Mathematical Statistics
Stockholm University

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Thomas Höglund

Research Report 2003:12

ISSN 1650-0377

Postal address:

Mathematical Statistics
Dept. of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden

Internet:

http://www.math.su.se/matstat

Mathematical Statistics
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Research Report **2003:12**,
http://www.math.su.se/matstat

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September 2003

Abstract

We derive a formula which in the case of a call shows a volatility smile. Our model assumption is geometric Brownian motion with constant volatility and drift.

KEY WORDS: Option pricing, volatility smile

AMS 2000 subject classification: 91B28

JEL Classification: G13

*Postal address: Mathematical Statistics, Stockholm University, SE-106 91, Sweden. E-mail: hoglund@math.su.se.

1 Introduction

The volatility smile has attracted some attention during the last decades. See Skiadopoulos (2001) which contains 105 references, and in which different approaches to the problem are described.

A common feature of many of these approaches is the assumption that geometric Brownian motion is not a sufficiently accurate model. This may be true, but we shall here show that the volatility smile appears even if we accept this model.

We derive a formula that takes into account the fact that one does not trade continuously when replicating an option. We assume that one trades only when the asset price has moved a few percent. The number of trades will then be random, and hence also the price. This random price is replaced by its expectation. The Black-Scholes formula is the limiting case when one trades continuously.

When comparing our formula with the Black-Scholes formula in the case of a call one finds that the former gives a higher price than the latter when the asset price is lower than the strike price, i.e. when the call is out-of-the-money. This phenomenon is more pronounced when the time to maturity is short, or the volatility low. This is thus in accordance with the empirical findings of Rubinstein (1985). See conclusion 1 p. 474.

2 The idea

Consider a replicating portfolio which at time t consists of c_t EUR in cash and a number a_t of an asset whose price is S_t. The portfolio is rebalanced at time $\tau_0, \tau_1, \tau_2, ...$ by selling or buying assets. The goal of the trading is to get the portfolio value $f(S_T)$ at a given time T. Here f is a given target function. By considering the present values of the asset and the portfolio one can assume that the interest rate equals zero. Also assume that the asset price follows a geometric Brownian motion with constant volatility, σ, and drift, ν.

We shall choose to trade when the asset price reaches the levels of the binomial tree

$$
\begin{array}{ccccc}
0 & \tau_1 & \tau_2 & \tau_3 & \cdot\ \cdot \\
 & & & & \cdot \\
 & & & \cdot & \\
 & & & S_0u^3 & \\
 & & S_0u^2 & & \\
 & S_0u & & S_0u^2d & \\
S_0 & & S_0ud & & \\
 & S_0d & & S_0ud^2 & \\
 & & S_0d^2 & & \\
 & & & S_0d^3 & \\
 & & & \cdot & \\
 & & & & \cdot
\end{array}
$$

Here $u = e^\delta$ and $d = 1/u$, and δ is a given positive number. Typically δ is a few percents. Let $\tau_0 = 0$. Then τ_k is the first time after τ_{k-1} for which

$$S_{\tau_k}/S_{\tau_{k-1}} = u \text{ or } S_{\tau_k}/S_{\tau_{k-1}} = d, \tag{2.1}$$

for $k = 1, 2, ...$ The numbers $\tau_1, \tau_2, ...$ are thus not known in advance but determined by the movements of the price of the asset.

Let $N = N_\delta(T)$ denote the number of τ_k in the interval $(0, T]$.

Consider the portfolio which at time τ_N takes the value $f(S_{\tau_N})$. Let $F_k(s)$ stand for the value of this portfolio at time τ_k when $S_{\tau_k} = s$.

Proceeding as in Cox, Ross and Rubinstein (1979) we get

$$F_k(s) = M_f(N - k, s),$$

where

$$M_f(r, s) = \sum_{i=0}^{r} b_q(r, i) f(su^i d^{r-i}) \text{ and } b_q(r, i) = \binom{r}{i} q^i (1 - q)^{r-i}, \tag{2.2}$$

and hence in particular $F_N(s) = f(s)$. Here $q = \frac{1-d}{u-d}$.

The number of assets at time t then equals $a_{k-1}(S_{\tau_{k-1}})$ for $\tau_{k-1} < t < \tau_k$. Here

$$a_{k-1}(s) = \frac{F_k(su) - F_k(sd)}{s(u - d)}. \tag{2.3}$$

The amount of cash therefore equals $C_{k-1}(S_{\tau_{k-1}})$ for $\tau_{k-1} < t < \tau_k$,, where

$$C_k(s) = F_k(s) - a_k(s)s. \tag{2.4}$$

Let $N_\delta(t,T)$ denote the number of τ_k in the interval $(t,T]$.

The value of the portfolio at time t thus equals the random variable

$$P_t^\delta = M_f(N_\delta(t,T), S_{\tau_{N_\delta(t)}}) + a_{N_\delta(t)}(S_{\tau_{N_\delta(t)}})(S_t - S_{\tau_{N_\delta(t)}}). \tag{2.4}$$

Hence in particular

$$P_T^\delta = f(S_{\tau_{N_\delta(T)}}) + a_{N_\delta(T)}(S_{\tau_{N_\delta(T)}})(S_T - S_{\tau_{N_\delta(T)}}). \tag{2.5}$$

The difference $P_T^\delta - f(S_{\tau_{N_\delta(T)}})$ can be shown to be small when δ is. See the proof of Corollary 3 in Höglund (2003). (In this case it is of the order $O(\delta^2)$.) It is also shown in the above paper that the value of this portfolio converges to

$$P_t \to I_f(\sigma^2(T-t), S_t)$$

in probability as $\delta \to 0$, provided f is regular. Here

$$I_f(v,s) = \int_{-\infty}^{\infty} f(se^{-\frac{v}{2}+\sqrt{v}z})\phi(z)dz \text{ where } \phi(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}. \tag{2.6}$$

In the case $f(s) = \max(0, s-K)$ this is the Black-Scholes formula for a call.

Another way to get rid of the randomness but without destroying the smile is to replace the portfolio value and the number of assets in the portfolio at time t by the expectations $F(T-t, S_t)$ and $a(T-t, S_t)$, respectively. Here

$$F(T,s) = EM_f(N_\delta(T), s) \tag{2.7}$$

and

$$a(T,s) = \frac{F(T, se^\delta) - F(T, se^{-\delta})}{s(e^\delta - e^{-\delta})}. \tag{2.8}$$

This is what we shall do in this paper.

3 Details

The Laplace transform, $\Psi(\theta) = E[e^{-\theta\tau_1}]$, of τ_1 satisfies

$$\Psi(\theta) = \frac{1}{\cosh(\sqrt{2\theta})} \tag{3.1}$$

when $\nu = 0$ and $\delta/\sigma = 1$, and in general

$$E[e^{-\theta\tau_1}] = \cosh(\beta)\Psi(\alpha\theta + \frac{1}{2}\beta^2)$$

where $\alpha = \delta^2/\sigma^2$ and $\beta = \delta\nu/\sigma^2$. See formula 3.01 p. 233 in Borokin and Salmimen (1996). Therefore

$$E\tau_1^k = \alpha^k \cosh(\beta)\Psi^{(k)}(\beta^2/2) = \alpha^k\Psi^{(k)}(0)(1 + O(\beta^2))$$

If the drift over the interest rate, ν, equals 0.05, $\sigma = 0.5$ and $\delta = 0.05$, then $\beta = 0.01$ and hence $\beta^2 = 0.0001$. We shall therefore let $\beta = 0$.

Lemma 1 *Assume that $\beta = 0$, then*

$$P(N_\delta(T) = n) = p_n(T\sigma^2/\delta^2),$$

for $n = 0, 1, \ldots$. Here

$$p_n(t) = G_n(t) - G_{n+1}(t), \text{ and } G_n(t) = P(\zeta_1 + \ldots\zeta_n \leq t), \tag{3.2}$$

where $\zeta_1, \zeta_2, \ldots$ are independent with $E[e^{-\theta\zeta_k}] = \Psi(\theta)$ for $k = 1, 2, \ldots$.

Proof. Let $\zeta = \tau_1/\alpha$, then the Laplace transform of ζ equals $\Psi(\theta)$, and τ_n has the same distribution as $(\zeta_1 + \ldots\zeta_n)\alpha$. Also

$$P(N_\delta(T) = n) = P(\tau_n \leq T) - P(\tau_{n+1} \leq T).$$

From which the lemma follows.

The portfolio value (2.7) thus satisfies

$$F(T, s) = f_\delta(T\sigma^2/\delta^2, s), \tag{3.3}$$

where

$$f_\delta(t,s) = \sum_{n=0}^{\infty} p_n(t) M_f(n,s). \tag{3.4}$$

Recall that $M_f(n,s)$ depends on δ via q. The portfolio value is thus a function of s and the two parameters $\sigma^2 T/\delta^2$ and δ.

If $\sigma = 0.5$, $\delta = 0.05$ and $T = 1/4$ i.e. three months, then $T\sigma^2/\delta^2 = 25$. We shall not use the exact expression for $f_\delta(t,s)$, but instead approximate it when t is large.

Define $\tilde{f}_\delta(t,s)$ by

$$\tilde{f}_\delta(t,s) = \sum_{n=0}^{\infty} \tilde{p}_n(t) M_f(n,s). \tag{3.5}$$

Here

$$\tilde{p}_n(t) = \phi(\frac{t-n}{\sqrt{n2/3}})\frac{1}{\sqrt{n2/3}}, \text{ where } \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{3.6}$$

Proposition 2 *If there are positive constants A and B such that*

$$|f(s)| \leq A + Bs \tag{3.7}$$

for all $s > 0$, then

$$f_\delta(t,s) = \tilde{f}_\delta(t,s) + O(t^{-1/3}), \tag{3.8}$$

as $t \to \infty$.

Note that the assumption (3.7) is satisfied when f is the target function of a linear combination of puts and calls.

The error term in (3.8) can be improved to $O(t^{-1/2})$.

See the appendix for a proof of the proposition.

4 Call options

We shall here consider a call with strike price K, remaining time to maturity T years, and volatility σ. The interest rate is assumed to be constant $= r$. Let $k = e^{-rT}K$ denote the present value of K, write $\omega = \sigma\sqrt{T}$ for the volatility under the remaining time to maturity, and put $s = S_0/k$.

The value of this call is according to the Black-Scholes formula $k\Gamma_1(s,\omega)$, where

$$\Gamma_1(s,\omega) = s\Phi(d_+) - \Phi(d_-), \text{ and } d_\pm = \frac{\ln s}{\omega} \pm \frac{\omega}{2}. \tag{4.1}$$

Proposition 3 *If $f(s) = \max(0, s-1)$, then*

$$M_f(n,s) = sB_q(n,x) - B_{1-q}(n,x), \tag{4.2}$$

where $x = \frac{n}{2} + \frac{\ln s}{2\delta}$ and

$$B_q(n,x) = \sum_{i \le x} b_q(n,i).$$

Proof. $M_f(n,s) = sS_1 - S_2$, where

$$S_1 = \sum_{i \ge y} b_q(n,i)u^i d^{n-i}, \quad S_2 = \sum_{i \ge y} b_q(n,i),$$

and $y = \frac{n}{2} - \frac{\ln s}{2\delta}$. We have $uq = 1-q$, $d(1-q) = q$, and

$$\sum_{i \ge y} b_p(n,i) = \sum_{i \le n-y} b_{1-p}(n,i).$$

The proposition follows.

It follows from (2.2) that $M_f(n,s)$ is a continuous function of s in this case. Note that $B_q(n,x)$ and $B_{1-q}(n,x)$ are constant for $k \le x < k+1$, i.e. for $e^{(2k-n)\delta} \le s < e^{(2k-n+2)\delta}$, $k = 0, \pm 1, \pm 2, \ldots$. Also $B_q(n,x)$ increases with s. It follows that $M_f(n,s)$ is a continuous, increasing, convex, and piece wise linear function of s.

The price of the call is thus given by

$$k\Gamma_2(s, \omega^2/\delta^2, \delta),$$

where $\Gamma_2(s, t, \delta)$ is as in (3.4) with $f(s) = \max(0, s - 1)$

We shall compare $\Gamma_1(s, \omega)$ with $\tilde{\Gamma}_2(s, \omega^2/\delta^2, \delta)$, where $\tilde{\Gamma}_2$ is the approximation of Proposition 2.

We shall consider two values of δ: 0.02, 0.04, and three values of ω: 0.35, 0.25 and 0.14. If $\sigma = 0.5$ the latter values correspond to $T = 6$, 3, and 1 months, respectively.

Let us start with the case $\delta = 0.04$, $\omega = 0.25$. If one plots Γ_1 and $\tilde{\Gamma}_2$ for $0.5 \leq s \leq 2$ it is not possible to to see any difference. This is so because the difference $\tilde{\Gamma}_2 - \Gamma_1$ varies between 0.00001 and 0.0009 in this case, and the price of the calls varies between approximately 0 and 1. In Figure 1 the two functions are plotted for s between 0.95 and 1.05.

In Figures 2 and 3 the quotient $\tilde{\Gamma}_2/\Gamma_1$ is plotted for $\omega = 0.25$ and 0.35 in the two cases $\delta = 0.04$ respectively 0.02.

The above mentioned piece wise linearity of Γ_2 is seen in the plots.

The difference between the two figures reflects the fact that the difference between the two methods decreases with δ.

When $\omega = 0.14$, $\delta = 0.04$, and $s = 0.5$ the quotient equals 3. Figure 4 shows the quotient over a narrower interval when $\omega = 0.14$, and $\delta = 0.02$ and 0.04.

In figures 5 and 6 are the implied volatilities plotted as functions of s. By the implied volatility we here mean the volatility for which the Black-Scholes formula take the same value as ours. In Figure 6 we have chosen a wider interval in order to be able to see a smile.

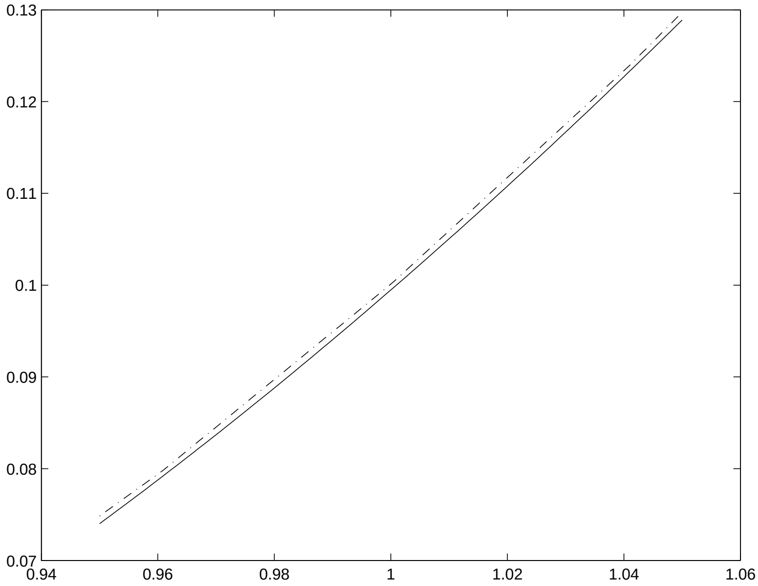

Figure 1: The value of a call as a function of s according to the Black-Scholes formula '–', and our formula, '-.'.

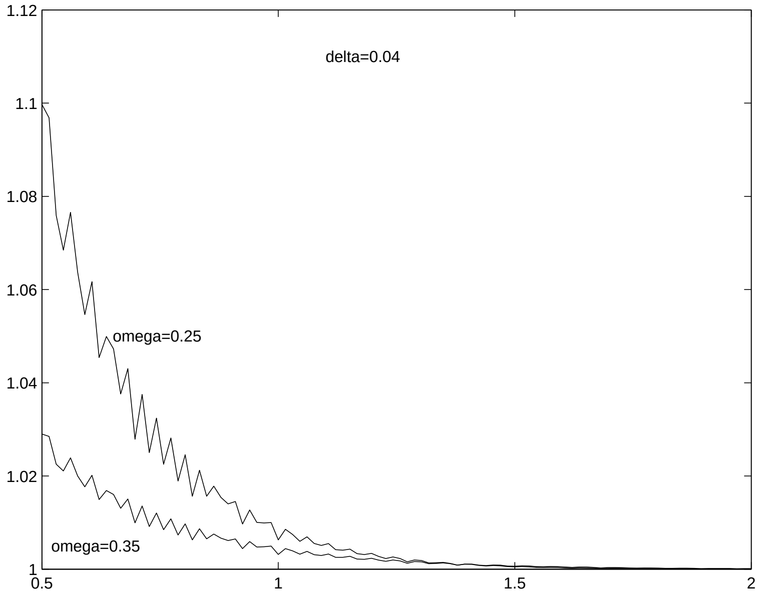

Figure 2: The quotients as functions of s when omega=0.25 and 0.35 in the case delta=0.04.

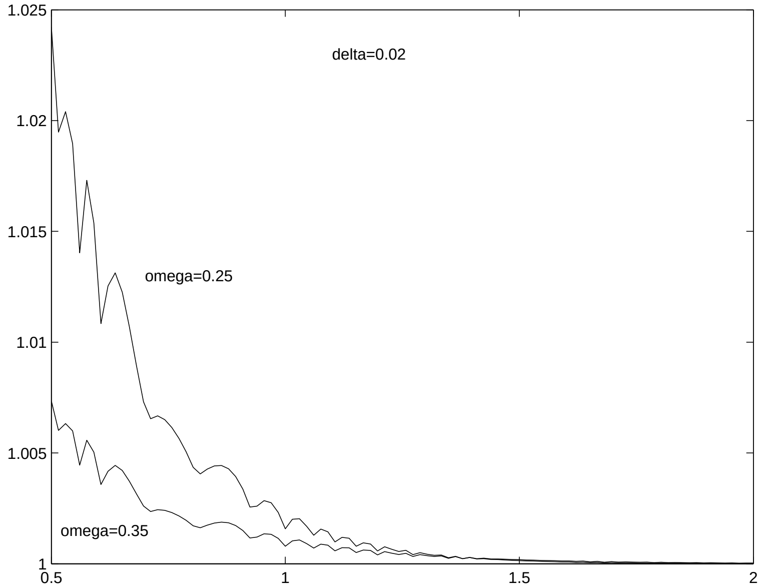

Figure 3: The quotients as functions of s when omega=0.25 and 0.35 in the case delta=0.02.

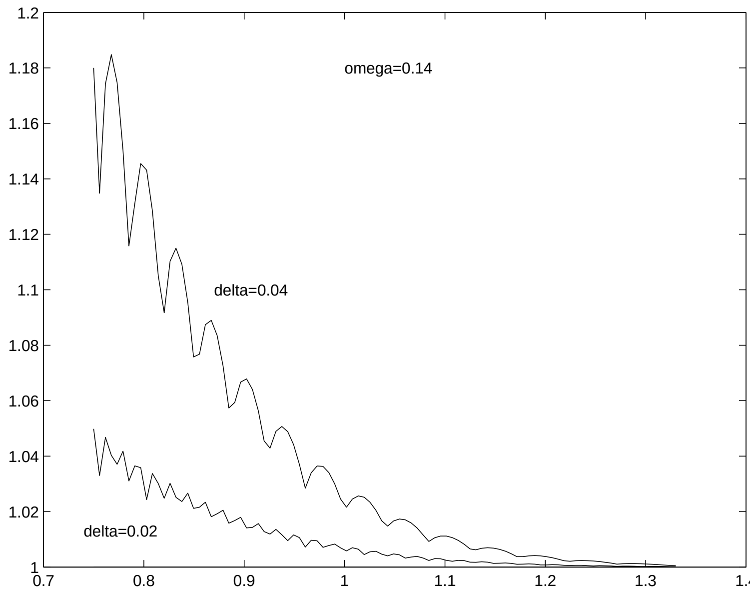

Figure 4: The quotients as functions of s when omega=0.14 in the cases delta=0.02 and 0.04.

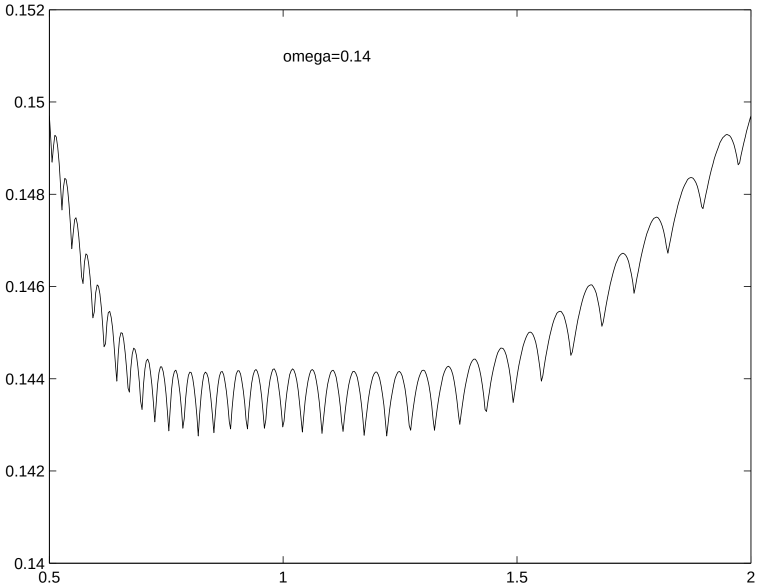

Figure 5: The implied volatility as a function of s when omega=0.14 and delta=0.04.

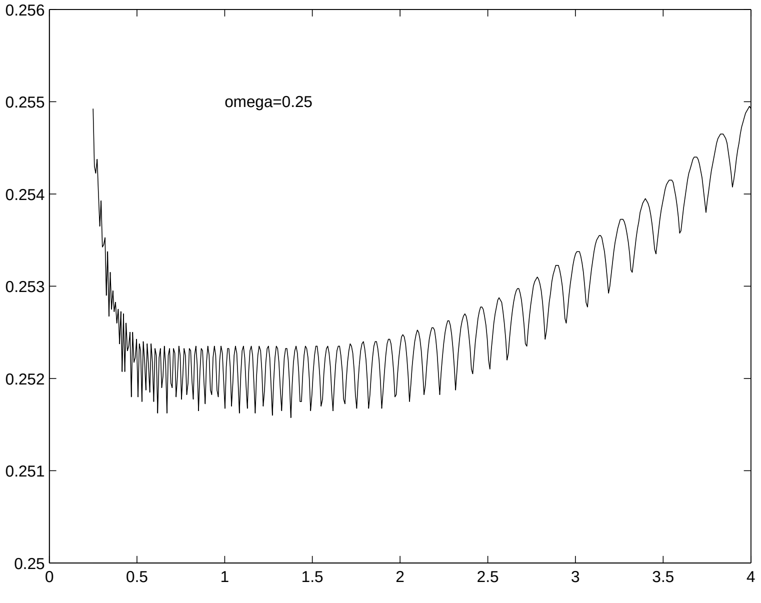

Figure 6: The implied volatility as a function of s when omega=0.25 and delta=0.04

Appendix. Proof of Proposition 2

Lemma 4

$$p_n(t) = \tilde{p}_n(t) + O(1/n), \qquad (A.1)$$

as $n \to \infty$. *Here* $\tilde{p}_n(t)$ *is as in (3.6).*

Proof. The mean value and variance of the variable ζ are 1 and 2/3, respectively. The Edgeworth expansion for G_n therefore equals

$$G_n(t) = \Phi(t_n) + \phi(t_n)R(t_n)/\sqrt{n} + O(1/n),$$

where R is a second degree polynomial and $t_n = (t-n)/\sqrt{n2/3}$. See Feller (1966), Ch. XVI.5, Thm. 3.

Thus by (3.2)

$$p_n(t) = \Phi(t_n) - \Phi(t_{n+1}) + \phi(t_n)R(t_n)/\sqrt{n} - \phi(t_{n+1})R(t_{n+1})/\sqrt{n+1} + O(1/n). \qquad (A.2)$$

Also

$$t_{n+1} = t_n - 1/\sqrt{n2/3} + \frac{a_n t_n + b_n}{n}, \qquad (A.3)$$

where $a_n \to -1/2$ and $b_n \to 0$, as $n \to \infty$.

By Taylor's formula

$$\Phi(t_n) - \Phi(t_{n+1}) = -\phi(t_n)(t_{n+1} - t_n) - \frac{1}{2}\phi'(\tau_n)(t_{n+1} - t_n)^2, \qquad (A.4)$$

where τ_n lies between t_n and t_{n+1}.

The first term to the right in (A.4) equals $\tilde{p}_n(t) + O(1/n)$, and the second term equals $O(1/n)$ if $|t_n| = O(\sqrt{n})$. Otherwise, if $|t_n|$ is large, then by (A.3) $|t_{n+1}|$ and hence also $|\tau_n|$ is of the same order as $|t_n|$. Therefore the second term to the right in (A.4) equals $O(1/n)$ also in this case.

It follows in a similar way that also the remainder of the right hand side of (A.2) equals $O(1/n)$.

Lemma 5 *If (3.7) is satisfied, then* $|M_f(n,s)| \leq A + Bs$.

Proof Let $b_q(n,i)$ be as in (2.2). Then

$$|M_f(n,s)| \leq \sum_{i=0}^{n} b_q(n,i)(A + Bsu^i d^{n-i}) =$$

$$\sum_{i=0}^{n}(Ab_q(n,i) + Bsb_{1-q}(n,i)) = A + Bs.$$

Here we used the identities $qu = 1 - q$ and $(1-q)d = q$.

Proof of Proposition 2. Let $N_\pm = [t \pm t^{2/3}]$. It follows from Lemma 5 and 4 that

$$|\sum_{n=N_-}^{N_+} p_n(t)M_f(n,s) - \sum_{n=N_-}^{N_+} \tilde{p}_n(t)M_f(n,s)| \leq$$

$$(A+Bs)Const\frac{N_+ - N_- + 1}{N_-} = O(t^{-1/3}).$$

Also

$$|\sum_{n>N_+} p_n(t)M_f(n,s)| \leq (A+Bs)\sum_{n>N_+}(G_n(t) - G_{n+1}(t)) = (A+Bs)G_{N_++1}(t),$$

and it follows from Chebyshev's inequality that

$$G_{N_++1}(t) \leq \frac{2}{3}\frac{N_+ + 1}{(N_+ + 1 - t)^2} = O(t^{-1/3}).$$

In the same way we get

$$\sum_{n<N_-} p_n(t)M_f(n,s) = O(t^{-1/3}),$$

and it should be clear that the two sums

$$\sum_{n>N_+} \tilde{p}(n,s) \text{ and } \sum_{n<N_-} \tilde{p}_n(t)M_f(n,s)$$

are of smaller order than this.

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