Mathematical Statistics
Stockholm University

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Research Report 2003:13

ISSN 1650-0377

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Mathematical Statistics
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Research Report **2003:13**,
http://www.math.su.se/matstat

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September 2003

Abstract

A portfolio is rebalanced in such a way that the weights are kept fixed. We express the value of this portfolio at a given time as a function of the values of the assets at that time.

KEY WORDS: Rebalancing, portfolios, constant proportions

AMS 2000 subject classification: 91B28

JEL Classification: G11

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1 Introduction

Consider a portfolio consisting of m assets with prices $S_1(t), ..., S_m(t)$ at time t. The portfolio is rebalanced in such a way that the weights of the assets are kept on given preassigned levels $w_1, ..., w_m$, respectively. We shall assume that the portfolio is rebalanced as soon as the prices have changed and neglect the transaction costs.

In this paper we show that the value, $P(t)$, of the portfolio at time t then equals

$$P(t) = P(0)e^{tL} \left(\frac{S_1(t)}{S_1(0)}\right)^{w_1} \cdot ... \cdot \left(\frac{S_m(t)}{S_m(0)}\right)^{w_m} \tag{1}$$

where

$$L = \frac{1}{2}(\sum_{i=1}^{m} w_i \sigma_{i,i} - w \cdot Qw). \tag{2}$$

Here $w = (w_1, ..., w_m)$,

$$\sigma_{i,j} = \text{Cov}\Big(\ln\left(S_i(1)/S_i(0)\right), \ln\left(S_j(1)/S_j(0)\right)\Big),$$

and Q is the covariance matrix

$$Q = (\sigma_{i,j}).$$

In Section 2 we reformulate the statement, in Section 3 we prove it under the usual Gaussian assumption, and in Section 4 we verify the result empirically. Finally, in Section 5 we give the asymptotic distribution of the value of the rebalanced portfolio after long time.

2 A reformulation

Assume that the portfolio is rebalanced at time $0, \Delta t, 2\Delta t, ..., n\Delta t$, where $n\Delta t = t$, and write $P_n(t)$ for the value of this portfolio at time t. Then

$$P_n(t) = P(0)\Pi_{k=1}^{n}\left(1 + R_P(k)\right),$$

where

$$R_P(k) = \frac{P_n\left(k\Delta t\right) - P_n\left((k-1)\Delta t\right)}{P_n\left((k-1)\Delta t\right)} = \sum_{i=1}^{m} w_i R_i(k) \tag{3}$$

and where

$$R_i(k) = \frac{S_i(k\Delta t) - S_i((k-1)\Delta t)}{S_i((k-1)\Delta t)}.$$

Therefore

$$\ln P_n(t) = \ln P(0) + \sum_{k=1}^{n} R_P(k) - \frac{1}{2}\sum_{k=1}^{n} R_P(k)^2 + \sum_{k=1}^{n} e(R_P(k)), \tag{4}$$

where

$$e(x) = \ln(1+x) - x + \frac{x^2}{2} = O(|x|^3). \tag{5}$$

Similarly

$$P(t) = P(0)e^{tL}\Pi_{i=1}^{m}\Pi_{k=1}^{n}(1+R_i(k))^{w_i},$$

and hence

$$\ln P(t) = \ln P(0) + tL + \sum_{i=1}^{m}\sum_{k=1}^{n} w_i R_i(k) - \frac{1}{2}\sum_{i=1}^{m}\sum_{k=1}^{n} w_i R_i(k)^2 + \sum_{i=1}^{m}\sum_{k=1}^{n} w_i e(R_i(k)). \tag{6}$$

The next proposition now follows from (4), (6), (3) and (2).

Proposition 1

$$\ln P_n(t) - \ln P(t) = -\frac{1}{2}\Sigma_1 + \frac{1}{2}\Sigma_2 + \Sigma_3 - \Sigma_4. \tag{7}$$

Here $\Sigma_j = \sum_{k=1}^{n} U_j(k)$ *for* $j = 1, 2, 3, 4$*, where*

$$U_1(k) = R_P(k)^2 - \Delta t w \cdot Q w,\ U_2(k) = \sum_{i=1}^{m} w_i (R_i(k)^2 - \Delta t \sigma_{i,i})$$

$$U_3(k) = e(R_P(k)), \; and \, U_4(k) = \sum_{i=1}^{m} w_i e(R_i(k)).$$

3 Gaussian growth

We shall here make the following assumption:

<u>Gaussian assumption</u>

$$S_i(t) = S_i(0)e^{\nu_i t + X_i(t)},$$

where the process $X(t) = (X_1(t), ..., X_m(t))$ has independent increments, and $X(t) - X(s)$ is normally distributed with zero expectation and variance $|t-s|Q$.

Theorem 2 *The Gaussian assumption implies that $P_n(t) \to P(t)$ in probability, as $n \to \infty$, and $\Delta t \to 0$ in such a way that $n\Delta t = t$.*

Proof Define $Z_i(k)$ by

$$X_i(k\Delta t) - X_i((k-1)\Delta t) = \sqrt{\Delta t} Z_i(k).$$

Then the random variables $Z(k) = (Z_1(k), ..., Z_m(k))$, $k = 1, ..., n$ are independent and normally distributed with zero expectation and covariance matrix Q.

We shall now show that the sums Σ_j of Proposition 1 are small, the details will be given only for Σ_2 and Σ_4. The remaining sums will be left to the reader. Σ_1 is similar to Σ_2 and Σ_3 to Σ_4.

$$\Sigma_2 :$$

We have

$$R_i(k) = e^{G_i(k)} - 1,$$

where

$$G_i(k) = \nu_i \Delta t + \sqrt{\Delta t} Z_i(k).$$

Also $e^x - 1 = x + r_1(x)$, where $|r_1(x)| \le |x|^2 e^{|x|}$, and hence

$$R_i(k)^2 = G_i(k)^2 + r_2(G_i(k)),$$

where $r_2(x) \le Const|x|^3 e^{2|x|}$.

It follows that

$$ER_i(k)^2 = EG_i(k)^2 + O(\sqrt{\Delta t}^3) = \Delta t \sigma_{i,i} + O(\sqrt{\Delta t}^3).$$

The variables $U_1(1), ..., U_2(n)$ are identically distributed and hence

$$E\Sigma_2 = nEU_2(1) = O(n\sqrt{\Delta t}^3) = O(\sqrt{\Delta t}). \tag{8}$$

In a similar way way we get

$$\mathrm{Var}(U_2(k)) = \mathrm{Var}\left(U_2\left(1\right)\right) = (\Delta t)^2 \mathrm{Var}\left(\sum_{i=1}^{m} w_i G_i\left(1\right)^2\right) + O\left(\sqrt{\Delta t}^5\right) =$$

$$(\Delta t)^2 \mathrm{Var}\left(\sum_{i=1}^{m} w_i Z_i\left(1\right)^2\right) + O\left(\sqrt{\Delta t}^5\right) = O\left((\Delta t)^2\right).$$

The variables $U_2(1), ..., U_2(n)$ are independent and identically distributed and hence

$$\mathrm{Var}(\Sigma_2) = n\mathrm{Var}\left(U_2\left(1\right)\right) = nO\left((\Delta t)^2\right) = O\left(\Delta t\right). \tag{9}$$

It follows from (8) and (9) that $\Sigma_2 \to 0$ in probability.

Σ_4

Note that $e(x)$ is large not only when x is large, but also when x is close to -1. But the inequality $|e(x)| \le Const.|x|^3$ holds for $x \ge -\frac{1}{2}$. Therefore let

$$B_n = \{R_i(k) \ge -\frac{1}{2} \text{ for } i = 1, ..., m \text{ and } k = 1, ..., n\}.$$

In B_n we thus have

$$|\Sigma_4| \leq Const. \sum_{i=1}^{m} \sum_{k=1}^{n} |R_i(k)|^3.$$

Therefore

$$P(|\Sigma_4| > \epsilon) \leq P\left(Const. \sum_{i=1}^{m} \sum_{k=1}^{n} |R_i(k)|^3 > \epsilon\right) + P(B_n^c). \qquad (10)$$

The first expression to the right is by Markov's inequality dominated by

$$Const. \frac{\sum_{i=1}^{m} \sum_{k=1}^{n} E|R_i(k)|^3}{\epsilon} = Const. \frac{\sum_{i=1}^{m} nE|R_i(1)|^3}{\epsilon}.$$

Also by Boole's and Markov's inequalities

$$P(B_n^c) \leq \sum_{i=1}^{m} \sum_{k=1}^{n} P\left(R_i(k) < -\frac{1}{2}\right) \leq 8 \sum_{i=1}^{m} \sum_{k=1}^{n} E|R_i(k)|^3 = 8 \sum_{i=1}^{m} nE|R_i(1)|^3.$$

In the same way as above we get

$$E|R_i(1)|^3 = O(\sqrt{\Delta t}^3).$$

The expression (10) is therefore dominated by $Const.\sqrt{\Delta t}(1 + 1/\epsilon)$, and therefore $\Sigma_4 \to 0$ in probability.

4 Empirical verification

The Gaussian model fits data fairly well when t is large, but deviates significantly when t is small, which it is in the proof. It is therefore not immediately clear that (1) gives the value of a frequently rebalanced portfolio. In this section we shall however verify (1) empirically.

We shall here consider the development of the five stocks AstraZeneca, Ericsson, Hennes, Skandia, and Skanska on Stockholmsbörsen during the period October 7 1996 to October 31 2000. In total 1025 trading days.

We shall consider three different portfolios: the portfolio with equal weights, the minimum variance portfolio, and the market neutral portfolio that has minimal variance. By a market neutral portfolio we mean a portfolio whose returns are uncorrelated with the returns of the market. The market is in this case Affärsväldens generalindex, an index that covers about 300 Swedish stocks.

The first 769 days will be used to estimate the weights of the different portfolios, and the remaining 256 days to check if the portfolio which is rebalanced every r days converges to the portfolio (1) as $r \to 0$.

Let $G(s,t) = (G_1(s,t), ..., G_m(s,t))$, where $G_i(s,t) = \ln(S_i(t)/S_i(s))$ stands for the growth of asset i in the time interval $(s,t]$. Let $G(t) = G(0,t)$, then $G(s,t) = G(t) - G(s)$.

We shall need the property

$$\mathrm{Var}(G(t, t + \Delta t)) = \Delta t \mathrm{Var}(G(1))$$

in order to be able to estimate the covariance matrix, and hence the value of the portfolio (1).

We shall therefore make an assumption about the moments of $G(t)$. For convenience we shall consider the one-dimensional random variables $x \cdot G(t) = \sum_{i=1}^{m} x_i G_i(t)$.

Moment assumption

The following hold for each $x \in \mathbf{R}^m$:

a) *The first two moments of* $x \cdot G(t)$ *exist, and are continuous functions of* t.

b) *For each pair* t_1, t_2 *the first two moments of the increments* $x \cdot G(t_2 + s) - x \cdot G(t_1 + s)$ *are the same for all* s.

c) *The increments* $x \cdot G(t_2) - x \cdot G(t_1)$, $x \cdot G(t_3) - x \cdot G(t_2)$ *are uncorrelated for all* $t_1 < t_2 < t_3$.

Assumption c is an empirical fact although the increments are not independent; the absolute values are positively correlated. See Ding, Granger&Engle (1993) or Rydberg (2000).

Lemma 3 *The Moment assumption implies*

$$E[G(t)] = tE[G(1)] \text{ and } \mathrm{Var}(G(t)) = t\mathrm{Var}(G(1)). \quad (11)$$

Proof. Put $e(t) = Ex \cdot G(t)$, $v(t) = \mathrm{Var}(x \cdot G(t))$. Assumption b) implies that $e(t + s) = e(t) + e(s)$, and Assumption b) and c) that $v(t + s) = v(t) + v(s)$.

These identities imply together with assumption a) that $e(t) = e(1)t$ and $v(t) = v(1)t$.

We thus have

$$EG(t) = t\nu \text{ and } \mathrm{Var}\,(G(t)) = tQ, \tag{12}$$

where $\nu = EG(1)$, and $Q = \mathrm{Var}(G(1))$.

We shall estimate Q with $\hat{Q} = (\hat{\sigma}_{i,j})$, where

$$\hat{\sigma}_{i,j} = 250\frac{1}{n}\sum_{k=1}^{n}(G_i((k-1)\Delta t, k\Delta t) - \bar{G}_i)^2,\ \bar{G}_i = \frac{\sum_{k=1}^{n} G_i((k-1)\Delta t, k\Delta t)}{n}.$$

Here $n = 768$, and $\Delta t = 1$ day $= 1/250$ year.

We shall assume that the starting values of all portfolios are 1. Let for $i = 1, ..., 256$, $\Pi_0(i)$ denote the value of the portfolio (1) with $Q = \hat{Q}$ after day i, and $\Pi_r(i)$ the value of the portfolio which is rebalanced every r days, $r = 1, 2, ...$

There is little need to rebalance if the weights change only marginally. The effect of rebalancing is thus best illustrated by choosing a period during which the weights change considerably. The first 100 of the last 256 days is such a period. Ericsson and Skandia went up about 140%, Hennes 55% and AstraZeneca and Skanska were in comparision almost unchanged.

We shall measure the distances

$$d(r) = \frac{\sum_{i=1}^{100} |\Pi_r(i) - \Pi_0(i)|}{100},\ r = 1, 2, ...$$

Figure 1 shows the values of $d(r)$ for $r = 1, 2, ..., 100$ days for the portfolio with equal weights. Figure 2 shows the development of the portfolios Π_0, Π_1 and Π_{100} during the first 100 days. The latter portfolio is thus never rebalanced.

The figures 3 to 6 are the corresponding plots for the minimum variance portfolio, and the market neutral minimum variance portfolio. The weights of these portfolios are

$$(0.35, 0.01, 0.17, -0.01, 0.47),\ (0.60, -0.58, 0.27, -0.20, 1.04)$$

respectively.

The minimum variance portfolio has the heavy weights 0.35 and 0.47 in the low volatility stocks AstraZeneca and Skanska whereas the high volatility stocks Ericsson and Skandia with high beta values have the low weights 0.01

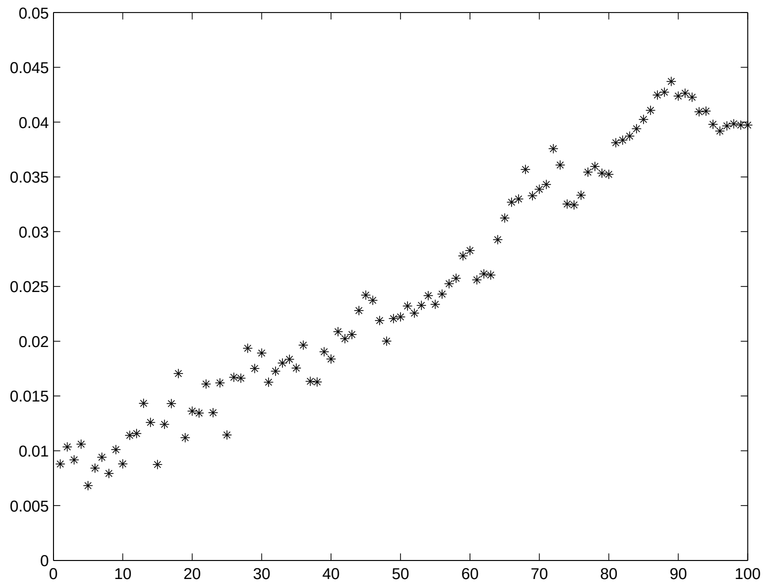

Figure 1: Equal weights. Distance between the portfolios that are rebalanced continuously and every k days, as a function of k.

and -0.01 of opposite signs. Therefore one may expect that there is little need of rebalancing. That it is so is seen in Figure 3.

The quotients $d(1)/d(100)$ are 0.22, 0.44, 0.04 for the three portfolios.

5 Asymptotic distribution of the portfolio value

It is an empirical fact that the growth of stock prices under long time periods are approximately gaussian. It follows from (1) that

$$\ln(P(t)/P(0)) = tL + \sum_{i=1}^{m} w_i G_i(t),$$

and hence the following result holds.

Theorem 4 *Assume that the moment assumption holds, and that the growth $G(t)$ is asymptotically normally distributed, as $t \to \infty$. Then $\ln(P(t)/P(0))$ is asymptotically normally distributed with expectation $t(L + w \cdot \nu)$, and variance $w \cdot Qw$. Here $\nu = EG(1)$ and $Q = Var(G(1))$.*

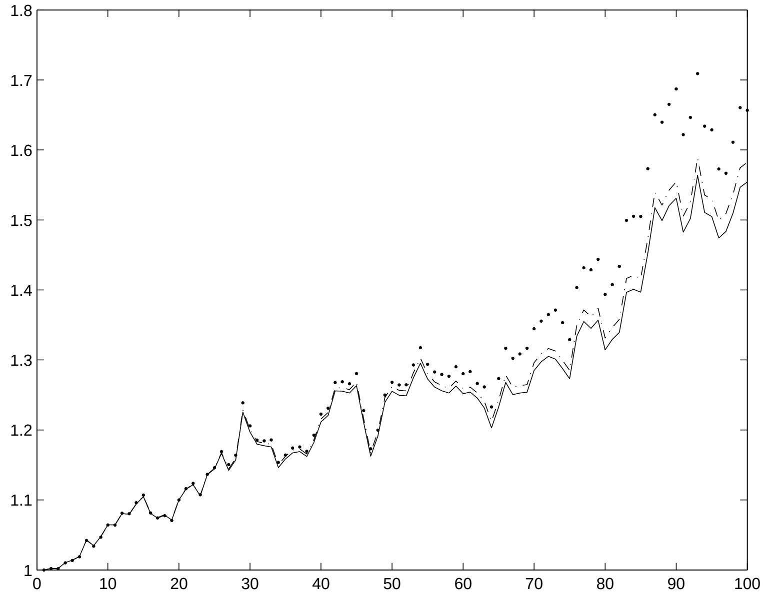

Figure 2: Equal weights. Development of the portfolios that are rebalanced continuously, '–', daily,'-.', and never,'.'.

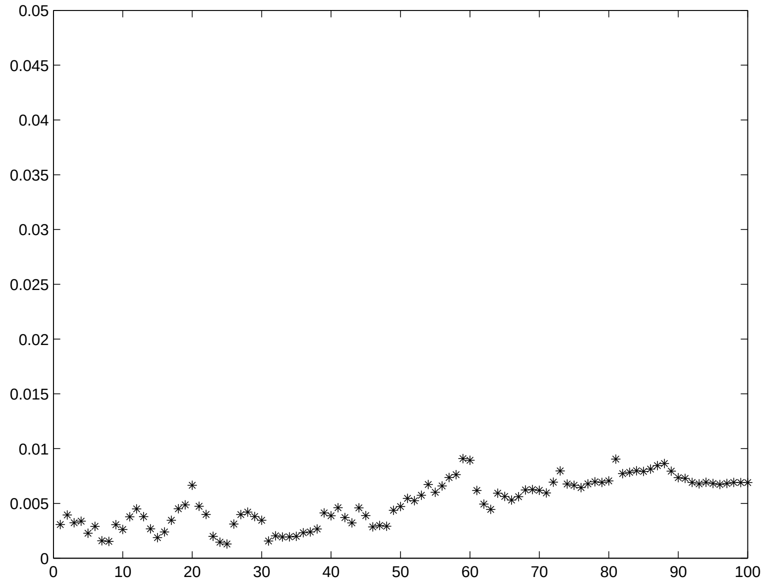

Figure 3: Minimum variance. Distance between the portfolios that are rebalanced continuously and every k days, as a function of k.

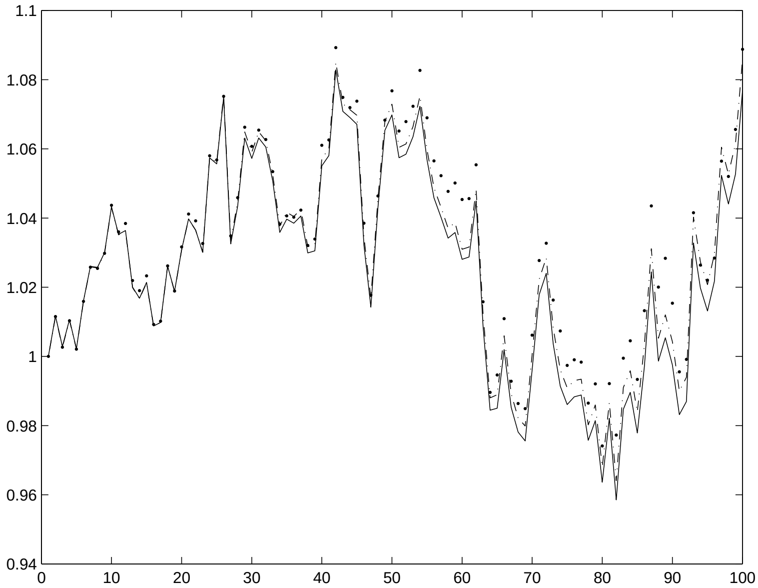

Figure 4: Minimum variance. Development of the portfolios that are rebalanced continuously, '–', daily,'-.', and never,'.'.

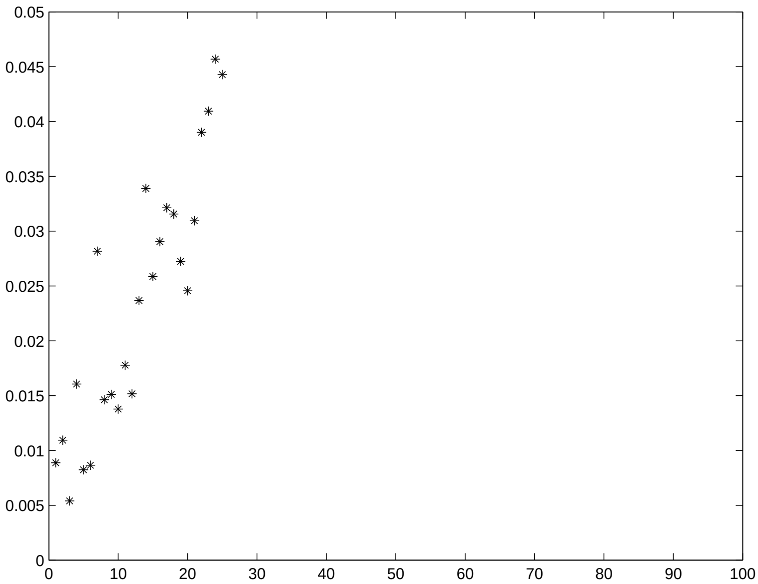

Figure 5: Market neutral. Distance between the portfolios that are rebalanced continuously and every k days, as a function of k.

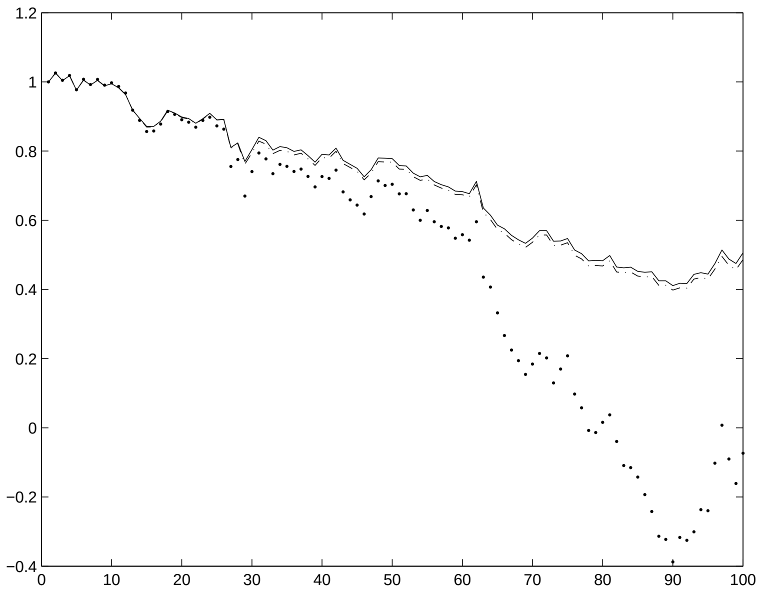

Figure 6: Market neutral. Development of the portfolios that are rebalanced continuously, '–', daily,'-.', and never,'.'.

References

Ding, Z., Granger, C.W.J.&Engle, R.F. A long memory property of stock market returns and a new model. Journal of Empirical Finance, **1**, 83-106 (1993)

Rydberg, T.H. Realistic statistical modelling of financial data. International Statistical Review **68**, 233-258 (2000)