Modeling VIX Futures and Pricing VIX Options in the Jump Diffusion Modeling

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Abstract

In this thesis, a closed-form solution for the price of options on VIX futures is derived by developing a term-structure model for VIX futures. We analyze the VIX futures by the Merton Jump Diffusion model and allow for stochastic interest rates in the model. The performance of the model is investigated based on the daily VIX futures prices from the Chicago Board Option Exchange (CBOE) data. Also, the model parameters are estimated and option prices are calculated based on the estimated values. The results imply that this model is appropriate for the analysis of VIX futures and is able to capture the empirical features of the VIX futures returns such as positive skewness, excess kurtosis and decreasing volatility for long-term expiration.

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1. Introduction

The Chicago Board Options Exchange (CBOE) introduced VIX futures and VIX options contracts for the first time in March 2004 and February 2006 respectively. Since 2004, the CBOE Futures Exchange has experienced a steady progress in trading VIX futures contracts. This growth is a consequence of accepting the volatility as a trading instrument and asset class by market participants. Currently, average daily volume for the VIX futures contracts is equivalent to the futures markets which have been around for decades. Since VIX futures and options are two derivatives having VIX as the underlying and the VIX index is also translated as the expected movement in the S&P 500 index over the next 30-day period, the price of VIX futures and options are based on the expected volatility of the S&P 500 over the 30 day period. As a result, Lin and Chang (2009) stated that pricing options on VIX futures is more appropriate than on VIX itself. VIX futures and options are exchange-traded derivatives and provide the opportunity to the investors to trade the volatility. Furthermore, they are considered as a useful tool to hedge the portfolio against future movements in volatility.

The VIX options offer the ability to hedge an equity portfolio better than other index options, even products that trade based on a portfolio's benchmark index directly. The VIX futures returns have some important empirical features such as excess kurtosis and positive skewness. Therefore, a proper model should be proposed to capture all these characteristics.

A large number of studies have been currently concentrated on VIX futures and options pricing. These studies can be divided into two different categories. In the first category, different models were developed for the VIX index in order to determine the price of VIX futures and options (Psychoyios, & Skiadopoulus, (2007); Dopoyet, Diagler, & Chen, (2011), Psychoyios, Dotsis, & Markellos, (2009 & 2010)). Also, some studies derived the price of VIX futures and options based on the model for instantaneous variance of S&P 500 Index , evaluating the VIX futures from the S&P 500 price dynamics, (Lin, (2007, Lu, Zhu, (2010); Zhang, Shu, & Brenner, (2010); Zhang and Zhu, (2006); Zhu and Zhang, (2007); Sepp, (2008)). In the study by Psychoyios, Dotsis, & Markellos, (2009), the VIX index is modelled by the mean-reverting logarithmic diffusion model with jump. They evaluated the performance based on the empirical study and conclude that the behavior of VIX can be properly modelled. Later on in 2010, they performed a comparison between the two continuous time diffusion and jump diffusion models and study the behavior of the models to capture the dynamics of implied volatility over time. Based on their empirical investigation, they concluded that adding jump is crucial to correctly capture the dynamics. As they expected, the model considering jump have a superior performance in predicting the price of the VIX futures.

In the second category, more efforts have been carried out to model the VIX futures considering their dynamics exogenously instead of focusing on the VIX itself (Huskaj and Nossman (2013), Lin (2013)). Huskaj and Nossman (2013) investigated the term-structure model for VIX futures. Their model was a one factor model where the VIX futures prices follow the Normal Inverse Gaussian process (NIG). They illustrated that this model leads to a better fit than by just assuming a Wiener process in the VIX futures dynamics.

In the present study, we develop a closed-form solution for the price of options on VIX futures by considering a term-structure model for VIX futures. We model the VIX futures by the Merton Jump

Diffusion model and allow for stochastic interest rates in the model. The performance of the model is investigated based on the daily VIX futures prices from the Chicago Board Option Exchange (CBOE) for the period March 2004 to December 2010. Also, the model parameters are estimated and option prices are calculated based on the estimated values. The results imply that this model is appropriate for the analysis of VIX futures and is able to capture the empirical features of the VIX futures returns such as positive skewness, excess kurtosis and decreasing volatility for long-term expiration. In fact, the main purpose of this thesis is to find an analytic formula for option price. Moreover, to investigate the influence of adding jump to the diffusion model to capture the empirical characteristics of VIX futures returns. Indeed, we modeled the VIX futures instead of VIX itself as in previous literatures.

The rest of the thesis is organized as follows: in section 2, some of the concepts and theorems in finance and probability theory are provided. In section 3, the model and its assumptions are described. Also, the Heath-Jarrow-Morton drift condition is derived. In section 4, the theoretical results for option pricing are provided for both having stochastic and constant interest rate in the model. Finally in section 5 and 6 the empirical results and conclusion will be expressed respectively.

2. Concepts in Finance and Probability Theory

In this section, some of the definitions and theorems related to this thesis that will be used in the following sections are presented in their general forms.

VIX Index:

VIX is a symbol for the CBOE Market Volatility Index and is a measure for the volatility of S&P 500 index option. It represents the market's expectation of the movements in the S&P 500 over the next 30-day period. It is stated that there is an inverse relationship between the movement direction of the SPX index and the VIX index. VIX can be calculated theoretically by using a formula provided by the CBOE.

$$VIX_t^2 = \frac{2}{\tau}\sum_i \frac{\Delta K_i}{K_i^2}\, Q(K_i) - \frac{1}{\tau}\left(\frac{F_t(t+\tau)}{K_0} - 1\right)^2$$

Where,

$\tau = {}^{30}/_{365}$, $Q(K_i)$ is the price of the out-the-money S&P 500 index option with strike price K_i. K_0 stands for the highest exercise price less than the index forward price $F_t(t+\tau)$.

It should be noticed that VIX index is quoted as percentage rather than a dollar amount. [9]

VIX options:

A VIX option is an option using the CBOE Volatility as the underlying asset. This is the first exchange-traded option giving individual investors the ability to trade market volatility. [14]

Futures Contracts:

A futures contract with expiration date T, on VIX as underlying is a financial derivative with the following properties: [1]

(1) At every point of time t 0$\leq t \leq T$, there exists a quoted price $F(t; T, VIX)$ in the market, known as the futures price at t, for delivery at T.
(2) During any arbitrary time interval (s, t] the holder of the contract receives the amount $F(t; T, VIX) - F(s; T, VIX)$
(3) At any point of time t prior to delivery, the spot price of the futures contract is equal to zero.

Also, by a proposition presented in [1], if market prices are obtained from the fixed risk neutral martingale measure $\mathbb{Q}$. Then, the futures price process is given by:

$$F(t; T, VIX) = E_t^Q[VIX_T]$$

Note, futures prices are $\mathbb{Q}$-martingales.

The Likelihood Process:

The following definition can be found in [1].
Consider a filtered probability space (Ω, $\underline{\mathbb{F}}$, $\mathbb{P}$, $\mathbb{F}$) on a compact interval [0, T]. Suppose now L_T is some nonnegative integrable random variable in $\mathcal{F}_T$. Define a new measure $\mathbb{Q}$ on $\mathcal{F}_T$ by setting

$$d\mathbb{Q} = L_T \, d\mathbb{P} \qquad on\ \mathcal{F}_T$$

And if

$$E^P[L_T] = 1$$

the new measure will also be a probability measure. The likelihood process $\{L_t : 0 \leq t \leq T\}$ for the measure transformation from $\mathbb{P}$ to the new probability measure $\mathbb{Q}$ is defined as:

$$L_t = \frac{dQ}{dP}, \text{ on } \mathcal{F}_t$$

Where

L_t is a $\mathbb{P}$-martingale and $\mathbb{Q} \ll \mathbb{P}$.

Girsanov Theorem in the jump diffusion model:

The following theorem is stated in [3], Consider the filtered probability space (Ω, $\underline{\mathbb{F}}$, $\mathbb{P}$, $\mathbb{F}$) and assume that $N^1, \dots \dots N^k$ are optional counting process with predictable intensities $\lambda^1, \dots \dots \lambda^k$. Assume furthermore that $W^1, \ldots \ldots . W^d$ are standard independent $\mathbb{P}$-Wiener processes. Let $h^1, \ldots \ldots . h^k$ be predictable process with

$$h_t^i < -1, \quad i = 1, \ldots \ldots k, \qquad P - a.s,$$

And let $g^1, \ldots \ldots g^d$ be optional processes. The likelihood process L_t is defined as:

$$\begin{Bmatrix} dL_t = L_t \sum_{i=1}^{d} g_t^i \, dW_t^i + L_{t-} \sum_{j=1}^{k} h_t \{dN_t^i - \lambda_t^i dt\} \\ L_0 = 1 \end{Bmatrix} \tag{1}$$

Then,

$$dW_t^i = g_t^i dt + dW_t^{Q,i}, \quad i = 1, \ldots . . d \tag{2}$$

$$\lambda_t^{Q,i} = \lambda_t^i (1 + h_t^i), \qquad i = 1, \ldots \ldots . k \tag{3}$$

Where $W^{Q,1}, \ldots \ldots \ldots W^{Q,d}$ are $\mathbb{Q}$ -wiener processes and $\lambda_t^{Q,i}$ is the $\mathbb{Q}$-intensity of N^i.

3. VIX Futures Model

In the present section, first the VIX futures model is presented and it is followed by deriving the Heath-Jarrow-Morton drift condition.

3.1. VIX Futures Model

Consider a filtered probability space (Ω, $\underline{\mathbb{F}}$, $\mathbb{P}$, $\mathbb{F}$) that carries a 2-multidimensional standard Wiener process W_t consisting of two independent scalar Wiener process, and a Poisson process N_t (with constant intensity λ^P). The compensated Poisson process under $\mathbb{P}$, $\widetilde{N_t}$ is defined as $\widetilde{N}_t = N_t - \lambda^P t$ and is a P-martingale. Also, it is assumed, the model has stochastic interest rate.

The futures contracts are written on VIX with different maturities. The price of VIX futures at time t with maturity T is denoted by $F(t,T)$. Short rate is presented by $r(t)$= $f(t,t)$, where $f(t,T)$ is forward rate. Furthermore, the bond market is considered and we denote the price at time t of a zero coupon bond with expiration date T by $P(t,T)$. The relationship between forward rate and T-bond is defined as:

$$f(t,T) = -\frac{\partial}{\partial \mathrm{T}} \ln \mathrm{P}(t,T)$$

The money account is also expressed as $B(t)$= exp $(\int_0^t r_s ds)$. It is assumed that the market is free of arbitrage and for the money account as numeraire, the probability measure $\mathbb{Q}$ is a martingale measure.
The dynamics of a VIX futures contract with maturity T under the physical probability measure $\mathbb{P}$ is assumed to be:

$$\frac{dF(t,T)}{F(t_-,T)} = \alpha(\mathrm{t},\mathrm{T})dt + \sigma(t,T)dW_t + (y_t - 1)dN_t \tag{4}$$

Which can also be written as:

$$\frac{dF(t,T)}{F(t_{-},T)} = (\alpha(\mathrm{t},\mathrm{T}) + m\lambda^{P})dt + \sigma(t,T)dW_t + (y_t - 1)d\tilde{N}_t \tag{5}$$

Where

$m = E[(y_t - 1)]$ is mean of relative jump size. In fact, $(y_t - 1)$ is relative price jump size which is a log-normally distributed random variable.

$(y_t - 1) \sim i.i.d.\, log - normal(m, e^{2\mu+\delta^2}\left(e^{\delta^2} - 1\right))$

Also, $\sigma(t,T)$ is a 2-dimentional vector known to be a deterministic volatility of futures prices. $\alpha(\mathrm{t},\mathrm{T})$ is interpreted as deterministic mean rate of return of futures prices between jumps and $(\alpha(\mathrm{t},\mathrm{T}) + m\lambda^{P})$ is mean rate of return including jumps. Also, N_t, W_t, y_t are independent in the model.

Moreover, the dynamics of the short rate and the dynamics of the T-bond under the assumption of existing and non-existing jump in their $\mathbb{P}$-dynamics are assumed to be:

$$dr(t) = \alpha^{r}(t)dt + \sigma^{r}(t)dW_t \tag{6}$$

$$dr_t = \alpha^{r}(t)dt + \sigma^{r}(t)dW_t + (g_t - 1)d\tilde{N}_t \tag{7}$$

$$P(t,T) = \alpha^{p}(t,T)P(t,T)dt + \sigma^{p}(t,T)P(t,T)dW_t \tag{8}$$

$$P(t,T) = \alpha^{p}(t,T)P(t,T)dt + \sigma^{p}(t,T)P(t,T)dW_t + (H_t - 1)P(t_{-},T)d\tilde{N}_t \tag{9}$$

Where

$(g_t - 1)$ and $(H_t - 1)$ are relative price jump size for the short rate and T-bonds respectively. Also, $\alpha^{r}(t)$ in (6) and (7) is the drift term and $\sigma^{r}(t)$ is a 2-dimentional deterministic volatility vector of the short rate. $\alpha^{p}(t,T)$ and $\sigma^{p}(t,T)$ in the equation (8) are deterministic mean rate of return of T-bond prices and the 2-dimentional volatility vector of T-bond prices respectively. $\alpha^{p}(t,T)$ in (9) is deterministic total mean rate of return of T-bond prices.

3.2. Heath-Jarrow-Morton Drift Condition

In order to derive the HJM drift condition, transformation from the probability measure $\mathbb{P}$ to $\mathbb{Q}$ is performed. By inserting equation (2) into (4), compensating the Poisson process N under probability measure $\mathbb{Q}$ and using (3), the $\mathbb{Q}$ dynamics of the VIX futures price is defined as:

$$\frac{dF(t,T)}{F(t_{-},T)} = [\alpha(\mathrm{t},\mathrm{T}) + \sigma(t,T)\cdot g_t]dt + \sigma(t,T)dW_t^{Q} + (y_t - 1)(dN_t - (1 + h_t)\lambda^{P}dt) + (y_t - 1)(1 + h_t)\lambda^{P}dt = [\alpha(\mathrm{t},\mathrm{T}) + \sigma(t,T)\cdot g_t + m(1 + h_t)\lambda^{P}]dt + \sigma(t,T)dW_t^{Q} + (y_t - 1)d\tilde{N}_t^{Q}$$

Where $d\tilde{N}_t^{Q}$ is Martingale increment under $\mathbb{Q}$, g_t is 2-dimensional Girsanov Kernel and $(\,\cdot\,)$ is a symbol for the scalar product of the two vectors.

Since the futures price is a $\mathbb{Q}$-martingale, the drift term has to be equal to zero.

$$[\alpha(\mathrm{t},\mathrm{T}) + \sigma(t,T)\cdot g_t + m(1+h_t)\lambda^P] = 0$$

$$\alpha(\mathrm{t},\mathrm{T}) + m\lambda^P = -\sigma(t,T)\cdot g_t - mh_t\lambda^P$$

$$\alpha(\mathrm{t},\mathrm{T}) + m\lambda^P = \sigma(t,T)\cdot \varphi_t + m\gamma_t$$

Where φ_t denotes the 2-dimentional vector of market price of diffusion risk and γ_t denotes the market price of jump risk. Market price of diffusion risk and market price of jump risk are related to their Girsanov kernel g_t and h_t [3] as follows

$$g_t = -\varphi_t$$

$$h_t = -\frac{\gamma_t}{\lambda^P}$$

Hence, The HJM drift condition is:

$$\alpha(\mathrm{t},\mathrm{T}) + m\lambda^P = \sigma(t,T)\cdot \varphi_t + m\gamma_t$$

Therefore:

The $\mathbb{Q}$ -dynamics of a VIX futures contract with expiration T is:

$$\frac{dF(t,T)}{F(t_{-},T)} = \sigma(t,T)dW_t^Q + (y_t - 1)d\widetilde{N}_t^Q \tag{10}$$

Which based on the definition of $d\widetilde{N}_t^Q$, it can also be written as:

$$\frac{dF(t,T)}{F(t_{-},T)} = -\,\lambda^Q m\; dt + \sigma(t,T)dW_t^Q + (y_t - 1)dN_t \tag{11}$$

Where $m = E[(y_t - 1)] = e^{\mu + \frac{\delta^2}{2}} - 1$.

4. Option Pricing

In this section, the option price formula is derived for three cases. First, pricing options under the assumption of having stochastic interest rate without existing jump in its dynamics, second, stochastic interest rate with jump and the last case is pricing formula with constant interest rate.

4.1. Stochastic Interest Rate without Jump

Since the short rate is stochastic in the model; the T-forward measure is used to derive the option price formula. In fact, by changing the numeraire from the money account in the probability measure $\mathbb{Q}$ to the T-bond in $\mathbb{Q}^{\mathbb{T}}$, the $\mathbb{Q}^{\mathbb{T}}$ dynamics of VIX futures price is obtained. In order to have the $\mathbb{Q}^{\mathbb{T}}$ dynamics of VIX futures prices, the likelihood process is defined as:

$$L_t^T = \frac{P(t,T)}{P(0,T)B(T)},$$

$$L_t^T = \frac{d\mathbb{Q}^{\mathbb{T}}}{d\mathbb{Q}}, \text{ on } \mathcal{F}_t$$

The L_t^T-dynamic is obtained by applying the Ito formula to L_t^T and based on the assumption of not having jump in the short rate, the value of h_t in the equation (1) is equal to zero.

$$dL_t^T = \sigma^p(t,T)L_t^T dW_t^Q$$

$$dW_t^Q = \sigma^p(t,T)dt + dW_t^T$$

Therefore, by transforming from $\mathbb{Q}$ to $\mathbb{Q}^{\mathbb{T}}$ the intensity does not change ($\lambda^T = \lambda^Q(1 + h_t) = \lambda^Q$) and by applying the Girsanov theorem to the equation (11):

$$\frac{dF(t,T)}{F(t_{-},T)} = \sigma(t,T) \cdot (\sigma^p(t,T)dt + dW_t^T) - \lambda^Q m dt + (y_t - 1)dN_t =$$

$$(-\lambda^Q m + \sigma(t,T) \cdot \sigma^p(t,T))dt + \sigma(t,T)dW_t^T + (y_t - 1)dN_t$$

For simplicity in derivation, define the scalar product $\sigma(t,T) \cdot \sigma^p(t,T) = \alpha^F(t,T)$ and the intensity $\lambda^Q = \lambda^T = \lambda$.

Hence, the $\mathbb{Q}^{\mathbb{T}}$-dynamics of VIX futures price and its price formula are:

$$\frac{dF(t,T)}{F(t_{-},T)} = (-\lambda m + \alpha^F(t,T))dt + \sigma(t,T)dW_t^T + (y_t - 1)dN_t \quad (12)$$

$$F(T,T) = F(t,T)exp[\left(-\lambda m\tau + \int_t^T \alpha^F(s,T)ds - {}^1\!/_2 \int_t^T \| \sigma(s,T) \|^2 ds\right) + \int_t^T \sigma(s,T)dW_s^T + \sum_{k=0}^{N_\tau} Y_k] \quad (13)$$

Where

$Y_k = log(y_t) \sim i.i.d. N(\mu, \delta^2)$ and $\tau = T - t$.

The detailed derivation of the formula (13) is presented in the Appendix.

Theorem: The price at time t of a European call option with maturity date T and strike price K, written on the terminal futures price of futures contract $F(T,T)$ following jump diffusion model, at any time t$\leq T$ is given by:

$$\boldsymbol{C(t,T) = P(t,T)\sum_{j\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^j}{j!}\left[F(t,T)\, exp\left(-\lambda m\tau + \int_t^T \alpha^F(s,T)ds + j\mu + j\,\frac{\delta^2}{2}\right)\Phi(d_1) - K\Phi(d_2)\right]}$$

Where

$\Phi(.)$ is the cumulative distribution function of the standard normal distribution and

$$d_2 = \frac{\ln\left(\frac{F(t,T)}{K}\right) + \left[-\lambda m\tau + \int_t^T \alpha^F(s,T)ds - {}^1/_2 \int_t^T \|\sigma(s,T)\|^2 ds\right] + j\mu}{\sqrt{\int_t^T \|\sigma(s,T)\|^2 ds + j\delta^2}}$$

$$d_1 = d_2 + \sqrt{\int_t^T \parallel \sigma(s,T) \parallel^2 \, ds + j\delta^2}$$

Proof

An arbitrage-free price of a European call option with maturity T, written on the terminal futures price of a futures contract F (T,T) and strike price K at any time $t \leq T$ with information $\mathcal{F}_t$ is given by:

$$C(t,T) = P(t,T)E^T[\max(\mathrm{F}\,(\mathrm{T}\,,\mathrm{T}) - \mathrm{K}\,,0) |\, \mathcal{F}_t\,)] \tag{14}$$

By inserting the equation (13) into (14) and condition on the number of jumps as:

$$N_\tau = j \quad , \quad j = 0,1,2,\ldots$$

The equation (14) is expressed as:

$$C(t,T) = P(t,T)\, E^T[\max(\mathrm{F}\,(\mathrm{T}\,,\mathrm{T}) - \mathrm{K}\,,0) |\, \mathcal{F}_t\,)] = P(t,T)E^T\left[(F(T,T) - K)\, I_{F_{(T,T)>K}}\middle|\mathcal{F}_t\right] =$$
$$P(t,T)\, \sum_{j\geq 0} Q^T(N_\tau = j)\left\{E^T\left[F(t,T)exp\left[\left(-\lambda m\tau + \int_t^T \alpha^F(s,T)\, ds - {}^1/_2 \int_t^T \parallel \sigma(s,T) \parallel^2 \, ds\right) + \right.\right.\right.$$
$$\left.\left.\left.\int_t^T \sigma(s,T)dW_s^T + \sum_{k=1}^j Y_k\right] I_{F_{(T,T)>K}} |\, \mathcal{F}_t\,, N_\tau = j\,\right] - E^T\left[K I_{F_{(T,T)>K}} \,|\, \mathcal{F}_t\,,\; N_\tau = j\right]\right\} \tag{15}$$

Notice, $\int_t^T \sigma(s,T)dW_s^T$ is normally distributed with zero mean and variance $\int_t^T \parallel \sigma(s,T) \parallel^2 \, ds$ and $\sum_{k=0}^j Y_k \sim i.i.d.\, N(j\mu, j\delta^2)$. Hence,

X$=\int_t^T \sigma(s,T)dW_s^T + \sum_{k=0}^j Y_k \sim N(j\mu, \beta + j\delta^2)$ where $\beta = \int_t^T \parallel \sigma(s,T) \parallel^2 \, ds.$

Random variable X can also be presented as:

X$\overset{\mathrm{d}}{\equiv} j\mu + \sqrt{\beta + j\delta^2}Z$ where Z is standard normal distributed. (Z$\sim N(0,1)$)

In order to calculate the equation (15), each part of it, is computed separately. The first expectation in (15) is obtained as:

$E^T\left[F(t,T)exp\left[\left(-\lambda m\tau+\int_t^T \alpha^F(s,T)ds-{}^1\!/_2\int_t^T \parallel \sigma(s,T)\parallel^2 ds\right)+\int_t^T \sigma(s,T)dW_s^T+\right.\right.$
$\left.\left.\sum_{k=0}^{j} Y_k\right] I_{F_{(T,T)>K}}|\mathcal{F}_t\,,\ N_\tau=j\right]=E^T\left[F(t,T)exp\left[\left(-\lambda m\tau+\int_t^T \alpha^F(s,T)ds-{}^1\!/_2\int_t^T \parallel \sigma(s,T)\parallel^2 ds\right)+\right.\right.$
$\left.\left.X\right] I_{F_{(T,T)>K}}|\mathcal{F}_t\,,\ N_\tau=j\right]=$
$E^T\left[F(t,T)\exp\left[-\lambda m\tau+\int_t^T \alpha^F(s,T)ds-{}^1\!/_2\int_t^T \parallel \sigma(s,T)\parallel^2 ds+j\mu+\sqrt{\beta+j\delta^2}Z\right] I_{F_{(T,T)>K}}\middle|\mathcal{F}_t\,,\ N_\tau=j\right]=$
$F(t,T)\exp\left[-\lambda m\tau+\int_t^T \alpha^F(s,T)ds-{}^1\!/_2\int_t^T \parallel \sigma(s,T)\parallel^2 ds+j\mu\right]E^T\left[\exp\left(\sqrt{\beta+j\delta^2}Z\right) I_{F_{(T,T)>K}}\mid \mathcal{F}_t\,,\right.$
$\left.N_\tau=j\right]=F(t,T)\exp\left[-\lambda m\tau+\int_t^T \alpha^F(s,T)ds-{}^1\!/_2\int_t^T \parallel \sigma(s,T)\parallel^2 ds+j\mu\right]\int_{-d_2}^{\infty}(e^{\sqrt{\beta+j\delta^2}z}\,f(z)\,dz=$
$F(t,T)\exp\left[-\lambda m\tau+\int_t^T \alpha^F(s,T)ds-{}^1\!/_2\int_t^T \parallel \sigma(s,T)\parallel^2 ds+j\mu\right]\int_{-d_2}^{\infty}(e^{\sqrt{\beta+j\delta^2}z}\,\frac{1}{\sqrt{2\pi}}e^{-z^2/2})dz.$ (16)

Where $f(z)=\frac{1}{\sqrt{2\pi}}e^{-z^2/2}$ is the density function of a standard normally distributed variable Z. Also, in order to find the integration interval, the indicator function $I_{F_{(T,T)>K}}$ implies that we need to find the area that $F(T,T)>K$.

$[F(T,T)|N_\tau=j\,]>K$ Implies:

$\sqrt{\beta+j\delta^2}Z>\ln\left(\frac{K}{F(t,T)}\right)-[-\lambda m\tau+\int_t^T \alpha^F(s,T)ds-{}^1\!/_2\int_t^T \parallel \sigma(s,T)\parallel^2 ds+j\mu]$

$Z>\frac{\ln\left(\frac{K}{\mathrm{F(t,T)}}\right)-[-\lambda m\tau+\int_t^T \alpha^F(s,T)-{}^1\!/_2\int_t^T\|\sigma(s,T)\|^2 ds+j\mu]}{\sqrt{\beta+j\delta^2}}=-d_2$

Therefore $Z>-d_2$ where

$d_2=\frac{\ln\left(\frac{F(t,T)}{K}\right)+\left[-\lambda m\tau+\int_t^T \alpha^F(s,T)ds-{}^1\!/_2\int_t^T\|\sigma(s,T)\|^2 ds\right]+j\mu}{\sqrt{\beta+j\delta^2}}$

The integral in the formula (16) is obtained as:

$$\int_{-d_2}^{\infty}(e^{\sqrt{\beta+j\delta^2}z}\,\frac{1}{\sqrt{2\pi}}e^{-z^2/2})dz=\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty}e^{\sqrt{\beta+j\delta^2}z-\frac{z^2}{2}}\,dz=\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty}e^{\frac{-1}{2}[z^2-2\sqrt{\beta+j\delta^2}z\,]}\,dz$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty}e^{\frac{-1}{2}[(z-\sqrt{\beta+j\delta^2})^2-(\beta+j\delta^2)]}\,dz=\frac{1}{\sqrt{2\pi}}e^{1/2(\beta+j\delta^2)}\int_{-d_2}^{\infty}e^{\frac{-1}{2}\left[z-\sqrt{\beta+j\delta^2}\right]^2}\,dz$$
$$=e^{1/2(\beta+j\delta^2)}[1-\Phi\left(-d_2-\sqrt{\beta+j\delta^2}\right)]=e^{1/2(\beta+j\delta^2)}\Phi(d_1)$$

Where $\Phi(.)$ is the cumulative distribution function of the standard normal random variable and

$d_1=d_2+\sqrt{\beta+j\delta^2}=\frac{\ln\left(\frac{F(t,T)}{K}\right)+\left[-\lambda m\tau+\int_t^T \alpha^F(s,T)ds+{}^1\!/_2\int_t^T\|\sigma(s,T)\|^2 ds\right]+j\mu+j\delta^2}{\sqrt{\beta+j\delta^2}}$

Consequently, the equation (16) equals:

$$F(t,T)\exp\left[-\lambda m\tau+\int_t^T \alpha^F(s,T)ds-{}^1\!/_2\int_t^T \parallel\sigma(s,T)\parallel^2 ds+j\mu\right]e^{{}^1\!/_2(\beta+j\delta^2)}\Phi(d_1)$$

Also, the second expectation in (15) is computed as:

$KE^T\left[I_{F(T,T)>K}\mid \mathcal{F}_t\,,\ N_\tau=j\right]=K\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz= K\Phi(d_2)$

Thus, the equation (15) is written as:

$C(t,T) = P(t,T)\sum_{j\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^j}{j!}\Big\{E^T\Big[F(t,T)exp\Big[\Big(-\lambda m\tau+\int_t^T\alpha^F(s,T)ds-{}^1\!/_2\int_t^T \parallel\sigma(s,T)\parallel^2 ds\Big)+\int_t^T\sigma(s,T)dW_s^T+\sum_{k=0}^{j}Y_k\Big]I_{F(T,T)>K}|\mathcal{F}_t\,,\ N_\tau=j\Big]-E^T\Big[KI_{F(T,T)>K}|\,\mathcal{F}_t\,,N_\tau=j\Big]\Big\}=$
$P(t,T)\sum_{j\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^j}{j!}\ \Big[F(t,T)\exp\Big[-\lambda m\tau+\int_t^T\alpha^F(s,T)ds-{}^1\!/_2\int_t^T \parallel\sigma(s,T)\parallel^2 ds+j\mu\Big]e^{{}^1\!/_2(\beta+j\delta^2)}\Phi(d_1)\Big]-P(t,T)\sum_{j\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^j}{j!}K\Phi(d_2)$

Therefore, the option price formula is:

$$\boldsymbol{C(t,T)=\ P(t,T)\sum_{j\geq 0}\frac{e^{-\lambda\tau}(\lambda\tau)^j}{j!}\left[F(t,T)\,exp\left(-\lambda m\tau+\int_t^T\alpha^F(s,T)ds+j\mu+j\frac{\delta^2}{2}\right)\Phi(d_1)-K\Phi(d_2)\right]}$$

Where

$\Phi(.)$ is the cumulative distribution function of the standard normal distribution and

$d_2=\frac{\ln\left(\frac{F(t,T)}{K}\right)+\left[-\lambda m\tau+\int_t^T\alpha^F(s,T)ds-{}^1\!/_2\int_t^T\|\sigma(s,T)\|^2ds\right]+j\mu}{\sqrt{\int_t^T\|\sigma(s,T)\|^2ds+j\delta^2}}$

$d_1=d_2+\sqrt{\int_t^T \parallel\sigma(s,T)\parallel^2 ds+j\delta^2}$ ■

4.2. Stochastic Interest Rate with Jump

In the case that jump exists in the bond market, the $\mathbb{Q}$ dynamics of a futures contracts maturing at T is defined as:

$$\frac{dF(t,T)}{F(t_{-},T)} = -\lambda^{Q} m dt + \sigma(t,T) dW_t^{Q} + (y_t - 1) dN_t \tag{17}$$

It is assumed that both bond and VIX futures markets follow the same Poisson process with the same intensity. The likelihood process and its dynamic are obtained as:

$$L_t^T = \frac{P(t,T)}{P(0,T)B(T)},$$

$$L_t^T = \frac{dQ^T}{dQ}, \text{ on } \mathcal{F}_t$$

$$dL_t^T = \sigma^p(t,T) L_t^T dW_t^Q + (H_t - 1) L_{t-} d\tilde{N}_t^Q$$

In this case h_t in the formula (3) equals $h_t = (H_t - 1)$ and the intensity of the Poisson process under $\mathbb{Q}^{\mathbb{T}}$ is $\lambda^T = H_t \lambda^Q$. By inserting the equation (2) into (17) and compensating for the Poisson process N under $\mathbb{Q}^{\mathbb{T}}$, the $\mathbb{Q}^{\mathbb{T}}$ dynamics of VIX futures is presented as:

$$\frac{dF(t,T)}{F(t_{-},T)} = [\sigma(t,T) \cdot \sigma^p(t,T) - \lambda^Q m] dt + \sigma(t,T) dW_t^T + (y_t - 1) dN_t$$

Where N_t is Poisson process with intensity $\lambda^T = H_t \lambda^Q$ and for simplicity $[\sigma(t,T) \cdot \sigma^p(t,T)]$ is defined as $\alpha^F(t,T)$. Although the method of derivation is the same as the previous case, the solution is different. In particular, the $\mathbb{Q}^{\mathbb{T}}$ $-$intensity is used when we calculate the $\mathbb{Q}^{\mathbb{T}}$ probability for $N_\tau = j$.

$$C(t,T) = P(t,T)\, E^T[\max(\mathrm{F}(\mathrm{T},\mathrm{T}) - \mathrm{K}, 0) |\, \mathcal{F}_t\,)] = P(t,T) E^T\left[(F(T,T) - K)\, I_{F_{(T,T)>K}} \middle| \mathcal{F}_t\right] =$$
$$P(t,T) \sum_{j\geq 0} Q^T(N_\tau = j) \Big\{ E^T \Big[F(t,T) exp \Big[\Big(-\lambda^Q m\tau + \int_t^T \alpha^F(s,T) ds - {}^1\!/_2 \int_t^T \parallel \sigma(s,T) \parallel^2 ds \Big) +$$
$$\int_t^T \sigma(s,T) dW_s^T + \sum_{k=0}^{j} Y_k \Big] I_{F_{(T,T)>K}} | \mathcal{F}_t, N_\tau = j \Big] - E^T \Big[K\, I_{F_{(T,T)>K}} \mid \mathcal{F}_t, N_\tau = j \Big] \Big\} =$$
$$P(t,T) \sum_{j\geq 0} \frac{e^{-\lambda^T \tau} (\lambda^T \tau)^j}{j!} \Big\{ E^T \Big[F(t,T) exp \Big[\Big(-\lambda^Q m\tau + \int_t^T \alpha^F(s,T) ds - {}^1\!/_2 \int_t^T \parallel \sigma(s,T) \parallel^2 ds \Big) +$$
$$\int_t^T \sigma(s,T) dW_s^T + \sum_{k=0}^{j} Y_k \Big] I_{F_{(T,T)>K}} | \mathcal{F}_t, N_\tau = j \Big] - E^T \Big[K\, I_{F_{(T,T)>K}} \mid \mathcal{F}_t, N_\tau = j \Big] \Big\}.$$

Hence, by computing the above expectations, the price of a European call option at time t, with expiration date T is defined as:

$$\boldsymbol{C(t,T) = P(t,T) \sum_{j\geq 0} \frac{e^{-\lambda^T \tau} (\lambda^T \tau)^j}{j!} \left[F(t,T)\, exp\left(-\lambda^Q m\tau + \int_t^T \alpha^F(s,T) ds + j\mu + j\frac{\delta^2}{2} \right) \Phi(d_1) - K\Phi(d_2) \right]}$$

Where

$\Phi(.)$ is the cumulative distribution function of the standard normal distribution and

$$d_2 = \frac{\ln\left(\frac{F(t,T)}{K}\right) + \left[-\lambda m\tau + \int_t^T \alpha^F(s,T)ds - {}^1\!/_2 \int_t^T \|\sigma(s,T)\|^2 ds\right] + j\mu}{\sqrt{\int_t^T \|\sigma(s,T)\|^2 ds + j\delta^2}}$$

$$d_1 = d_2 + \sqrt{\int_t^T \| \sigma(s,T) \|^2 \, ds + j\delta^2} \qquad \blacksquare$$

4.3. Constant Interest rate

In case of having constant interest rate, the option price is obtained under the risk neutral probability measure $\mathbb{Q}$ and having the bank account as numeraire:

$$\frac{dF(t,T)}{F(t_-,T)} = \sigma(t,T)dW_t^Q + (y_t - 1)d\widetilde{N}_t^Q = -\lambda^Q m\, dt + \sigma(t,T)dW_t^Q + (y_t - 1)dN_t$$

In this case the $\alpha^F(t,T) = 0$ in the formula (12).

$$\boldsymbol{C(t,T) = e^{-r(T-t)} \sum_{j\geq 0} \frac{e^{-\lambda^Q\tau}(\lambda^Q\tau)^j}{j!} \left[F(t,T)\, exp\left(-\lambda^Q m\tau + j\mu + j\frac{\delta^2}{2}\right) \Phi(d_1) - K\Phi(d_2)\right]} \blacksquare$$

Where

$\Phi(.)$ is the cumulative distribution function of a standard normal random variable.

$$d_2 = \frac{\ln\left(\frac{F(t,T)}{K}\right) + \left[-\lambda^Q m\tau - {}^1\!/_2 \int_t^T \|\sigma(s,T)\|^2 ds\right] + j\mu}{\sqrt{\int_t^T \|\sigma(s,T)\|^2 ds + j\delta^2}} \quad , \quad d_1 = d_2 + \sqrt{\int_t^T \| \sigma(s,T) \|^2 \, ds + j\delta^2}$$

5. Empirical Discussion and Results

5.1. Data

The daily settlement prices of VIX futures with expiration up to six months over the period March 2004 to December 2010 were gathered from the CBOE website. We only considered contracts with maturity up to six months since longer contracts are less liquid. This resulted in a total of 7121 observations. The VIX Special Opening Quote prices were multiplying by ten prior to March 26, 2007 in order to determine its final settlement value. Since that date, the final settlement values for VIX futures have been based on the actual underlying index level instead of ten times the underlying index level. Hence, we divided the settlement prices from 2004 to March 26, 2007 by ten to be able to work with prices for the whole period. The opening hours of the VIX futures markets are on business days from 7:20 A.M. to 13:15 P.M. while the majority of futures markets are open almost 24 hours a day.

5.2. Empirical Properties of VIX Futures

As I mentioned in the introduction, the VIX futures returns have some important characteristics such as positive skewness, excess kurtosis and a decreasing volatility term structure for long term expirations. These characteristics are illustrated in table 1 where the four moments of the VIX futures logarithmic returns (mean, standard deviation, skewness and kurtosis) for all sample data and three expiration categories are calculated. Positive and significant values of skewness and kurtosis admit the existence of these features. Therefore, it is stated that the VIX futures return are not normally distributed and a more appropriate and flexible term-structure model is needed to capture these features of VIX futures returns. Also, it is observed that volatility of VIX futures return decreases as there is more time left to maturity. Furthermore, from the values in the table, it is clear that mean returns of VIX futures are positive for long-term and negative for short-term VIX futures contracts.

Table I

Descriptive Statistics for the VIX Futures Returns

Time-to Maturity	All	1-2 months	3-4 months	5-6 months
Mean	-0.0226	-0.37	-0.0282	0.11
Standard deviation	0.3663	0.29516	0.2681	0.251
Skewness	0.4322	0.0025	0.0079	0.5851
Kurtosis	9.775	5.33456	5.18602	4.89958

*In this table the descriptive statistics for the logarithmic returns of VIX futures obtained from their settlement prices is provided. The Settlement prices are from the period March 26, 2004 to December 1, 2010 and the number of data for the whole period is 7121. The standard deviation is annualized by a factor $\sqrt{252}$ and average return is on daily based and multiplied by 100. It is observed in the table, the value of skewness and kurtosis are significantly high and positive and there is the least volatility for long-term expiration.

The desired candidate model for VIX futures returns is jump diffusion model. The Kernel density of VIX futures returns for the data and the model with normal distribution are provided in Figure 1. In figure 2, the Kernel estimate of the logarithmic VIX futures returns together with the MJD model are observed. From the figure 2, it is clear that the jump diffusion model provides a god fit for the sample and has a better performance compared to a case without jump.

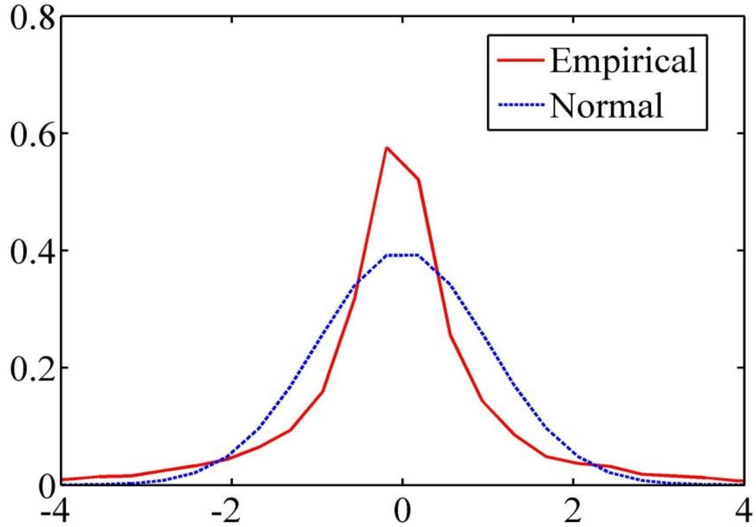

FIGURE 1

Kernel Estimate of VIX Futures returns and Normal

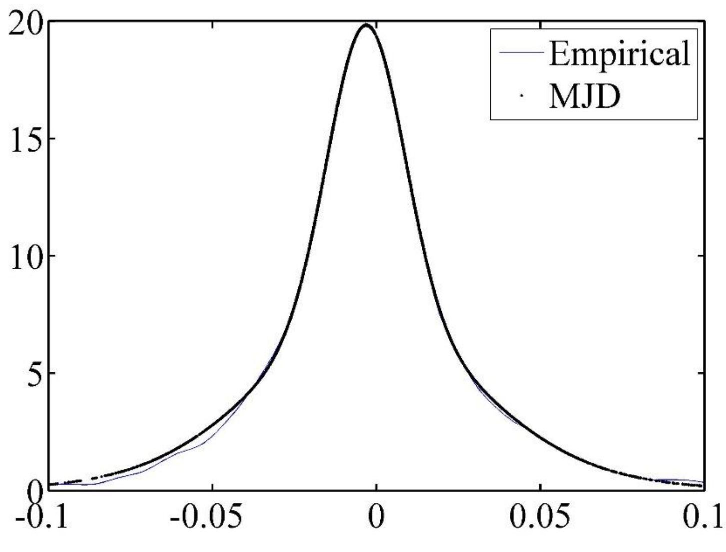

FIGURE 2

Kernel Estimate of Logarithmic VIX Futures returns and MJD

5.3. Parameters Estimation

There exist different methods for the purpose of parameter estimation. Although Maximum Likelihood Estimation is one of the most popular methods, in the case of jump diffusion model, it does not work well and it is not a careful numerical optimization. The reason is that the maximum likelihood is very sensitive to the initial values and by really small changes in those values, the likelihood function cannot be converges easily[2]. Therefore, to estimate the parameters of the model, the Non-Linear Least Square (NLS) method is used and they are estimated under the assumption that the model has constant interest rate and one dimensional Wiener process. Also, it is assumed that the Girsanov Kernel h_t (in the equation (3)) is equal to zero. Therefore, based on the relationship between the market price of jump risk and its Kernel, the market price of jump risk is zero in our estimation. During the process of estimation by NLS, it was observed that by changing the initial values, the model converges to different estimated values. Consequently, it was clear, there are more than one local minimum that minimize the error between the data and the model. In fact, the global minimum should be considered to estimate the parameters.

The Merton Jump Diffusion model is the mixture of N normally distributed terms and the mean, variance and weight of j'th stochastic variable in the mixture are

$m_j = \left(\alpha - \frac{\sigma^2}{2}\right)\tau + j\mu,\ s_j^2 = \sigma^2\tau + j\delta^2$ and $w_j = \frac{e^{-\lambda\tau}(\lambda\tau)^j}{j!}$ respectively. The sufficiently large N is chosen and it should be noted that the selected N depends on λ. The numerical studies using daily observations demonstrate that there is no significant difference in estimates from N=20. For this study the number of jumps is considered to be N=140.The volatility in the model in the equation (4) is specified as $\sigma(t,T) = \sigma_1 e^{-\sigma_2(T-t)}$ where σ_1 and σ_2 are nonnegative[3]. Also, the market price of risk is assumed to be constant not time dependent. In table 2, the estimated values of the six parameters σ_1, σ_2 ,μ, δ, λ and φ of the model are observed where μ and δ are the mean and standard deviation of logarithmic jump size, λ is the P-intensity and φ is market price of diffusion risk.

Table II

Estimation Results for the VIX Futures Models

Models	$\widehat{\sigma_1}$	$\widehat{\sigma_2}$	$\hat{\mu}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\varphi}$
Merton Jump Diffusion	0.1836	0.033	0.0003	0.0285	252.0681	-3.3139
Normal case	0.3561	0.1257	-	-	-	-2.1312

*The models parameters are estimated using daily logarithmic returns of VIX futures prices with maturity up to six months over the period March 26, 2004 to December 2010. The number of data is 7121.

2 Some empirical researches have applied method other than Maximum Likelihood Estimation. Duncan and Randal (2009) is one of the studies used EM algorithm for estimation.
3 This volatility function was suggested by Hilliard-Reise (1998)

Since the intensity is the expected number of jumps, its larger value results in occurring jump more frequently. Moreover, the sign of $\hat{\mu}$ (the mean of logarithmic jump size) determines if returns are positive or negative skewed. From the table, it is observed that $\hat{\mu}$ is positive for our data which admits the positive skewness feature of the VIX futures returns. Table illustrates that market price of risk has a negative sign for both MJD model and Normal model which is consistent with the results in the study by Nossman & Wilhelmsson (2008).

Figure 3 illustrates the changes in the call option values in both the MJD model and the standard model without jump for different time to maturities. The following assumptions are considered, namely interest rate, r =0.075 and current VIX futures price, F=30

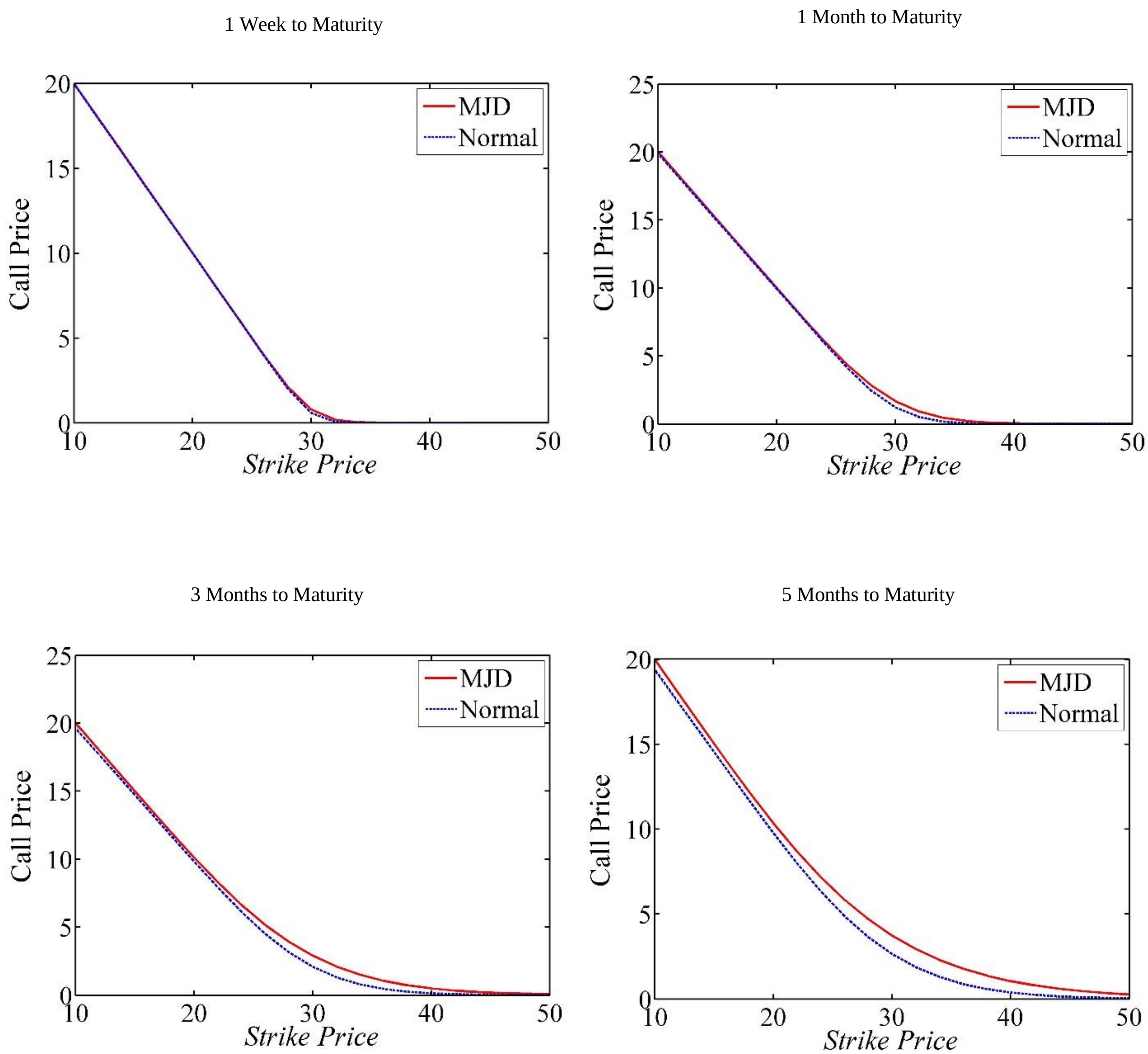

Figure 3

MJD Call Price vs. Normal Call Price

The changes in the price of call options with respect to the strike price are illustrated for both models in figure 3. The figures demonstrate that the MJD call prices have greater values than the standard model for both in-the-money and out-the-money options. Also, it is observed, by increasing maturity these results still hold. This conclusion is consistent with the results in the research by Matsuda (2004) who compared the price of stock call options in the MJD model and the Black-Scholes model. Moreover, figures illustrate that by increasing expiration time the price difference between the MJD call price and the Black call price increases.

6. Conclusion

It is around a decade that VIX futures and options have been presented to the market and are trading in a large volume today. The literatures on VIX futures and options are growing speedily. A large number of researches have been done to reveal different characteristics of VIX futures. Some of the researches focus on modeling the VIX index and try to find an appropriate distribution for VIX futures returns while other researchers specified the VIX futures dynamics exogenously in their studies. In this thesis, in the theory part, the VIX futures were modeled by the Merton jump diffusion model and a closed-form solution for the price of options on VIX futures was derived for both stochastic and constant interest rate cases in the model. In the empirical part, by using the historical VIX futures prices from the CBOE data, the behaviors of the VIX futures returns were investigated and the model parameters were estimated. The descriptive statistics of the data illustrated that the VIX futures returns are positive skewed and have excess kurtosis. Therefore, it is clear that the VIX futures returns are not normally distributed. Also, we calculated the price of the VIX call options for both the MJD model and the standards model using the estimated parameters. The results implied that the MJD lead to greater values than the other model for both in-the-money and out-the-money options. Hence, it is concluded that adding jump to the diffusion process is crucial to capture the features of the data. In fact, the jump diffusion model is well approximated and presents better performance compared to the standard case.

In order to extend this study, the performance of the model can be assessed by using the market VIX options. Also, by applying different models to the VIX futures and investigating the performance of that model in future researches, the most appropriate and fit model can be revealed.

7. Reference

[1] Björk, T. (2009). "Arbitrage Theory in Continuous Time". (3rd Ed.). New York: Oxford University Press.

[2] Björk, T., & Landen, C. (2000). "On the Term Structure of Futures and Forward Prices". SSE/EFI Working Paper Series in Economics and Finance, No. 417.

[3] Björk, T. (2011). "An Introduction to Point Process from a Martingale Point of View". "Lecture note". KTH.

[4] Black, F., & Scholes, M. (1973). "The pricing of options and corporate liabilities". Journal of Political Economy, 81, 637‑659

[5] Dotsis, G., Psychoyios, D., & Skiadopoulus, G. (2007). "An empirical comparison of continuous-time models of implied volatility indices". Journal of Banking and Finance, 31, 3584‑3603.

[6] Haskay, B. and M. Nossman, (2013), "A Term Structure Model for VIX Futures". Journal of Futures Markets, Vol. 33, Iss. 5, PP. 421-442

[7] Hilliard, J., and Reis, J. (1998). "Valuation of commodity futures and options under stochastic convenience yields, interest rates, and jump diffusion in the spot". JFQA 33, 1, 61-86.

[8] Honore, P. (1998). "Pitfalls in Estimating Jump Diffusion Model". Social Science Research Network.

[9] Lin, Y. N. (2007). "Pricing VIX futures: Evidence form integrated physical and risk neutral probability measures". The Journal of Futures Markets, 27, 1175‑1217.

[10] Lin, Y. N., & Chang, C. (2009). "VIX option pricing". The Journal of Futures Markets, 29, 523‑543.

[11] Lu, Z. J., & Zhu, Y. Z. (2010). "Volatility components: The term structure dynamics of VIX futures". The Journal of Futures Markets, 30, 230‑256.

[12] Merton, R.C., (1975). "Option Pricing When Underlying Stock returns are Discontinuous". Journal of Financial Economics, 3, 125-144. Q North-Holland Publishing Company

[13] Psychoyios, D., Dotsis, G., & Markellos, R. N. (2009). "A jump diffusion model for VIX volatility options and futures". Review of Quantitative Finance and Accounting, 35, 245‑269.

[14] Rhoads, R. (2011). "Trading VIX Derivatives: Trading and Hedging Strategies Using VIX Futures, Options and Exchange Traded Notes". New Jersey: John Wiley & Sons

[15] Wang, Z., & Daigler, R. T. (2011). “The performance of VIX option pricing models: Empirical evidence beyond simulation”. The Journal of Futures Markets, 31, 251‐281.

[16] Zhang, J. E., Shu, J., & Brenner, M. (2010). “The new market for volatility trading”. The Journal of Futures Markets, 30, 809‐833.

[17] Zhang, J. E., & Zhu, Y. Z. (2006). “VIX futures”. The Journal of Futures Markets, 26, 521‐531

Appendix

A.1. Futures Price Formula

It was stated in section three that the model has the following $\mathbb{P}$-dynamics:

$$\frac{dF(t,T)}{F(t_{-},T)} = \alpha(\mathrm{t},\mathrm{T})dt + \sigma(t,T)dW_t + (y_t - 1)dN_t$$

Define function $g(t,T) = \ln(F(t,T))$ and by applying the Ito formula to this function:

$$dg(t,T) = \frac{1}{F(t,T)}dF(t,T) - \frac{1}{F^2(t,T)}(F(t,T))^2 + dN_t\left[\ln\left(F(t_{-},T) + F(t_{-},T)(y_t - 1)\right) - \ln(F(t-,T))\right] = \left(\alpha(\mathrm{t},\mathrm{T}) - {}^{1}\!/_{2} \parallel \sigma(s,T) \parallel^2\right)dt + \sigma(t,T)dW_t + \ln(y_t)\,dN_t = \left(\alpha(\mathrm{t},\mathrm{T}) - {}^{1}\!/_{2} \parallel \sigma(s,T) \parallel^2\right)dt + \sigma(t,T)dW_t + Y_t dN_t$$

Where $Y_k = \ln(y_k)$

By integrating over the interval [t, T]:

$$g(T,T) = \ln(F(T,T)) = g(t,T) + \int_t^T (\alpha(\mathrm{s},\mathrm{T}) - {}^{1}\!/_{2} \parallel \sigma(s,T) \parallel^2)\,ds + \int_t^T \sigma(s,T)dW_s + \sum_{k=0}^{N_{T-t}} Y_t$$

Hence,

$$F(T,T) = F(t,T)\exp\left[\int_t^T \alpha(\mathrm{s},\mathrm{T})ds - {}^{1}\!/_{2}\int_t^T \parallel \sigma(s,T) \parallel^2 ds + \int_t^T \sigma(s,T)dW_s + \sum_{k=0}^{N_{T-t}} Y_k\right] = F(t,T)\exp\left[\int_t^T \alpha(\mathrm{s},\mathrm{T})ds - {}^{1}\!/_{2}\int_t^T \parallel \sigma(s,T) \parallel^2 ds + \int_t^T \sigma(s,T)dW_s * \prod_{k=0}^{N_{T-t}} y_k\right]$$

Therefore, the price of futures contract with expiration T at time t is calculated by the following formula:

$$F(T,T) = F(t,T)\exp\left[\int_t^T \alpha(\mathrm{s},\mathrm{T})ds - {}^{1}\!/_{2}\int_t^T \parallel \sigma(s,T) \parallel^2 ds + \int_t^T \sigma(s,T)dW_s + \sum\nolimits_{k=0}^{N_{T-t}} Y_k\right]$$