When to Accept a Sequence of Gambles that Follows a Large Deviation Principle

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Abstract

Recently Ross showed the applicability of Samuelson's 'fallacy of large numbers' to be limited. He also found a class of utility functions that eventually will accept a partial sum of individually rejected 'good' gambles. We study when a sequence of gambles, that initially are rejected, eventually are accepted. Acceptance is found to be when a sequence of gambles follows a large deviation principle and the utility function is nonsatiated and bounded from below in a certain way. The number of needed gambles for acceptance is also computed.

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1 Introduction

The following example illustrates 'when to accept a sequence of gambles'. Think of the game, you win \$ 2 or lose \$ 1 with equal chance. The expectation is positive, but the guiding utility rejects the offer. Then one would suspect that, if you are offered to play the game a large number of times, then by the law of large numbers, the probability of losing has decreased so that the sum of games is accepted. It is not obvious, as it first seems, that the partial sum is accepted. For example the probability might not decrease fast enough compared to the negative tail of the utility function, to be accepted. 'When to accept a sequence' is all about the competition between the rate of decrease in the negative tail of the utility function and the probability of a sequence of gambles.

Decision-makers are assumed to act according to maximised expected utility, von Neumann-Morgenstern utilities, opting for maximal satisfaction. An offer to gamble which increases the current expected utility is accepted while a decrease in expected utility is rejected. Therefore a gamble X is accepted at the wealth level w if, $E[U(w+X)] > U(w)$, see for example Ingersoll or Luca-Raiffa [9, 11].

Kelly [10] suggested the logarithm as utility function. The growth rate of a portfolio selected according to the logarithm outperforms asymptotically any other strategy. Therefore it is sometimes argued to be the only rational way to make a decision because of its asymptotic properties.

Samuelson [15, 16] and Merton and Samuelson [13] called the asymptotic argument a 'fallacy of large numbers', as in every practical case the 'large number' is finite. Samuelson shows that if a utility function, $U(x)$, rejects a gamble X at all wealth levels, then it will reject any arbitrary partial sum, $S_n = \sum_{i=1}^{n} X_i$, where the X_i are independent identically distributed replications of X. The point of view of Samuelson and Merton leads to a real world paradox since the probability of losing is decreasing, Aurell *et al.* [1, 2]. Furthermore, Ross shows that the theorem of Saumelson is limited; the only utility functions that reject the same gambles at all wealth levels are the risk-neutral function and the exponential $U(x) = x$ or $-e^{-ax}$. Ross also finds a class of utility functions, exponentially bounded from below, that are eventually accepted for 'good' gambles, [14]. However the definition of eventual acceptance differs from the one stated here.

The Eventual Acceptance Property (EAP) is here defined as a pair property between the utility and gambles. A class of utility-gamble pairs that have EAP is found in which the utility function has to be nonsatiated with a negative tail that decreases at a slower rate than the tail-probability of the sequence of gambles. The sequence does not have to be a partial sum.

To estimate the tail-probability, large deviation techniques are used. Furthermore, an expression is computed for the number of gambles needed for acceptance.

This article is organised as follows: In section 2 the object studied, EAP is defined and compared to Ross' definition. When to accept a partial sum of gambles, independent or dependent gambles, is studied in section 3 together with some required results from large deviation theory. The EAP is then in section 4 generalised to any sequence that follows a large deviation principle with a rate function. (The rate function does not need to be convex.) In section 5 the actual number of games needed to accept a sequence of gambles is computed. Section 6 contains two technical lemmas, relaxing technical conditions in the general theorem, when the rate function is convex and continuous. Section 7 summarises the findings of this article.

2 Eventual Acceptance Property

The Eventual Acceptance Property (EAP) is the essential property which is studied.

Definition 1 (EAP) *If $U(x)$ is a utility function and S_n a sequence of random variables, then the pair $(U(x), \{S_n\}_{i=1}^{\infty})$, has the Eventual Acceptance Property if there exists a finite n such that S_n is accepted.*

The definition is made to stress the dependent relation between the utility function and the gamble. The sequence in mind can be a partial sum of, in some sense 'nice' random variables, such that, even if all of the single gambles are rejected, the partial sum will eventually be accepted.

Note that definition 1 does not require the sequence to be a partial sum of independent random variables, nor does the definition require $E[S_{n+1}-S_n] > 0$. This makes it possible to study, for example a Markov chain of gambles with negative conditional expected gambles but an over all expected positive gamble, or functions of the partial sum.

The class of sequences of independent random variables $\{X_i\}_{i=1}^{\infty}$, with expected value $E[X_i] = \mu_i$ and $\mu = \inf \mu_i > 0$ is called 'good'. Ross defines a utility function to have EAP if for each such sequence of 'good' random variables there exists a finite number such that the partial sum is accepted [14] .

Example: To illustrate the difference between the definition made here and by Ross, take X_i independent identically distributed $\mathrm{N}(\mu, \sigma^2)$ and the nonsatiated utility function $U(x) = 1 - e^{-ax}$, $a > 0$. Let the partial sum be

$S_n = \sum_{i=1}^n X_i$ then,

$$E[U(S_n)] = 1 - E[e^{-aS_n}] = 1 - \left(E[e^{-aX_1}]\right)^n = 1 - \exp\left(n(-\mu a + a^2\sigma^2/2)\right). \tag{1}$$

If $2\mu a < \sigma^2 a^2$ then, as the number of gambles grows, the right hand side of equation (1) approaches minus infinity. Since for every a, a pair (μ, σ^2) can be found, such that $2\mu a < \sigma^2 a^2$. The EAP definition by Ross is therefore violated. However the EAP defined here would be satisfied for every utility-gamble pair with $2\mu a > \sigma^2 a^2$.

3 Large Deviation Principle and EAP

In the theory of large deviations one usually looks at the distribution of S_n/n when n is large. By the law of large numbers S_n/n is concentrated around the mean when it exists, and it is large deviations from it, that are studied.

Definition 2 (Large Deviation Principle) *The sequence S_n/n is said to follow a large deviation principle (LDP) with rate function $I(x)$ if,*

1. *The function I(x) is lower semi continuous.*
2. *For each real number a the level set $\{x \in \mathbb{R} : I(x) \leq a\}$ is compact.*
3. *For each closed subset $F \subset \mathbb{R}$*

$$\limsup_{n\to\infty} \frac{1}{n} \log P\left(\frac{S_n}{n} \in F\right) \leq -\inf_{x\in F} I(x). \tag{2}$$

4. *For each open subset $G \subset \mathbb{R}$*

$$\liminf_{n\to\infty} \frac{1}{n} \log P\left(\frac{S_n}{n} \in G\right) \geq -\inf_{x\in G} I(x). \tag{3}$$

Usually one writes $P(\frac{S_n}{n} \approx x) \approx e^{-nI(x)}$ for n large enough.

To find a class of utility-gambles pairs that has the EAP, we use large deviation techniques. The question of eventual acceptance arose from partial sums of independent identically distributed random variables with positive mean, so a natural starting point of the analysis is to state and prove a theorem for that class.

Consider the sum

$$S_n = \sum_{i=1}^{n} X_i, \tag{4}$$

of independent identically distributed random variables with positive expected value and cumulant function $g(t) = \log(E[e^{tX_i}])$. Cramér's theorem states that S_n/n satisfies a LDP if $g(t) < \infty$ for all t [4]. The condition $g(t) < \infty$ can be replaced by a requirement that $g(t)$ is steep, Bucklew [3].

In Cramér's theorem the rate function is defined as,

$$I(x) = \sup_t (tx - g(t)). \tag{5}$$

Moreover the rate function satisfies therefore three important properties. First the rate function is convex. Secondly the rate function attains its minimum at $\mu = E[X]$ and $I(x) \geq I(\mu) = 0$ for all x. Third if X has no upper or lower bound, then there exists a $t \in \mathbb{R}$ such that $I(x) = tx - g(t)$ for every value of x and t solves the equation $g'(t) = x$.

Theorem 1 *Let $\{X_i\}_{i=1}^{\infty}$ be independent identically distributed random variables, with $E[X_1] > 0$ and $g(t) < \infty$ for every t. Denote the partial sum by $S_n = \sum_{i=1}^{n} X_i$.*

Sufficient conditions for the pair $(U(x), \{S_n\}_{i=1}^{\infty})$ to have the EAP are that $U(x)$ is nonsatiated and there are constants $C > 0$ and $\gamma > 0$ such that for every $x \leq 0$,

$$|U(x)| \leq Ce^{-\gamma x}.$$

The constant γ satisfies $g(-\gamma) < 0$ that is, $\tau < -\gamma < 0$ where τ is the negative root to the equation $g(\tau) = 0$.

The last sentence of the theorem can be rephrased in terms of the rate function. The constant τ is equal to $I'(x_\tau)$. This is true since the cumulant function is connected to the rate function by the Legendre transform. The choice of τ is such that $g(\tau) = 0$. Let $x_\tau = g'(\tau)$, then the Legendre transformation is $I(x_\tau) = \tau x_\tau - g(\tau) = \tau x_\tau$. Hence $I(x_\tau) = I'(x_\tau)x_\tau$ a fact that also can be concluded from the convexity of $I(x)$. One can also see that $\tau = I'(x_\tau)$.

The conditions for the gambles in theorem 1 assure that the conditions in Cramér's theorem are fulfilled, therefore S_n/n follows a LDP with a convex rate function $I(x)$. The use of large deviation techniques is not necessary in the proof at this stage, but will in the general case.

The theorem loosely states that if the sequence obeys the LDP and the rate function decreases faster than the negative tail of the utility function, then the pair has the EAP. It is an exponential competition between the tail of the utility function and the rate function, and when the rate function is the winner, then the pair is EAP.

Proof. Since utility functions are not unique in the information they carry, they can be normalised for every level of wealth without any lost of generality, such that $U(0) = 0$. Only affine transformations $aU(x) + b$, $a > 0$, preserve equivalence between two von Neumann-Morgenstren utility functions, Ingersoll [9]. Therefore to prove the EAP, one has to show that there exist a n such that S_n satisfy, $E[U(S_n)] > U(0)$. Divide the expected utility into a positive and a negative part,

$$E[U(S_n)] = \int_{-\infty}^{\infty} U(nx)dF_n(x) = \int_{-\infty}^{0} U(nx)dF_n(x) + \int_{0}^{\infty} U(nx)dF_n(x).$$

In the equation the measure $F_n(x)$ denotes the probability distribution function of S_n/n. The last integral is positive for a n sufficiently large, because of the assumptions of nonsatiation and positive expected value. According to nonsatiation there exists a constant a such that $U(0) < U(a)$ and $a < nE[X_1]$ leads to $0 < P(S_n \in [a, \infty)) = 1 - F_n(a/n)$. Therefore choose n such that $a < nE[X_1]$,

$$0 \geq E[U(S_n)] \geq \int_{-\infty}^{0} U(nx)dF_n(x) + U(a)(1 - F_n(\frac{a}{n})). \tag{6}$$

The utility-gamble has the EAP if the first integral on the right hand side in equation (6) is converging to zero. By the condition $|U(x)| \leq Ce^{-\gamma x}$ and integrating over real line the first integral is,

$$E[U(S_n), S_n \leq 0] \geq -CE[e^{-\gamma S_n}] = -Ce^{ng(-\gamma)}. \tag{7}$$

If $g(-\gamma) < 0$ then equation (7) converges to zero as n increases. The convexity of $g(t)$, $0 = g(0)$ and $0 < g'(0) = E[X_i]$ assure that $g(t) < 0$ for $\tau < t < 0$, where $\tau < 0$ is a point where $0 = g(\tau)$. Therefore the condition $\tau < -\gamma < 0$ assures $g(-\gamma) < 0$. Graphically this is easily realised, use the figure on page 8. This completes the proof. □

Example: Exponential-, gamma- and normal distributed random variables are examples of random variables with steep cumulant functions which therefore have convex rate functions.

The structure of the proof of theorem 1 is only dependent on LDP, convexity of the cumulant function, (convexity of the rate function) nonsatiation of the utility function, and that for some finite number of gambles there is a strictly positive probability to be above the nonsatiation point for the partial sum of gambles. Theorem 1 could therefore be extended to dependent variables.

A more general large deviation result is the Gärtner-Ellis theorem, [8, 7] which guarantees a convex rate function. Define $g_n(t) = n^{-1} \log\left(E\left[e^{tS_n}\right]\right)$.

Then there is a convex rate function, $I(x) = \sup_t(tx - g(t))$, if $g(t) = \lim_{n\to\infty} g_n(t)$ exists for all $t \in \mathbb{R}$, where we allow infinity both as a limit value and as an element of the sequence $\{g_n(t)\}_{n=1}^{\infty}$.

Now let us make use of the Gärtner-Ellis theorem, to state an theorem which is applicable to sequences of dependent gambles. Some of the gambles may even conditionally have negative expected value.

Theorem 2 *Assume the utility function $U(x)$ is nonsatiated, that is there is an a such that $U(0) < U(a)$. Let $\{S_n\}_{n=1}^{\infty}$, be a sequence of random variables satisfying the conditions of Gärtner-Ellis theorem and that there is a N such that for $n \geq N$, $P(S_n > a) > 0$.*

Sufficient conditions for $(U(x), \{S_n\}_{i=1}^{\infty})$ to have the EAP are that there are constants $C > 0$ and a $\gamma > 0$ such that for every $x \leq 0$,

$$|U(x)| \leq Ce^{-\gamma x},$$

The constant γ satisfies $g(-\gamma) < 0$ that is, $\tau < -\gamma < 0$ where τ is the negative root to the equation $g(\tau) = 0$.

The ingredients are the same as before to have the EAP. The gamble sequence need to fulfil the LDP and its rate function must dominate the negative tail of a nonsatiated utility function. The proof of theorem 2 is skipped since it can be constructed almost identically as the proof of theorem 1 and it follows directly from the general theorem 3, lemma 2 and lemma 3.

Example: The EAP definition in this article makes it possible to study gambles, which are described by a Markov-model. Theorem 2 handles this case. Assume that we have an ergodic Markov chain with two states, 1 and 2. We get \$ 2 for every visit to state 1 and we lose \$ 1 in state 2. Denote what we earn at time i by X_i and the current state by Y_i. The Markov chain Y_i is characterised by the transition matrix,

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 \\ 7/24 & 17/24 \end{pmatrix}.$$

Given that the process is in state 1 the conditional expectation is $E[X_{i+1}|Y_i = 1] = 1/2$ and if it is in state 2, $E[X_{i+1}|Y_i = 1] = -1/8$. This gamble is not a 'good' one, but it is an interesting one for the definition of EAP stated here.

The Markov chain is ergodic and therefore it has a limit distribution. The limit distribution π is the solution to $\pi\mathbf{P} = \pi$. The solution is $\pi = (7/19, 12/19)$. The expected value of the game under the limit distribution is, $\lim_{i\to\infty} E_\pi[X_i] = 2/19$. The asymptotic game is a positive one. Therefore the partial sums $S_n = \sum_{i=1}^{n} X_i$ have positive drift, although some of the individual games have negative conditional expectation.

The function $g(\gamma)$ for a Markov chain is found as the logarithm of largest eigenvalue to the matrix,

$$\mathbf{T} = \begin{pmatrix} 1/2e^{2\gamma} & 1/2e^{-\gamma} \\ 7/24e^{2\gamma} & 17/24e^{-\gamma} \end{pmatrix}.$$

Eigenvalues λ are found by solving the equation $\det(\mathbf{T} - \lambda\mathbf{I}) = 0$. Hence the largest eigenvalue is,

$$\lambda_{\max}(\gamma) = \frac{1/2e^{2\gamma} + 17/24e^{-\gamma}}{2} + \sqrt{\frac{(1/2e^{2\gamma} - 17/24e^{-\gamma})^2}{4} + 7/24e^{2\gamma} \cdot 1/2e^{-\gamma}}.$$

The cumulant function is therefore equal to $g(\gamma) = \log(\lambda_{\max}(\gamma))$. By solving the equation $g(\tau) = \log(\lambda_{\max}(\tau)) = 0$ the EAP utility-gamble pair can be found. By numerical calculations $\tau = -0.0685$ see figure on page 8. Therefore every nonsatiated utility function such that $|U(x)| < Ce^{-0.0685x}$ when $x < 0$ will eventually be accepted.

Figure 1: The cumulant function $g(t)$ for the Markov chain example. Notice the points where $g(t) = 0$, $t = 0$ and $t = -0.0685$.

All large deviations results used in this section can be found in Bucklew [3].

4 Main Result

In a more general situation where for example the sequence studied is a function of the partial sum, $f(\sum X_i)$, the rate function no more has to be convex.

In large deviation theory the contraction principle can be used to find the rate functions for a function of a sequence that follows the LDP. Assume that $f(\cdot)$ is a continuous function with inverse $f^{-1}(\cdot)$. If the sequence S_n/n is LDP with rate function $I(x)$ then the contraction principle can be found by using,

$$P\left(f\left(\frac{S_n}{n}\right)\approx y\right) = P\left(\frac{S_n}{n}\approx f^{-1}(y)\right).$$

Therefore the rate function $J(y)$ for $f(S_n/n)$ is given by,

$$J(y) = \inf_{x:f(x)=y} I(x).$$

Example: If X_i are independent identically distributed $N(\mu,\sigma^2)$ then the rate function for the estimated mean is $I(x) = (x-\mu)^2/2\sigma^2$. If the gamble sequence of interest is $f(S_n/n) = b - (S_n/n)^4$, where b is a constant, then the inverse of $y = f(x) = b - (x)^4$ is $f^{-1}(y) = \pm(b-x)^{1/4}$. Assume that it is the positive root that minimises the rate function for this y. Then the rate function for this sequence is $J(y) = I((b-y)^{1/4}) = ((b-y)^{1/4} - \mu)^2/2\sigma^2$, $y \leq b$, which is not convex. From this simple example one understands that a rate function that is not convex is not uncommon and quite natural.

Theorem 3 *The pair $(U(x), \{S_n\}_{n=1}^{\infty})$ have the EAP if the following sufficient conditions hold:*

1. $\frac{S_n}{n}$ *follows the LDP with rate function $I(x)$.*
2. *$U(x)$ is nonsatiated, that is, there is an a such that $U(0) < U(a)$.*
3. *There is an N such that for every $n \geq N$, $P(S_n > a) \geq \delta > 0$.*
4. $\lim_{n\to\infty} \inf_{x<0} U(nx)e^{-nI(x)} = 0.$
5. $\lim_{M\to\infty} \limsup_{n\to\infty} \log\left(E\left[-U(S_n)\mathbf{1}_{\{U(S_n)\leq -e^{nM}\}}\right]\right) = -\infty.$

It can be shown that the technical conditions 4 and 5 in theorem 3 are equivalent to the condition $|U(x)| \leq Ce^{-\gamma x}$, $x \leq 0$ and $\tau < -\gamma < 0$, when $I(x)$ is convex and continuous, see lemma 2 and 3 in the section 6.

Example: To see that the condition $|U(x)| \leq Ce^{-\gamma x}$ when $x \leq 0$ from theorem

2 is not enough when $I(x)$ is not convex, take the rate function $I(x) = ((b-x)^{1/4} - \mu)^2/2\sigma^2$, from the example above and $\mu = 0$. Take the utility function to be $|U(x)| = Ce^{-\gamma x}$ then,

$$\lim_{n\to\infty} \inf_{x<0} -Ce^{-n\gamma x} e^{-n\frac{(b-x)^{1/2}}{2\sigma^2}} = -\infty.$$

Varadans integral lemma is needed to prove theorem 3.

Lemma 1 (Varadhan) *If $\phi : X \to \mathbb{R}$ is an upper semi continuous function for which one of the tail conditions holds,*

$$\lim_{M\to\infty} \limsup_{n\to\infty} \log \left(E\left[e^{n\phi(\frac{S_n}{n})} \mathbf{1}_{\phi(S_n)\geq M} \right] \right) = -\infty$$

or for some $\gamma > 1$,

$$\limsup_{n\to\infty} \log \left(E\left[e^{n\gamma\phi(\frac{S_n}{n})} \right] \right) < \infty,$$

and the large deviations upper bound holds with a good rate function, then,

$$\limsup_{n\to\infty} \frac{1}{n} \log \left(E\left[e^{n\phi(\frac{S_n}{n})} \right] \right) \leq \sup_{x\in X} \left(\phi(x) - I(x) \right)$$

For formulation of the lemma and further details see [5].
Proof. Without loss of generality, let the utility functions be normalised, for every level of wealth such that $U(0) = 0$. Therefore to prove the EAP, one has to show that there exist a n such that S_n satisfy, $E[U(S_n)] > U(0)$. The probability distribution function of S_n/n is denoted $F_n(x)$. Split the integral in two,

$$E[U(S_n)] = \int_{-\infty}^{\infty} U(nx)dF_n(x) = \int_{-\infty}^{0} U(nx)dF_n(x) + \int_{0}^{\infty} U(nx)dF_n(x).$$

The second integral is for n large enough positive, because the existence of an a such that $U(0) < U(a)$ (nonsatiation), and that there is a N such that for every $n \geq N$, $P(S_n > a) \geq \delta > 0$. Therefore,

$$E[U(S_n)] \geq \int_{-\infty}^{0} U(nx)dF_n(x) + U(a)\delta, \tag{8}$$

for every $n \geq N$. Heuristically one would expect that,

$$\int_{-\infty}^{0} U(nx)dF_n(x) \approx \int_{-\infty}^{0} U(nx)e^{-nI(x)}dx \approx \inf_{x<0} U(nx)e^{-nI(x)} \to 0, \tag{9}$$

as $n \to \infty$ by assumption.

Varadhan's Integral lemma verifies this conclusion. By the existence theorem for utility functions Ingersoll [9], U(x) is continuous, and therefore lower semi continuous. Together with the fact that S_n/n obeys the LDP with rate function $I(x)$ and by substituting $-U(x) = e^{n\phi(x/n)}$ the conditions in Varadhans lemma are fulfilled. Equation (8) is therefore,

$$\geq \inf_x U(nx)e^{-nI(x)} + U(a)\delta \to U(a)\delta > 0, n \to \infty.$$

This concludes that S_n is eventually accepted. $\square$

5 Number of gambles needed

A natural question to ask is for what number of gambles is the sequence accepted? For a convex rate function which 'dominates' the negative tail of the utility function, an explicit formula for a sufficient number of gambles needed can be derived.

Theorem 4 *Assume the conditions of theorem 2 hold i.e., the pair* $(U(x), \{S_n\}_{n=1}^{\infty})$ *have the EAP. Let,*

$$N_1 = \log\left(\frac{C}{U(a)}\right)\frac{1}{\gamma x_\gamma + I(x_\gamma)} = \log\left(\frac{U(a)}{C}\right)\frac{1}{g(-\gamma)},$$

where x_γ *solves* $\gamma = -I'(x_\gamma)$*. Let* N_2 *be the smallest* n *such that* $a/n < \mu$*. A sufficient* N *for acceptance of* S_n *for every* $n \geq N$ *is* $N = \max(N_1, N_2)$*.*

Notice that if $g(\tau) = 0$, for some $\tau < 0$ then,

$$\lim_{-\gamma \to \tau} \log\left(\frac{U(a)}{C}\right)\frac{1}{g(-\gamma)} = \infty.$$

The conclusion is that when the utility function is close to the rate function in the rate of decrease in the negative tail, then a big number of gambles is needed.

Proof. From the proof of theorem 1 we know that,

$$E[U(S_n)] \geq \inf_{x<0} -Ce^{-n(\gamma x + I(x))} + U(a) \sup_{x \geq a/n} e^{-nI(x)} > 0. \tag{10}$$

Since the pair has the EAP a x that minimises $-Ce^{-n(\gamma x + I(x))}$ can be found. Differentiate with respect to x and put the first order derivative equal to zero leads to the equation,

$$C\left(\gamma + I'(x)\right)e^{-n(\gamma x + I(x))} = 0.$$

Therefore the equation to be solved is,

$$-\gamma = I'(x),$$

which has the solution x_γ. If $n \geq N_2$ then $\sup_{x \geq a/n} -nI(x) = 0$ and hence equation (10) can be solved with x_γ,

$$\log\left(\frac{C}{U(a)}\right)\frac{1}{\gamma x_\gamma + I(x_\gamma)} \leq n.$$

Recall the Legendre transform, which states that $g(-\gamma) = -\left(-\gamma x_\gamma - I(x_\gamma)\right)$ and

$$\log\left(\frac{U(a)}{C}\right)\frac{1}{g(-\gamma)} \leq n.$$

which ends the proof. □

6 Relaxing the Technical Conditions

The essence of the two lemmas in this section is that the technical conditions 4 and 5 in theorem 3 are equivalent to the condition $|U(x)| \leq Ce^{-\gamma x}$, $x \leq 0$ and $I'(x_\tau) < -\gamma < 0$, when $I(x)$ is convex and continuous. The point x_τ is found by $g(\tau) = 0$ and $x_\tau = g'(\tau)$, also $I'(x_\tau) = \tau$.

Lemma 2 *Let the rate function $I(x)$ be convex and $I(\mu) = 0$ for $0 < \mu$. If and only if for every $x \leq 0$,*

$$|U(x)| \leq Ce^{-\gamma x}$$

for $I'(x_\tau) = \tau < -\gamma < 0$, then,

$$\lim_{n\to\infty} \inf_{x<0} U(nx)e^{-nI(x)} = 0.$$

Proof. As before assume that $U(x)$ is normalised for every level of wealth such that $U(0) = 0$. First we prove the if part. The utility function $U(x)$ is not decreasing and $U(0) = 0$, therefore

$$0 \geq \inf_{x<0} U(nx)e^{-nI(x)}.$$

By the assumption $|U(x)| \leq Ce^{-\gamma x}$ and $U(x) \leq 0$ for $x < 0$,

$$\geq \inf_{x<0} -Ce^{-\gamma nx}e^{-nI(x)} = -C\sup_{x<0} e^{-nx(\gamma + \frac{I(x)}{x})}.$$

The convexity of the rate function guarantees that it is bounded from below by a line: That is $I(x) \geq I'(x_\tau)x$, $x_\tau < 0$, $I(\mu) = 0$ for $0 < \mu$ and the convexity of $I(x)$, $I'(x_\tau) < 0$. Therefore,

$$\geq -C \sup_{x<0} e^{-nx(\gamma + I'(x_\tau))} \to 0,$$

as $n \to -\infty$ by choosing $0 < \gamma$ such that $I'(x_\tau) < -\gamma$. The convexity of $I(x)$ assures that $I'(x_\tau) \leq I'(0) < 0$ which can be used as an easy check if γ is small enough. Alternatively one can use the Legendre transform instead of the convexity of $I(x)$ to show the last step.

The proof of the only if part. By the continuity of $U(x)$, $U(0) = 0$ and $\lim_{n\to\infty} \inf_{x<0} U(nx)e^{-nI(x)} = 0$, assures that $\inf_{0\leq n} \inf_{x<0} U(nx)e^{-nI(x)} = -C > -\infty$ exists and

$$\inf_{0\leq n} \inf_{x<0} -|U(nx)|e^{-nI(x)} = -\sup_{0\leq n} \sup_{x<0} |U(nx)|e^{-nI(x)}.$$

For every x and n is $-\sup_{x,n} f(n,x) \leq -f(x,n)$,

$$\leq -|U(nx)|e^{-nI(x)} \leq -|U(nx)|e^{-nI'(x_\tau)x},$$

where the second inequality is due to the convexity of $I(x)$. Exchange $y = nx$,

$$\leq -|U(y)|e^{-yI'(x_\tau)} \leq -|U(y)|e^{y\gamma}.$$

where $I'(x_\tau) < -\gamma < 0$. Conclude that, $\sup_n \sup_{x<0} |U(nx)|e^{-nI(x)} = C$ and that,

$$|U(y)| \leq Ce^{-y\gamma},$$

for $y < 0$. □

Lemma 3 *Let the rate function $I(x)$ be convex, continuous and $I(\mu) = 0$ for $0 < \mu$. If and only if for every $x \leq 0$,*

$$|U(x)| \leq Ce^{-\gamma x}$$

for $I(x_\tau) < -\gamma < 0$, then,

$$\lim_{M\to\infty} \limsup_{n\to\infty} \log\left(E\left[-U(S_n)\mathbf{1}_{\{U(S_n)\leq -e^{nM}\}}\right]\right) = -\infty.$$

Proof. Start by proving the if part. It is equivalent to show that $E\left[-U(S_n)\mathbf{1}_{\{U(S_n)\leq -e^{nM}\}}\right] \to 0$,

$$0 \leq E\left[|U(S_n)|\mathbf{1}_{\{U(S_n)\leq -e^{nM}\}}\right] \leq$$

by assumption is $|U(S_n)| \leq Ce^{-\gamma S_n}$,

$$E\left[|U(S_n)|\mathbf{1}_{\{U(S_n)\leq -e^{nM}\}}\right] \leq CE\left[e^{-\gamma S_n}\mathbf{1}\right].$$

Use the definition of upper integral, written as a sum,

$$\leq C\sum_i \sup_{x\in\Delta x_i} e^{-\gamma nx}P(S_n\in\Delta x_i)\mathbf{1}$$

where $\Delta x_i = [x_i, x_{i+1})$ is some partition of the negative real line. $U(x)$ is not decreasing and therefore has its lowest value for $U(x)$ in x_i and $e^{-nI(x_{i+1})} \geq P(S_n \in \Delta x_i)$. By continuity $x_i \to x_{i+1}$ the difference is small, therefore using the convexity of $I(x)$, we get $I(x) \geq I'(x_\tau)x$ and

$$= C\sum_i e^{-\gamma n x_i}e^{-nI(x_i)}\mathbf{1} \leq C\sum_i e^{-nx_i(\gamma + I'(x_\tau))}.$$

The sum is convergent if $I'(x_\tau) < -\gamma < 0$, and as $M, n \to \infty$, the sum converges to 0.

The proof of the only if part. The facts that $U(0) = 0$, $U(x)$ is continuous and that $\lim_{M\to\infty}\limsup_{n\to\infty} E\left[-U(S_n)\mathbf{1}\right] = 0$ assure the existence of $\sup_n E\left[|U(S_n)|\right] = C < \infty$. Use the definition of a lower integral,

$$E[|U(S_n|] \geq \sum |U(S_n)|P(S_n\in\Delta x_i)$$

The LDP is used to find a upper bound for the probability,

$$\geq \sum_i |U(nx_i)|e^{-nI(x_i)} \geq |U(nx)|e^{-nI(x)}.$$

Exchange $y = nx$,

$$C = \sup_n E[|U(S_n)|] \geq |U(y)|e^{-y\frac{I(x)}{x}} \geq |U(y)|e^{-yI'(x_\tau)},$$

which ends the proof. □

7 Conclusions

In this article we examined the Eventual Acceptance Property. EAP was defined as a pair property between the utility function and a sequence of gambles. This definition does not require positive expectation for every gamble.

The class of utility-gamble pairs to have EAP depended on three things: First, that the utility function was not decreasing and nonsatiated. Secondly that the sequence of gambles followed the LDP. Third, that the negative tail of the utility function was decreasing more slow than the rate function. The technical conditions for this domination of the rate function, when it was convex, was that the utility function was bounded from below by an exponential function. The exponential function was related to the first order derivative of the rate function. The large deviation techniques made it possible to study for example Markov chains with conditional negative expectation.

When the rate function was not convex the technical conditions for EAP to hold was less transparent. Rate functions that is not convex could for example arise by transformations of the partial sum.

We proved that the two technical conditions 4 and 5 in the general theorem 3, could be substituted by the condition in theorem 2. This was possible when the rate function of the gambles were convex and continuous.

When the conditions for EAP to hold were fulfilled with a convex rate function, a closed form formula for a sufficient number of gambles needed to accept the gamble was derived.

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