[c^{\alpha/2}]})}\middle|\, \mathbf{Z} \cap \mathbf{A}\right] \\ & \times E_\theta\left[e^{i\zeta N_{[c^{\alpha/2}]}}\middle|\, \mathbf{Z} \cap \mathbf{A}\right].\end{aligned}$$

The stopping times are asymptotically independent, that is

$$\lim_{c\to\infty} E_\theta\left[e^{i\zeta N}\right] = \lim_{c\to\infty} E_\theta\left[e^{i\zeta(N-\tilde{N})}\right] E_\theta\left[e^{i\zeta(\tilde{N}-N_{[c^{\alpha/2}]})}\right] E_\theta\left[e^{i\zeta N_{[c^{\alpha/2}]}}\right].$$

Proof. The pair ΔN_j and Z_j are by construction independent of another pair ΔN_i and Z_i, where $i \neq j$, since ΔS_{N_i} is independent of ΔS_{N_j}. Furthermore, the overshoot and the auxiliary stopping time are asymptotically independent. This implies that $N_{[c^{\alpha/2}]}$ and $\sum_{j=1}^{[c^{\alpha/2}]} Z_j$ are asymptotically independent.

The dependence between $\tilde{N}-N_{[c^{\alpha/2}]}$ and $N_{[c^{\alpha/2}]}$ is according to equation (8) only through $\sum_{j=1}^{[c^{\alpha/2}]} Z_j$, if $S_{N_{[c^{\alpha/2}]}}$ and $S_{\tilde{N}} - S_{N_{[c^{\alpha/2}]}}$ are independent. These partial sums are independent because $N_{[c^{\alpha/2}]} < \tilde{N}$ on $\mathbf{A}$. Thus, $\tilde{N} - N_{[c^{\alpha/2}]}$ and $N_{[c^{\alpha/2}]}$ are conditionally independent.

The overshoot $Z_{\tilde{N}}$ determines if $N = \tilde{N}$ or not, which is asymptotically independent of $\tilde{N}$. The partial sum $S_N - S_{\tilde{N}}$ and $S_{\tilde{N}} - S_{N_{[c^{\alpha/2}]}}$ are independent on $\mathbf{A}$. Therefore since $N-\tilde{N}$ and $\tilde{N}-N_{[c^{\alpha/2}]}$ only depend through the overshoot they are conditionally independent. □

Lemma 2.6 *Let $\delta > 0$, $0 < \alpha < 1$ and $\delta < |\zeta| < \pi$. Then the characteristic function*

$$\left|E_\theta\left[e^{i\zeta N}\right]\right| \leq |1 - D\delta|^{[c^{\alpha/2}]} + 2\exp\left(-(c-c^\alpha)\gamma(b(t) - b'(t)t) + c^{\alpha/2} g_\theta^Z(\gamma)\right),$$

where $\gamma > 0$ is such that $g_\theta^Z(\gamma) < \infty$ and $b(t) - b'(t)t > 0$.

Proof. The law of total probability and that the absolute value of a characteristic function is less than or equal to one give

$$\left|E_\theta\left[e^{i\zeta N}\right]\right| \leq \left|E_\theta\left[e^{i\zeta N}, \mathbf{A}\right]\right| + P_\theta(\mathbf{A}^*),$$

where the set $\mathbf{A}$ is defined in Lemma 2.5 and $\mathbf{A}^*$ is its complement. We have by Lemma 2.5 and $P(\mathbf{A}) \leq 1$ that when we condition on $\mathbf{Z}$, then

$$\begin{aligned} \left|E_\theta\left[e^{i\zeta N}\right]\right| \leq & \left|E_\theta\left[E_\theta\left[e^{i\zeta(N-\tilde{N})}\middle|\mathbf{Z}\cap\mathbf{A}\right] E_\theta\left[e^{i\zeta(\tilde{N}-N_{[c^{\alpha/2}]})}\middle|\mathbf{Z}\cap\mathbf{A}\right]\right.\right. \\ & \left.\left. \times E_\theta\left[e^{i\zeta N_{[c^{\alpha/2}]}}\middle|\mathbf{Z}\cap\mathbf{A}\right]\right]\right| + P_\theta(\mathbf{A}^*). \end{aligned}$$

The increments of $N_{[c^{\alpha/2}]} = \sum_{j=1}^{[c^{\alpha/2}]} \Delta N_j$ are independent identically distributed, see the discussion after Definition 2.3 and thus,

$$E_\theta[\exp(i\zeta N_{[c^{\alpha/2}]})|\mathbf{Z}\cap\mathbf{A}] = E_\theta[\exp(i\zeta\Delta N_1)|\mathbf{Z}\cap\mathbf{A}]^{[c^{\alpha/2}]}.$$

Both the auxiliary stopping times and N are lattice random variables. It is well established that the absolute value of the characteristic function of a lattice random variable full fills $\max_{\delta<|\zeta|\leq\pi}|f(\zeta)| \leq 1 - D\delta^2$, where D is some constant, Chung [3]. Benedicks [1] shows how a $D > 0$ can be determined from the distribution. The absolute value of a characteristic function is less or equal to one, so that

$$\begin{aligned}\left|E_\theta\left[e^{i\zeta N}\right]\right| &\le E_\theta\left[\left|E_\theta\left[e^{i\zeta\Delta N_1}\middle|\mathbf{Z}\cap\mathbf{A}\right]\right|^{[c^{\alpha/2}]}\right]+P(\mathbf{A}^*)\\ &\le |1-D\delta^2|^{[c^{\alpha/2}]}+P(\mathbf{A}^*).\end{aligned}$$

The lemma follows from the large deviation estimate of $P(\mathbf{A}^*)$ derived by Lemma A.3 and Lemma A.4. □

Theorem 2.7 *Assume that the barrier $b(s)$ is convex, three times continuously differentiable, satisfies $b(0)>0$ and that $\mu s<b(s)$ for all s. The rate of the stopping time N is $R(t)=\theta b(t)-g(\theta)t$ and $R'(t)=\theta b'(t)-g(\theta)<0$, where $g'(\theta)=b(t)/t$ and $t<T$.*

Then the asymptotic distribution of the first passage time is given by

$$\lim_{c\to\infty} c^{1/2}e^{cR(t)}P\left(N\le ct\right)=\frac{E_\theta\left[e^{-\theta Z_l}\right]}{\left(1-e^{R'(t)}\right)\sqrt{2\pi}a},$$

where $a^2=g''(\theta)t^3(b(t)-b'(t)t)^{-2}$ and Z_l is the overshoot over the barrier $cb(t)+b'(t)(n-ct)$.

Proof: Introduce the indicator function $\mathbf{1}=\mathbf{1}(N\le ct)$ and tilt the distribution so that

$$c^{1/2}e^{cR(t)}P\left(N\le ct\right)=c^{1/2}E_\theta\left[\mathbf{1}\exp\left(cR(t)+g(\theta)N-\theta S_N\right)\right].$$

Expand S_N up to order three around ct,

$$S_N=cb(t)+b'(t)(N-ct)+b''(t)\frac{(N-ct)^2}{2c}+b'''(\xi)\frac{(N-ct)^3}{3!c^2}+Z, \tag{9}$$

where ξ is between t and N/c. We will skip the remainder since it converges to zero by Lemma 2.1 ($b'''(s)$ is bounded for all $s\le T$).

The first order derivative, $R'(t)=\theta b'(t)-g(\theta)\ne 0$, is easily derived from $R(s)=sI(b(s)/s)$ and the properties of the rate function, see the proof of Lemma 1 in Hammarlid [5]. The first order derivative is negative because the rate $R(s)$ is convex and attains its minimum at T and $t<T$. After some computations we get that

$$c^{1/2}e^{cR(t)}P\left(N\le ct\right)=c^{1/2}E_\theta\left[\mathbf{1}\exp\left(-R'(t)(N-ct)-\theta\frac{b''(t)}{2c}(N-ct)^2-\theta Z\right)\right].$$

The part $\exp(-\theta Z)$ can be handled separately because the overshoot Z and N are asymptotically independent, see Hammarlid [5]. The expectation is by definition

$$\begin{aligned}c^{1/2}E_\theta\left[\mathbf{1}\exp\left(-R'(t)(N-ct)-\theta b''(t)\frac{(N-ct)^2}{2c}\right)\right]= \qquad (10)\\ =c^{1/2}\sum_{n=1}^{ct}\exp\left(-R'(t)(n-ct)-\theta b''(t)\frac{(n-ct)^2}{2c}\right)P_\theta(N=n).\end{aligned}$$

The characteristic function of N, for a fixed c, is denoted by $f_c(\zeta) = \sum_{n=1}^{\infty} e^{i\zeta n} P_\theta(N = n)$. The stopping time N is an integer lattice variable, for which the following inversion formula holds,

$$P_\theta(N = n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\zeta n} f_c(\zeta) d\zeta,$$

see for example Chung [3]. Insert the inversion formula in (10), change order of summation and integration in accordance to Fubini's theorem

$$\begin{aligned} c^{1/2} E_\theta \left[\mathbf{1} \exp\left(-R'(t)(N - ct) - \theta \frac{b''(t)(N - ct)^2}{2c} \right) \right] &= \\ = \frac{c^{1/2}}{2\pi} \int_{-\pi}^{\pi} e^{R'(t)ct} \sum_{n=1}^{ct} \exp\left(-n(R'(t) + i\zeta) - \theta \frac{b''(t)(n - ct)^2}{2c} \right) f_c(\zeta) d\zeta \\ = \frac{c^{1/2}}{2\pi} \int_{-\pi}^{\pi} k_c(\zeta) e^{-ict\zeta} f_c(\zeta) d\zeta, \end{aligned} \tag{11}$$

where

$$k_c(\zeta) = \exp\left(ct(R'(t) + i\zeta)\right) \sum_{n=1}^{ct} \exp\left(-n(R'(t) + i\zeta) - \theta \frac{b''(t)(n - ct)^2}{2c} \right).$$

We will now show that $\lim_{c\to\infty} k_c(c^{-1/2}\zeta) = \left(1 - e^{R'(t)}\right)^{-1}$. The barrier is convex and $b''(t) > 0$. The θ that solves $g'(\theta) = b(t)/t$ have to be positive. Therefore, ignore the quadratic part of the sum and we take the absolute value of the terms. Then the resulting geometric series is an upper bound, that is

$$\left| k_c(c^{-1/2}\zeta) \right| \le e^{R'(t)ct} e^{-R'(t)} \frac{1 - e^{-R'(t)ct}}{1 - e^{-R'(t)}}.$$

We can therefore put $c^{-1/2}\zeta = 0$ in the computations by the Bounded convergence theorem and we have

$$\lim_{c\to\infty} k_c(c^{-1/2}\zeta) = \lim_{c\to\infty} k_c(0) \le \frac{e^{-R'(t)}}{e^{-R'(t)} - 1}.$$

We delete the sum up to $c(t - \varepsilon)$, to construct a lower bound of the limit,

$$\begin{aligned} e^{ctR'(t)} \sum_{n=1}^{ct} \exp\left(-nR'(t) - \theta \frac{b''(t)(n - ct)^2}{2c} \right) &\ge \\ \ge \exp\left(ctR'(t) - \theta c \frac{b''(t)\varepsilon^2}{2} \right) \sum_{n=c(t-\varepsilon)}^{ct} e^{-nR'(t)} \\ = \exp\left(-\theta c \frac{b''(t)\varepsilon^2}{2} \right) \frac{e^{R'(t)c\varepsilon} - e^{R'(t)}}{1 - e^{-R'(t)}}. \end{aligned}$$

The choice of $\varepsilon = c^{-3/4}$ and the fact that $R'(t)$ is negative leads to

$$\lim_{c\to\infty} \sum_{n=1}^{ct} \exp\left(-nR'(t) - \theta \frac{b''(t)(n - ct)^2}{2c} \right) \ge$$

$$\geq \lim_{c\to\infty} \exp\left(-\theta\frac{b''(t)}{2c^{1/2}}\right)\frac{e^{R'(t)c^{1/4}}-e^{-R'(t)}}{1-e^{-R'(t)}}$$
$$= \frac{e^{-R'(t)}}{e^{-R'(t)}-1}.$$

The limit of $k_c(\zeta/c^{1/2})$ is sandwiched between two equal limits and hence

$$\lim_{c\to\infty} k_c(\zeta/c^{1/2}) = \frac{e^{-R'(t)}}{e^{-R'(t)}-1} = \left(1-e^{R'(t)}\right)^{-1}. \quad (12)$$

Let us now return to the integral in equation (11). The integral can be split into two regions of integration, one central region $\{|\zeta| \leq \delta\}$ where $\delta > 0$, is assumed small, and the complement $\{|\zeta| > \delta\}$. The idea is to exchange the characteristic function with the characteristic function of a normally distributed random variable in the central part and show that the integral over the complement vanishes when c goes to infinity.

We start analyzing the central part. The expansion of the logarithm of a characteristic function on $\{|\zeta| \leq \delta\}$,

$$\log\left(f_c(\zeta)\right) = i\zeta ct - \frac{1}{2}\zeta^2 ca^2 + O(i\zeta^3),$$

where $a^2 = t^3 g''(\theta)(b(t)-b'(t)t)^{-2}$ by Theorem 1 in Hammarlid [5]. Plug in the expansion of the characteristic function in the integral over the central part and substitute $\zeta = \xi/c^{1/2}$,

$$\frac{c^{1/2}}{2\pi}\int_{|\zeta|\leq\delta} e^{-ict\zeta} f_c(\zeta)k_c(\zeta)d\zeta = \frac{1}{2\pi}\int_{|\xi|\leq c^{1/2}\delta} e^{-a^2\xi^2/2}k_c(\xi/c^{1/2})d\xi + c^{1/2}O(i\delta^4).$$

Let us chose $\delta = c^{-1/7}$, then by the Dominated convergence theorem and the fact that the exponential part is the density of the normal distribution we have

$$\lim_{c\to\infty}\frac{1}{2\pi}\int_{|\xi|\leq\delta c^{1/2}} e^{-a^2\xi^2/2}k_c(\xi/c^{1/2})d\xi + c^{1/2}O(c^{-4/7}) = \left(\left(1-e^{R'(t)}\right)\sqrt{2\pi}a\right)^{-1}. \quad (13)$$

Left to show is that the second integral,

$$\left|\frac{c^{1/2}}{2\pi}\int_{\delta<|\zeta|<\pi} e^{-ict\zeta} f_c(\zeta)k_c(\zeta)d\zeta\right| \leq \text{constant}\int_{\delta<|\zeta|<\pi} c^{1/2}\left|f_c(\zeta)\right|d\zeta$$

converges to zero. Assume that $\gamma > 0$ is such that $g_\theta^Z(\gamma) < \infty$. Then there is constants D computed from the distribution so that

$$|f_c(\zeta)| \leq \left|1-D\delta^2\right|^{[c^{\alpha/2}]} + P(\mathbf{A}^*),$$

where $P(\mathbf{A}^*) \leq 2\exp\left(-(c-c^\alpha)\gamma(b(t)-b'(t)t) + c^{\alpha/2}g_\theta^Z(\gamma)\right)$, by Lemma 2.6. Substitute the characteristic function by this bound,

$$\int_{\delta<|\zeta|<\pi} c^{1/2}\left|f_c(\zeta)\right|d\zeta \leq 2\pi c^{1/2}\left(\left|1-D\delta^2\right|^{[c^{\alpha/2}]} + P(\mathbf{A}^*)\right).$$

If chose $\alpha = 13/21 < 2/3$. Then since $\delta = c^{-1/7}$

$$\int_{c^{-1/7}<|\zeta|<\pi} c^{1/2}\,|f_c(\zeta)|\,d\zeta \leq 2\pi c^{1/2}\left(\left|1 - Dc^{-12/42}\right|^{c^{[13/42]}} + P(\mathbf{A}^*)\right) \longrightarrow 0$$

as c goes to infinity. □

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A

In this appendix we show that the tilted probability of $N_{[c^{\alpha/2}]} > N$ converges to zero, $c^{-\alpha}\sum_{j=1}^{k} Z_j \to 0$ as c goes to infinity and that the tail of the distribution of the auxiliary stopping times, properly scaled, decay exponentially.

First note that $c^{-\alpha/2}\Delta N_j \overset{a.s}{\to} t$ as c goes to infinity, by corollary 2.2. Lemma 2.1 is also applicable on ΔN_j. The recursive formulation of $N_{[c^{\alpha/2}]}$, equation (5), involved the sum of the overshoots. We need therefore to know the asymptotic behavior of the sum of overshoots.

Lemma A.1 *Fix $\varepsilon > 0$ and assume that $g'(\theta) = b(t)/t$. Let $\alpha > 0$ and $\{Z_j\}_{j=1}^{[c^{\alpha/2}]}$ be the sequence of overshoots over the auxiliary barriers. Then under the tilted distribution there is a constant $\gamma > 0$ so that,*

$$P_\theta\left(\sum_{j=1}^{[c^{\alpha/2}]} Z_j > c^\alpha \varepsilon\right) \leq \exp\left(-c^\alpha \gamma\varepsilon + c^{\alpha/2} g_\theta^Z(\gamma)\right),$$

where γ is such that,

$$g_\theta^Z(\gamma) = \log\left(E_\theta\left[e^{\gamma Z_j}\right]\right) < \infty.$$

Furthermore, $c^{-\alpha}\sum_{j=1}^{[c^{\alpha/2}]} Z_j \xrightarrow{a.s} 0$, as $c \to \infty$.

Proof. The overshoots are independent identically distributed by construction and by the same type of argument leading to the Chernoff bound, see Bucklew [2] or Hammarlid [5], we have that

$$P_\theta\left(\sum_{j=1}^{k} Z_j > c^\alpha \varepsilon\right) \leq \exp\left(-c^\alpha \gamma\varepsilon + c^{\alpha/2} g_\theta^Z(\gamma)\right). \tag{14}$$

For the moment, assume that there is a γ such that $g_\theta^Z(\gamma)$ is finite and let $\varepsilon = c^{-\alpha/4}$, then

$$\sum_{c=1}^{\infty} P_\theta\left(\sum_{j=1}^{k} Z_j > c^\alpha \varepsilon\right) \leq \sum_{c=1}^{\infty} \exp\left(-c^{3\alpha/4}\gamma + c^{\alpha/2} g_\theta^Z(\gamma)\right) < \infty.$$

Hence $c^{-\alpha}\sum_{j=1}^{k} Z_j \overset{a.s}{\to} 0$, as c goes to infinity by Borel-Cantelli lemma.

We now prove that there is a $\gamma > 0$ such that $g_\theta^Z(\gamma) < \infty$. Without loss of generality, assume that $b'(t) = 0$, since we can always change drift. Define the stopping times

$$\begin{aligned} N_+ &= \inf\{n : n \geq 1, S_n > 0\} \\ N_- &= \inf\{n : n \geq 1, S_n < 0\}. \end{aligned}$$

and denote by $M_\theta(\lambda) = E_\theta\left[\exp\left(\lambda X_1\right)\right]$ and $M_\theta^{\pm}(\lambda) = E_\theta\left[\exp\left(\lambda S_{N_\pm}\right)\right]$. Then by Theorem 8.41 in Siegmund [9],

$$1 - M_\theta(\lambda) = \left(1 - M_\theta^+(\lambda)\right)\left(1 - M_\theta^-(\lambda)\right), \qquad \text{for} \qquad |\lambda| < \infty. \tag{15}$$

We now use the law of total probability to split the moment generating function,

$$M_\theta^-(\lambda) = \sum_{n=1}^{\infty} E_\theta\left[\exp\left(\lambda S_n\right), N_- = n\right] \leq \sum_{n=1}^{\infty} P_\theta(S_n < 0).$$

The drift is greater than zero under the tilted distribution, which implies that $I_\theta(0) > 0$. Hence

$$M_\theta^-(\lambda) \leq \sum_{n=0}^{\infty} \exp\left(-nI_\theta(0)\right) = \left(1 - \exp\left(-I_\theta(0)\right)\right)^{-1}.$$

For some $\bar{\lambda}$ we have that $M_\theta(\lambda)$ exists when $|\lambda| < \bar{\lambda}$. These facts and equation (15) give that $M_\theta^+(\lambda)$ exists. Define the sequence of ladder variables as $W_k = \sum_{j=1}^{k} S_{N_+}(j)$, where $S_{N_+}(j)$ are independent identically distributed uprisings. Then the overshoot $Z_1 = W_k - c^{\alpha/2}b(t)$ for some k.

The idea now is to show that $P(Z_1 > z) \leq \text{constant} \cdot e^{-\lambda z}$ for some $\lambda > 0$. The distribution of S_{N_+} and W_k is written $F_\theta(ds)$ and $F_\theta^{k*}(dw)$ respectively and

$$P(Z_1 > z) = \sum_{k=1}^{\infty} \int_{w \leq cb(t)} F_\theta^{k*}(dw)\left(1 - F_\theta(cb(t) - w + z)\right).$$

We can, since $M_\theta^+(\lambda)$ exists for $\lambda < \bar{\lambda}$, estimate

$$P(S_{N_+} > z) \leq \exp\left(-\lambda z + g_\theta^+(\lambda)\right),$$

where $g_\theta^+(\lambda) = \log\left(M_\theta^+(\lambda)\right)$. Thus

$$P(Z_1 > z) \leq \exp\left(-\lambda z\right) \sum_{k=1}^{\infty} \int_{w \leq cb(t)} F_\theta^{k*}(dw) \exp\left(-\lambda(cb(t) - w) + g_\theta^+(\lambda)\right),$$

where the sum part is uniformly bounded in c by the renewal theorem. □

It is easy to believe that there are equivalent results of exponential decay, like Lemma 2.1, for $\tilde{N}$ and $N_{[c^{\alpha/2}]}$.

Lemma A.2 *Fix arbitrary $\varepsilon > 0$ and let $\beta > 0$, $\eta > 0$ and $\alpha > 0$. Then under the tilted distribution there is a constant $\gamma > 0$ such that $g_\theta^Z(\gamma) < \infty$ and*

$$P_\theta\left(\left|\frac{\left(N_{[c^{\alpha/2}]} - c^\alpha t\right)^\eta}{c^\beta}\right| > \varepsilon\right) \quad \leq \quad 2c^{\alpha/2} t \exp\left(-c^{2\beta/\eta - \alpha}\gamma\varepsilon^{2/\eta} + c^{\alpha/2} g_\theta^Z(\gamma)\right).$$

where $g'(\theta) = b(t)/t$ *and* $g_\theta^Z(\gamma)$ *is defined in Lemma A.1. If* $4\beta > 3\alpha\eta$ *then,*

$$\frac{(N_{[c^{\alpha/2}]} - ct)^\eta}{c^\beta} \xrightarrow{a.s} 0, \text{ as } c \to \infty.$$

Proof. The set $\{\omega : |N_{[c^{\alpha/2}]}/c^\alpha - t| > \varepsilon\}$ is the union of the two disjoint sets

$$A' = \left\{\omega : S_n \geq c^\alpha b(t) + b'(t)(n - c^\alpha t) + \sum_{j=1}^{[c^{\alpha/2}]-1} Z_j, \text{ for some } n \leq c^\alpha(t-\varepsilon)\right\}$$
$$A'' = \left\{\omega : S_n < c^\alpha b(t) + b'(t)(n - c^\alpha t) + \sum_{j=1}^{[c^{\alpha/2}]-1} Z_j, \text{ for all } n \leq c^\alpha(t+\varepsilon)\right\},$$

and $P_\theta(|N_{[c^{\alpha/2}]}/c^\alpha - t| > \varepsilon) = P_\theta(A') + P_\theta(A'')$. These probabilities can be dominated by large deviation estimates.

Fix $\delta > 0$ such that $(b(t-\varepsilon)+\delta)(t-\varepsilon)^{-1} < b(t)/t$, for example $\delta = \varepsilon^2$, where ε is small enough. Denote the set $D = \left\{\sum_{j=1}^{[c^{\alpha/2}]-1} Z_j < c^\alpha \delta\right\}$ and $P_\theta(A') \leq P_\theta(A', D) + P_\theta(D^*)$ where D^* is the complement to D. By Booles inequality and the Chernoff bound we have that,

$$\begin{aligned} P_\theta(A', D) &\leq E_\theta\left[c^\alpha t \exp\left(-c^\alpha \inf_{0<s\leq t-\varepsilon} sI_\theta\left(\frac{b(s) + c^{-\alpha}\sum_{j=1}^{[c^{\alpha/2}]-1} Z_j}{s}\right)\right), D\right] \\ &\leq c^\alpha t \exp\left(-c^\alpha \inf_{0<s\leq t-\varepsilon} sI_\theta\left(\frac{b(s)+\delta}{s}\right)\right). \end{aligned}$$

The last inequality is true because $I_\theta(x)$ is decreasing for $x \leq b(t)/t$.

The tilted rate function satisfies $I_\theta(b(t)/t) = 0$ and the first order derivative $I'_\theta(b(t)/t) = 0$. The choice $\delta = \varepsilon^2$ gives after a Taylor expansion that,

$$\inf_{0<s\leq t-\varepsilon} sI_\theta\left(\frac{b(s)+\delta}{s}\right) = \frac{(b'(t)t - b(t))^2}{2g''(\theta)t^3}\varepsilon^2 + O(\varepsilon^3).$$

Hence, $P(A', D) \leq c^\alpha t \exp\left(-c^\alpha d\varepsilon^2\right)$, for some $d < (b'(t)t - b(t))^2 \left(2g''(\theta)t^3\right)^{-1}$ and we have an estimate of $P(D^*)$ by Lemma A.1. Let γ be such that $\gamma \leq d$ and $g_\theta^Z(\gamma) < \infty$ then,

$$P_\theta(A') \leq c^\alpha t \exp\left(-c^\alpha\gamma\varepsilon^2\right) + \exp\left(-c^\alpha\gamma\varepsilon^2 + c^{\alpha/2} g_\theta^Z(\gamma)\right).$$

Almost identical calculations give

$$P_\theta(A'') \leq \exp\left(-c^\alpha\gamma\varepsilon^2\right) + \exp\left(-c^\alpha\gamma\varepsilon^2 + c^{\alpha/2} g_\theta^Z(\gamma)\right).$$

This together with the fact that the second term in the upper bound of $P(A')$ (or $P(A'')$) dominates the first imply that,

$$P_\theta(|N_{[c^{\alpha/2}]}/c^\alpha - t| > \varepsilon) \leq 2c^\alpha t \exp\left(-c^\alpha\gamma\varepsilon^2 + c^{\alpha/2} g_\theta^Z(\gamma)\right).$$

Take $\varepsilon > 0$, then by the last equation,

$$P_\theta\left(\left|\frac{(N_{[c^{\alpha/2}]} - c^\alpha t)^\eta}{c^\beta}\right| > \varepsilon\right) = P_\theta\left(\left|\frac{N_{[c^{\alpha/2}]}}{c^\alpha} - t\right| > c^{-\alpha}\left(c^\beta \varepsilon\right)^{1/\eta}\right)$$
$$\leq 2c^\alpha t \exp\left(-c^{2\beta/\eta-\alpha}\gamma\varepsilon^{2/\eta} + c^{\alpha/2} g_\theta^Z(\gamma)\right).$$

If $4\beta > 3\alpha\eta$ then the right hand side converges to zero as c goes to infinity.

Let $\upsilon > 0$ and $4(\beta - \upsilon) > 3\alpha\eta$ then

$$\sum_{c=1}^\infty P_\theta\left(\left|\frac{(N_{[c^{\alpha/2}]} - ct)^\eta}{c^\beta}\right| > c^{-\upsilon}\right) \leq 2\sum_{c=1}^\infty c^\alpha t \exp\left(-c^{(2\beta-\alpha\eta-2\upsilon)/\eta}\gamma + c^{\alpha/2} g_\theta^Z(\gamma)\right),$$

is convergent. Hence the convergence is, by Borel-Cantelli lemma, almost sure.
□

One concern is the possibility that $N_k > N$ or $N_{[c^{\alpha/2}]} > \tilde{N}$, that is that the process hits the auxiliary barrier after the true barrier. The construction is such that the probability of this event is vanishing.

Lemma A.3 *Fix* $0 < \varepsilon$. *Let* $g'(\theta) = b(t)/t$ *and* $0 < \alpha < 1$. *Then under the tilted distribution*

$$P_\theta(N_{[c^{\alpha/2}]} > N) \leq \exp\left(-(c - c^\alpha)\gamma(b(t) - b'(t)t) + c^{\alpha/2} g_\theta^Z(\gamma)\right),$$

for some $\gamma > 0$ *that fulfills* $g_\theta^Z(\gamma) < \infty$. *The cumlant generating function* $g_\theta^Z(\gamma)$ *is defined in Lemma A.1 and* $b(t) - b'(t)t > 0$.

Lemma A.4 *Fix* $0 < \varepsilon$. *Let* $g'(\theta) = b(t)/t$ *and* $0 < \alpha < 1$. *Then under the tilted distribution*

$$P_\theta(N_{[c^{\alpha/2}]} > \tilde{N}) \leq \exp\left(-(c - c^\alpha)\gamma(b(t) - b'(t)t) + c^{\alpha/2} g_\theta^Z(\gamma)\right),$$

for some $\gamma > 0$ *that fulfills* $g_\theta^Z(\gamma) < \infty$. *The cumlant generating function* $g_\theta^Z(\gamma)$ *is defined in Lemma A.1.*

The proof of Lemma A.4 and Lemma A.3 is almost identical. Therefore is only the first of the two Lemmas proved.

Proof. The linear barrier, defined by the first two terms of the Taylor expansion of $cb(n/c)$, is passed before the original barrier. We have therefore by the definition of $N_{[c^{\alpha/2}]}$ and by Lemma A.1 that

$$P(N_{[c^{\alpha/2}]} > N) \leq P\left(\sum_{j=1}^{[c^{\alpha/2}]} Z_j > (c - c^\alpha)(b(t) - b'(t)t)\right) \quad (16)$$
$$\leq \exp\left(-(c - c^\alpha)\gamma(b(t) - b'(t)t) + c^{\alpha/2} g_\theta^Z(\gamma)\right), \quad (17)$$

where γ is such that $g_\theta^Z(\gamma) < \infty$ by Lemma A.3. The fact $t \leq T$ assure that $b(t) > b'(t)t$. □

The auxiliary stopping time $\tilde{N}$ is very close to N, and the same type of technique that is used to derive Lemma 2.1, see Hammarlid [5], can be used for $\tilde{N}$. We state the following lemma without proof.

Lemma A.5 *Let $\eta > 0$ and $\alpha > 0$ such that $\alpha < 2\eta$. Then under the tilted distribution for arbitrary $\varepsilon > 0$,*

$$P_\theta\left(\left|\frac{(\tilde{N}-ct)^\alpha}{c^\eta}\right| > \varepsilon\right) \le 2ct\exp\left(-c^{2\eta/\alpha-1}d\varepsilon^{2/\alpha}\right),$$

where $g'(\theta) = b(t)/t$. Also, $c^{-\eta}(\tilde{N}-ct)^\alpha \xrightarrow{a.s} 0$, as $c \to \infty$.

Lemma A.6 *Fix $\eta > 0$ and $0 < \alpha < 2/3$. Then under the tilted distribution, where $g'(\theta) = b(t)/t$, there are $\gamma > 0$ and $\nu > 0$ such that $0 < 1 - 3/2\alpha - 2\nu$ and*

$$P_\theta\left(\left|(\tilde{N}-ct)^2 - (\tilde{N} - N_{[c^{\alpha/2}]} - (c-c^\alpha)t)^2\right| > c\eta\right) \le$$
$$\le 2\left(\exp\left(-c^{\alpha/2+\nu}\gamma + c^{\alpha/2}g_\theta^Z(\gamma)\right) + ct\exp\left(-c^{1-3/2\alpha-2\nu}\gamma\right)\right),$$

for all $c > 3\eta^{-2/\nu}$. Furthermore,

$$\lim_{c\to\infty} c^{-1}\left((\tilde{N}-ct)^2 - (\tilde{N} - N_{[c^{\alpha/2}]} - (c-c^\alpha)t)^2\right) \overset{a.s}{=} 0.$$

Proof. The trick of the proof is to split the sample space into two sets, the likely event A and the unlikely event A^*. On A the distance between the two quadratic terms is converging to zero and $\lim_{c\to\infty} P(A^*) = 0$.

Fix $\varepsilon = c^{-3\alpha/4-\nu}$ and $\delta^2 = c^{-\alpha/2+\nu}$ to define the set

$$A = \{\omega : |N_{[c^{\alpha/2}]} - c^\alpha t| < c^\alpha\delta\}\bigcap\{|\tilde{N} - ct| < c\varepsilon\}. \tag{18}$$

On the set A, by the conjugate rule,

$$\begin{aligned} c^{-1}&\left|(\tilde{N}-ct)^2 - (\tilde{N} - N_{[c^{\alpha/2}]} - (c-c^\alpha)t)^2\right| = \\ &= c^{-1}\left|N_{[c^{\alpha/2}]} - c^\alpha t\right|\left|2(\tilde{N}-ct) - N_{[c^{\alpha/2}]} + c^\alpha t\right| \\ &\le c^{2\alpha-1}\delta^2 + 2c^\alpha\delta\varepsilon \\ &\le c^{3\alpha/2+\nu-1} + 2c^{-\nu/2} \\ &\le 3c^{-\nu/2} < \eta. \end{aligned} \tag{19}$$

for $c > 3\eta^{-2/\nu}$. Therefore, by the law of total probability,

$$P_\theta\left(\left|(\tilde{N}-ct)^2 - (\tilde{N} - N_{[c^{\alpha/2}]} - (c-c^\alpha)t)^2\right| > c\eta\right) \le P(A^*), \qquad \text{when } c \ge 3\eta^{-2/\nu}.$$

We have by Lemma A.2 and Lemma A.5 that there is a $\gamma \le d$ so that

$$\begin{aligned} P_\theta(A^*) &\le 2c^\alpha t\exp\left(-c^\alpha\gamma\delta^2 + c^{\alpha/2}g_\theta^Z(\gamma)\right) + 2ct\exp\left(-c\gamma\varepsilon^2\right) \\ &= 2c^\alpha t\left(\exp\left(-c^{\alpha/2+\nu}\gamma + c^{\alpha/2}g_\theta^Z(\gamma)\right) + 2ct\exp\left(-c^{1-3/2\alpha-2\nu}\gamma\right)\right). \end{aligned}$$

Let $\eta = c^{-\nu/4}/3$ then since $c > 3\eta^{-2/\nu} = 3c^{1/2}$ is fulfilled for every $c \ge 3$ we have a convergent sum,

$$\sum_{c=1}^{\infty} P_\theta\left(c^{-1}\left|(\tilde{N}-ct)^2 - (\tilde{N} - N_{[c^{\alpha/2}]} - (c-c^\alpha)t)^2\right| > c^{-\nu/4}/3\right)$$
$$\le 2\sum_{c=1}^{\infty} c^\alpha t\exp\left(-c^{\alpha/2+\nu}\gamma + c^{\alpha/2}g_\theta^Z(\gamma)\right) + ct\exp\left(-c^{1-3/2\alpha-2\nu}\gamma\right).$$

Therefore by the Borel-Cantelli theorem we have almost sure convergence. □

Lemma A.7 *When* $0 < \alpha < 2/3$ *and* $g'(\theta) = b(t)/t$ *then under the tilted distribution,*

$$N - \tilde{N} \xrightarrow{P_\theta} 0, \qquad \text{as } c \to \infty.$$

Proof. Fix $\nu > 0$ such that $1 - 3\alpha/2 - 2\nu > 0$ and let $\delta^2 = c^{-\alpha/2+\nu}$ and $\varepsilon = c^{-3\alpha/4+\nu}$. Define the set

$$A = \{\omega : |N_{[c^{\alpha/2}]} - c^\alpha t| < c^\alpha \delta\} \bigcap \{\omega : |N - ct| < c\varepsilon\} \bigcap \{\omega : |\tilde{N} - ct| < c\varepsilon\}.$$

The auxiliary barrier $c\tilde{b}(n/c)$, Definition 2.4, is just in front of the true barrier on the set A since by the conjugate rule,

$$\begin{aligned}
cb(N/c) - \hat{b}(N/c) &= \frac{b''(t)}{2c}\left((N-ct)^2 - (N - N_{[c^{\alpha/2}]} - (c - c^\alpha)t)^2\right) + h(c) \\
&= \frac{b''(t)}{2c}\left((N_{[c^{\alpha/2}]} - c^\alpha t)(2(N-ct) - N_{[c^{\alpha/2}]} + c^\alpha t)\right) + h(c) \\
&\geq -\frac{b''(t)}{2c}\left(c^{2\alpha}\delta^2 + 2c^{1+\alpha}\varepsilon\delta\right) + h(c) \\
&= -\frac{b''(t)}{2}\left(c^{1-3\alpha/2-\nu} + 2c^{-\nu/2}\right) + h(c) \\
&\geq -\frac{3b''(t)}{2}c^{-\nu/2} + h(c) = 0,
\end{aligned}$$

since $h(c) = 3b''(t)c^{-\nu/2}/2$, see Definition 2.4. Therefore we have that $\tilde{N} \leq N$ on A.

By equation (19) we see that the stopping times are not equal on the set A when,

$$\begin{aligned}
\{\omega : \tilde{N} < N\} &= \left\{\omega : c\hat{b}(\tilde{N}/c) \leq S_{\tilde{N}} < cb(\tilde{N}/c)\right\} \\
&= \left\{\omega : 0 \leq Z_{\tilde{N}} < \frac{b''(t)}{2c}\left(\left(\tilde{N} - N_{[c^{\alpha/2}]} - (c - c^\alpha)t\right)^2 - (\tilde{N} - ct)^2\right) + h(c)\right\} \\
&\subseteq \{\omega : 0 \leq Z_{\tilde{N}} < 2h(c)\}.
\end{aligned}$$

The probability of the complement of A

$$P_\theta(A^*) \leq 2c^\alpha t\left(\exp\left(-c^{\alpha/2+\nu}\gamma + c^{\alpha/2}g_\theta^Z(\gamma)\right) + 2ct\exp\left(-c^{1-3/2\alpha-2\nu}\gamma\right)\right),$$

by Lemma A.2, Lemma A.5 and Lemma 2.1. This and the law of total probability imply that

$$\begin{aligned}
\lim_{c\to\infty} P_\theta(N \neq \tilde{N}) &= \lim_{c\to\infty} P_\theta(N > \tilde{N}, A) + \lim_{c\to\infty} P_\theta(N \neq \tilde{N}, A^*) \\
&\leq \lim_{c\to\infty} P_\theta(0 \leq Z_{\tilde{N}} < 2h(c), A) + \lim_{c\to\infty} P_\theta(A^*) \\
&= \lim_{c\to\infty} F^c_{Z_{\tilde{N}}}(2h(c)),
\end{aligned}$$

where $F^c_{Z_{\tilde{N}}}(z)$ is the distribution function of the overshoot for a fixed c. The limit distribution $F_{Z_{\tilde{N}}}(z)$ is continuous, see Hammarlid [5]. Therefore for fixed $\rho > 0$ there is a c such that $2h(c) < \rho$ and $F^c_{Z_{\tilde{N}}}(h(c)) \leq F^c_{Z_{\tilde{N}}}(\rho)$. The limit $\lim_{c\to\infty} F^c_{Z_{\tilde{N}}}(2h(c)) \leq F_{Z_{\tilde{N}}}(\rho) \leq \rho$, but ρ is arbitrary. □

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