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An analysis of two modifications of the Petersburg game and of another game allowing arbitrage

Anders Martin-Löf*

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Abstract

Two modifications of the Petersburg game are considered.
1: Truncation, so that the player has a finite capital at his disposal, and
2: A cost of borrowing capital, so that the player has to pay interest on the capital needed. In both cases limit theorems for the total net gain are derived, so that it is easy to judge if the game is favourable or not. Another similar game is also analysed in the same way.

Key words: Petersburg game, free lunch, arbitrage possibility, limited capital, cost of capital, profitability

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Introduction.

The well known Petersburg game is performed by tossing a fair coin until heads turns up for the first time. Let T be this time. It has a geometric distribution: $P(T=k) = 2^{-k}$, k=1,2,....
Paul's gain in one game is $X=2^T$, and this quantity has an infinite expectation, so that the ordinary law of large numbers does not provide a recipe for what is a fair price for playing several successive independent such games. Also this game allows the well known doubling strategy: As long as tails has come up double the stakes. Then the total amount spent is $1+2+4+...+2^{T-1}=2^T-1$, so the net gain is $2^T-(2^T-1)=1$ with probabiliy one, and we have a money machine providing an arbitrage possibility. This "paradox" of course depends on the fact that Paul has an unbounded amount of money available for free to use as stakes.

In this paper we want to analyse two natural modifications of this game:

1.*Truncation*. Paul has only a finite amount of capital, e.g. 2^c available. This means that he is ruined if T>c, which event has probability 2^{-c}. Otherwise he gains 1 as before and can continue to play. Then the expected gain is $1(1-2^{-c})-(2^c-1)2^{-c}$, so now the game is fair as has been pointed out e.g. in Aase (2001).

A natural strategy for this game is to continue playing until the inevitable ruin occurs. The total gain V_c in this game is easy to analyse, and in Theorem 3 we show a simple limit theorem for $2^{-c} V_c$ as $c \to \infty$.

2. *Introduction of a cost of the capital needed*. Paul can borrow money for the stakes without limit, but he has to pay interest at a constant rate on the loans needed. In this case the expected

present value of the net gain in a single game is zero as has also been pointed out in Aase (2001), so again the game is fair. Now Paul can continue to play forever, and we can consider the present value of all future costs and gains. We prove a limit theorem for these suitably normalised when the rate of interest d goes to zero and hence the discount factor r=1/(1+d) goes to one. This involves a limit distribution very similar to that found in Martin-Löf (1985) for the total gain $S_N=X_1+...+X_N$ in a large number of independent games as $N=2^n \to \infty$, and as in that case a simple asymptotic formula for the tail of the limit distribution can be obtained, Theorem 5.

Before studying the above problems for the Petersburg game however we first consider the same problems for a simpler game, the "Stockholm game", which is easier to analyse. This game is obtained if Paul does not increase the stakes, but puts up the amount one in each game as long as the accumulated losses are non-negative. The successive losses X_k are then ordinary Bernoulli-variables with $P(X_k=+1) = P(X_k=-1) = 1/2$, and the accumulated ones, $S_n=X_1+...+X_n$, $S_0=0$, form a random walk as long as $S_n>0$. However when $S_{n-1}=0$ and $X_n=-1$ Paul gains $Y_n=1$ and has $S_n=0$. Then the game can continue as before, and S_n is a random walk with a reflecting boundary at zero where S_n has transition probabilities 1/2 to 1 and 0 respectively. The first time T when $Y_n=1$ is the same as the first hitting time of -1 for a free random walk, so we know that $P(T<\infty)=1$ and we again have a game with an arbitrage possibility. If the game is continued indefinitely the successive times when $Y_n=1$ form a renewal process whose inter-arrival times have the generating function

$$E(s^T)=(1-(1-s)^{1/2})/s$$

as shown e.g., in Feller (1968). This means that E(T)= +∞, so we do not get a money machine making gains at a positive rate in the long run.

For this game we can also introduce the two modifications truncation and a cost of the capital, and then we get fair games. In Theorem 1 and 2 we prove limit theorems for the total gain until ruin and the total discounted gain respectively. In this game the latter has a simple normal limit distribution as $r \to 1$.

1.1. The truncated Stockholm game.

Let us put an upper barrier at $S_n=c$, so that Paul is ruined and looses c when this happens for the first time and the game stops. The probability of a gain before ruin is the same as the probability of $S_n=-1$ before $S_n=c$ starting with $S_0=0$ for a free random walk, which is c/(c+1), and the complementary probability of ruin before a gain is hence 1/(c+1). This means that the total number of gains M_c before ruin has a geometric distribution with

$P(M_c = m) = (c/(c+1))^m / (c+1), \;\; m = 0,1,...,$

so that $P(M_c \geq m) = (c/(c+1))^m$.

The expected net gain in a game is hence 1(c/(c+1))-c(1/(c+1))=0, so we have a fair game.

The total net gain before ruin is $V_c=M_c-c$. This means that we get the following simple limit theorem for V_c/c as $c\to\infty$:

Theorem 1. When $c\to\infty$ M_c/c converges in distribution to that of U having an exponential distribution with $P(U>u)=e^{-u}$, u≥0, so that V_c/c converges to U-1.

Proof: $P(M_c / c > u) = (c/(c+1))^{cu} = (1-1/(c+1))^{cu} \to e^{-u}$ as $c\to\infty$.

We hence see that although ruin is certain the total gain can be positive e.g., $P(V_c > 0) \to P(U > 1) = 1/e$.

1.2. The Stockholm game with interest.

Recalling that the successive gains are $Y_n = I(S_n = 0)(1 - X_n)/2$ we have the balance equations $S_n = S_{n-1} + X_n + Y_n$, $S_0 = 0$. The discount factor per time unit is r=1/(1+d), so the present values of the gains minus the interest payments are:

$$V_n = \sum_1^n r^k Y_k - (r^{-1} - 1)\sum_1^n r^k S_{k-1}.$$

(The interest $dS_{k-1} = (r^{-1} - 1)S_{k-1}$ is paid at time k.) The last sum can be changed as follows:

$$\sum_1^n (r^{-1} - 1) r^k S_{k-1} = \sum_1^{n-1} r^k S_k - \sum_1^n r^k S_{k-1} = \sum_1^n r^k (X_k + Y_k) - r^n S_n,$$

so that we have $V_n = -\sum_1^n r^k X_k + r^n S_n$,

and as $n \to \infty$ $V(r) = V_\infty = -\sum_1^\infty r^k X_k$.

Since T the duration until $Y_n = 1$ for the first time is a stopping time and $S_T = 0$, we have $V_T = -\sum_1^T r^k X_k$, and hence $E(V_T) = 0$ according to Wald, so that the game is fair.

From the formula for V(r) we can get a normal limit theorem for V(r) when $r \to 1$ as follows:

Theorem 2. When $d \to 0$ and hence $r \to 1$ then $d^{1/2} V(r) \quad V$,

where V has a centered Gaussian distribution with variance 1/2.

Proof: Let $W_n = X_1 + \ldots + X_n$ be the free random walk generated by the X_k. Then $d^{1/2} W_{t/d} \quad W(t)$, a standard Wiener process as $d \to 0$,

and since $r^k = (1 + d)^{-k} \approx e^{-kd}$ we get

$$d^{1/2}V(r) = -d^{1/2}\sum_1^\infty r^k(W_k - W_{k-1}) = -(1-r)d^{1/2}\sum_1^\infty r^k W_k \approx -d\sum_1^\infty e^{-kd}W(kd)$$

$$-\int_0^\infty e^{-t}W(t)dt = -\int_0^\infty e^{-t}dW(t) \equiv V$$

using the fact that we have weak convergence to W(.).

The last quantity is Gaussian with variance

$$\int_0^\infty e^{-2t}dt = 1/2.$$

This result implies that the value of the net gain at time n, $(1+d)^n V_n$, behaves roughly like $e^{nd}(V)/d^{1/2}$ when n is large, so either it grows to +∞ or -∞ according to the sign of V.

2.1. **The truncated Petersburg game.**

The analysis of this game is entirely analogous to that in section 1.1.

In each game the probability of a gain before ruin is $1-2^{-c}$, and the probability of ruin before a gain is 2^{-c}, so now the number of gains M_c before ruin has a geometric distribution with

$P(M_c \geq m) = (1-2^{-c})^m$.

The total net gain before ruin is $V_c = M_c - (2^c - 1)$,

and again we get an exponential limit distribution as $c \to \infty$:

Theorem 3. When $c \to \infty$ $M_c 2^{-c}$ converges in distribution to U having an exponential distribution with $P(U>u) = e^{-u}$, $u \geq 0$, so that $V_c 2^{-c}$ converges to U-1.

Again we see that although ruin is certain there is a good chance of making a positive net gain before that happens.

2.2. **The Petersburg game with interest.**

Note: In this section the notation X_n, S_n etc. is independent of that in section 1 in order not to have too many symbols around.

Still the duration of each game T has a geometric distribution: $P(T=k)=2^{-k}$, k=1,2,... but now the present value of the gain is $r^T 2^T=(2r)^T$, which has a finite expectation:

$$\sum_1^\infty 2^{-k}(2r)^k = \sum_1^\infty r^k = r/(1-r).$$

If the doubling strategy is used the present value of the stakes is:

$$r + r^2 2 + r^3 2^2 + \ldots + r^T 2^{T-1} = r\sum_0^{T-1}(2r)^k = r((2r)^T - 1)/(2r-1),$$

and the expected value of this is:

$$r\sum_1^\infty 2^{-k}((2r)^k - 1)/(2r-1) = r(\sum_1^\infty r^k - 1)/(2r-1) = r(r/(1-r)-1)/(2r-1) = r/(1-r),$$

so the game is now fair.

We can obtain a formula for the present value of the net gain V_n analogous to the one in section 1.2. for the doubling strategy as follows. Let B_k be independent Bernoulli variables taking the values +1,-1 with probability 1/2, and let B_k=-1 define the instances of gain, and $Y_k=(1-B_k)/2$ the successive gains.

Moreover let U_k denote the accumulated debt +1, so that $U_k=2^k$ for k=0,1,...,T-1 and U_T=1 etc.

This can be expressed by the recursion $U_k = U_{k-1}(1+B_k) + Y_k,\ U_0 = 1,$ so that $U_k = U_{k-1} + U_{k-1}B_k + Y_k.$

The total value of the interest payments up to time n is then

$$\sum_1^n (r^{-1}-1)r^k(U_{k-1}-1) = \sum_1^{n-1} r^k(U_k - 1) - \sum_1^n r^k(U_{k-1}-1) =$$

$$\sum_1^n r^k(U_k - U_{k-1}) - r^n(U_n - 1) = \sum_1^n r^k(U_{k-1}B_k + Y_k) - r^n(U_n - 1),$$

so we see that

$V_n = -\sum_1^n r^k U_{k-1}B_k + r^n(U_n - 1)$ as in 1.2.

Since U_T=1 we get $V_T = -\sum_1^T r^k U_{k-1} B_k$, and we see again that $E(V_T)=0$ according to Wald.

Explicitly we see that

$$V_T = -\sum_1^{T-1} r^k 2^{k-1} + r^T 2^{T-1} = -r\sum_0^{T-2} (2r)^k + r(2r)^{T-1} =$$

$$-r((2r)^{T-1} - 1)/(2r-1) + r(2r)^{T-1} = (r-(1-r)(2r)^T)/(2r-1) \equiv V.$$

The total value V(r) can be decomposed as a sum of contributions from the successive games. Let T_1, T_2,... be the times when Y_k=+1, and T_0=0. They form a renewal process with T_i-T_{i-1} having the same distribution as T. From each such interval there is an independent contribution $V_i = (r-(1-r)(2r)^{(T_i - T_{i-1})})/(2r-1)$ all with the same distribution as V, and the total is given by

$$V(r) = \sum_1^{\infty} r^{T_{i-1}} V_i, .$$

This expression is useful for studying the distribution of V(r) as r→1. In order to do this it is convenient to consider the positive and negative contributions separately and put $V_i = V^+_i - V^-_i$ with

$$V_i^+ = r/(2r-1) \approx 1, \quad V_i^- = (1-r)(2r)^{(T_i - T_{i-1})}/(2r-1) \approx (1-r)2^{(T_i - T_{i-1})}$$

and correspondingly $V(r) = V^+(r) - V^-(r)$.

As in Martin-Löf (1985) we scale time by a factor $N=2^n \to \infty$ and put r=exp(-a/N) with $1 \le a < 2$. Then the renewal process has a deterministic limit: $(1/N)T_{Nt} \quad 2t$ since E(T)=2.

This implies that $(1-r)V^+(r)$ has a deterministic limit:

$$(1-r)V^+(r) \approx (a/N)\sum_0^{\infty} \exp(-aT_i/N) \quad a\int_0^{\infty} e^{-2at}dt = 1/2 \quad \text{as } n \to \infty.$$

In order to approximate $V^-(r)$ put $X_i = 2^{(T_i - T_{i-1})}$ and introduce the random walk generated by these variables: $S_k = X_1 + \ldots + X_k$.

In Martin-Löf (1985) it is shown that the following limit theorem holds: $(1/N)S_{Nt} - nt \quad S(t)$, where S(t) is a Lévy process which can be represented as follows: $S(t) = \sum_{-\infty}^{+\infty} Z_k(t)2^k$, where $Z_k(t)$ for k>0 are independent Po-processes with mean $t2^{-k}$ and for k≤0 are centered such processes.

Its characteristic function thus has the following representation:

E(exp(izS(t))) = exp(tl(z)) with

$$l(z) = \sum_{-\infty}^{0} 2^{-k}(\exp(iz2^k) - 1 - iz2^k) + \sum_{1}^{\infty} 2^{-k}(\exp(iz2^k) - 1) \equiv \int_0^\infty (\exp(izx) - 1 - izc(x))L(dx),$$

where the Lévy measure L(dx) has masses 2^{-k} at the positions $x=2^k$ and the centering c(x) = x for x ≤ 1 and = 0 for x ≥ 2.

From this follows that the two-dimensional random walk (S_k,T_k) obeys the following limit theorem:

$((1/N)T_{Nt},(1/N)S_{Nt} - nt) \quad (2t,S(t))$ as n→∞.

Since we see that

$$V^-(r) - na\int_0^\infty e^{-2at}dt \qquad a\int_0^\infty e^{-2at}dS(t), \text{ or}$$

$$V^-(r) - n/2 \qquad a\int_0^\infty e^{-2at}dS(t) \equiv aU.$$

it is therefore interesting to study the distribution of

$$U = \int_0^\infty e^{-2at}dS(t)$$

and especially to try to estimate the tail of its distribution. This can in fact be done in a way similar to that used for the study of the distribution of S(t) in Martin-Löf (1985) because of the fact that U has a Lévy representation quite similar to that of S(t):

Lemma1. If S(t) is a Lévy process with

$$E(\exp(izS(t))) \equiv \exp(tl(z)) = \exp(t\int_0^\infty (\exp(izx) - 1 - izc(x))L(dx)).$$

then $U = \int_0^\infty e^{-2at}dS(t)$ has also a Lévy distribution with

$E(\exp(izU)) \equiv \exp(g(z)/2a)$, and

$$g(z)=\int_0^\infty(\exp(izx)-1-izxc'(x))(\bar{L}(x)/x)dx,$$ where

$$\bar{L}(x)=\int_x^\infty L(dy),\quad \int_1^\infty(\bar{L}(x)/x)dx=\int_1^\infty(\log x)L(dx)$$ is finite,

and c(x) is sufficiently smooth with compact support and c´(0)=1.

Proof: Since S(t) has independent increments with

E(exp(izdS(t))) = exp(l(z)dt) we have

$$E(\exp(izU))=\exp(\int_0^\infty l(ze^{-2at})dt).$$

Putting $u=e^{-2at}$ the exponent is

$$(1/2a)\int_0^1 l(zu)du/u\equiv g(z)/2a,$$ and

$$g(z)=\int_0^1 du/u\int_0^\infty(\exp(izux)-1-izuc(x))L(dx)=\int_0^1 du/u\int_0^\infty(izux)(\exp(izux)-c'(x))\bar{L}(x)dx=$$

$$\int_0^\infty\bar{L}(x)dx\int_0^1(iz)(\exp(izux)-c'(x))du=\int_0^\infty(\exp(izx)-1-izxc'(x))(\bar{L}(x)/x)dx.$$

The Lévy measure of U is hence $(\bar{L}(x)/x)dx$ and the centering is xc´(x).

In the case we consider we have

$$L(dx)=\sum_k 2^{-k}\delta(x-2^k)dx,$$

so that for $2^k<x<2^{k+1}$ we have

$$\bar{L}(x)=\sum_{k+1}^\infty 2^{-j}=2^{-k},$$ and hence

$$g(z)=\sum_k 2^{-k}\int_{2^k}^{2^{k+1}}(\exp(izx)-1-izxc'(x))dx/x=\sum_k 2^{-k}\int_1^2(\exp(iz2^kx)-1-iz2^kxc'(2^kx))dx/x.$$

Here the centering can be modified because

$$\int_1^2 2^kc'(2^kx)dx=c(2^{k+1})-c(2^k),$$ so

$$\sum_k 2^{-k}\int_1^2 2^kc'(2^kx)dx=\sum_k 2^{-k}(c(2^{k+1})-c(2^k))=\sum_k 2^{-k}c(2^k),$$

and we finally get:

$$g(z)=\sum_k 2^{-k}\int_1^2(\exp(iz2^kx)-1-izxc(2^k))dx/x=$$

$$\sum_{k\le 0}2^{-k}\int_1^2(\exp(iz2^kx)-1-iz2^kx)dx/x+\sum_{k>0}2^{-k}\int_1^2(\exp(iz2^kx)-1)dx/x$$

if c(x)=x for x≤1 and c(x)=0 for x≥2 as it is for S(t).

This means that U has a representation as a sum similar to the one for S(t) as a sum of Po-processes:

Theorem 4. Let Wk, k>0 be independent compound Po-variables with Lévy measures $(2^{-k}/2a)(dx/x)$ for 1<x<2, so that $E(\exp(izW_k) = \exp((2^{-k}/2a)\int_1^2(\exp(izx)-1)dx/x),$ and W_k, k≤0 the corresponding centered variables with $E(\exp(izW_k) = \exp((2^{-k}/2a)\int_1^2(\exp(izx)-1-izx)dx/x).$

Then $U=\int_0^\infty e^{-2at}dS(t)$ can be represented as $U=\sum_k 2^k W_k,$ and hence has a Lévy distribution with characteristic function exp((1/2a)g(z)) defined above:

$$E(\exp(izU)) =$$

$$\exp((1/2a)\sum_{-\infty}^{0}2^{-k}\int_1^2(\exp(iz2^k x)-1-iz2^k x)dx/x+(1/2a)\sum_1^\infty 2^{-k}\int_1^2(\exp(iz2^k x)-1)dx/x).$$

This representation is useful for getting an estimate of the right tail of the distribution of U quite analogous to Theorem 3 in Martin-Löf (1985). Let us outline the derivation of this estimate:

We want to estimate $2^m P(U>x2^m)$ with $1<x\le 2$ as $m\to\infty$, and for this purpose we can use the "semistability" of g(z), which says that

$$g(z2^{-m}) = \sum_{-\infty}^{0}2^{-k}\int_1^2(\exp(izx2^{k-m})-1-izx2^{k-m})dx/x+\sum_1^\infty 2^{-k}\int_1^2(\exp(izx2^{k-m})-1)dx/x =$$

$$2^{-m}(\sum_{-\infty}^{-m}+\sum_{-m+1}^{\infty}) = 2^{-m}(\sum_{-\infty}^{0}+\sum_1^\infty+izm) = 2^{-m}(g(z)+imz).$$

This means that the characteristic function of $2^{-m}U$ can be evaluated as follows:

$$E(\exp(iz2^{-m}(U-m/2a))) = \exp(g(z2^{-m})/2a-izm/2a) =$$

$$\exp(2^{-m}g(z)/2a)\approx 1+2^{-m}g(z)/2a.$$

Using the continuity theorem for "quasi-characteristic" functions proved in Feller (1971) ch. XVII we can conclude from this that

$$2^m P(2^{-m}(U-m/2a)>x)\to(1/2a)\int_x^\infty(\bar{L}(y)/y)dy.$$

But since $m2^{-m}\to 0$ this means that $2^m P(U>x)$ has the same limit.

For $1 < x \le 2$ it can be evaluated as follows, since as we have seen $\bar{L}(y) = 2^{-k}$ when $2^k < x < 2^{k+1}$.

$$\int_x^\infty = \int_x^2 dy/y + \sum_1^\infty 2^{-k} \int_{2^k}^{2^{k+1}} dy/y = (\log 2 - \log x) + (\log 2)\sum_1^\infty 2^{-k} = (2\log 2 - \log x),$$

and we have proved the following result:

Theorem 5. When $m \to \infty$ and $1 < x \le 2$ we have

$2^m P(U - m/2a > 2^m x) \to (2\log 2 - \log x)/\ 2a = (\log 2 / 2a)(2 - Logx),$

where Log denotes log to the base two.

3. Conclusions.

In order to evaluate the doubling strategy we remember that the total discounted gain V(r)=V$^+$(r)-V$^-$(r) and that

$a2^{-n}V^+(r) \to 1/\ 2$ and $V^-(r) - n/2 \quad aU$ when $r = \exp(-a2^{-n})$

and $n \to \infty$, and U defined in Theorem 4 can be estimated using Theorem 5. Since $V^+(r) \approx 2^n / 2a$ and $V^-(r) \approx n/2 + aU$ we see that V$^+$(r) dominates V$^-$(r) and the game seems to be quite favourable with P(V(r)<0) ≈ const.2^{-n} if the asymptotic formula is valid for m=n. This is in contrast to the Stockholm game where $(1-r)^{1/2}V(r)$ has a symmetric Gaussian limit distribution. This difference reflects the fact that the time between gains T has E(T) finite in the first case and infinite in the second case.

If we consider the total gain in a Petersburg game with interest but without the doubling strategy the present value of the total gain V(r) is the same as V$^-$(r) except for the factor (1-r), so our limit theorem says that in this case

$(1-r)V(r) \quad aU.$

Using Theorem 4 we can hence conclude that if

$P(U \ge x2^m) \approx 2^{-m}(2\log 2 - \log x)/\ 2a = s,$

a desired safety limit, then $x2^{n+m}$ is a premium which is sufficient to cover the total gain with probability $\approx 1-s$. (neglecting the centering $n/2$ which is very small compared to 2^n.)

References.

(1) Aase, K. On the St. Petersburg Paradox, Scand. Actuarial J. 2001; 1: 69-7

(2) Feller, W. An introduction to probability theory and its applications, vol.1, 3rd ed. Wiley 1968 and vol.2, 2nd ed. Wiley 1971.

(3) Martin-Löf, A. A limit theorem which clarifies the "Petersburg paradox". J.Appl.Prob. 22, 634-643, 1985.

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