Mathematical Statistics
Stockholm University

Asymptotic shape in a continuum growth model

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Research Report 2001:10
Licentiate Thesis

ISSN 1650-0377

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Research Report **2001:10**,
http://www.matematik.su.se/matstat

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October 2001

Abstract

A continuum analogue of the Richardson model is introduced. The state at time t, S_t, is a subset of $\mathbb{R}^d$ and consists of a connected union of unit balls, which emerge from outbursts at their center points. An outburst occurs somewhere in S_t after an exponentially distributed time with expected value $|S_t|^{-1}$ and the location of the outburst is uniformly distributed over S_t. The main result is that if S_0 is a unit ball around the origin then the diameter of S_t grows linearly and S_t/t has a non-random shape as $t \to \infty$. Due to rotation invariance the asymptotic shape must be a Euclidean ball.

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Acknowledgement

I thank my supervisor Olle Häggström for proposing the problem and for excellent guidance whenever I need it. I also thank Anders Martin-Löf for many stimulating discussions and for encouraging me in my studies.

Stockholm, October 2001
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Contents

1 Introduction

The Richardson model, introduced in Richardson (1973), is one of the simplest models for an interacting particle system. It describes a Markov process whose state at time t, S_t, is a subset of $\mathbb{Z}^d$. Each site in $\mathbb{Z}^d$ is in either of two states, denoted 0 and 1, and S_t consists of the sites that are in state 1 at time t. A site in state 0 is transferred to state 1 at a rate proportional to the number of nearest neighbors in state 1, and once in state 1 it never returns to state 0. Thus, if sites in state 1 are thought of as infected sites and sites in state 0 as uninfected this dynamics defines a pure growth model. The set of infected sites increases to cover all of $\mathbb{Z}^d$ and most of the work that has been put into the study of the model concerns *how* the set grows. The main result, first proved in Richardson (1973), states that if S_0 consists of a single site then S_t/t has a non-random shape as $t \to \infty$. Generalizations of Richardson's result can be found in Cox and Durrett (1981), Kesten (1986) and Boivin (1990). Apart from the fact that the asymptotic shape is convex and compact, not much is known about its qualitative features. This lack of information about the asymptotic shape is shared with other models for interacting particle systems that incorporates lattice structures – see e.g. Durrett (1985) for an overview – and the problems with characterizing the shape revolve around the fact that the lattices are not rotation invariant.

In this paper we introduce a continuum analogue of the Richardson model. The state at time t, still denoted S_t, is a subset of $\mathbb{R}^d$ instead of $\mathbb{Z}^d$ and the process should be thought of as describing the spread of some kind of infection in a continuous medium. As in the Richardson model, the set S_t specifies the region infected at time t. The growth takes place by way of outbursts in the infected region and it is initiated by an outburst at the origin at time zero. Given the development of the infection up to time t, the time until an outburst occurs somewhere in S_t is exponentially distributed with parameter $|S_t|$ and the location of the outburst is uniformly distributed over S_t. When an outburst occurs at an infected point it causes a ball of fixed size around the outburst point to be infected and the total infected region is enlarged by the amount of this ball that was not previously infected. Consequently, the infected region is a connected union of Euclidean balls, which by rescaling can be assumed to be unit balls.

The main result in this paper is a shape theorem for the continuum model. An essential advantage of the continuum model as compared to the Richardson model is that it possesses rotational invariance, which requires the asymptotic shape to be a Euclidean ball. Indeed, the shape theorem asserts that for large t the infected area is approximately a ball with radius proportional to t. To formulate the theorem, let $B(x, r)$ denote a ball with radius r around the point $x \in \mathbb{R}^d$ and let S_t denote the infected region at time t in the d-dimensional continuum model.

Theorem 1.1 (Shape theorem) *For any dimension d there is a real number $\mu > 0$ such that for any ε with $0 < \varepsilon < \mu^{-1}$ almost surely*

$$(1-\varepsilon)B(0,\mu^{-1}) \subset \frac{S_t}{t} \subset (1+\varepsilon)B(0,\mu^{-1})$$

for all sufficiently large t.

Another example of a continuum model with a Euclidean ball as asymptotic shape is described in Howard and Newman (1997).

In the case $d = 2$ the infected region in our continuum model corresponds to an area in the plane and the shape theorem asserts that if we observe the infected area from above, moving away from the plane linearly in time, then it will asymptotically appear to us as a circle with radius μ. It is indeed advisable to keep the case $d = 2$ in mind throughout reading this paper (except in Section 3). In the proofs we will frequently employ geometrical constructions and these are easiest to understand in two dimensions.

The rest of the paper is organized as follows. We start by describing the model more thoroughly in Section 2. Section 3 treats the case $d = 1$, which turns out to be particularly simple. In Section 4 we define some important quantities and prove two technical lemmas. Section 5 contains results concerning the growth in a fixed direction and Section 6 is devoted to the proof of the shape theorem. A simulation of the model can be found in Section 7.

2 Description of the model

In this section we construct the model more formally by defining a Markov process whose state at time t, S_t, is a subset of $\mathbb{R}^d$. The process may be thought of as describing the spread of an infection (with no recoveries) or the growth of a germ colony in a continuous medium. Points in S_t will be referred to as infected.

The aim is to define the model in such a way that the growth takes place by way of outbursts in the infected region, an outburst at a point x infecting all points within distance one from x. It is of course desirable for the time from t until an outburst takes place somewhere in S_t to decrease as $|S_t|$ increases, since as $|S_t|$ increases so does the number of points where an outburst can occur. Moreover it is preferable for the growth to exhibit a Markovian behavior in the sense that it should hold that

$$P(A \subset S_t |\, S_{s_1}, \ldots, S_{s_k}) = P(A \subset S_t |\, S_{s_k}) \tag{1}$$

for all $s_1 < \ldots < s_k < t$ and all $A \in \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel sigma field of $\mathbb{R}^d$.

To formulate a model that possesses these features, consider a Poisson process on $\mathbb{R}^{d+1}$ with unit intensity, that is, consider a point process N on $\mathbb{R}^{d+1}$ with

$$P(N(A) = k) = \frac{|A|^k e^{-|A|}}{k!} \quad \text{for } A \in \mathcal{B}(\mathbb{R}^{d+1})$$

and with $N(A_1)$ and $N(A_2)$ independent for disjoint sets A_1 and A_2. The extra dimension represents the time dimension and the points of N are hence denoted (X_k, T_k), where $X_k \in \mathbb{R}^d$ and the last coordinate T_k gives the location on the time axis. At time zero the unit ball around the origin, denoted by B_0, is infected, i.e. we imagine that there is an infected point at the origin where an outburst takes place, infecting the surrounding unit ball, as we start observing the process. The idea now is to follow the cylinder $B_0 \times \mathbb{R}$ upwards along the time axis until a point in the Poisson process is found. An outburst then takes place at this point generating a new infection ball B_1 and the new infected area is given by $B_0 \cup B_1$. Scanning within the cylinder $(B_0 \cup B_1) \times \mathbb{R}$ further upward along the time axis we eventually hit a new Poisson point, representing a new outburst and corresponding enlargement of the infected region. And so on.

To make this description more formal, let $N_{S\times\mathbb{R}}$, $S \in \mathbb{R}^d$, denote the restriction of N to $S \times \mathbb{R}$. The growth of the infected area takes place at time points $\{T_n\}$ by aid of outbursts at points $\{X_n\}$ obtained from the following recursion:

> Let $X_0 = 0$, $T_0 = 0$ and define $B_n = \{y \in \mathbb{R}^d;\ |X_n - y| \leq 1\}$, i.e. B_n is a unit ball in $\mathbb{R}^d$ centered at X_n.
>
> Given $\{X_i;\ i \leq n\}$ and $\{T_i;\ i \leq n\}$, the time T_{n+1} is defined as
>
> $$T_{n+1} = \inf_k \{T_k;\ T_k > T_n \text{ and } (X_k, T_k) \in N_{\cup_{i=0}^n B_i \times \mathbb{R}}\}$$
>
> and X_{n+1} is the unique point in $\mathbb{R}^d$ such that $(X_{n+1}, T_{n+1}) \in N_{\cup_{i=0}^n B_i \times \mathbb{R}}$.
>
> *Remark:* The uniqueness of X_{n+1} is due to the fact that the probability of having two or more points at the same t-level in the Poisson process equals zero.

From the sequence $\{X_n\}$ a new sequence $\{S_{(n)}\}$ is constructed by defining $S_{(n)} = \cup_{i=0}^n B_i$. The infected region at time t is now given by

$$S_t = S_{(n)} \quad \text{for } t \in [T_n, T_{n+1}).$$

Let us introduce the notation $\Delta_n = T_n - T_{n-1}$, $n \geq 1$, for the successive times between the outbursts. By construction of the model and properties of the Poisson process

$$\Delta_{n+1} | \mathcal{F}_n \sim \text{Exp}(|S_{(n)}|),$$

where $\mathcal{F}_n = \sigma(X_0, \ldots, X_n, T_0, \ldots, T_n)$. Given S_t, the memoryless property of the exponential distribution implies that the time until an outburst occurs somewhere in S_t is exponentially distributed with parameter $|S_t|$. The location of the outburst, X_n – where n is such that $t \in [T_n, T_{n+1})$ – is uniformly distributed over S_t and the new infected region is given by $S_t \cup B_n$. Furthermore, the model clearly possesses the Markovian property described in (1). Hence we have arrived at a model exhibiting the desired properties.

In the present formulation of the model the development of S_t is determined by a strictly increasing sequence $\{T_n\}$, specifying the time points of the outbursts, and a sequence $\{X_n\} \subset \mathbb{R}^d$, specifying the locations of the outbursts. The sequences are constructed recursively as described above. To guarantee that the model is defined for all t one detail remains to be checked: We have to make sure that the sequence $\{T_n\}$ does not have a finite limit point T_∞, since this would cause problems defining S_t for $t > T_\infty$. The following proposition is what we need:

Proposition 2.1 *Almost surely, $T_n \to \infty$ as $n \to \infty$.*

Proof: Since $T_n - T_0 = \sum_{k=1}^{n} \Delta_k$ and $T_0 = 0$ it suffices to show that $\sum_{k=1}^{\infty} \Delta_k = \infty$ with probability one. As pointed out above

$$\Delta_k | \mathcal{F}_{k-1} \sim \mathrm{Exp}(|S_{(k-1)}|).$$

Due to properties of the Poisson process, given $\mathcal{F}_{k-1}$ the increment Δ_k can be written as

$$\Delta_k = \frac{k}{|S_{(k-1)}|} E_k,$$

where $\{E_k\}$ are independent, $E_k \sim \mathrm{Exp}(k)$. Let $v = v(d)$ denote the volume of a unit ball in $\mathbb{R}^d$. A trivial upper bound for $|S_{(k-1)}|$ is given by $|S_{(k-1)}| \leq kv$, implying that $\Delta_k \geq E_k/v$. Hence

$$\sum_{k=1}^{\infty} \Delta_k \geq \frac{1}{v} \sum_{k=1}^{\infty} E_k \quad \text{a.s.} \tag{2}$$

and we have reduced the problem to showing that $\sum_{k=1}^{\infty} E_k = \infty$ with probability one. To this end, introduce $\tilde{E}_k = E_k - E[E_k] = E_k - 1/k$. Using the fact that $\sum_{k=1}^{\infty} E[\tilde{E}_k^2] = \sum_{k=1}^{\infty} 1/k^2 < \infty$, Kolmogorov's three series theorem implies that $\sum_{k=1}^{\infty} \tilde{E}_k$ converges almost surely. Thus

$$\sum_{k=1}^{\infty} E_k = \sum_{k=1}^{\infty} \tilde{E}_k + \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

since $\sum_{k=1}^{\infty} 1/k = \infty$. The proposition now follows from (2). □

3 The one-dimensional case

There is a big difference in the amount of work required to prove the shape theorem in two or more dimensions as compared to the one-dimensional case. To illustrate this we give in this section a separate proof for the case $d = 1$.

In the one-dimensional growth model the infected region corresponds to an interval on the real line. The interval is enlarged as an outburst occurs at an infected point within distance one from one of the endpoints. We will call an outburst *right(left)-effective* if it takes place somewhere in the infected interval within distance one from the right(left) endpoint. First consider the right endpoint. Let T_n^r denote the time for the n:th right-effective outburst ($T_0^r := 0$) and write $S_{(n)}^{\,r}$ for the location of the right endpoint after this outburst ($S_{(0)}^{\,r} := 1$). For any n, given $S_{(n)}^{\,r}$ the next right-effective outburst occurs after a time that is exponentially distributed with parameter one and the location of the outburst is uniformly distributed on $[S_{(n)}^{\,r} - 1, S_{(n)}^{\,r}]$ so that the expected change at the endpoint is 1/2 unit. Thus, T_n^r can be written as a sum of n iid exponential variables with expected value one and $S_{(n)}^{\,r}$ can be written as $S_{(0)}^{\,r}$ plus a sum of n iid uniform variables with expected value 1/2. Hence, by the strong law of large numbers, almost surely

$$\frac{1}{n}\left(T_n^r, S_{(n)}^{\,r}\right) \to (1, \frac{1}{2}) \quad \text{as } n \to \infty$$

and it follows that $S_{(n)}^{\,r}/T_n^r \to 1/2$ almost surely as $n \to \infty$. Now, let S_t^r denote the location of the right endpoint at time t and let n_t be such that $t \in [T_{n_t}^r, T_{n_t+1}^r)$. Since clearly $T_{n_t}^r/t \to 1$ as $t \to \infty$ we have

$$\lim_{t\to\infty} \frac{S_t^r}{t} = \lim_{t\to\infty} \frac{T_{n_t}^r}{t} \cdot \frac{S_{(n_t)}^{\,r}}{T_{n_t}^r} = \frac{1}{2}.$$

Analogously it can be shown that $\lim_{t\to\infty} S_t^l/t = -1/2$, where S_t^l denotes the left endpoint of the infected interval at time t. Thus, in one dimension the shape theorem with $\mu^{-1} = 1/2$ is an easy consequence of the strong law of large numbers.

4 Definitions and two lemmas

In this section we introduce notation and define a number of quantities needed to prove the shape theorem for $d \geq 2$. We also formulate and prove two auxiliary results which will be used extensively in the rest of the paper.

To begin with let $T(x)$ denote the time when the point x is infected, i.e.

$$T(x) = \inf\{t;\ x \in S_t\}.$$

Our first result is a lemma bounding the time it takes for the infection to travel between two points x and y. The bound is expressed as a sum of independent exponential random variables where the number of terms is proportional to $|x-y|$.

Lemma 4.1 *For any $x, y \in \mathbb{R}^d$ there exist iid exponential variables $\{E_k\}$ with parameter $\lambda = \lambda(d)$ such that*

$$|T(x) - T(y)| \leq \sum_{k=1}^{2\lceil |x-y| \rceil} E_k.$$

Proof: (The proof is based on geometrical arguments which are best manifested in two dimensions. Therefore the proof is formulated here for $d = 2$. The case $d \geq 3$ is analogous.)

Fix $x, y \in \mathbb{R}^2$ and assume, without loss of generality, that x and y are located on the x-axis with $y < x$, that is, assume $y = (y', 0)$ and $x = (x', 0) = (y' + |x-y|, 0)$. The idea of the proof is to consider small balls located consecutively on the line from y to x and let the infection wander from one ball to another starting at y until it reaches x, see Figure 1. Given that a ball is infected, the time until a neighboring ball is infected is exponentially distributed and hence, after some work, we will arrive at a bound for $|T(x) - T(y)|$ expressed as a sum of exponential variables.

To begin with, consider a small ball with radius c around the point y and let $S_{T(y)} \cap B(y, c)$ denote the part of this ball that is infected at time $T(y)$. It is clear that $|S_{T(y)} \cap B(y, c)|$ is minimized if y is infected by a point z at unit distance. In that case, for small c,

$$|S_{T(y)} \cap B(y, c)| \geq |B(y, c) \cap B(z, 1)| \geq \frac{\pi c^2}{4}.$$

Next, write B_k for the ball of radius $c/2$ around the point $(y' + 0.5 \cdot k, 0)$, see Figure 1. If c is sufficiently small, say $c \leq 0.1$, then B_1 is contained in the infected area as soon as an outburst has occurred somewhere in $B(y, c)$. As explained above, at time $T(y)$ the area of the infected part of $B(y, c)$ is greater than $\pi c^2/4$ and thus the time from time $T(y)$ until an outburst has occurred within it can be dominated by a random variable $E_0 \sim \text{Exp}(\pi c^2/4)$. Now let E_1 denote the time from $T(y) + E_0$ until an outburst occurs somewhere in B_1 and consider B_2, that is, consider the ball of radius $c/2$ around the point $(y' + 1, 0)$. Clearly B_2 is infected by the time an outburst has occurred in B_1, that is, $B_2 \subset S_{T(y)+E_0+E_1}$ and since $B_1 \subset S_{T(y)+E_0}$ we have that $E_1 \sim \text{Exp}(\pi c^2/4)$.

The idea how to continue should now be clear. Let us formalize it as follows: According to the reasoning above $B_1 \subset S_{T(y)+E_0}$, where $E_0 \sim \text{Exp}(\pi c^2/4)$.

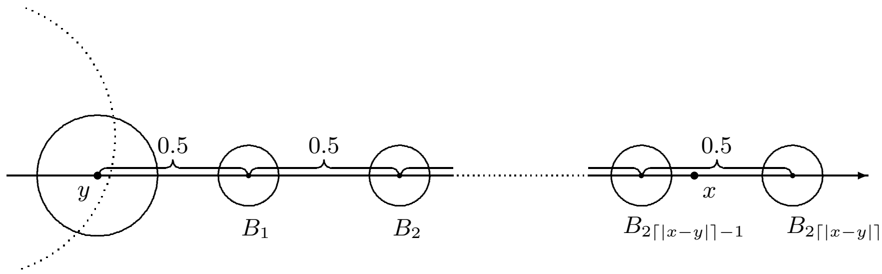

Figure 1: A chain of small balls located 0.5 units apart on the line segment joining y and x is constructed. The infection enters the chain at time $T(y)$ as indicated to the left in the figure and then wanders from one ball to another until it reaches x.

Define E_k, $k \geq 1$, recursively as the time until an outburst occurs in B_k counting from time $T(y) + E_0 + \ldots + E_{k-1}$. Since

$$B_k \subset S_{T(y)+E_0+\ldots+E_{k-1}}$$

we have that $E_k \sim \text{Exp}(\pi c^2/4)$, where $\pi c^2/4$ is the area of B_k. When an outburst has occurred in $B_{2\lceil |x-y|\rceil -1}$ it is clear that x must be infected, i.e.

$$x \in S_{T(y)+E_0+\ldots+E_{2\lceil |x-y|\rceil -1}}.$$

Hence

$$|T(x) - T(y)| \leq \sum_{k=0}^{2\lceil |x-y|\rceil -1} E_k$$

and the lemma is proved. □

Remark: Choosing $y = 0$ Lemma 4.1 asserts that

$$T(x) \leq \sum_{k=1}^{2\lceil |x|\rceil} E_k.$$

Moreover, it follows from the proof of the lemma that the bound for $T(x)$ is valid also for the time until a small cube around x is infected. By construction of the variables $\{E_k\}$, at time $\sum_{k=0}^{2\lceil |x|\rceil -1} E_k$ an outburst has occurred somewhere in a ball with radius $c/2$, $c \leq 0.1$, centered at a point within distance $1/2$ from x. Thus a cube centered at x will be contained in the infected area at time $\sum_{k=0}^{2\lceil |x|\rceil -1} E_k$ if the side length of the cube is chosen small enough. This observation will be useful in proving Lemma 6.1.

We now introduce some auxiliary quantities that will be of use later. Let $S_t^{(x,s)}$, $t \geq s$, denote the set of points that can be reached from x within time t if a new process is started at x at time s. That is, at time s all infection except a unit ball around x is erased. The infection then evolves in time according to the same rules as for the original process, using the same $(d+1)$-dimensional Poisson process. This gives rise to a new process, emanating from x, whose state at time t, $t \geq s$, is given by $S_t^{(x,s)}$. At a first glance this new process might appear a bit artificial, but it will turn out to be useful in proving that the asymptotic speed of the growth in a fixed direction exists almost surely. Now let

$$\tilde{T}(x) = \inf\{t;\ B(x,1) \subset S_t\}$$

and

$$\tilde{T}(x,y) = \inf\{t;\ B(y,1) \subset S^{(x,\tilde{T}(x))}_{\tilde{T}(x)+t}\}.$$

In words $\tilde{T}(x)$ is the time when the entire unit ball around x is infected and $\tilde{T}(x,y)$ is the time it takes for the infection to invade the entire unit ball around y if a new process is started at x at time $\tilde{T}(x)$. Since a process started at x at time $\tilde{T}(x)$ uses Poisson points located in a region of $\mathbb{R}^d$ that is disjoint from the region containing the points used by the original process up to time $\tilde{T}(x)$, the quantity $\tilde{T}(x,y)$ is independent of $\tilde{T}(x)$ and has the same distribution as $\tilde{T}(y-x)$. It is clear that if a point is contained in the region infected at time $\tilde{T}(x)+t$ in the process started at x at time $\tilde{T}(x)$ then it is also contained in the region infected at time $\tilde{T}(x)+t$ in the original process, i.e.

$$S^{(x,\tilde{T}(x))}_{\tilde{T}(x)+t} \subset S_{\tilde{T}(x)+t}$$

and hence

$$\tilde{T}(y) \leq \tilde{T}(x) + \tilde{T}(x,y). \tag{3}$$

Moving on towards another useful observation, note that the time when the entire unit ball around a point x is infected is trivially greater than the time when the single point x is infected, implying that $\tilde{T}(x) - T(x)$ is always greater than zero. An upper bound is given in the following lemma:

Lemma 4.2 *For any $x \in \mathbb{R}^d$ there exist iid random variables $E_1, \ldots, E_{k_0}$, $E_k \sim Exp(\lambda)$ for some $\lambda = \lambda(d)$, such that*

$$0 \leq \tilde{T}(x) - T(x) \leq \sum_{k=1}^{k_0} E_k.$$

The number of terms, k_0, depends only on d.

Proof: Again the proof is formulated here for $d = 2$. We will use a geometric argument to prove that in this case

$$\tilde{T}(x) - T(x) \leq \sum_{k=1}^{8} E_k,$$

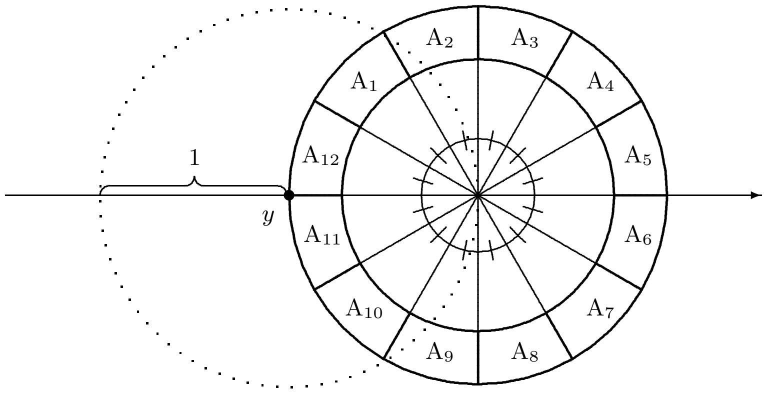

Figure 2: The unit ball around x and its front zone $F = \bigcup_{k=1}^{12} A_k$. The unit ball around y is indicated with a dotted circle.

where $E_k \sim \text{Exp}(\pi(2c-c^2)/12)$ for some $c \leq 0.1$. It will be clear that similar arguments can be constructed in higher dimensions.

Fix $x \in \mathbb{R}^2$ and let $B(x,1)$ denote the unit ball centered at x. Our first goal is to realize that at the time when x is infected it is possible to find at least one point y on the boundary of $B(x,1)$ such that $B(x,1) \cap B(y,1)$ is contained in the infected area. If x receives the infection from a point $y \in \partial B(x,1)$ this is obvious, since in this case trivially $B(y,1) \subset S_{T(x)}$. If x is infected by a point z in the interior of $B(x,1)$, consider the point $y \in B(z,1)$, belonging to the line passing through x and z and with $|x-y| = 1$. It is clear that $B(x,1) \cap B(y,1) \subset S_{T(x)}$. Hence, at time $T(x)$ we can always find a point y such that $|x-y| = 1$ and $B(x,1) \cap B(y,1) \subset S_{T(x)}$.

Let F be a front annulus of width $c \in (0,1)$ in $B(x,1)$, that is, F is a ring constituted by the outermost parts of $B(x,1)$, see Figure 2. Formally

$$F = B(x,1) \backslash B(x, 1-c).$$

Divide F into twelve disjoint pieces $A_1, \dots, A_{12}$ of equal area as shown in Figure 2. The division of F is obtained by using the lines having angle $k\pi/6$ ($k = 1, \dots, 12$) at x. For reasons that will become apparent in what follows we enumerate the pieces in a clockwise sense with A_1 being the piece defined by the lines having angle $4\pi/6$ and $5\pi/6$ respectively at x.

Define E_1 as the time until an outburst occurs in A_1 counting from time $T(x)$. The piece A_1 is contained in $B(y,1) \cap B(x,1)$ implying that A_1 is infected at time $T(x)$. Hence E_1 is exponentially distributed with parameter $|A_1| = \pi(2c-c^2)/12$. Now, if c is chosen small, say $c \leq 0.1$, then the neighboring piece A_2 must be infected when an outburst has occurred in A_1, that is, $A_2 \subset S_{T(x)+E_1}$. Thus, for such c, the time from time $T(x)+E_1$ until an outburst occurs in A_2 is exponentially distributed with parameter $|A_2|$.

In general, let E_k ($k = 1, \dots, 12$) denote the time until an outburst occurs in A_k counting from time $T(x)+E_1+\dots+E_{k-1}$. Since $A_k \subset S_{T(x)+E_1+\dots+E_{k-1}}$ we have $E_k \sim \text{Exp}(|A_k|)$. Furthermore, the variables $\{E_k\}$ are constructed on disjoint intervals on the time axis and consequently they are independent. When an outburst has occurred in each of the areas $A_1, \dots, A_8$ it is clear that the entire unit ball around x must be infected. Hence,

$$B(x,1) \subset S_{T(x)+\sum_{k=1}^{8} E_k}$$

and we can conclude that

$$\tilde{T}(x) - T(x) \leq \sum_{k=1}^{8} E_k$$

as desired. □

We close this section by anticipating that in what follows it will be convenient with a special notation for the quantity $\tilde{T}(mx, nx)$. Thus, let

$$\tilde{T}(mx, nx) = \tilde{T}_{m,n}(x).$$

Since $\tilde{T}(nx) = \tilde{T}_{0,n}(x)$ the subadditivity property (3) translates into

$$\tilde{T}_{0,n}(x) \leq \tilde{T}_{0,m}(x) + \tilde{T}_{m,n}(x).$$

5 Growth in a fixed direction

The proof of the shape theorem basically consists of two parts:

1. Show that S_t grows linearly in each fixed direction and that the asymptotic speed of the growth in each direction is an almost sure constant. By rotation invariance of $\mathbb{R}^d$ and the model this constant must be the same for all directions.

2. Show that the linear growth of S_t is preserved when all directions are considered simultaneously.

This section is devoted to Step 1. The first task is to prove two basic results concerning the growth of the infection in a fixed direction. The first result asserts that $\lim_{n\to\infty} \tilde{T}(nx)/n$ is an almost sure constant and also gives a characterization of the constant:

Proposition 5.1 *For each $x \in \mathbb{R}^d$ we have*

(a) $\mu(x) := \lim_{n\to\infty} E[\tilde{T}(nx)]/n = \inf_{n\geq 1} E[\tilde{T}(nx)]/n$;

(b) $\lim_{n\to\infty} \tilde{T}(nx)/n = \mu(x)$ *a.s.*

Remark: We will employ the convention that limits over n are taken over the positive integers while limits over t, which will occur later on in the paper, are taken over all positive reals.

The second result states that the sequence $\{T(nx)/n\}$ asymptotically exhibits the same behavior as $\{\tilde{T}(nx)/n\}$:

Proposition 5.2 *For each $x \in \mathbb{R}^d$, almost surely $\lim_{n\to\infty} T(nx)/n = \mu(x)$.*

Being the limit of the sequence $\{T(nx)/n\}$, the number $\mu(x)$ indicates the time it takes for the infection asymptotically to travel the distance $|x|$ in direction x and consequently it should be interpreted as a measure of the inverse asymptotic speed of the growth of the infected area in direction x.

To prove Proposition 5.1 we will invoke a theorem by Liggett, concerning the existence of an asymptotic average for sequences of stochastic variables possessing certain subadditivity and stationarity properties. The theorem is not immediately applicable to the sequence $\{\tilde{T}(nx)/n\}$, since one of the assumptions of the theorem fails. However, it turns out that the proof of the theorem can be modified so that it becomes valid also for $\{\tilde{T}(nx)/n\}$. Furthermore, Proposition 5.2 will turn out to be a consequence of Proposition 5.1 and Lemma 4.2. We will give the proof of Proposition 5.2 followed by the proof of Proposition 5.1.

Proof of Proposition 5.2: Fix $x \in \mathbb{R}^d$. By Proposition 5.1,

$$P\left(\lim_{n\to\infty} \frac{\tilde{T}(nx)}{n} = \mu(x)\right) = 1. \tag{4}$$

It remains to show that this result holds also if $\tilde{T}(nx)$ is replaced by $T(nx)$. By Lemma 4.2, for each n there exist iid random variables $E_1, \ldots, E_{k_0}$, $E_k \sim \text{Exp}(\lambda)$, such that

$$\tilde{T}(nx) - T(nx) \leq \sum_{k=1}^{k_0} E_k := R_n.$$

Thus

$$\limsup_{n\to\infty} \left(\frac{\tilde{T}(nx)}{n} - \frac{T(nx)}{n}\right) \leq \limsup_{n\to\infty} \frac{R_n}{n} \tag{5}$$

and it is enough to show that $\limsup_{n\to\infty} R_n/n$ is almost surely equal to zero (note that the R_n:s are not necessarily independent). By Markov's inequality and the fact that $R_n \sim \text{Gamma}(k_0, \lambda)$, for $\theta \in (0, \lambda^{-1})$ we have

$$e^{\theta x} P(R_n > x) \leq E[e^{\theta R_n}] = (1 - \lambda\theta)^{-k_0}$$

which implies that

$$P(R_n > x) \le e^{-f_x(\theta)}$$

where $f_x(\theta) = \theta x + k_0 \log(1-\lambda\theta)$. The above bound for $P(R_n > x)$ is valid for all $\theta \in (0, \lambda^{-1})$ and in particular it is valid for $\theta = (2\lambda)^{-1}$. From this it follows that

$$P(R_n > x) \le c_1 e^{-c_2 x},$$

where $c_1 = e^{k_0 \log 2}$ and $c_2 = (2\lambda)^{-1}$, and therefore, for all $\varepsilon > 0$,

$$\sum_{n=0}^{\infty} P(R_n > n\varepsilon) \le \sum_{n=0}^{\infty} c_1 e^{-c_2 n\varepsilon} < \infty.$$

By the Borel-Cantelli lemma this implies that $\lim_{n\to\infty} R_n/n = 0$ almost surely. Thus, by (4) and (5),

$$P\left(\lim_{n\to\infty} \frac{T(nx)}{n} = \mu(x)\right) = P\left(\lim_{n\to\infty} \frac{\tilde{T}(nx)}{n} = \mu(x)\right) = 1$$

as desired. □

We now state the aforementioned theorem by Liggett which will be our main tool in proving Proposition 5.1. The theorem is a sharpened version of Kingman's subadditive ergodic theorem – the subadditivity and stationarity assumptions are relaxed without weakening the conclusions.

Theorem 5.1 *Let $\{X_{m,n}\}$ be a collection of random variables indexed by integers satisfying $0 \le m < n$. Suppose $\{X_{m,n}\}$ has the following properties:*

(i) $X_{0,n} \le X_{0,m} + X_{m,n}$.

(ii) *The distribution of $\{X_{m,m+k};\ k \ge 1\}$ does not depend on m.*

(iii) *For each $k \ge 1$, $\{X_{nk,(n+1)k};\ n \ge 1\}$ is a stationary sequence.*

(iv) $E[X_{0,1}^+] < \infty$.

Then:

(v) $\gamma := \lim_{n\to\infty} E[X_{0,n}]/n = \inf_{n\ge 1} E[X_{0,n}]/n$.

(vi) $X := \lim_{n\to\infty} X_{0,n}/n$ *exists a.s.*

(vii) $E[X] = \gamma$.

If the stationary processes in (iii) *are ergodic, then $X = \gamma$ a.s.*

A brief outline of the structure of the proof can be found below. For more detail we refer to Liggett (1985).

The reason for introducing the variables $\{\tilde{T}_{m,n}(x)\}$ should now be clear; $\{\tilde{T}_{m,n}(x)\}$ is an attempt to create a collection of random variables that fits into the assumptions of Theorem 5.1 when x is held fixed. But is the attempt successful? At a first glance it might appear so: The assumptions (i), (ii) and (iv) of the theorem are clearly satisfied by $\{\tilde{T}_{m,n}(x)\}$. At a close look though, it turns out that (iii) fails, that is, the sequences $\{\tilde{T}_{nk,(n+1)k}(x);\ n \geq 1\}$ are not stationary. Indeed, for fixed k the distribution of $\tilde{T}_{nk,(n+1)k}(x)$ is independent of n, but the joint distributions are not. To see this, note that $\{\tilde{T}_{nk,(n+1)k}(x);\ n \geq 1\}$ can be viewed as a collection of timers keeping track of the time it takes for the infection to invade the unit ball around $(n+1)kx$ starting from the unit ball around nkx at time $\tilde{T}_{0,nk}(x)$. Loosely speaking, the reason the joint distributions are not shift-invariant is that the relation between the starting times of the timers is changed by the shift, causing the dependence structure in the sequence $\{\tilde{T}_{nk,(n+1)k}(x);\ n \geq 1\}$ to change with n. As an example, $\tilde{T}_{0,k}(x)$ and $\tilde{T}_{k,2k}(x)$ are clearly independent but $\tilde{T}_{k,2k}(x)$ and $\tilde{T}_{2k,3k}(x)$ are dependent: If $\tilde{T}_{0,k}(x) < \tilde{T}_{0,2k}(x) < \tilde{T}_{0,k}(x) + \tilde{T}_{k,2k}(x)$, the timer $\tilde{T}_{2k,3k}(x)$ starts while the timer $\tilde{T}_{k,2k}(x)$ is still running, indicating that the variables $\tilde{T}_{k,2k}(x)$ and $\tilde{T}_{2k,3k}(x)$ uses partly the same Poisson points. Thus, $\tilde{T}_{k,2k}(x)$ and $\tilde{T}_{2k,3k}(x)$ are not independent and therefore $(\tilde{T}_{k,2k}(x), \tilde{T}_{2k,3k}(x))$ does not have the same distribution as $(\tilde{T}_{0,k}(x), \tilde{T}_{k,2k}(x))$. The conclusion is that Theorem 5.1 can not immediately be applied to $\{\tilde{T}_{m,n}(x)\}$. However, it turns out that it is possible to modify Liggett's proof slightly so that it becomes valid also for $\{\tilde{T}_{m,n}(x)\}$. The modification requires some knowledge about the structure of the proof and this knowledge is provided in the following brief sketch:

Write

$$\bar{X} = \limsup_{n\to\infty} \frac{X_{0,n}}{n}$$

and

$$\underline{X} = \liminf_{n\to\infty} \frac{X_{0,n}}{n}.$$

The proof of Theorem 5.1 is broken up into three steps:

(L1) $\gamma = \lim_{n\to\infty} E[X_{0,n}]/n = \inf_{n\geq 1} E[X_{0,n}]/n$;

(L2) $E[\bar{X}] \leq \gamma$, and if the stationary processes in (iii) are ergodic, then $\bar{X} \leq \gamma$ almost surely;

(L3) $E[\underline{X}] \geq \gamma$.

From (L2) and (L3) it follows that $E[\underline{X}] \geq E[\bar{X}]$. This implies that $\underline{X}$ and $\bar{X}$ are equal, since trivially $\underline{X} \leq \bar{X}$. Hence, once (L2) and (L3) are established it is clear that $X := \lim_{n\to\infty} X_{0,n}/n$ exists with probability one. It also follows from (L2) and (L3) that $E[X] = \gamma$ and by (L1) $\gamma < \infty$. Furthermore, if $\bar{X} \leq \gamma$ – which, according to (L2), for example is the case if the sequences $\{X_{nk,(n+1)k};\ n \geq 1\}$ are ergodic – then we can deduce that $X = \gamma$ almost surely.

For a proof of (L1)-(L3) we refer to Liggett (1985). The essential task for us is to identify the parts of the proof that makes use of the assumption (iii). Since the variables $\{\tilde{T}_{m,n}(x)\}$ satisfy all the assumptions of Theorem 5.1 except (iii), these are the parts that have to be modified. The following table shows how the assumptions are used in the different steps:

Step	Assumptions used
(L1)	(i), (ii)
(L2)	(i)-(iv)
(L3)	(i), (ii), (iv)

Since the proofs of (L1) and (L3) does not use (iii) we can immediately conclude that (L1) and (L3) holds for $\tilde{T}_{0,n}(x)$, that is,

$$\mu(x) := \lim_{n\to\infty} \frac{E[\tilde{T}_{0,n}(x)]}{n} = \inf_{n\geq 1} \frac{E[\tilde{T}_{0,n}(x)]}{n} \tag{6}$$

and

$$E\left[\liminf_{n\to\infty} \frac{\tilde{T}_{0,n}(x)}{n}\right] \geq \mu(x). \tag{7}$$

In proving (L2) Liggett uses the assumption (iii) and thus a modification of Liggett's proof is necessary to establish that (L2) holds for $\tilde{T}_{0,n}(x)$. The modification is described in the proof of Proposition 5.1, which we are now ready to present.

Proof of Proposition 5.1: Part (a) of the proposition is established in (6). Using (7), part (b) will follow if we can show that

$$\limsup_{n\to\infty} \frac{\tilde{T}_{0,n}(x)}{n} \leq \mu(x) \quad \text{a.s.} \tag{8}$$

To achieve this, fix $\delta > 0$ and choose k large so that

$$\frac{E[\tilde{T}_{0,k}(x)]}{k} \leq \mu(x) + \delta.$$

We will show that for all j,

$$\limsup_{n\to\infty} \frac{\tilde{T}_{0,nk+j}(x)}{nk+j} \leq \frac{E[\tilde{T}_{0,k}(x)]}{k} \quad \text{a.s.} \tag{9}$$

which yields

$$\limsup_{n\to\infty} \frac{\tilde{T}_{0,n}(x)}{n} \leq \frac{E[\tilde{T}_{0,k}(x)]}{k} \quad \text{a.s.}$$

Hence, once (9) has been established it follows from the choice of k that

$$\limsup_{n\to\infty} \frac{\tilde{T}_{0,n}(x)}{n} \leq \mu(x) + \delta$$

and since $\delta > 0$ was arbitrary this implies (8). To prove (9), fix j and use subadditivity to get

$$\frac{\tilde{T}_{0,nk+j}(x)}{nk+j} \leq \frac{n}{nk+j} \cdot \frac{\tilde{T}_{0,nk}(x)}{n} + \frac{n}{nk+j} \cdot \frac{\tilde{T}_{nk,nk+j}(x)}{n}. \tag{10}$$

The distribution of $\tilde{T}_{nk,nk+j}(x)$ depends only on j and has a finite first moment. Thus

$$\sum_{n=1}^{\infty} P(\tilde{T}_{nk,nk+j}(x) > n\varepsilon) < \infty$$

for all $\varepsilon > 0$. Using the Borel-Cantelli lemma this implies that

$$\lim_{n\to\infty} \frac{\tilde{T}_{nk,nk+j}(x)}{n} = 0 \quad \text{a.s.} \tag{11}$$

Now, if we can show that

$$\limsup_{n\to\infty} \frac{\tilde{T}_{0,nk}(x)}{n} \leq E[\tilde{T}_{0,k}] \tag{12}$$

then (9) will follow from (10) and (11). To establish (12) it is necessary to introduce an auxiliary sequence $\{\tilde{T}'_{(i-1)k,ik}(x);\ i \geq 1\}$ defined recursively as follows:

Let $\tilde{T}'_{0,k}(x) = \tilde{T}_{0,k}(x)$. For $i \geq 2$, given $\{\tilde{T}'_{(l-1)k,lk}(x);\ l \leq i-1\}$, define

$$\tilde{T}'_{(i-1)k,ik}(x) = \inf\{t;\ B(ikx,1) \subset S^{((i-1)kx,\ \Phi^k_{i-1})}_{\Phi^k_{i-1}+t}\},$$

where $\Phi^k_{i-1} = \sum_{l=1}^{i-1} \tilde{T}'_{(l-1)k,lk}(x)$.

Remember that $S^{(x,s)}_t$ is the area infected at time t, $t \geq s$, in a process started at time s emanating from the unit ball around x. Thus $\tilde{T}'_{(i-1)k,ik}(x)$ is the time when the unit ball around ikx is infected in a process started at $(i-1)kx$ at time Φ^k_{i-1}. As with the sequence $\{\tilde{T}_{(i-1)k,ik}(x)\}$, the sequence $\{\tilde{T}'_{(i-1)k,ik}(x)\}$ can be interpreted as a collection of timers indicating the time it takes for the infection to invade the unit ball around ikx starting from the unit ball around $(i-1)kx$ but – and this is the crucial point – in the sequence $\{\tilde{T}'_{(i-1)k,ik}(x)\}$ timer number i does not start until timer number $i-1$ has stopped. This indicates that the variables $\{\tilde{T}'_{(i-1)k,ik}(x)\}$ are independent and identically distributed with expected value $E[\tilde{T}_{0,k}(x)]$. Hence, by the strong law of large numbers

$$\frac{1}{n}\sum_{i=1}^{n}\tilde{T}'_{(i-1)k,ik}(x) \to E[\tilde{T}_{0,k}(x)] \quad \text{as } n \to \infty. \tag{13}$$

Furthermore it is readily seen that

$$\tilde{T}_{0,nk}(x) \leq \sum_{i=1}^{n}\tilde{T}'_{(i-1)k,ik}(x).$$

Dividing this inequality by n and using (13) we obtain (12). Thereby (9) is established and the proposition follows. □

It is now confirmed that for each $x \in \mathbb{R}^d$ the limit of the sequences $\{T(nx)/n\}$ and $\{\tilde{T}(nx)/n\}$ is an almost sure constant $\mu(x)$ given by the common value of $\lim_{n\to\infty} E[\tilde{T}(nx)]/n$ and $\inf_{n\geq 1} E[\tilde{T}(nx)]/n$. Since $E[\tilde{T}(x)] < \infty$, clearly $\mu(x) < \infty$, that is, the infection grows at least linearly in time in each direction. In the following proposition we prove that $\mu(x) > 0$ for $x \neq 0$, which guarantees that the infection does not grow faster than linearly.

Proposition 5.3 *For each $x \in \mathbb{R}^d$, $x \neq 0$, we have $0 < \mu(x) < \infty$.*

Proof: Fix $x \in \mathbb{R}^d$. We will show that there are constants $c_1, c_2, c_3 \in (0,\infty)$ such that

$$P(T(x) \leq c_1|x|) \leq c_2 e^{-c_3|x|}. \tag{14}$$

The proposition follows from this and the fact that

$$\lim_{n\to\infty}\frac{E[T(nx)]}{n} = \mu(x), \tag{15}$$

which is an easy consequence of Lemma 4.2. Substituting x by nx in (14) yields

$$P\left(\frac{T(nx)}{n} \leq c_1|x|\right) \leq c_2 e^{-c_3 n|x|}$$

which implies that

$$\frac{E[T(nx)]}{n} \geq c_1\left(1 - c_2 e^{-c_3 n|x|}\right).$$

Using (15) this gives that $\mu(x) \geq c_1$.

To verify (15) note that by Lemma 4.2

$$0 \leq \tilde{T}(nx) - T(nx) \leq R_n,$$

where R_n is a sum of k_0 independent exponential variables with parameter λ. Thus

$$0 \leq \frac{E[\tilde{T}(nx)]}{n} - \frac{E[T(nx)]}{n} \leq \frac{k_0 \cdot \lambda}{n}$$

and it follows that

$$\lim_{n\to\infty} \frac{E[T(nx)]}{n} = \lim_{n\to\infty} \frac{E[\tilde{T}(nx)]}{n} = \mu(x).$$

To prove (14), partition $\mathbb{R}^d$ into cubes centered at the points $\alpha\mathbb{Z}^d$ and with vertices $(\alpha/2, \dots, \alpha/2) + \alpha\mathbb{Z}^d$. The side length α should be chosen small, say $\alpha \leq 0.1$. Let C_y, $y \in \alpha\mathbb{Z}^d$, denote the cube centered at y. We say that $(C_{x_1}, \dots, C_{x_k})$ is a *path* from the origin to the point x if $|x_1| < 1 + \alpha$, $|x_k - x| < 1 + \alpha$ and $|x_i - x_{i-1}| < 1 + \alpha$ for all $i = 2, \dots, k$. The number of cubes in a path will be referred to as the *length* of the path. Finally we call a path $(C_{x_1}, \dots, C_{x_k})$ *open* at time t if there exist a sequence of outburst points $x'_1 \dots x'_k$, $x'_i \in C_{x_i}$, such that the times of the outbursts $\{T_{x'_i}\}$ form an increasing sequence with $T_{x'_k} < t$. In words, a path is open if an outburst has occurred in each of its cubes in the "correct" time order, that is, first an outburst took place in the first cube, then an outburst took place in the second cube, and so on.

To see the use of these concepts, note that if x is infected at time t then there is a chain of outburst points $x'_1, \dots, x'_k$ within distance one from each other, $x'_1 \in B(0,1)$ and $x'_k \in B(x,1)$, and with $T_{x'_{i-1}} < T_{x'_i}$ and $T_{x'_k} < t$. This implies that there is an open path to x at time t. Hence if we let

$$F = \{\exists \text{ an open path to } x \text{ at time } a|x|\},$$

where a is a real number that will be specified later, then

$$P(T(x) \leq a|x|) \leq P(F).$$

Furthermore, if

$$F_k = \{\exists \text{ an open path of length } k \text{ to } x \text{ at time } a|x|\},$$

then

$$P(F) \leq \sum_{k=\lfloor |x| \rfloor}^{\infty} P(F_k).$$

The sum starts at $\lfloor |x| \rfloor$ since a path must have length at least $\lfloor |x| \rfloor$ to reach x.

Next, consider a given path $(C_{x_1}, \dots, C_{x_k})$ from the origin to x. Define x_0 to be the origin and let $x_{k+1} = x$. For $i = 1, \dots, k+1$, let z'_i denote the center of the line segment from x_{i-1} to x_i and chose $z_i \in \alpha\mathbb{Z}^d$ so that $|z'_i - z_i| \leq \alpha$. Merge the two sequences $\{x_i\}$ and $\{z_i\}$ into a new sequence $\{y_i\}$ by picking every second element from the sequence $\{z_i\}$ and every second element from $\{x_i\}$ starting at $i = 1$ with $y_1 = z_1$. Formally

$$y_i = \begin{cases} z_{(i+1)/2} & \text{if } i \text{ is odd,} \\ x_{i/2} & \text{if } i \text{ is even.} \end{cases}$$

By choice of the points $\{z_i\}$ two consecutive points in the sequence $\{y_i\}$ lie within distance $(1 + 3\alpha)/2$ from each other. To each cube C_{y_i} we associate a random variable E_i defined as follows: Let E_0 be the time from time zero until an outburst occurs in C_{y_1}. Since C_{y_1} is contained in the unit ball around the origin it is infected at time zero and hence $E_0 \sim \text{Exp}(\alpha^d)$, where α^d is the volume of C_{y_1}. For $i = 2, \ldots, 2k+1$, let E_i be the time from time $E_1 + \ldots + E_{i-1}$ until an outburst occurs in C_{y_i}. Since $|y_i - y_{i-1}| < (1+3\alpha)/2$ and $\alpha \leq 0.1$, the cube C_{y_i} is contained in the infected area at time $E_1 + \ldots + E_{i-1}$ implying that $E_i \sim \text{Exp}(\alpha^d)$. Note that $x \in S_{E_1+\ldots+E_{2k+1}}$. Thus the time it takes for the infection to reach x using the path $(C_{x_1}, \ldots, C_{x_k})$ can be dominated by $\sum_{i=1}^{2k+1} E_i$. Let us introduce the notation

$$\Gamma_{2k+1} := \sum_{i=1}^{2k+1} E_i.$$

Now, for each k the probability that there exists an open path of length k to x at time $a|x|$ is dominated by the expected number of such paths, that is,

$$P(F_k) \leq \beta_k \cdot P(\Gamma_{2k+1} \leq a|x|)$$

where β_k is the number of paths of length k to x. Let $b = b(d)$ be any number that dominates the number of α-cubes contained in a ball of radius $1 + 2\alpha$ in $\mathbb{R}^d$. Then clearly $\beta_k \leq b^k$.

To sum up, we have deduced that

$$P(T(x) \leq a|x|) \leq \sum_{k=\lfloor |x| \rfloor}^{\infty} b^k \cdot P(\Gamma_{2k+1} \leq a|x|) \tag{16}$$

and what remains is to find an upper bound for $P(\Gamma_{2k+1} \leq a|x|)$. To this end, note that if k is such that $2k + 1 \geq |x|$ then trivially

$$P(\Gamma_{2k+1} \leq a|x|) \leq P(\Gamma_{2k+1} \leq a(2k+1)).$$

By Markov's inequality

$$e^{-\theta a(2k+1)} P(\Gamma_{2k+1} \leq a(2k+1)) \leq E[e^{-\theta \Gamma_{2k+1}}]$$

for $\theta > 0$, and since $\Gamma_{2k+1} \sim \text{Gamma}(2k+1, \alpha^d)$ we have $E[e^{-\theta\Gamma_{2k+1}}] = (1 + \theta\alpha^d)^{-(2k+1)}$. Thus

$$P(\Gamma_{2k+1} \leq a(2k+1)) \leq e^{-(2k+1) f_a(\theta)},$$

where $f_a(\theta) = \log(1+\theta\alpha^d) - a\theta$. For fixed a, the best choice of θ is $\theta_a = a^{-1} - \alpha^{-d}$ and in this case

$$f_a(\theta_a) = \log(a^{-1}\alpha^d) - 1 + a\alpha^{-d}.$$

Thus we have arrived at the estimate

$$P(\Gamma_{2k+1} \leq a(2k+1)) \leq e^{-(2k+1)f_a(\theta_a)}$$

valid for all k such that $2k+1 \geq |x|$. In particular the estimate is valid for $k \geq \lfloor |x| \rfloor$. Substituting it into (16) yields

$$\begin{aligned} P(T(x) \leq a|x|) &\leq \sum_{k=\lfloor |x| \rfloor}^{\infty} b^k \cdot e^{-(2k+1)f_a(\theta_a)} \\ &= e^{-f_a(\theta_a)} \sum_{k=\lfloor |x| \rfloor}^{\infty} e^{-k(2f_a(\theta_a)-\log b)}. \end{aligned} \tag{17}$$

Since $f_a(\theta_a) \to \infty$ as $a \to 0$ we can chose a small so that $2f_a(\theta_a) - \log b > 0$. For such an a the sum in (17) converges and thereby (14) is established. □

The next step is to prove that the discrete limits in Proposition 5.1 and Proposition 5.2 can be replaced by continuous ones. We know by now that the sequences $\{T(nx)/n\}$ and $\{\tilde{T}(nx)/n\}$ converge to $\mu(x) \in (0,\infty)$, that is, the inverse speed obtained by observing the growth at discrete points located $|x|$ units apart along a straight line through the origin and x is asymptotically equal to $\mu(x)$. The following proposition asserts that the same asymptotic speed is obtained if we move away from the origin in a continuous fashion, observing the growth along the entire line.

Proposition 5.4 *For each $x \in \mathbb{R}$ we have*

(a) $\lim_{t\to\infty} T(tx)/t = \mu(x)$;

(b) $\lim_{t\to\infty} \tilde{T}(tx)/t = \mu(x)$;

where the limits are taken along $t \in \mathbb{R}^+$.

Proof: Fix $x \in \mathbb{R}^d$. To prove (a) we start by showing that

$$\lim_{n\to\infty} \frac{T(ncx)}{nc} = \mu(x) \quad \text{for all } c \in \mathbb{R}, \tag{18}$$

that is, moving away from the origin in direction x using steps of arbitrary length yields the same limit. To this end, introduce the notation

$$\bar{T}_{nc} = \frac{T(ncx)}{nc}.$$

First assume that $c \in \mathbb{Q}$. In this case we have $c = k/m$ for some integers k and m. The sequence $\{\bar{T}_{nk}\}$ is a subsequence of $\{\bar{T}_n\}$ and hence

$$\lim_{n\to\infty} \bar{T}_{nk} = \lim_{n\to\infty} \bar{T}_n = \mu(x) \quad \text{a.s.}$$

However, $\{\bar{T}_{nk}\}$ is also a subsequence of $\{\bar{T}_{nc}\}$ – obtained by considering only those points where n is a multiple of m – implying that the sequence $\{\bar{T}_{nc}\}$ must tend to the same limit as $\{\bar{T}_{nk}\}$, i.e.

$$\lim_{n\to\infty} \bar{T}_{nc} = \lim_{n\to\infty} \bar{T}_{nk} = \mu(x) \quad \text{a.s.}$$

Now assume that $c \in \mathbb{R}\backslash\mathbb{Q}$. Let $\delta > 0$ be small and pick $q_\delta \in \mathbb{Q}$ such that $0 < c - q_\delta < \delta$. The aim is to show that $\lim_{n\to\infty} \bar{T}_{nc} = \mu(x)$ and as we have just verified that $\lim_{n\to\infty} \bar{T}_{nq} = \mu(x)$ for all $q \in \mathbb{Q}$ it suffices to prove that

$$\limsup_{\delta\to 0} \limsup_{n\to\infty} |\bar{T}_{nc} - \bar{T}_{nq_\delta}| = 0 \quad \text{a.s.}$$

Since $q_\delta < c$, trivially $|\bar{T}_{nc} - \bar{T}_{nq_\delta}| \leq |T(ncx) - T(nq_\delta x)|(nc)^{-1}$ and, since $|ncx - nq_\delta x| \leq n\delta|x|$, by Lemma 4.1

$$|T(ncx) - T(nq_\delta x)| \leq \sum_{k=1}^{2\lceil n\delta|x|\rceil} E_k, \tag{19}$$

where $\{E_k\}$ are iid exponential random variables with parameter λ. Thus it is enough to show that

$$\limsup_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{nc} \sum_{k=1}^{2\lceil n\delta|x|\rceil} E_k = 0 \quad \text{a.s.} \tag{20}$$

To achieve this, write

$$\frac{1}{nc} \sum_{k=1}^{2\lceil n\delta|x|\rceil} E_k = \frac{2\lceil n\delta|x|\rceil}{nc} \cdot \frac{1}{2\lceil n\delta|x|\rceil} \sum_{k=1}^{2\lceil n\delta|x|\rceil} E_k.$$

Trivially

$$\frac{2\lceil n\delta|x|\rceil}{nc} \longrightarrow \frac{2\delta|x|}{c} \quad \text{as } n \to \infty$$

and by the strong law of large numbers

$$\frac{1}{2\lceil n\delta|x|\rceil} \sum_{k=1}^{2\lceil n\delta|x|\rceil} E_k \longrightarrow \frac{1}{\lambda} \quad \text{a.s. as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} \frac{1}{nc} \sum_{k=1}^{2\lceil n\delta|x|\rceil} E_k = \delta \cdot \frac{2|x|}{c\lambda}.$$

Letting $\delta \to 0$ we obtain (20) and thereby (18) is established.

To complete the proof of part (a) we use (18) to show that

$$\limsup_{t\to\infty} \frac{T(tx)}{t} \le \mu(x) \le \liminf_{t\to\infty} \frac{T(tx)}{t}. \tag{21}$$

As for the first inequality in (21), let $\varphi_{nc}(x)$ denote the time from the point ncx is infected until the entire line segment between ncx and $(n+1)cx$ is infected. At time $T(ncx)$ we can find a point z such that $|z-ncx| \le 1/2$ and $B(z, 0.1) \subset S_{T(ncx)}$. Furthermore, if c is chosen so that $c|x|$ is small enough, then the entire line segment between ncx and $(n+1)cx$ will be contained in the infected area as soon as an outburst has occurred in $B(z, 0.1)$. Thus, for small c we have $\varphi_{nc}(x) \le E_n$, where $E_n \sim \mathrm{Exp}(\lambda)$ and λ is the volume of $B(z, 0.1)$. Since

$$\sum_{n=1}^{\infty} P(\varphi_{nc}(x) > n\varepsilon) \le \sum_{n=1}^{\infty} e^{-\lambda nc} < \infty$$

the Borel-Cantelli lemma gives

$$\lim_{n\to\infty} \frac{\varphi_{nc}(x)}{n} = 0 \quad \text{a.s.} \tag{22}$$

Now, for $t \in \mathbb{R}$, let n_t be such that $t \in [n_t c, (n_t+1)c)$. Clearly $T(tx) \le T(n_t cx) + \varphi_{n_t c}(x)$. Hence

$$\begin{aligned}\limsup_{t\to\infty} \frac{T(tx)}{t} &\le \limsup_{t\to\infty} \frac{T(n_t cx) + \varphi_{n_t c}(x)}{n_t c} \\ &= \lim_{n\to\infty} \frac{T(ncx) + \varphi_{nc}(x)}{nc} \\ &= \mu(x),\end{aligned} \tag{23}$$

where the last equality follows from (18) and (22).

To prove the second inequality in (21), let $\psi_{nc}(x)$ denote the time from the first point that is infected on the line segment between ncx and $(n+1)cx$ is infected until the entire segment is infected. As for $\varphi_{nc}(x)$ it can be seen that $\psi_{nc}(x) \le E_n$, where $E_n \sim \mathrm{Exp}(\lambda)$ and λ is the volume of a ball with radius 0.1. Thus, using the Borel-Cantelli lemma we obtain

$$\lim_{n\to\infty} \frac{\psi_{nc}(x)}{n} = 0 \quad \text{a.s.} \tag{24}$$

Since $T(tx) + \psi_{n_t c}(x) \ge T(n_t cx)$ we have

$$\begin{aligned}
\liminf_{t\to\infty} \frac{T(tx)}{t} &\geq \liminf_{t\to\infty} \frac{T(n_t cx) - \psi_{n_t c}(x)}{(n_t+1)} \\
&= \lim_{n\to\infty} \frac{T(ncx) - \psi_{nc}(x)}{(n+1)c} \qquad (25)\\
&= \mu(x),
\end{aligned}$$

where the last equality is a consequence of (18) and (24). Thereby part (a) is established.

Part (b) is proved along the same lines as part (a). First we prove that

$$\lim_{n\to\infty} \frac{\tilde{T}(ncx)}{nc} = \mu(x) \quad \text{for all } c \in \mathbb{R}, \qquad (26)$$

and then we use this to show that

$$\limsup_{t\to\infty} \frac{\tilde{T}(tx)}{t} \leq \mu(x) \leq \liminf_{t\to\infty} \frac{\tilde{T}(tx)}{t}. \qquad (27)$$

The proof of (26) can be copied more or less word by word from the proof of (18). However, instead of the bound for $|T(ncx) - T(nq_\delta x)|$ in (19) we need a bound for $|\tilde{T}(ncx) - \tilde{T}(nq_\delta x)|$. Clearly

$$\begin{aligned}
|\tilde{T}(ncx) - \tilde{T}(nq_\delta x)| &\leq |\tilde{T}(ncx) - T(ncx)| + |T(ncx) - T(nq_\delta x)| \\
&+ |\tilde{T}(nq_\delta x) - T(nq_\delta x)|.
\end{aligned}$$

By Lemma 4.2 the first and last term can be dominated by sums of k_0 iid exponential random variables with parameter λ_1 and by Lemma 4.1 the second term can be dominated by a sum of $2\lceil n\delta|x|\rceil$ iid exponential variables with parameter λ_2. Furthermore, by consulting the proofs of Lemma 4.1 and Lemma 4.2 it can be seen that the variables in the different sums can be constructed so that they are independent and $\lambda_1 = \lambda_2 = \lambda$. Thus we arrive at the bound

$$|\tilde{T}(ncx) - \tilde{T}(nq_\delta x)| \leq \sum_{k=1}^{2(\lceil n\delta|x|\rceil + k_0)} E_k,$$

where $\{E_k\}$ are iid exponential random variables with parameter λ.

To prove (27) we will need a bound for the time from the unit ball around an arbitrary point on the line segment between ncx and $(n+1)cx$ is infected until the unit balls of all points on the line segment are infected. To obtain such a bound, let l_{ncx} denote the line segment between ncx and $(n+1)cx$ and write $\tilde{T}(l_{ncx})$ for the time when all points on l_{ncx} has their unit balls infected, i.e.

$$\tilde{T}(l_{ncx}) = \inf\{t;\ B(z,1) \subset S_t \text{ for all } z \in l_{ncx}\}.$$

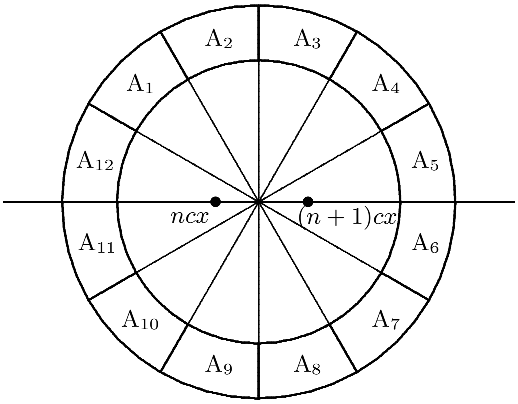

Figure 3: The front zone F of $B(z_0, 1)$ divided into pieces $A_1, \ldots, A_{12}$.

Assume that at time t_0 there is a point $z_0 \in l_{ncx}$ such that $B(z_0, 1) \subset S_{t_0}$. For small c we will derive an upper bound for $|\tilde{T}(l_{ncx}) - t_0|$ expressed as the maximum of a number of iid exponential random variables. In two dimensions such a bound is easily obtained using a geometric construction similar to the one employed in the proof of Lemma 4.2. Namely, let F be a front stripe of width 0.1 in $B(z_0, 1)$, that is, $F = B(z_0, 1) \backslash B(z_0, 0.9)$. Divide F into twelve disjoint pieces $A_1, \ldots, A_{12}$ of equal area as shown in Figure 3 and let E_k $(k = 1, \ldots, 12)$ be the time from time t_0 until an outburst occurs in A_k. By construction the areas $A_1, \ldots, A_{12}$ are all infected at time t_0, implying that $E_k \sim \text{Exp}(\lambda)$, where $\lambda = |A_k|$. Furthermore, since the A_k:s are disjoint, the variables $\{E_k\}$ are independent. Now, if c is small so that $c|x|$ is small enough – say $c|x| \leq 0.05$ – then all points on l_{ncx} must have their unit balls infected by the time an outburst has occurred in each of the areas A_k $(k = 1, \ldots, 12)$. Hence $B(z, 1) \subset S_{t_0 + \max\{E_1, \ldots, E_{12}\}}$ for all $z \in l_{ncx}$ and we have proved that $|\tilde{T}(l_{ncx}) - t_0| \leq \max\{E_1, \ldots, E_{12}\}$ for all times t_0 such that there exists a point on l_{ncx} whose entire unit ball is infected at time t_0. If we allow the parameter λ and the number of variables in the maximum to depend on d the above reasoning easily generalizes to $d \geq 3$. We obtain

$$|\tilde{T}(l_{ncx}) - t_0| \leq \max\{E_1, \ldots, E_{k_0}\}. \tag{28}$$

Now, to prove the first inequality in (27) let c be small enough to ensure (28) and let $\tilde{\varphi}_{nc}(x)$ be the time when the unit balls of all points on l_{ncx} are infected counting from the time when the unit ball around ncx is infected, that is, $\tilde{\varphi}_{nc}(x) = \tilde{T}(l_{ncx}) - \tilde{T}(ncx)$. By (28) there exist iid random variables $E_1, \ldots, E_{k_0}$, $E_k \sim \text{Exp}(\lambda)$, such that

$$\tilde{\varphi}_{nc}(x) \leq \max\{E_1, \ldots, E_{k_0}\} := R_n. \tag{29}$$

Since

$$P(R_n \le x) = P(E_k \le x,\ k = 1, \ldots, 12) = (1 - e^{-\lambda x})^{k_0}$$

it follows that

$$P(R_n > x) \le a e^{-\lambda x}$$

for some constant $a \in \mathbb{R}$ and therefore, for all $\varepsilon > 0$,

$$\sum_{n=0}^{\infty} P(R_n > n\varepsilon) \le \sum_{n=0}^{\infty} a e^{-\lambda n \varepsilon} < \infty.$$

Thus, by (29) and the Borel-Cantelli lemma,

$$\lim_{n\to\infty} \frac{\tilde{\varphi}_{nc}(x)}{n} = 0 \quad \text{a.s.} \tag{30}$$

Let n_t be such that $t \in [n_t c, (n_t+1)c)$. Since clearly $\tilde{T}(tx) \le \tilde{T}(n_t cx) + \tilde{\varphi}_{nc}(x)$ the first part of (27) follows from (26) and (30) using the same reasoning as in (23).

To prove the last inequality in (27), let z_{ncx} be the first point on l_{ncx} whose unit ball is infected and let $\tilde{\psi}_{nc}(x)$ be the time from this occurs until the infection has invaded the unit balls of all points on l_{ncx}, that is, $\tilde{\psi}_{nc}(x) = \tilde{T}(l_{ncx}) - \tilde{T}(z_{ncx})$. As for $\tilde{\varphi}_{nc}(x)$ it can be seen that

$$\lim_{n\to\infty} \frac{\tilde{\varphi}_{nc}(x)}{n} = 0 \quad \text{a.s.} \tag{31}$$

Using the fact that $\tilde{T}(tx) + \tilde{\psi}_{n_t c}(x) \ge \tilde{T}_{n_t cx}$ and the results in (26) and (31) the last part of (27) is obtained as in (25). Thereby part (b) is established. □

It follows from Proposition 5.4 that $\mu(cx) = c\mu(x)$. This implies that $\mu(x) = |x|\mu(\hat{x})$, where $\hat{x}$ is the unit vector in direction x, that is, $\hat{x} = x/|x|$. Due to rotational invariance of $\mathbb{R}^d$ and the model it is clear that $\mu(\hat{x}) = \mu(\hat{y})$ for all $x, y \in \mathbb{R}^d$. Thus we can define a constant

$$\mu := \mu((1, 0, \ldots, 0))$$

representing the asymptotic time it takes for the infection to travel a unit vector in arbitrary direction. By Proposition 5.3, $\mu \in (0, \infty)$. We end up with the simple relation

$$\mu(x) = |x|\mu$$

valid for all $x \in \mathbb{R}^d$.

To summarize the results obtained in the present section, we have deduced that there is a real number $\mu > 0$ such that for each fixed $x \in \mathbb{R}^d$ almost surely

$$\lim_{t\to\infty} \frac{\tilde{T}(tx)}{t} = \lim_{t\to\infty} \frac{T(tx)}{t} = |x|\mu. \tag{32}$$

6 Proof of the shape theorem

The shape theorem asserts that $S_t \approx tB(0,\mu^{-1})$ for large t. To see that this – in view of (32) – is indeed what to expect, consider a point $x \in B(0,\mu^{-1})$. That x belongs to $B(0,\mu^{-1})$ is the same as to say that $|x| \leq \mu^{-1}$ or equivalently $|x|\mu \leq 1$. For such an x we have, by (32),

$$\lim_{t\to\infty} \frac{T(tx)}{t} \leq 1 \quad \text{a.s.}$$

This means that almost surely $T(tx) \leq t$ for large t, which implies that almost surely $tx \in S_t$ for large t. This is the intuition to keep in mind when pondering upon the shape theorem. However, the almost sure convergence in (32) holds only for a fixed x and the shape theorem is a statement concerning uncountably many x. Thus, some work remains before the theorem is fully proved: We have to make sure that the above intuition is valid also when all $x \in B(0,\mu^{-1})$ are considered simultaneously. To this end we need the following lemma, which asserts that with high probability the infected region in a process emanating from a point y will eventually contain a ball centered at y with radius proportional to time.

Lemma 6.1 *For small $\delta > 0$ there is a constant $c \in (0,\infty)$ and a time s_0 such that for any $y \in \mathbb{R}^d$ and $s' \geq 0$ we have*

$$P(B(y,s\delta) \not\subset S_{s'+s}^{(y,s')}) \leq e^{-cs}$$

if $s > s_0$.

Proof of Lemma 6.1: Due to shift invariance of the model it suffices to prove the lemma for $y = 0$ and $s' = 0$, that is, it suffices to show that

$$P(B(0,s\delta) \not\subset S_s) \leq e^{-cs} \tag{33}$$

for large s. To achieve this, partition $\mathbb{R}^d$ into cubes centered at the points $\alpha\mathbb{Z}^d$ and with vertices $(\alpha/2,\ldots,\alpha/2)+\alpha\mathbb{Z}^d$. Let $\hat{T}_\alpha(x)$, $x \in \alpha\mathbb{Z}^d$, denote the time when the entire cube centered at x is infected and let a be an arbitrary positive constant bounded away from zero, say $a \geq 1/2$. For small α and δ we will derive the estimate

$$P(\hat{T}_\alpha(x) > a|x|/\delta) \leq e^{-c'a|x|}, \tag{34}$$

where c' is a positive constant. Given this estimate the lemma is readily established: Let $C(s\delta+\alpha)$ denote the cube with side length $2(s\delta+\alpha)$ centered at the origin and write $C_\alpha(s\delta+\alpha) = \alpha\mathbb{Z}^d \cap C(s\delta+\alpha)$. Note that $B(0,s\delta)$ is contained in the union of all α-cubes whose center is in $C_\alpha(s\delta+\alpha)$. Thus, if $B(0,s\delta) \not\subset S_s$ then all α-cubes with center in $C_\alpha(s\delta+\alpha)$ can not be infected at time s, that is,

$$\begin{aligned} P(B(0,s\delta) \not\subset S_s) &\leq P\left(\bigcup_{x \in C_\alpha(s\delta+\alpha)} \{\hat{T}_\alpha(x) > s\} \right) \\ &\leq \sum_{x \in C_\alpha(s\delta+\alpha)} P(\hat{T}_\alpha(x) > s). \end{aligned}$$

Trivially

$$P(\hat{T}_\alpha(x) > s) = P\left(\hat{T}_\alpha(x) > \frac{s\delta}{|x|} \cdot |x|/\delta\right).$$

For $x \in C_\alpha(s\delta+\alpha)$ we have $s\delta/|x| \geq s\delta/(s\delta+\alpha)$ and since $s\delta/(s\delta+\alpha) \to 1$ as $s \to \infty$ there is s_1 such that $s\delta/|x| \geq 1/2$ for $s > s_1$. Hence, if $s > s_1$ and $x \in C_\alpha(s\delta+\alpha)$ it follows from (34) that

$$P\left(\hat{T}_\alpha(x) > \frac{s\delta}{|x|} \cdot |x|/\delta\right) \leq e^{-c's\delta}$$

and, consequently, for $s > s_1$ we have

$$P(B(0,s\delta) \not\subset S_s) \leq \sum_{x \in C_\alpha(s\delta+\alpha)} e^{-c's\delta}.$$

Furthermore, the number of lattice points contained in $C(s\delta+\alpha)$ equals $(2\lfloor s\delta\alpha^{-1}\rfloor + 3)^2$ and thus

$$\begin{aligned} \sum_{x \in C_\alpha(s\delta+\alpha)} e^{-c's\delta} &\leq (2s\delta\alpha^{-1} + 3)^2 e^{-c's\delta} \\ &= e^{-s(c'\delta - g(s))}, \end{aligned}$$

where $g(s) = 2\log(2s\delta\alpha^{-1}+3)/s$. Since $g(s) \to 0$ as $s \to \infty$ there is s_2 such that $g(s) \leq c'\delta/2$ for $s > s_2$. Define $s_0 = \max\{s_1, s_2\}$. Then, for $s > s_0$ we have

$$P(B(0,s\delta) \not\subset S_s) \leq e^{-sc'\delta/2}$$

as desired.

It remains to prove (34). Fix $x \in \alpha\mathbb{Z}^d$. By Lemma 4.1 and the remark following its proof, if α is small - say $\alpha \leq 0.1$ - we have

$$\hat{T}_\alpha(x) \leq \sum_{k=1}^{2\lceil |x| \rceil} E_k,$$

where $\{E_k\}$ are iid exponential variables with parameter λ. Thus it suffices to find $c' > 0$ such that

$$P\left(\sum_{k=1}^{2\lceil |x|\rceil} E_k > a|x|/\delta\right) \le e^{-c'a|x|}. \tag{35}$$

To this end, write $\lceil |x|\rceil = m$ and introduce the notation

$$\Gamma_{2m} := \sum_{k=1}^{2m} E_k.$$

Using Markov's inequality and the fact that $\Gamma_{2m} \sim \text{Gamma}(2m, \lambda)$ we obtain

$$e^{\theta a|x|\delta^{-1}} P\left(\Gamma_{2m} > a|x|\delta^{-1}\right) \le E[e^{\theta\Gamma_{2m}}] = (1-\lambda\theta)^{-2m}$$

for $\theta \in (0, \lambda^{-1})$. Thus

$$P\left(\Gamma_{2m} > a|x|\delta^{-1}\right) \le \exp\left\{-a|x|\left(\theta\delta^{-1} + \frac{2m}{a|x|}\log(1-\theta\lambda)\right)\right\}. \tag{36}$$

We may assume that $|x| > 1-\alpha$, since for $x \in B(0, 1-\alpha)$ the α-cube centered at x is contained in $B(0,1)$, implying that the left hand side in (34) equals zero and hence (34) is trivially true in this case. For $|x| > 1-\alpha$ the quotient $m/|x| = \lceil |x|\rceil/|x|$ is bounded by 2. Substituting this in (36) and also using the fact that $a \ge 1/2$ yields

$$P\left(\Gamma_{2m} > a|x|\delta^{-1}\right) \le e^{-a|x|f_\delta(\theta)},$$

where

$$f_\delta(\theta) = \theta\delta^{-1} + 8\log(1-\lambda\theta).$$

Now, $f_\delta(0) = 0$ and $f'_\delta(0) = \delta^{-1} - 8\lambda$. Thus, if δ is small so that $f'_\delta(0) > 0$, then we can pick θ small and get $f_\delta(\theta) > 0$. This proves (35). □

Finally – equipped with the above lemma and the results from Section 4 – we are ready to prove the shape theorem.

Proof of Theorem 1.1: Fix $\varepsilon \in (0, \mu^{-1})$. We will prove the theorem in two steps:

(i) There is almost surely a time T_1 such that $(1-\varepsilon)tB(0,\mu^{-1}) \subset S_t$ for $t > T_1$.

(ii) There is almost surely a time T_2 such that $S_t \subset (1+\varepsilon)tB(0,\mu^{-1})$ for $t > T_2$.

As for (i) we will show

(i') There is almost surely a time T_1' such that $(1-\varepsilon/2)tB(0,\mu^{-1}) \subset S_t$ for $t > T_1'$, $t \in \mathbb{N}$.

From (i') it follows that $(1-\varepsilon/2)\lfloor t \rfloor B(0,\mu^{-1}) \subset S_{\lfloor t \rfloor}$ for $t > T_1'+1$ and since $S_{\lfloor t \rfloor} \subset S_t$ for all t we obtain

$$(1-\varepsilon/2)\lfloor t \rfloor B(0,\mu^{-1}) \subset S_t.$$

But for large t the ball with radius $(1-\varepsilon/2)\lfloor t \rfloor \mu^{-1}$ contains the ball with radius $(1-\varepsilon)t\mu^{-1}$: Since $\lfloor t \rfloor/t \uparrow 1$ as $t \to \infty$ there is t_0 such that $\lfloor t \rfloor/t \geq 1-\varepsilon/2$ for $t > t_0$. For $t > t_0$ we have $(1-\varepsilon/2)\lfloor t \rfloor/t \geq (1-\varepsilon)$, implying that $(1-\varepsilon/2)\lfloor t \rfloor \mu^{-1} \geq (1-\varepsilon)t\mu^{-1}$ and hence

$$(1-\varepsilon)tB(0,\mu^{-1}) \subset (1-\varepsilon/2)\lfloor t \rfloor B(0,\mu^{-1}).$$

Consequently $(1-\varepsilon)tB(0,\mu^{-1}) \subset S_t$ if $t > \max\{T_1'+1, t_0\}$ and hence (i) follows from (i').

To prove (i'), note that since $(1-\varepsilon/2)B(0,\mu^{-1})$ is compact there are points $x_1,\ldots,x_n \in (1-\varepsilon/2)B(0,\mu^{-1})$ such that

$$(1-\varepsilon/2)B(0,\mu^{-1}) \subset \bigcup_{i=1}^{n} B(x_i, \delta\varepsilon/4),$$

where $\delta > 0$ is chosen small enough to ensure Lemma 6.1. Clearly

$$(1-\varepsilon/2)tB(0,\mu^{-1}) \subset \bigcup_{i=1}^{n} B(tx_i, t\delta\varepsilon/4). \tag{37}$$

By (32), for each i almost surely $\lim_{t\to\infty} \tilde{T}(tx_i)/t = |x_i|\mu$ and since $|x_i| \leq (1-\varepsilon/2)\mu^{-1}$ we obtain $\lim_{t\to\infty} \tilde{T}(tx_i)/t \leq 1-\varepsilon/2$. This implies that almost surely $\tilde{T}(tx_i) \leq t(1-\varepsilon/4)$ for each i if t is large, that is,

$$B(tx_i, 1) \subset S_{t(1-\varepsilon/4)} \tag{38}$$

for large t. Furthermore, by Lemma 6.1,

$$\sum_{t\in\mathbb{N}} P\left(B(tx_i, t\delta\varepsilon/4) \not\subset S_t^{(tx_i, t(1-\varepsilon/4))}\right) \sim \sum_{t\in\mathbb{N}} e^{-ct\varepsilon/4} < \infty.$$

Thus, by the Borel-Cantelli lemma there are only finitely many integer times for which the cancelled inclusion above holds, that is, for large integer times we have almost surely

$$B(tx_i, t\delta\varepsilon/4) \subset S_t^{(tx_i, t(1-\varepsilon/4))}. \tag{39}$$

Now, for each i, let T_i be such that both (38) and (39) hold for $t > T_i$ and define $T_1' = \max\{T_i\}$. For $t > T_1'$ we have $B(tx_i, t\delta\varepsilon/4) \subset S_t$ for all i and

$t \in \mathbb{N}$. Using (37) this implies that $(1-\varepsilon/2)tB(0,\mu^{-1}) \subset S_t$ for all integer times larger than T_1', as desired.

Moving on to (ii), let

$$R = B(0, 2\mu^{-1}) \backslash (1+\varepsilon)B(0,\mu^{-1}).$$

We will show that almost surely $tR \cap S_t = \emptyset$ for large t, that is, the region tR does not contain any infected points when t is large. Since $(1+\varepsilon)tB(0,\mu^{-1})$ is surrounded by tR the assertion (ii) follows from this.

First, let δ be small enough to ensure Lemma 6.1 and pick $x_1, \dots, x_n \in R$ such that

$$R \subset \bigcup_{i=1}^{n} B(x_i, \delta\varepsilon/4). \tag{40}$$

Since $x_i \notin (1+\varepsilon)B(0,\mu^{-1})$ we have $|x_i| \geq (1+\varepsilon)\mu^{-1}$ and thus, by (32), $\lim_{t\to\infty} T(tx_i)/t \geq 1+\varepsilon$. Hence, for large t it holds that almost surely $T(tx_i)/t \geq t(1+\varepsilon/2)$ for each i, implying that

$$P\left(tx_i \in S_{t(1+\varepsilon/2)} \text{ for some } i = 1, \dots, n\right) = 0 \tag{41}$$

if t is large. The idea of the proof is that if tR contains infected points for large t, then with high probability some point tx_i will be infected within time $t\varepsilon/2$ and this conflicts with (41). To formalize this intuition, let

$$p = P\left(tR \cap S_t \neq \emptyset \text{ for arbitrarily large } t\right)$$

and assume for contradiction that $p > 0$. For fixed t write

$$E_t = \{t' \geq t;\ t'R \cap S_{t'} \neq \emptyset\}$$

and define

$$T_t = \begin{cases} \inf E_t & \text{if } E_t \neq \emptyset, \\ \infty & \text{if } E_t = \emptyset. \end{cases}$$

Note that $P(T_t < \infty) \geq p > 0$ for each t. Consequently we can condition on the event that $T_t < \infty$ and pick y_t uniformly on $T_tR \cap S_{T_t}$. Since $y_t \in S_{T_t}$, by Lemma 4.2 the unit ball around y_t is infected at time $T_t + E$, where E is a sum of a number of iid exponential variables. Using the fact that $P(E < T_t\varepsilon/4)$ tends to one as t becomes large we obtain that

$$P\left(B(y_t, 1) \subset S_{T_t(1+\varepsilon/4)} |\ T_t < \infty\right) \to 1 \quad \text{as } t \to \infty. \tag{42}$$

Furthermore, if t is large we have, by Lemma 6.1,

$$P\left(B(y_t, T_t\delta\varepsilon/4) \subset S_{T_t(1+\varepsilon/2)}^{(y_t, T_t(1+\varepsilon/4))} |\ T_t < \infty\right) \geq 1 - e^{-cT_t\varepsilon/4}$$

implying that

$$P\left(B(y_t, T_t\delta\varepsilon/4) \subset S_{T_t(1+\varepsilon/2)}^{(y_t, T_t(1+\varepsilon/4))} | \, T_t < \infty\right) \to 1 \quad \text{as } t \to \infty. \tag{43}$$

Combining (42) and (43) we obtain

$$P\left(B(y_t, T_t\delta\varepsilon/4) \subset S_{T_t(1+\varepsilon/2)} | \, T_t < \infty\right) \to 1 \quad \text{as } t \to \infty.$$

Now, by (40) T_tR is covered by the balls $B(T_tx_i, T_t\delta\varepsilon/4)$, that is, every point in T_tR is within distance $T_t\delta\varepsilon/4$ from some point T_tx_i. Hence, since $y_t \in T_tR$ we can find at least one point x_i such that $T_tx_i \in B(y_t, T_t\delta\varepsilon/4)$ and consequently

$$P\left(T_tx_i \in S_{T_t(1+\varepsilon/2)} \text{ for some } i = 1, \ldots, n | \, T_t < \infty\right) \to 1 \quad \text{as } t \to \infty.$$

Pick t large so that the above probability is greater than 1/2 and use the fact that $P(T_t < \infty) \geq p$ for each t to obtain

$$\begin{aligned} & P\left(\exists t' \geq t \text{ such that } t'x_i \in S_{t'(1+\varepsilon/2)} \text{ for some } i = 1, \ldots, n\right) \\ \geq \; & P\left(T_tx_i \in S_{T_t(1+\varepsilon/2)} \text{ for some } i = 1, \ldots, n | \, T_t < \infty\right) P\left(T_t < \infty\right) \\ > \; & p/2. \end{aligned}$$

This contradicts (41). Hence we must have $p = 0$, that is, almost surely $tR \cap S_t = \emptyset$ for large t, as desired.

At this point (i) and (ii) are established and all that remains is to note that for $t > \max\{T_1, T_2\}$ we have

$$(1-\varepsilon)tB(0, \mu^{-1}) \subset S_t \subset (1+\varepsilon)tB(0, \mu^{-1}).$$

The shape theorem is proved. □

7 Simulations

The pictures in this section are the result of a computer simulation of the model in two dimensions. Remember that the development of the infection is described by a sequence $\{X_n\} \subset \mathbb{R}^2$, specifying the outburst points, and a strictly increasing sequence $\{T_n\}$, specifying the time points of the outbursts. In Figure 4 the locations of all outbursts that have occurred up to time t are plotted on the scale $1/t$ for $t = 23$, $t = 35$ and $t = 75$, that is, in each picture the points $\{X_n/t\}$ are plotted for all n such that $T_n \leq t$. The area S_t/t is obtained as the union of all circles with radius $1/t$ centered at the outburst points. These circles are not included in the plots, since already in the first picture ($t = 23$) their radius (1/23) would barely exceed the size of the dots used to mark the outburst points. Thus the collections of points in Figure 4 are approximately equal to S_t/t. As can be seen the shape of this area approaches a circle as t grows.

Figure 4: Outburst points scaled by time for $t = 23$, $t = 35$ and $t = 75$.

In Figure 5 the distance from the origin to the point in the infected area that is located furthest away from the origin is plotted against time. Since S_t is approximately a ball with radius $t\mu^{-1}$ if t is large, the asymptotic slope in the plot gives an estimate of μ^{-1} in two dimensions. To help estimate the slope, the line $y = x$ is included in the plot. We obtain $\mu^{-1} \approx 1$.

Figure 5: Largest distance to the origin plotted against time.

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