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Abstract

We give a derivation which is natural from a trading point of view.

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Let $S_t, \ \ t \geq 0$, denote the price of a stock at time t. Assume that

$$S_t = S_0 e^{L_t}$$

where $L_t = \mu t + \sigma W_t$ and W_t is a Wiener process.

We shall here consider a portfolio which at time t consists of $\$c_t$ in cash and a number d_t of the stock. We change the portfolio at time $t_0, t_1, t_2, ...$ by selling or buying stocks.

In the original derivation of the Black-Scholes formula (Black and Scholes 1973) the authors assumed that $\Delta t_k = t_k - t_{k-1}$, $k = 1, 2, ...$ are deterministic. The movements of the stock $\Delta L_{t_k} = L_{t_k} - L_{t_{k-1}}$ are then random.

In this note we shall instead trade when the stock has moved a fixed amount $\Delta L_{t_k} = \pm\delta$. In this case $t_1, t_2, ...$ are random.

By considering the present values of the stock and the portfolio we can assume that the interest rate equals zero.

Define $t_0 = 0$ and

$$t_k = \inf\{t > t_{k-1}; |L_t - L_{t_{k-1}}| \geq \delta\},$$

for $k = 1, 2, ...$, and let

$$N_\delta(t) = \max\{n; t_n \leq t\}.$$

This assumption thus implies that the stock moves along a binomial tree and that $t_1, t_2, ...$ and $N_\delta(t)$ are random variables.

Let f be a real valued, continuous function defined on $(0, \infty)$. We are looking for a trading strategy for f. That is, a sequence of portfolios that end up close to $f(S_T)$ at time T. We shall choose the trading strategy that ends up at $f(S_{t_{N_\delta(T)}})$ at time $t_{N_\delta(T)}$.

Let $f_n(s)$ stand for the value of the portfolio at time t_n if $S_{t_n} = s$. It follows in the same way as in Cox, Ross and Rubinstein (1979) that

$$f_n(s) = M_f(N_\delta(T) - n, s)$$

where

$$M_f(r, s) = \sum_{i=0}^{r} \binom{r}{i} P^i Q^{r-i} f(se^{\delta i} e^{-\delta(r-i)}).$$

Here

$$P = \frac{1-e^{-\delta}}{e^{\delta}-e^{-\delta}} = \frac{1}{1+e^{\delta}}, \quad Q = 1-P.$$

The stock holding at time t then equals $d_{n-1}(S_{t_{n-1}})$ for $t_{n-1} \leq t < t_n$. Here

$$d_{n-1}(s) = \frac{f_n(se^{\delta}) - f_n(se^{-\delta})}{s(e^{\delta}-e^{-\delta})}.$$

The value of the portfolio at time t thus equals the random variable

$$P_t^{\delta} = f_{N_\delta(t)}(S_{t_{N_\delta(t)}}) + d_{N_\delta(t)}(S_{t_{N_\delta(t)}})(S_t - S_{t_{N_\delta(t)}}).$$

(The quantities $S_{t_{N_\delta(t)}}$ and $N_\delta(t)$ are known at time t but $N_\delta(T)$ is still random.)

The next theorem shows that this random portfolio value tends to the constant $I(S_t, \sigma\sqrt{T-t})$ as $\delta \to 0$. Here

$$I(s,\rho) = \int_{-\infty}^{\infty} f(se^{-\frac{\rho^2}{2}+\rho z}) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz.$$

Theorem *Assume that there are positive constants C and m such that*

$$|f(s)| \leq C(s^m + s^{-m})$$

for all $s > 0$, then

$$Prob(|P_t - I(s,\sigma\sqrt{T-t})| > \epsilon \ \ |S_t = s) \to 0$$

for each $\epsilon > 0$, as $\delta \to 0$.

If f is a linear combination of calls and puts, then $|f(s)| \leq A + Bs$. So the growth condition on f is certainly satisfied in this case.

The portfolio given by $f(s) = Cs^{\alpha}$, where α is a real number, is the portfolio that always has the amount $\alpha f_t(S_t)$ in the stock (and hence $(1-\alpha)f_t(S_t)$ on the account). Here $f_t(S_t) = I(S_t, \sigma\sqrt{T-t})$ stands for the value of the portfolio at time t. The portfolio borrows money if $\alpha > 1$, it borrows stocks if $\alpha < 0$.

The Theorem will be a consequence of two lemmas.

Lemma 1

$$M_f(r,s) \to I(s,\rho)$$

as $\delta \to 0$ and $r \to \infty$ in such a way that $\delta^2 r \to \rho^2$. The convergence is uniform in s on compacts.

Proof Let X_r denote a random variable which is binomially distributed with parameters r and P. Then

$$M_f(r,s) = Ef(se^{\delta(2X_r - r)}).$$

We have

$$E[\delta(2X_r - r)] = \delta r(2P-1) = -\frac{r\delta^2}{2}(1+O(\delta^2))$$

$$\mathrm{Var}[\delta(2X_r - r)] = \delta^2 4rPQ = -\frac{r\delta^2}{2}(1+O(\delta^2)).$$

Let I_K stand for the indicator of the set $[-K,K]$; $I_K(x) = 1$ if $-K \le x \le K$, $I_K(x) = 0$ otherwise. It follows from the central limit theorem that

$$E[f(se^{\delta(2X_r-r)})I_K(\delta(2X_r - r))] \to E[f(se^{-\frac{\rho^2}{2}+\rho Z})I_K(-\frac{\rho^2}{2}+\rho Z)]$$

where Z is Gaussian with mean 0 and variance 1.

It is clear that

$$|E[f(se^{-\frac{\rho^2}{2}+\rho Z})(1 - I_K(-\frac{\rho^2}{2}+\rho Z))]|$$

can be made arbitrarily small by choosing K large, and it follows from Appendix 1 that

$$E[|f(se^{\delta(2X_r-r)})|(1-I_K(\delta(2X_r - r)))] \le 2C(s^m + s^{-m})\exp[-\frac{K}{2}(\frac{K}{\delta^2 r} - 2m + O(\delta))],$$

and hence also this tail can be made arbitrarily small by choosing K large.

Lemma 2

$$\delta^2(N_\delta(T) - N_\delta(t)) \to \sigma^2(T-t)$$

in probability as $\delta \to 0$.

Proof Let $\tau_k = t_k - t_{k-1}$ for $k = 1, 2, ...$ These random variables are independent with the common distribution

$$F_\delta(t) = \text{Prob}(\max_{0 \le s \le t} |\mu s + \sigma W(s)| \ge \delta) = G_{\mu\delta/\sigma^2}(t\sigma^2/\delta^2)$$

where

$$G_\epsilon(t) = \text{Prob}(\max_{0 \le u \le t} |\epsilon u + W(u)| \ge 1).$$

This is so because

$$\max_{0 \le s \le t} |\mu s + \sigma W(s)|/\delta$$

has the same distribution as

$$\max_{0 \le s \le t} |(\mu/\delta)s + W(\sigma^2 s/\delta^2)|.$$

(Let $u = \sigma^2 s/\delta^2$.)

Also $G_\epsilon(s) \to G_0(s)$ as $\epsilon \to 0$. It follows that

$$\text{Prob}(\sigma^2\tau_k/\delta^2 \le t) \to G_0(t)$$

as $\delta \to 0$.

$N_\delta(T) - N_\delta(t)$ has the same distribution as $N_\delta(T-t)$ and we have

$$\tau_1 + ... + \tau_{N_\delta(T-t)} \le T - t \le \tau_1 + ... + \tau_{N_\delta(T-t)} + \tau_{N_\delta(T-t)+1}$$

that is

$$\frac{\tau_1\sigma^2/\delta^2 + ... + \tau_{N_\delta(T-t)}\sigma^2/\delta^2}{N_\delta(T-t)} \le \frac{(T-t)\sigma^2/\delta^2}{N_\delta(T-t)} \le \frac{\tau_1\sigma^2/\delta^2 + ... + \tau_{N_\delta(T-t)+1}\sigma^2/\delta^2}{N_\delta(T-t)}.$$

The extreme members tends to $\nu = \int tG_0(dt)$ as $\delta \to 0$ and hence

$$\delta^2 N_\delta(T-t) \to \sigma^2(T-t)/\nu$$

$\delta \to 0$.

An explicit expression for $G_0(t)$ is known and a consequence of this is $\nu = 1$. But this is overkill. A direct proof can be found in Appendix 2.

Proof of the Theorem We have $S_t e^{-\delta} < S_{t_{N_\delta(t)}} < S_t e^\delta$. It therefore follows from Lemma 1 and 2 that $f_{N_\delta(t)}(S_{t_{N_\delta(t)}})$ tends to the desired limit. Also

$$|P_t^\delta - f_{N_\delta(t)}(S_{t_{N_\delta(t)}})| = |d_{t_{N_\delta(t)}}(S_t - S_{t_{N_\delta(t)}})| \le$$

$$|f_{N_\delta(t)+1}(S_{t_{N_\delta(t)}}e^{\delta}) - f_{N_\delta(t)+1}(S_{t_{N_\delta(t)}}e^{-\delta})|.$$

This is so because

$$e^{-\delta} - 1 < (S_t - S_{t_{N_\delta(t)}})/S_{t_{N_\delta(t)}} < e^{\delta} - 1$$

and

$$\frac{\max(|e^{-\delta}-1|, e^{\delta}-1)}{e^{\delta}-e^{-\delta}} = \frac{1}{1+e^{-\delta}} < 1.$$

The Theorem now follows from Lemma 1 and 2.

Appendix 1

We shall only consider the right tail. The other is analogous. Let $I'_K(x) = 1$ for $x > K$, $= 0$ for $x \le K$, and put

$$R = E(e^{m\delta(2X_r - r)} I'_K(\delta(2X_r - r))$$

Then

$$R = A^r \sum_{k>t} \binom{r}{k} \theta^k (1-\theta)^{r-k}.$$

Here

$$A = e^{m\delta}P + e^{-m\delta}Q, \quad \theta = \frac{e^{m\delta}P}{A}, \quad t = \frac{1}{2}(r + K/\delta).$$

We shall use the inequality

$$\sum_{k \ge t} \binom{r}{k} \theta^k (1-\theta)^{r-k} \le e^{-2r(\frac{t}{r}-\theta)^2}$$

valid for $\frac{t}{r} \ge \theta$. (Höglund 1974, p. 144.)

Calculations show that

$$e^{m\delta}P = \frac{1}{2}(1 + \delta(m - \frac{1}{2}) + \frac{\delta^2}{2}m(m-1) + O(\delta^3))$$

$$e^{-m\delta}Q = \frac{1}{2}(1 - \delta(m - \frac{1}{2}) + \frac{\delta^2}{2}m(m-1) + O(\delta^3))$$

$$A = 1 + \frac{\delta^2}{2}m(m-1) + O(\delta^3)) = e^{\frac{\delta^2}{2}m(m-1) + O(\delta^3)}$$

and

$$\theta = \frac{1}{2}(1 + \delta(m - \frac{1}{2}) + O(\delta^3)).$$

Put $v = \delta^2 r$, then

$$\frac{t}{r} - \theta = \frac{\delta}{2}(\frac{K}{v} - m + \frac{1}{2} + O(\delta^2)) > 0$$

for $K > mv$ and δ small.

We thus have

$$\log R \leq \frac{v}{2}m(m-1) - 2r(\frac{t}{r} - \theta)^2 + O(\delta) =$$

$$-\frac{v}{2}((\frac{K}{v})^2 - 2\frac{K}{v}(m - \frac{1}{2}) + \frac{1}{4} + O(\delta))$$

$$< -\frac{K}{2}(\frac{K}{v} - 2m + O(\delta)).$$

Appendix 2

Let $t_0, t_1, t_2, ..., \tau_1, \tau_2, ...$ and $N(t)$ be as in the beginning of this paper but with $\mu = 0$, $\sigma = 1$ and $\delta = 1$. Then

$$\tau_1 + ... + \tau_{N(t)} \leq t < \tau_1 + ... + \tau_{N(t)+1}.$$

Taking expectations we get

$$EN(t)E\tau \leq t < (EN(t) + 1)E\tau, \text{ i.e. } \frac{t}{EN(t)+1} < E\tau \leq \frac{t}{EN(t)}.$$

Define $X_k = W_{t_k} - W_{t_{k-1}}$ for $k = 1, 2, ..., N(t)$, and $Y = W_t - W_{t_{N(t)}}$. Then $Y, X_1, X_2, ...$ are independent, and X_k take the values ± 1 with probability $\frac{1}{2}$, and $|Y| < 1$. We have

$$W_t = \sum_{k=1}^{N(t)} X_k + Y$$

and hence

$$t = EW_t^2 = EN(t)EX^2 + EY^2 = EN(t) + EY^2.$$

Therefore $t - 1 < EN(t) \leq t$. Combining this with the earlier inequality we get

$$\frac{t}{t+1} < E\tau < \frac{t}{t-1}, \text{ for all } t > 1.$$

Therefore $E\tau = 1$.

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