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Methods for estimating premium risk for Solvency purposes

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Abstract

For an operating non-life insurer, premium risk is a key driver of uncertainty both from an operational and solvency perspective. Traditionally, the day-to-day operations of a non-life insurance company focus mainly on estimating the expected average outcomes both within pricing and reserving. In the new European solvency regulation Solvency II own stochastic models (Internal Models) for estimating the Solvency Capital Requirement (SCR) are allowed, subject to supervisory approval. Following the change in regulation, models for assessing the uncertainty and not only the expected value in insurance operations are gaining increasing interest within research and also among practitioner working with assessing the uncertainty for solvency purposes.

In regards of the solvency perspective of premium risk, a lot of different methods exist aimed to give a correct view on the capital needed to meet adverse tail outcomes related to premium risks. This thesis is a review of some of these models, with the aim to understand the assumptions and their impact, the practical aspects of the parameter estimation as such and possible extensions of the methods. In particular, the issue of limited time versus ultimate parameter estimation is given special attention.

A general conclusion is that it is preferable to use methods which explicitly model the claim outcomes in terms of underlying frequency and severity distributions. The clear benefit is that these methods provide more insight in the resulting volatility than a method that directly measures uncertainty on the financial results. In regards of the time perspective, the conclusion is that going from ultimate uncertainty to limited time uncertainty can be achieved by two main methods: using transformation methods based on reserving principles to transform ultimate estimates or by the use of data observed at the appropriate point in time.

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Abstract

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1. Introduction

This chapter gives a general introduction and background to the problem, with the ultimate goal to give the reader an understanding of how the specific topic of this thesis is of importance to ongoing developments within the regulatory area for insurance companies.

1.1 Background

The current regulatory solvency framework for insurance companies, Solvency I, has its roots in the 1970s. The solvency requirements within Solvency I are based on relatively simple factor based expressions in which mainly premiums and reserves are used to determine the sufficient level of capital needed. At least on the non-life side, the capital requirements coming out of these expressions are generally on the low side compared to for instance actual capitalization levels or requirement from models used by rating agencies. Also, a solvency requirement based on only premium and reserve volumes potentially miss large elements of risk, for instance related to investment assets or potentially excessive exposures to catastrophe risk or other heavy tailed risks. This has encouraged some financial supervisory authorities within the European region to develop their own solvency models, in order to identify companies which seem to be undercapitalized in relation to their risk profile. An example is the so called Traffic Light model used in the Swedish market, which is a risk-based solvency capital model trying to quantify the key risks of both non-life and life insurers. The resulting overall capital requirement is then compared to a capital base derived using an economic valuation of the balance sheet. The fact that several European countries has gone down the route to develop their own solvency regimes can be seen as a strong indicator of a consensus that current regulatory rules not being sufficient to reflect the capital need of insurance companies.

Having different ways of dealing with solvency issues within different countries within the EU is not desirable, since it potentially creates an uneven playing field for insurance companies in different countries. This as one of the key drivers, together with the lack of risk-based principles within Solvency I as such, has created a clear need for a more harmonized and risk-based solvency regulation within the EU. As a response to the above, the work of developing a new solvency framework, named Solvency II, has been going since the first half of the last decade. The main intentions with the new regulation is to have a harmonized regulation across EU, which as a foundation introduces risk-based capital requirements and principles around risk management that promotes holistic handling and management of risks within insurance companies.

Currently, Solvency II is expected to come into force by the 1th of January 2013 and will apply to all insurance companies within EU. It is even so that countries outside the EU have

the intention to implement the Solvency II regulation in national law, simply as a measure to both harmonize the playing field for companies and as a way to promote pure policyholder protection, and in addition even as a way to avoid potential financial instability resulting from insurers defaulting. Some examples of such countries are Switzerland and Norway but discussions are ongoing even in countries outside Europe, in for instance South Africa. For some more general information around Solvency II and sources to statements above, please see for instance Ayadi (2007) or Eling et al. (2007).

1.2 Solvency regulation in general

The intention with any solvency regulation is to ensure that the amount of excess capital, i.e. assets minus liabilities, is high enough to be able to meet large but still realistic fluctuations in the balance sheet without facing a default where the liabilities are larger than the assets. Introducing from the balance sheet the total assets as A and total liabilities as L , the solvency regulation thus wants to ensure, through excess capital requirements, that

$$P(A - L < 0) \leq \alpha \tag{1.1}$$

over a certain time horizon, where α is the suitably chosen confidence level. The expressions (1.1) is somewhat simplifying since there might, depending on the solvency regulation discussed, also be so called tiering limits coming into play and further limiting the excess capital. The idea behind tiering is to classify all components of the excess capital depending on quality, which is related to for instance availability and liquidity, and then limit the share of lower quality elements in the total capital base.

In an insurance company, fluctuations in assets and liabilities can stem from for instance adverse claim outcomes incurring, revaluation of historical but still outstanding claims due to new information, revaluation of assets and/or liabilities due to developments in financial markets, to name a few examples.

It is important to bear in mind that the stakeholders to an insurance company are mainly policyholders (customers) and the shareholders (owners). The main party to protect within a solvency regulation is usually the policyholder, since customers need to be certain that the insurance company is able to meet their obligations agreed in their policies, even under stressed scenarios. Also, it is a fact that insurance companies are important players on the financial markets. This means that governments, which in general are interested in financial stability, have interests in insurance company regulation in general and solvency matters in particular. By ensuring financial stability in general through solvency regulation also shareholders are protected from default situations, at least indirectly.

Note that from a policy holder perspective, having a counterparty with excessive amounts of capital is not desirable despite the lower default risk, since being over capitalized will

lead to higher nominal return requirements from shareholders, which ultimate will lead to higher premiums. Thus the level of required capital in a solvency framework is a balance between protection against possible defaults and the increased premiums coming with higher capitalization. This leads to the need to define some kind of confidence level within a solvency framework, corresponding to the probability of default over a certain time horizon. Setting this confidence level is thus a compromise between what a reasonable default probability is over a certain time horizon and what is reasonable from a pricing perspective.

1.3 Solvency II and Internal Models in a nutshell

The Solvency II regulation is based on a three pillar approach, with the following contents in each pillar:

- Pillar 1: Contains the quantitative risk-related part of the regulation, describing for instance the determination of the Solvency Capital Requirement (SCR) and the Minimum Capital Requirement (MCR). It also covers the principles around the determination of the capital base, confidence levels and valuation principles for assets and liabilities. One important ingredient is the allowance of so called Internal Models for determining the SCR.
- Pillar 2: Contains the principles around the practical possibilities for supervisory authorities to perform actual supervision and principles around risk management and governance. This pillar also establishes the allowance for additional capital requirements over the SCR that supervisory authorities can demand under certain circumstances, given that they see a strong reason for doing so.
- Pillar 3: This pillar deals with issues around the disclosure of information towards the supervisory authorities, policyholders and other stakeholders. It sets principles for the content and frequency of quantitative and qualitative data to report, and regulates what is public information and what is not.

The Solvency II directive establish that the confidence level in (1.1) should be set to 0,5% and the time horizon should be one year, i.e. the solvency regulation should make sure that a default event within one year, for a particular insurance company, occurs with a maximum frequency of once in 200 years. It is also explicitly stated that the capital base should be derived using economical principles, meaning that assets and liabilities should be to the extent possible valuated using market consistent values (Solvency II directive, 2009). For typical insurance liabilities financial markets cannot be used to determine this, but the approach of using probability weighted averages of discounted future cash-flow scenarios is instead promoted. Also, consistent with the derivation of the excess capital, the

uncertainty of the excess capital should be based on the revaluation of assets and liabilities on an economic basis.

To summarize, the principles for deriving the excess capital required and the actual capital base for a company should both be carried out with consistent and economic principles. Thus the intention with the Solvency II is to take a pure economic cash-flow based approach, in order to align the capital requirements and capital base determination with the industry practices in regards of risk management.

Note in the description of Pillar 1 that there is allowance for so called Internal Models. Normally within Solvency II, the standard approach is to determine the SCR using the so called Standard Formula, which is a predefined solvency model based on factors and scenarios to apply. The Standard Formula is a ‘one size fits all’ approach in the sense that the principles are supposed to apply to all companies. Since companies in practice more or less differ from the ‘average company’ the Standard Formula might either overstate or understate the true excess capital need. As a response to this, Pillar 1 includes principles around company specific solvency models, Internal Models, which may be designed by each company and be used for calculating the SCR after supervisory approval. Since own solvency models are allowed to be used, developing sound principles and methods for estimating uncertainty in both insurance and investment operations of an insurance company are essential to meet the supervisory standards in this area (Solvency II directive, 2009).

1.4 Insurance risks within a non-life company

A normal non-life insurance company faces the following insurance risks in their daily operations (Ohlsson & Lauzenings 2008):

- Premium risk. The risk of financial losses related to premiums earned during the period considered (typically the next year), i.e. claims incurring in the future. The risk in the losses relates to uncertainty in severity, frequency or even timing of claims incurring during the period, as well as to uncertainty related to operating expenses. This risk is typically defined to include both risks underwritten during the period and contracts which are unexpired at the start of the period and thus are subject to uncertainty.
- Reserve risk. The risk of financial losses related to policies for which premiums already have been earned (fully or partly), i.e. risk related to claims that has already incurred but which might be unsettled, reopened or even not yet known to the insurance company. This risk relates to uncertainty in both the amounts paid and the timing of these amounts.

- Catastrophe risk. The risk of financial losses related to unlikely events with high severity, where common examples include windstorms, landslide and earthquakes (natural catastrophes) and terrorist attacks, high severity motor liability events and large accidents (man-made catastrophes). Catastrophe risk is usually considered to be a part of premium risk, but due to its special and more extreme nature it is usually dealt with separately.

The topic of this thesis is the premium risk, which is perhaps the insurance risk which has been subject to the least attention within the academic world and among practitioners within the insurance industry. For instance, the estimation of uncertainty in reserves is a topic discussed in relatively many papers; a few recent papers can be found among the references (Björkwall et al. 2009 and England & Verrall 2006). The main topics of interest in recent research around reserve risk is typically the distinction between one-year and ultimate risk and the question of measuring uncertainty in a reserving setup which is not purely chain-ladder based (either with other methods or with smoothing of development factors). Catastrophe risk is typically handled using one of the following approaches or a combination of both:

- By using pure historical losses to estimate distributions for frequency and severity, which requires significant amount of data to be a sound statistical approach or a strong a priori view of the choice of distribution and/or tail behavior, as well as assumptions around correlation from an aggregated point of view.
- By using explicit catastrophe models for different catastrophe perils, that tries to simulate the actual events (windstorms etc.) occurring and their financial impact for specific portfolios of policies.

The aim of this thesis is to give an overview of methods available for estimating premium risk, as well as to discuss possible extensions of the methods as such. Practical aspects of the methods and their pros and cons will be playing a central role of the discussions. A goal is also to discuss the question of one-year versus ultimate premium risk, to at least form an opinion valid in the context of this thesis.

2. Premium risk in general

This chapter casts premium risk into a more formal framework. The goal is to introduce the concept of insurance risk and to make a clear distinction between premium and reserve risk and define what is included in each risk. The goal is also to discuss the question of one-year (or more generally, limited time) risk and ultimate risk, and especially the distinction between the concepts from a premium risk perspective.

2.1 The underwriting result at a glance

The underwriting result U during an accounting year is for a non-life insurance company defined as:

$$U = P - E - L \quad (2.1)$$

where P is the earned premium during the year, E are the operating expenses and L are the loss payments and changes in loss reserve during the year. Note that the term L in the formula above includes payments and changes in loss reserves which are due to loss adjustment expenses, i.e. costs which can be seen as being related to the claim handling and thus are considered a natural part of the loss as such. This goes for all formulas in this thesis unless otherwise mentioned. In general, the loss element L can be split in the following way:

$$L = L_{CY} + L_{PY} \quad (2.2)$$

where L_{CY} is the loss related to the current accident year (i.e. losses actually *incurring* during the accounting year in question) and L_{PY} is the loss related to previous accident year (i.e. for losses that has already *incurred* but which are not necessarily not fully settled or correctly reserved for). Further, note that each part of (2.2) will consist both of actual payments during the accounting year and changes in the reserve for the corresponding accident years, i.e. combining (2.1) and (2.2) with the split of payments and changes in reserves we end up with

$$U = P - E - (C_{CY} + R_{CY} + C_{PY} + R_{PY}) \quad (2.3)$$

where C and R are payments and changes in reserves, for the current accident year (CY) and previous accident years (PY) respectively. Of course, the relation between C and R in relative terms will vary significantly between different lines of business and between the current year and previous year parts of the result. For instance, property business will typically have a small proportion of reserves in relation to premiums (i.e. be short-tailed)

due to short time between claims incurring and actual payments. This will make the payments to be the driving part of the CY result and it will lead to the PY run-off having a less significant impact on the overall accounting result due to relatively limited reserve balances. The opposite is true for so called long-tailed lines, where general third party liability or motor third party liability are two good examples. For these lines, the reserve part of the current year result will be much more significant and also the PY results as such will be much more important, at least for a portfolio that has existed (“matured”) for a long enough time for PY reserves to accumulate.

Based on (2.3), we split the financial results into one part for the CY result and one part for the PY result:

$$U_{CY} = P - E - (C_{CY} + R_{CY}) \quad (2.4)$$

$$U_{PY} = -(C_{PY} + R_{PY}) \quad (2.5)$$

Including the premium in CY result is natural since the CY loss elements are directly connected to the premium earned during the year. Including the expenses under the CY result is not that obvious but note that, as defined earlier, the PY reserves (as the CY reserves) includes also claims adjustment expenses, so the reserves as such should be sufficient for “running off” the liabilities even in a situation where no new premium is earned. Note that (2.5), which are usually labeled the run-off result, will be assumed to have a mean value of zero. This is motivated by the fact that given that the a priori reserves for the payments in the current year are correct on average and given that the information about the payment as such does not bias the estimation of future payments, the average change in reserve will on offset the average payment during the year. A mean value of zero will of course only be the case if the reserves are really the probability weighted average of the future cash flows, but since that is a regulatory requirement it is a reasonable assumption. Based on (2.5) one realizes that reserve risk will essentially be both the risk of payments during the year not equal to what was assumed when the reserve was estimated, as well as risk in the estimation of cash flows after the accounting period, which are regularly subject to revaluation.

We will in this thesis let (2.4) define the premium risk and (2.5) define the reserve risk. Premium risk will thus be the risk of setting the premium too low on average, and/or the risk of expenses and losses being higher than the average outcome. As a side note it can be mentioned that the principles within Solvency II in regards of the valuation of the premium reserve will naturally introduce another uncertainty element in (2.4) related to the possible revaluation of the premium reserve as such. However by assuming in this thesis that we have one-year policies only, which is the case for the majority of the volume within non-life insurance, this problem is effectively avoided through the consideration of one-year

accounting periods. For further discussions around this topic, please see Ohlsson & Lauzeningsks 2008.

With (2.3) describing the financial result for an accounting year, it is natural to define the 99,5% percentile for the insurance risk based on the percentile for the expression since the change in excess capital for a company resulting from the insurance operations will be a direct consequence of this expression. Of course, this captures only the impact on the balance sheet due the pure insurance part of the cash flows, additional uncertainty usually stems from the investments operations, the discounting of the liability cash flows as such, possible uncertainty in non-insurance liabilities and so on, but that is beyond the scope of this thesis.

2.2 Deterministic parts of the underwriting result

Considering (2.3), we see that the uncertainty in the underwriting result could come from uncertainty in premium income, expenses and the losses. In the beginning of a financial year forecasts for all these variables will be available and the question boils down to which of these elements contribute mainly to the uncertainty, to pinpoint where we should spend our time when we try to model the volatility of the operations. It turns out in practice that the premium income and operating expenses within a non-life company can be well forecasted, especially for a direct insurer. The premiums are relatively easy to forecast since companies have control over the premiums that they charge customers, and deviations from forecasts are more related to changes in market shares and/or changes to the size of the market, i.e changes in exposure. Changes in the exposure on the other hand affect all variables in (2.4) to roughly the same degree, with operating expenses as at least partly a possible exception to that rule. Disregarding that, because premiums and losses are the largest parts of (2.4), this means that changes in underlying exposure is related to translation of the size of the company or line of business rather than uncertainty in profitability as such. Operating expenses are easy to forecast since there is not much uncertainty related to salaries, marketing costs, IT-costs and other possible parts of the operating expenses. Thus the major driver for the volatility of the underwriting result is rather the losses as such (see Ohlsson & Lauzeningsks 2008 and Gisler 2009).

This can be described in more formal terms. By adding and subtracting the expected current year underwriting result, $E[U] = E[P - E - (C_{CY} + R_{CY} + C_{PY} + R_{PY})]$, to (2.3) we get

$$U = (P - E - E[C_{CY} + R_{CY}]) + E[C_{CY} + R_{CY}] - (C_{CY} + R_{CY}) - (C_{PY} + R_{PY}) \quad (2.6)$$

where we have used that $E[U_{PY}] = E[C_{PY} + R_{PY}] = 0$. Reshuffling terms a bit and replacing the premiums and expenses with their expected values, as argued above, we get

$$U = E[P - E - (C_{CY} + R_{CY})] - (C_{CY} + R_{CY} - E[C_{CY} + R_{CY}]) - (C_{PY} + R_{PY}) \quad (2.7).$$

Since the first term, the expected underwriting result is a constant; we have under these assumptions a model where we have uncertainty in the current year underwriting result and in the run-off result.

Going forward in this thesis, (2.7) will be the starting point when considering the uncertainty, and only the volatility of the current year underwriting result, the premium risk, will be the area of interest.

2.3 Premium risk and the time horizon

When it comes to non-life insurance risk in general, the time horizon of one year specified within Solvency II is of great importance when it comes to assessing both reserve and premium risk. Historically within research and among practitioners, reserve risk has been considered from an ultimate horizon rather than a one-year or time-limited perspective (see Mack 1993 and Verrall & England 2000). This means that the methods considered has been focused on assessing the potential difference between the current reserve, which is an estimate of the future cash flow, and the actual cash flow itself. Note that whatever time horizon is considered, we always discuss the next account year. The difference between different time horizon relates rather to what kind of uncertainty in the *next account year* that should be taken into account, rather than how long accounting period we are considering.

Of course, during an actual accounting year, only the first year's cash flows will actually be observed. Thus there will be uncertainty related to the payment as such, but also due to the revaluation of the future cash flows as a result of the new information (new payments) observed during the year. This is the one year or more generally the limited time approach, where the uncertainty thus is related to the actual run-off result during a limited period of time rather than the difference between the reserve and the sum of the future cash flow over the full run-off period. As a consequence of the Solvency II regulation, the limited time approach to reserve risk has been subject to rapid development during recent years and the methods has been developed to give a more correct view of the uncertainty of the run-off results as an effect of actual reserving (see for instance Björkwall et al. 2009 and England & Verrall 2006). In practice, the uncertainty is usually assessed by simulating the payments during the year using a suitable method, after which the payments during the year are used together with the actual historical payments to set a reserve after one year. This will produce outcomes conditioned on the simulated payment, so by the definition of a conditional distribution one realizes that the unconditioned distribution of the run-off result will be produced by covering the whole sample space of payments during the first year. In general, the limited time uncertainty for reserve risk is expected to be smaller than the

ultimate uncertainty, since reserving means in essence estimating future stochastic payments with their modeled average amounts.

What does the issue of time horizon mean for the premium risk? In practice it means that we should try to mimic an actual accounting year in order to comply with the Solvency II definition of risk, i.e. define risk as the uncertainty in the CY result based on (2.5). Essentially we would then have one part of the risk related to the payment during the year and one part related to the reserve set at the end of the year. Traditional methods for estimating premium risk typically have the ultimate risk as the point of view, i.e. quantification of the uncertainty in the loss over the full cash flow. However, with premium risk as opposed the reserve risk the ultimate and limited time uncertainty are more closely related.

To see why, consider how we would go from an ultimate amount to a limited time loss of amount in the cases of either a short tailed or a long tailed line of business. If the line is short tailed, most payments will occur during the first year and we can actually approximate the uncertainty in the accounting result with the ultimate uncertainty. We would thus use the following approximation:

$$C_{CY} + R_{CY} \approx C_{CY}^{\infty} \tag{2.8}$$

where C_{CY}^{∞} is the sum of the future payments, or equivalently the total loss amount from an ultimate perspective. Note that since we assume best estimate reserving, (2.8) will turn into an equality if we take expected values of each side.

If the line of business instead is long tailed, only a relatively small amount of the payments will take place during the first accounting year. Now, during practical reserving for a long tailed line using for instance the Benktander-Hovinen method (see Dahl 2003 for a practical description of reserving methods), not much weight would be put on the ultimate loss consistent with the actual observed payment during the accounting period (in a Chain-Ladder respect), but rather on the a priori estimate of the ultimate loss. The a priori loss, by definition, is not updated in the light of the payment during the year but is estimated before observing the year. This would then lead to the uncertainty in the CY result being to a large degree being related to the payment during the year and the proportion paid during the first year rather than uncertainty in the reserve for future payments set during the end of the accounting period. The proportion of the total loss that is paid during year could possibly be seen as constant in a model setup, in which case the uncertainty in the payment during the year can be estimated using an ultimate approach which is then scaled down accordingly. This would make the current year risk for long tailed lines to possibly be smaller than the current year risk for short tailed lines, a fact that is discussed further in Ohlsson 2008 and AISAM-ACME 2007 and is seen as a direct consequence of the limited time approach of Solvency II.

The reasoning above *is* highly heuristic and would definitely *require* theoretical and numerical considerations to have any kind of value as a general approach. Of course, not all companies use for all long tailed lines reserving methods that behave principally like the Benktander-Hovinen method, or even for the majority of their reserves. Of course, even if they did, the Benktander-Hovinen does not put identically zero weight on the Chain-Ladder estimate based on the first payment. Also, the uncertainty in the proportion of the ultimate that is paid during the first year could definitely in some cases be substantial. The point with the reasoning above is rather to illustrate the fact that a model for the uncertainty in the ultimate loss amount could, for premium risk, rather easily be used as basis for deriving the (smaller) limited time uncertainty if so wanted. Possibly in a slightly different way for short and long tailed and possibly even done on a case to case basis depending on uncertainties in payment patterns and reserving methods used, but still in a relatively easy way to correspond to the actual account year risk. Approaches to do so could either be based on an explicit method for transforming the ultimate risk to a limited time risk, or through the selection of the data used, for instance by using historical ultimate estimates which are not the most updated ones but instead the ones estimated at the end of the accounting year (the data approach is further discussed in chapter 4.1).

This principal argumentation is used as a basis to support the fact that this thesis will not consistently consider any specific methods for the limited time uncertainty, but rather consider it to be a relatively straight forward matter to go from ultimate risk to limited time risk for premium risk. Approaches to do so will be mentioned in certain cases but not in general. Although usually not explicitly stated, this seems to be in line with the approach taken within the academic world (see for instance Gisler 2009) and also among practical implementations within the industry. The same argument of course does not hold for reserve risk, basically since the methods are far too complex to establish an explicit relation between the limited time and ultimate perspectives.

Thus we conclude that the concept of ultimate versus limited time risk is indeed relevant also for premium risk, but that we within this thesis limit ourselves to consider the ultimate premium risk while still conceptually state how to derive estimates for the corresponding one year quantities.

3. Methods for estimating premium risk

This chapter goes through a few popular methods for estimating premium risk, with emphasis on the assumptions in the underlying models and how estimators are derived. Possible extensions of the methods are also discussed, as well as an illustrational chapter on the possibility to split the modeling into several layers.

3.1 Non-parametric versus parametric methods

Before looking into a few methods for estimating premium risk, we consider the problem at hand. What we want to model is the uncertainty in the next account year, independent of if we have a limited-time or ultimate view on risk, as discussed earlier. This means that we are usually limited to considering yearly observations. Since it is important that the observations are to the extent possible outcomes of the same random variable over time, we can due to possible time dependencies and due to the fact that insurance portfolio characteristics change over time not use historical data that is too old. Also, in practice, relatively old data can be unpractical to come by and may be hard to use since people knowing about data quality and reasons for outliers might not still be around to contribute with their knowledge. In practice, the typical case is to have 5 – 25 yearly observations. Considering more granular observations in an estimation process is usually not practically possible due to seasonal effects during years being common within non-life insurance. Also, a typical assumption in these models is independence between observations, and shorter time horizons will make it harder to argue for that.

To conclude there are practice very few observations available, especially since we want to have models capturing also events out in the tail in the distributions, due to the percentile definition of capital requirements within Solvency II. Given the few observations, we are more or less forced to consider parametric methods where, one way or another, there are assumptions around the distribution of losses. While non-parametric methods definitely are valuable for cases where more data is available, the robustness in estimations will be insufficient when there are relatively few observations (see for instance Cox 2006).

This leads us to use parametric approaches to have some kind of statistical accuracy in estimations, which ultimate will mean that we have to make a priori assumptions around suitable distributions to use. Of course, care has to be taken when choosing distributions so that they are reasonable in terms of for instance sample spaces, tail behavior and robustness in parameter estimation.

We will now go through a few methods for estimating premium risk. Advantages and disadvantages with the methods are discussed as they are presented. We will start with going through three loss-ratio methods, and then three methods which models explicitly the

claims outcome through consideration of the frequency and the severity separately. The loss-ratio methods are commonly used within practical applications (see for instance European Commission 2010), while the more explicit methods are more of interest within the academic research (see for instance Johansson 2008 and Gisler 2009) but they still are usable in applications. Note that we will denote by $E[X]$ and $V[X]$ the expected value and the variance of a random variable X , respectively.

3.2 Method 1: Normal Loss ratio with proportional variance

The first method is a relatively simple method based on the historical so called loss ratios with Normal distribution assumption. We assume further that accounting years are independent, and that margins ('premium rates') does not change significantly over time. Within non-life insurance, one usually considers the loss normalized with the earned premium, i.e. the loss ratio LR is defined for year i by

$$LR_{CY,i} = C_{CY,i}^{\infty}/P_i \quad (3.1)$$

where P_i is the earned premium. We now introduce μ_{CY} as the expected loss ratio. The idea is to have the a priori view that the loss follows a Normal distribution with a variance that is proportional to the size of the total loss, i.e. that

$$C_{CY,i}^{\infty} \sim N(\mu_{CY}P_i, P_i\sigma^2) \quad (3.2)$$

and then estimate the standard deviation σ of this loss based on the observed historical loss and premiums. Assume that we have observed k financial years. The model can be expressed in terms of a mean and a variance part b , i.e.

$$C_{CY,i}^{\infty} = \mu_{CY}P_i + \sqrt{P_i}\sigma\varepsilon \quad (3.3)$$

where we have assumed that the average loss outcome is given by the exposure times the average loss ratio and where ε is a normal distribution with unit variance, i.e. $\varepsilon \sim N(0,1)$. Rewriting (3.3) we see that

$$\sigma\varepsilon = \frac{C_{CY,i}^{\infty} - \mu_{CY}P_i}{\sqrt{P_i}} \sim N(0, \sigma^2) \quad (3.4)$$

and we realize that since we have a normally distributed variable, we can simple estimate the standard deviation by (3.4) using the unbiased sample variance estimator for a normal distribution (see Lindgren 1993) as

$$\hat{\sigma}^2 = \frac{1}{k-1} \sum_{i=1}^k \frac{(C_{CY,i}^{\infty} - \hat{\mu}_{CY} P_i)^2}{P_i} \quad (3.5).$$

The average loss ratio is naturally estimated in a way to minimize the variance of (3.5), which means that we should use (see Gisler 2009)

$$\hat{\mu}_{CY} = \frac{\sum_{i=1}^k C_{CY,i}^{\infty}}{\sum_{i=1}^k P_i} \quad (3.6).$$

To have a more comparable (between insurance portfolios) measurement of uncertainty usually the standard deviation per premium is of interest, which for this method will be

$$\frac{STDEV(C_{CY,i}^{\infty})}{P} = \frac{\hat{\sigma}}{\sqrt{P}} \quad (3.7)$$

where we have introduced P as the expected premium for the next year. The expression (3.6) is in line with the assumption of the variance structure in (3.2); we have a model where the relative standard deviation of the loss will decrease as the portfolio gets larger, by the inverse of the square root of the volume.

As a side note it can be mentioned that this method is actually in line with one of the methods proposed for estimating company specific parameters for premium risk to be used within the Standard Formula (see European Commission 2010).

In this method as well as in some of the other methods described the assumption of having the variance as a function of the premium can be discussed, and actually the premium variable can be replaced with another other suitable exposure measure, after adjusting the formulas accordingly. As will be discussed later, the choice of ‘risk volume’ is important when it comes to premium risk, since for instance premium might not reflect the exposure appropriately over time due to changes in ‘premium rate’ and other factors. This as well as other data related issues are further discussed under Chapter 4.

3.3 Method 2: Normal Loss ratio with quadratic variance

The second method has a distribution assumption consistent with method 1, but where we have a variance which is quadratic to the total loss. This means that we have the situation where the risk per premium in relative terms cannot be reduced by further growing the portfolio.

$$C_{CY,i}^{\infty} \sim N(\mu_{CY} P_i, P_i^2 \sigma^2) \quad (3.8)$$

which can, in the same manner as the proportional variance model, be rewritten to

$$C_{CY,i}^{\infty} = \mu_{CY}P_i + P_i\sigma\varepsilon \quad (3.9).$$

The estimate for the standard deviation term in this case is found by breaking out the standard deviation times the unit normal term, and we arrive at

$$\sigma\varepsilon = \frac{C_{CY,i}^{\infty} - \mu_{CY}P_i}{P_i} \sim N(0, \sigma^2) \quad (3.10)$$

from which we get the estimator

$$\hat{\sigma}^2 = \frac{1}{k-1} \sum_{i=1}^k \frac{(C_{CY,i}^{\infty} - \hat{\mu}_{CY}P_i)^2}{(P_i)^2} \quad (3.11).$$

Minimizing the variance of this expression to find the expected loss ratio we arrive at the estimator

$$\hat{\mu}_{CY} = \frac{1}{k} \sum_{i=1}^k \frac{C_i}{P_i} \quad (3.12)$$

which is simply the average loss ratio. The standard deviation per premium within this variance structure becomes

$$\frac{STDEV(C_{CY,i}^{\infty})}{P} = \hat{\sigma} \quad (3.13)$$

and we thus have a method where the standard deviation per premium is not expected to decrease when we have a larger exposure.

In theory, other variance structures could also be considered under the Normal assumption, but due to reasons explained below the linear and quadratic variance structures are the most relevant ones.

3.4 Method 3: LogNormal Loss ratio with quadratic variance

The third method is based on the variance structure and assumptions in method 2, i.e. quadratic variance and independent accounting years, but where we instead assume a LogNormal distribution for the loss. We assume that

$$\ln(C_{CY,i}^{\infty}/P_i) \sim N(\pi, \varphi^2) \quad (3.14)$$

where the parameters have no premium dependence. We have as mentioned a variance structure similar to method 2, i.e. we have that

$$E[C_{CY,i}^{\infty}] = P_i \mu_{CY} \quad (3.15)$$

$$V[C_{CY,i}^{\infty}] = P_i^2 \sigma^2 \quad (3.16)$$

This structure is actually crucial to make the parameters in (3.14) independent of the premium, since exactly this variance structure will make the relative standard deviation of the loss independent of premium volume. This fact is exploited in this method, because it makes it possible to derive analytical expressions for estimators of the parameters in the proposed model.

Now, we know also that the expected value and variance of the ultimate loss can be expressed in terms of the two parameters

$$E[C_{CY,i}^{\infty}/P_i] = \mu_{CY} = e^{\pi + \varphi^2/2} \quad (3.17)$$

$$V[C_{CY,i}^{\infty}/P_i] = \sigma^2 = (e^{\varphi^2} - 1)e^{2\pi + \varphi^2} \quad (3.18)$$

where we note that the expected value and variance of the loss ratio does not depend on the premium, due to cancellation effects in the variance structure. Inverting (3.17) and (3.18) we get

$$\pi = \ln(\mu_{CY}) - \frac{1}{2} \ln(1 + \sigma^2/\mu_{CY}^2) \quad (3.19)$$

$$\varphi^2 = \ln(1 + \sigma^2/\mu_{CY}^2) \quad (3.20)$$

Note that we have due to the Normal property in (3.14) that MVUE²'s of the parameters in the Normal distribution are

$$\hat{\pi} = \frac{1}{k} \sum_{i=1}^k \ln(C_{CY,i}^{\infty}/P_i) \quad (3.21)$$

$$\hat{\varphi}^2 = \frac{1}{k-1} \sum_{i=1}^k (\ln(C_{CY,i}^{\infty}/P_i) - \hat{\pi})^2 \quad (3.22)$$

Inverting (3.19) and (3.20) back and combining with the estimators we finally get that

$$\hat{\mu}_{CY} = e^{\hat{\pi} + \frac{1}{2}\hat{\varphi}^2} \quad (3.23)$$

² MVUE stands for Minimum Variance Unbiased Estimates.

$$\hat{\sigma}^2 = \hat{\mu}_{CY}^2 (e^{\hat{\varphi}^2} - 1) \quad (3.24)$$

with the estimators for π and φ given by (3.21) and (3.22). Thus we have derived the estimators in a quadratic variance structure, similar to method 2, but with a Lognormal distribution assumption. This gives the same relative standard deviation per premium as in method 2, i.e.

$$\frac{STDEV(C_{CY,i}^{\infty})}{P} = \hat{\sigma} \quad (3.25)$$

where we however have other estimators for the parameters and obviously a different shape of the distribution. The change of distribution assumption as such, assuming a given standard deviation, effectively implies higher capital requirements since percentiles in the tail of the distribution will be further from the mean with a Lognormal distribution.

Considering also the linear variance structure under the Lognormal distribution is of course possible, but it unfortunately leads to expressions which needs to be solved numerically and those will not be considered in this thesis. As an approximation, the parameters estimated under the Normal distribution assumption can be used in a Lognormal distribution, after transformation using (3.19) and (3.20) modified to be based on a linear variance structure instead of a quadratic one.

Due to reasons discussed later, the linear and quadratic variance structures are the natural structures to consider due to their principal behavior in terms of the premium. It is also important to remember that under constant or nearly constant premium volume (exposure) historically, the estimated parameters should not differ to a large degree given that the estimate of the future premium income is in line with the historical figures.

3.5 Method 4: Compound Poisson with no parameter error

Method 1 – 3 deals with the general problem of estimating the uncertainty in the loss ratio based on the observed historical losses and premiums per accounting year. The methods do not really try to break down the uncertainty into parts, and thus does not really provide any insight into what actually are the drivers for the volatility as such. A straight forward way to do so is to consider the total loss explicitly as the sum of a random number of random variables.

We do so and as a starting point assume that the number of claims N follows a Poisson distribution with parameter λ , i.e. that $N \sim \text{Poisson}(\lambda)$ and we consider the distribution of the total loss

$$C_{CY,i}^{\infty} = \sum_{i=1}^N Y_i \quad (3.26)$$

where Y_i is the distribution of the severity of claim i . Claim distributions in this form is commonly referred to as collective models, see for instance Johansson 2008. We assume further that Y_i and Y_j are identically distributed and mutually independent $\forall i \neq j$ and independent of N . Under these assumptions the mean and variance can easily be expressed in terms of the Poisson parameter and the corresponding measures of the severity distribution (see for instance Gut 2009) by conditioning on the number of claims variable:

$$E[C_{CY,i}^{\infty}] = E[E[C_{CY,i}^{\infty}|N]] = E[NE[Y_i]] = \lambda E[Y_i] \quad (3.27)$$

$$V[C_{CY,i}^{\infty}] = E[V[C_{CY,i}^{\infty}|N]] + V[E[C_{CY,i}^{\infty}|N]] = E[NV[Y_i]] + V[NE[Y_i]] = \lambda(V[Y] + E[Y]^2) \quad (3.28)$$

Thus to use this method the first and second moments of the individual claim data is used, to be able to estimate the mean (the average claim size) and variance of the severity density. These can of course be estimated using standard estimators for the sample mean value and sample variance respectively. Also, we need an estimate of the frequency for the next account year that we want to model, which will then be used as the parameter in the Poisson distribution.

The premium for the next year can be expressed in terms of a risk premium, equal to the expected claims outcome $\lambda E[Y_i]$, times a constant larger than 1 (to make profit), c say. Introducing the coefficient of variation for the severity as $v = \frac{\sqrt{V[Y]}}{E[Y]}$ this means that the standard deviation per premium becomes

$$\frac{STDEV(C_{CY,i}^{\infty})}{P} = \frac{\sqrt{\lambda(V[Y]+E[Y]^2)}}{P} = \frac{\sqrt{\lambda(V[Y]+E[Y]^2)}}{c\lambda E[Y_i]} = \sqrt{\frac{\lambda(V[Y]+E[Y]^2)}{c^2\lambda^2 E[Y_i]^2}} = \frac{1}{c} \sqrt{\frac{1+v^2}{\lambda}} \quad (3.29).$$

Formulated like this, we see that we essentially have the same behavior as in method 1, i.e. we have a variance structure with a linear dependency in the volume, since the expected claims frequency is a linear function of the exposure given a homogenous portfolio of policies. Actually, this is a direct consequence of the choice of the Poisson distribution for the claim frequency, as can be seen from (3.28). The variance structure implies as in method 1 that the standard deviation per premium to the square root of the inverse of the volume. Note also that we have made assumptions about the severity distribution Y , only in terms of the expected value and variance. We see here the clear benefit of having a model explicitly in terms of the frequency and severity; we can in this setup among other things distinguish portfolios having the same volatility in terms of the historical LR, in terms of

identifying if the volatility is mainly due to portfolio size, high severity coefficient of variation or a combination of the two.

We might be interested in the exact or approximate density of the total loss $C_{CY,i}^\infty$, and in principle we would then need to specify the density of the severity. However in practice it turns out that deriving the distribution of the total loss will in most cases not lead to anything analytical, and in practice numerical methods has to be considered (see Johansson 2008). However in the description of method 6 an elegant recursive method, Panjer recursion, is presented. It can be used to numerically derive the distribution of the total loss is presented, and the method holds for a general class of frequency distributions and not only the Poisson distribution.

Thus, for this method to stand on its own we need a prior view on the distribution of the total loss, which parameters can be set to match the mean and standard deviation derived above. A typical choice is the Lognormal distribution which is the choice of total loss distribution, although on a more aggregated level, in the Standard Formula (European Commission 2010).

3.6 Method 5: Compound Poisson with frequency parameter error

One issue with methods with linear variance structures, method 1 and 4 being two examples, are the fact that they imply a relative standard deviation that is in terms of the portfolio size strictly decreasing and even converges to zero as the portfolio size goes to infinity. This means that however large the portfolio is, we can always make it larger to gain even further diversification effects. This actually conflicts with empirical loss ratio data, which suggest that even though portfolios indeed can be further diversified by growth, the positive effect (in terms of for instance relative standard deviation) of having a larger portfolio will become smaller and smaller as the volume increases (see AonBenfield 2009). Thus even method 2 and 3 can be questioned in this respect, but they are on the other end of the scale: they assume that the volatility is independent of premium volume, which only seems reasonable when it comes to very large portfolios. The idea of method 5 is to extend method 4 by introducing a parameter error in the frequency distribution, to achieve the principal behavior implied by data.

We introduce as in method 4 the total loss in terms of the frequency and severity

$$C_{CY,i}^\infty = \sum_{i=1}^N Y_i \tag{3.30}$$

We assume once again that Y_i and Y_j are identically distributed and mutually independent $\forall i \neq j$ and independent of N . Now we introduce a random variable θ , independent of the frequency and severity distributions. We assume $E[\theta] = 1$ and assume that the frequency is

Poisson distributed conditioning on the outcome of this variable, which is defined to be a multiplicative factor to the Poisson parameter λ . I.e. the frequency distribution will fulfill

$$N|\theta \sim \text{Poisson}(\lambda\theta) \quad (3.31)$$

which effectively mean that the variance of the frequency will be larger than λ given that we have $V[\theta] > 0$. We now derive what this means in terms of the relative standard deviation per premium and derive estimators for the additional parameter in this model. The mean of this distribution is straight forward to find by applying twice the general formula $E[X] = E[E[X|Y]]$ (see Gut 2009):

$$E[C_{CY,i}^\infty] = E[E[C_{CY,i}^\infty|\theta]] = E[E[E[C_{CY,i}^\infty|N]|\theta]] = E[E[\lambda E[Y_i]\theta|\theta]] = \lambda E[Y_i]E[\theta] = \lambda E[Y_i] \quad (3.32)$$

and we see that we have the mean value as in method 4. We now look at the variance of the total loss and apply twice the general formula $V[X] = E[V[X|Y]] + V[E[X|Y]]$ (see Gut 2009) and get

$$V[C_{CY,i}^\infty] = E[V[C_{CY,i}^\infty|\theta]] + V[E[C_{CY,i}^\infty|\theta]] = E[\theta\lambda(V[Y] + E[Y]^2)|\theta] + V[E[C_{CY,i}^\infty|\theta]] = \lambda(V[Y] + E[Y]^2) + V[\lambda E[Y]\theta] = \lambda(V[Y] + E[Y]^2) + \lambda^2 E[Y]^2 V[\theta] \quad (3.33)$$

Again introducing the coefficient of variation of the severity distribution as $v = \frac{\sqrt{V[Y]}}{E[Y]}$ and the premium as a risk premium times a factor c , i.e. $P = c\lambda E[Y]$, we get the standard deviation per premium

$$\frac{STDEV(C_{CY,i}^\infty)}{P} = \frac{\sqrt{\lambda(V[Y]+E[Y]^2)+\lambda^2 E[Y]^2 V[\theta]}}{P} = \frac{\sqrt{\lambda(V[Y]+E[Y]^2)+\lambda^2 E[Y]^2 V[\theta]}}{c\lambda E[Y_i]} = \sqrt{\frac{\lambda(V[Y]+E[Y]^2)+\lambda^2 E[Y]^2 V[\theta]}{c^2 \lambda^2 E[Y_i]^2}} = \frac{1}{c} \sqrt{\frac{1+v^2}{\lambda} + V[\theta]} \quad (3.34).$$

We see that we have a model which have one term with the principal behavior of a linear variance structure (decreasing with volume as the inverse of the square root of the volume) and one term behaving like the quadratic variance structure (constant with volume). This is more consistent with empirical data (see AonBenfield 2009) and the model can be interpreted as having one term corresponding to a pure random risk and a second part corresponding to a systematic risk or parameter risk. Figure 3.1 shows a principal diagram over this, with the uncertainty as a function of the volume. The line with random risk could be seen as a representative of the principal behavior of method 1 and 4, and the systematic risk can be seen as the behavior of method 2 and 3.

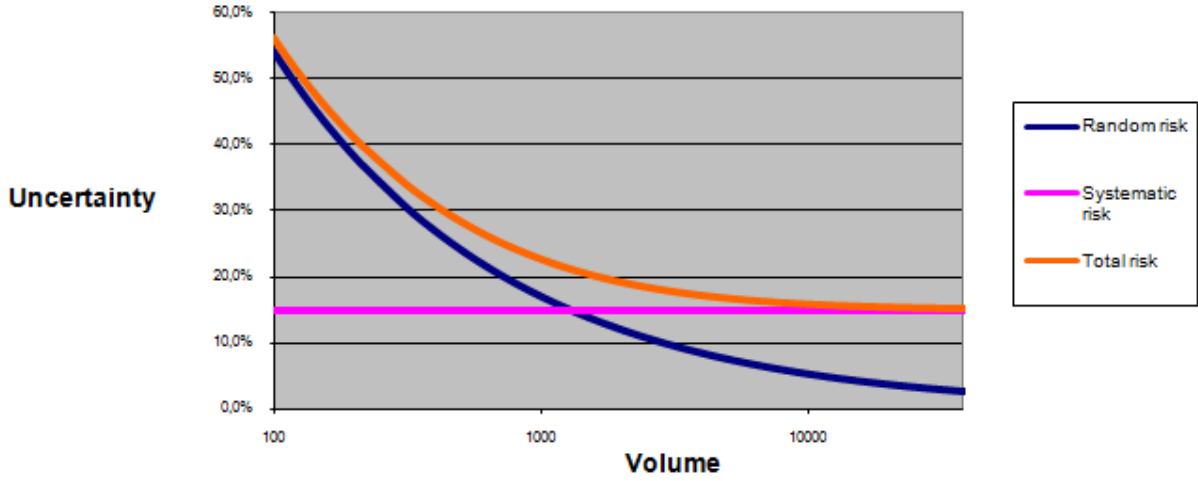


Figure 3.1: Uncertainty as a function of the volume, illustrating the principal behavior of method 5.

In regards of estimating parameters of this model, we can as in method 4 estimate the expected frequency and first and second moments of the severity distribution using standard estimators applied on historical claim data. Thus the only remaining parameter to estimate is the parameter error term $V[\theta]$. Assuming that we have annual (or whatever time horizon is considered) data of frequencies for year i , N_i , corresponding estimate of the frequency prior to observing the year v_j and earned premiums P_i and say that we have J observations, one can derive using a Bühlmann-Straub credibility model (see Gisler 2009) that an unbiased estimator is given by

$$\hat{V}[\theta] = \left(\frac{J}{dv_*}\right) \left(\frac{V_F}{\bar{F}} - 1\right) \quad (3.35)$$

where

$$d = \sum_{j=1}^J \frac{v_j}{v_*} \left(1 - \frac{v_j}{v_*}\right) \quad (3.36)$$

$$F_j = \frac{N_j}{v_j} \quad (3.37)$$

$$\bar{F} = \sum_{j=1}^J \frac{v_j}{v_*} F_j \quad (3.38)$$

$$V_F = \frac{1}{J-1} \sum_{j=1}^J v_j (F_j - \bar{F})^2 \quad (3.39)$$

$$v_* = \sum_{j=1}^J v_j \quad (3.40)$$

If the a priori estimates of the frequency per accident year are not available, one can assume for instance a linear model for the a priori frequency per exposure v_j/P_i . For details on data estimation, please see Chapter 4.

Looking at (3.35) it is obvious that we have one part measuring the variance of the frequency deviations divided by the average a priori frequency, and since the ratio of the variance and mean of a Poisson distribution is always 1 (3.35) will effectively tend asymptotically to 0 when we have no parameter error. As in method 4, the a priori view on the total loss distribution is still needed.

3.7 Method 6: Compound with a Panjer class frequency distribution

We will now consider a model similar to the model underlying method 4, the pure compound Poisson model, where we instead assume a general class of frequency distributions. We will also show that the unconditioned frequency distribution implied by method 5 with a parameter error is actually a special case of the consequently more general method 6. We consider again the total loss in terms of frequency and severity:

$$C_{CY,i}^{\infty} = \sum_{i=1}^N Y_i \quad (3.41)$$

We assume once again that Y_i and Y_j are identically distributed and mutually independent $\forall i \neq j$ and independent of N . Now we assume that N , which naturally has a positive discrete distribution, with the probability mass at point k defined recursively:

$$P[N = k] = p_k = \left(a + \frac{b}{k}\right)p_{k-1} \quad (3.42)$$

for $k \geq 1$ for some constants a and b fulfilling $a + b \geq 0$ and with p_0 defined so that the total density equals to 1, i.e. so that $p_0 = 1 - \sum_{i=1}^{\infty} p_i$. The family of distributions satisfying (3.42) is called the Panjer class, which consists of the Poisson, Binomial and the Negative Binomial distributions without any additional constraints on the their parameters (see Panjer 1980). As we discussed earlier, finding the exact distribution for (3.42) with a general severity distribution is not mathematically tractable, and this holds of course also to an even greater extent when we have a more general view on the frequency distributions.

Instead we will consider a numerical method that is valid for distributions of the Panjer class. We assume that we can discretize the continuous severity distribution on a lattice with width $h > 0$, and we introduce

$$f_k = P[X_i = hk] \quad (3.43)$$

with h chosen small enough to represent the continuous distribution. Defining $g_k = P[C_{CY,i}^\infty = hk]$ it can be shown (see Panjer 1980) that the recursive representation

$$g_k = \frac{1}{1-f_0a} \sum_{i=1}^k \left(a + \frac{bi}{k} \right) f_i g_{k-i} \quad (3.44)$$

where $g_0 = \frac{p_0}{(1-f_0a)^{1+b/a}}$ will actually lead to the computation of the distribution of the total loss, with the only approximation being related to the discretization of the severity distribution. This recursive method is called Panjer recursion and can be used to produce a numerical representation of the total claims distribution, which effectively will make it possible to numerically estimate the distribution for any severity distribution in combination with all frequency distributions in the Panjer class. An alternative method would of course be to simulate directly from (3.41) to form the empirical distribution, but note that most portfolios have a large expected number of claims which would make such a simulation very time consuming.

As mentioned earlier, it can be shown that method 6 can produce a distribution similar to the ideas in method 5, at least in terms of the expected values and relative standard deviations. This turns out to be achievable by choosing the Negative Binomial distribution as the frequency distribution. To see why, note that the split of random and systematic risk in method 5 was achieved by having a parameter error in the parameter in the Poisson distribution for the frequency. If we introduce a distribution for the Poisson parameter and can achieve any variance of this parameter error, we have a model which is as general as method 5 where the variance is a free parameter. If we now assume that $N|\lambda \sim \text{Poisson}(\lambda)$ and that $\lambda \sim \text{Gamma}(r, p/(1-p))$ we can show this.

The total distribution for N will be by the law of total probability

$$f_N(k) = \int_0^\infty f_{N|\lambda}(k) f_\lambda(\lambda) d\lambda = \int_0^\infty \frac{\lambda^k}{k!} e^{-\lambda} \lambda^{r-1} \frac{e^{-\lambda(1-p)/p}}{\Gamma(r) \left(\frac{p}{1-p}\right)^r} d\lambda = \{ \dots \} = \frac{\Gamma(r+k)}{k! \Gamma(r)} (1-p)^r p^k \quad (3.45)$$

which is a negative binomial distribution with parameters r and p . Now since $p \in (0,1)$ and $r > 0$ is allowed in the negative binomial distribution, we can achieve any positive parameters wanted in $N \sim \text{Gamma}(r, p/(1-p))$. To see why, note that the mean of the Gamma distribution is $rp/(1-p)$ and the variance is $r(p/(1-p))^2$. By choosing r and p so that $rp/(1-p) = 1$ we will have a variance of $p/(1-p)$, so by the choice of p we can obviously achieve any variance. Thus, by the proper choice of parameters in the Negative Binomial distribution, we can achieve the same kind of variance structure as in method 5. We have however in this method also the possibility to recursively find the approximate total loss distribution,

which is not possible with method 5 which will only produce an expected value and variance of the total loss distribution.

We will not consider any empirical results from method 6, since it is not a method to produce estimates for the considered distributions, but rather a way to achieve the total compound distribution given the a priori view of the frequency distribution combined with the chosen severity distribution.

3.8 Separating frequency claims and large claims

We mentioned earlier that catastrophe related claims are usually modeled separately from premium risk. The same concept can actually be applied for the premium risk as such, even when excluding catastrophe risk, to improve the accuracy of the methods proposed. One usually tries to separate *individual* claims in the total loss which has a large severity, so that the portfolio will consist of one part with large claims and one part with the rest, usually called frequency claims, which are estimated with the methods proposed in earlier subsection. In general, the total loss is split into several layers:

$$C_{CY}^{\infty} = \sum_{i=1}^m C_{CY,i}^{\infty} \quad (3.46)$$

where $C_{CY,i}^{\infty}$ is the random variable representing the loss in layer i . We define WLOG the bottom layer as the frequency claim layer and the layers above we define as the large claims layers (and we thus split based on severity). This way of splitting the losses has mainly three purposes:

- It can be used to improve the homogeneity of the portfolio modeled, in the sense that the resulting frequency and large claims portfolios might be closer to the usual assumption stating that all claims have the same severity distribution. Thus the split might lead to more correct modeling of the stochastic properties of the total loss distribution.
- Say that we have a model in which we simulate the total loss by, in each simulation, draw the frequency and then draw the number of losses from the severity distribution. To improve run-times, it is a good idea to try to characterize the total loss instead by its distribution and draw from that directly. However, if the insurance portfolio that we try to model has an excess of loss reinsurance program in place, we still have a need to simulate the very large claims (at least above the attachment of the reinsurance program) to take reinsurance into account properly. By having separated models for frequency claims and large claims, the run times when simulating the total loss can be improved through the simulation of the total

frequency claim loss through a single distribution, while maintaining the possibility to net down individual large claims in respect of reinsurance.

- Having large claims separated is beneficial from an ‘uncertainty forecasting’ perspective. To see why, assume that we have parameterized the frequency claims and the large claims models separately. If we now know that we will have a significantly different exposure to large claims for the next year in terms of the frequency, we can rather easily incorporate that into our solvency model. If we instead have a consolidated model for the whole loss result, it is much harder to correct for such an effect in a sound way.

Of course, in practice the modeling of large claims can be carried out in several different layers with different frequency and/or severity distributions. We are dealing with the following general situation of different modeling layers, expressed in the severity distributions:

$$C_{CY}^{\infty} = \sum_{i=1}^m \sum_{j=1}^{N_i} Y_{ij} \quad (3.47)$$

where m is the number of modeling layers, N_i is the frequency distribution for layer i and Y_{ij} is the severity distribution for claim j within layer i (i.e. conditioning on i all Y_{ij} are i.i.d). As usual, we assume that all claims within the same layer have the same severity distribution and that all variables Y_{ij} and N_i are mutually independent $\forall i, j$. Deciding which layers to model will be a question of standard statistical diagnostics during distribution fitting, using normal techniques for goodness of fit and similar, as well as practical considerations.

We now for simplicity assume that we have only one large layer, for claims above a threshold y . Given that the large claims limit is chosen so that the resulting frequency will be relatively small, we can argue for independence between frequency claim result (the lower layer) and the large claim result (usable during simulation) and we can argue for a pure Poisson distribution without any parameter error, by the law of rare events (see Resnick 1992). By differentiating the log-likelihood one can easily derive (see Lindgren 1993) from the Poisson density that the MVUE for each layer is equal to the empirical frequency, i.e. for the frequency ω_i for layer i we have

$$\widehat{\omega}_i = \frac{1}{k} \widehat{N}_i \quad (3.48)$$

where N_i is the observed number of claims in the layer and k is the number of observation years.

A Pareto distribution is commonly chosen for the severity (see Gisler 2004 and Rytgaard 1990), i.e.

$$F_{X_i}(x) = 1 - \left(\frac{x}{y}\right)^\alpha \quad (3.49)$$

If y is treated as a fixed quantity (i.e. it is not estimated based on the dataset), the unbiased M.L.E for a dataset $\{x_1, \dots, x_n\}$ can easily be derived by taking the derivative of the log-likelihood estimator and correcting for the bias in the resulting estimator, in which case we arrive at the MVUE (see Rytgaard 1990):

$$\hat{\alpha} = \left(\frac{1}{n-1} \sum_{i=1}^n \left(\frac{x_i}{y}\right)\right)^{-1} \quad (3.50)$$

For a more detailed review of estimators of Pareto distribution, please see Rytgaard 1990. This chapter is mainly here for illustrational purposes; no numerical examples on parameter estimation for large claims will be included since the frequency claim modeling is the main area of interest within the thesis. Still, this chapter may serve as guidance for practical implementations of premium risk or further theoretical considerations.

3.9 Possible extensions of the methods

Methods 1 – 3 are methods which estimate the uncertainty in Loss Ratio outcomes by estimating parameters in the a priori suggested distributions by considering the Loss Ratio as such. Possible extensions of the methods are to consider other distributions, if one has another a priori view than Normal or Lognormal for the Loss Ratios, after which corresponding estimators can be derived. Another possible extension is to consider estimating even higher moments of the distributions than just the first and the second. On the other hand, as discussed in chapter 3.1, the number of observations is typically low, meaning that higher moment estimators based on data might not be statistically sound to use but should instead be a natural consequence of the a priori choice of distribution and the lower moment estimators. Last but not least a possibility could be to consider a variance structure for the Loss Ratio directly similar to the one achieved in method 5, which is essentially a combination of the linear and quadratic variance structures. On the other hand, method 5 has these features and is a more explicit method; it gives as previously discussed a better understanding of the resulting uncertainty than a pure Loss Ratio based approach.

When it comes to method 4 & 5 they also have room for further extensions. One example is that both methods could be expanded to include a parameter error for the severity distribution, in order to make it a stochastic quantity as well. It however often turns out that it is more natural to consider the parameter error in the frequency only (see Gisler 2009),

especially since one of the more important drivers for severity stochastics is inflation which is usually modeled separately. Another possible extension of method 4 & 5 is to consider other frequency distributions than the Poisson distributions, but as discussed in chapter 3.7 this often does not lead to any analytical estimators. Method 6, which is a numerical method, is more general in that respect and it also allows for any severity distribution and the most common frequency distributions, and extending that method is not that straight forward and not very natural since most modifications will ruin the assumptions that need to be fulfilled for using the method in the first place. Note that method 6 requires the frequency to fulfill the iterative requirement given by (3.42) for $k \geq 1$. It can be interesting to note that the recursive method can be extended to hold also for the frequency above an arbitrary natural number j . In more formal terms: it can be shown that under the condition that the distribution for the frequency N fulfills

$$P[N = k] = p_k = \left(a + \frac{b}{k}\right)p_{k-1} \quad (3.51)$$

for $k \geq j$ with the usual assumption that the above holds for some constants a and b fulfilling $a + b \geq 0$. For details, please see Hess et al 2002.

4. Methods applied on data and estimation errors

In this chapter we discuss practical estimation in regards of handling of data, we discuss estimates from the different methods applied on a few datasets, and we discuss the topic of estimations error. The last topic is of particular interest since the number of observations available for non-life solvency modeling is typically limited.

4.1 One-year vs ultimate view in terms of data

In chapter 2.3 the premium risk was discussed in the light of the time horizon considered, and it was concluded that the time horizon considered is indeed a relevant choice when dealing with premium risk. We also concluded that going from ultimate to limited-time risk either does not have a large impact (short-tailed lines), or can be done in a rather straight forward way using ideas from reserving (long-tailed lines). An alternative way to deal with this issue was mentioned and is further presented here, where the idea is to correct for the time horizon in respect of the data rather than scaling down the ultimate estimates

Consider for instance the case where we use one of the methods 1, 2 or 3, which estimate the uncertainty directly from loss ratio outcomes. If we take the ultimate approach, the values to consider are simply the *latest estimates* of the ultimate loss ratios. This of course means in practice that all the historical accident years will not be equally developed, since older years typically will have a higher paid to ultimate ratio than more recent years. On the other hand, if we want to measure the ultimate risk we should use the latest estimates of the ultimate loss ratios, since it is the ultimate amounts that we want to measure volatility on, i.e. the quantity that we earlier denoted by C_{CY}^{∞} . As an alternative approach, we could instead use the ultimate loss ratio as they were estimated *at the end of the accident year in question*, i.e. if we look at accident year 1995 we consider the ultimate loss as it was estimated at the end of the year. This will lead to estimates consistent with the one-year view in Solvency II, since we concluded earlier that the quantity $C_{CY} + R_{CY}$ was the one of interest in this case. The same thinking can in a straight forward way be used in conjunction with the other methods as well, but applied on frequencies, severities and other parameters instead.

It is not obvious which of the two methods, which imply in practice either scaling down ultimate estimates or using the ‘right’ data quantities directly, is the preferable one. In many practical situations, the choice is likely to boil down to which of the two different data types actually is available. If both are available the basis for the decision might rather be related to the quality of the data, rather than being a direct consequence of theoretical considerations.

4.2 Issues with data and cleaning of outliers

When applying the methods described, it is highly important to make sure that the data is consistent with the assumptions made when deriving the estimators, to the extent possible. A few practical considerations are the following:

- Where applicable, historical figures need to be adjusted in respect of inflation. Usually CPI inflation is used, despite the fact that calendar year effects on insurance claim payments usually are not identical to the overall CPI inflation. The claim specific inflation is in most cases hard to quantify and thus in practice it could be preferable to remove only CPI inflation and let the overall uncertainty measures include any additional non-CPI inflation in historical data.
- As has been discussed, catastrophe events are usually modeled using separate models. This implies that historical data need to be cleaned in respect of catastrophe events, which will constitute outliers that can be removed with good conscience. In practice this is usually straight forward, since aggregated figures for catastrophe events are often needed for reinsurance purposes.
- For instance method 4 and 5 includes an explicit assumption on the severity distribution being the same for all claims for the coming year, which puts requirements on the homogeneity of the portfolio considered. In practice it is never possible to comply to this assumption completely, but it illustrates that it is important to estimate parameters such that a reasonable level of homogeneity is reached. Similarly, also assumptions around independence between claims need to be considered when grouping exposures together.
- As with more or less all risk models, it is important to realize that uncertainty estimated on historical data might not be representative for future outcomes. It could be due to parameter errors, but also due to introduction of new products and similar. The risk of estimation errors due to this can be mitigated by proper consideration and information around the difference between the historical portfolio and the portfolio in the modeling period.
- In some of the methods, premiums are suggested as exposure measures. In practice, premiums might be a poor measure of exposure due to changes in margins ('premium rate') over time, in which case other exposure measures should be considered. Using other exposure measures might not be as practical, but should as a good practice always be considered.

- Determining the data quality of relatively old data could be a rather cumbersome task, and uncertainty around the quality can definitely be enough of motivation to consider omitting parts of data series.

The above list is not a complete list of all thinkable issues that might arise during application of the methods presented in this thesis. On the other hand, it serves as a list of a few potential issues that one needs to overcome in practical implementations.

4.3 Methods applied on data

It is now time to apply the methods presented on a number of datasets with different characteristics. As discussed before, we always take the ultimate view in this thesis when it comes to practical estimation; although we have showed heuristically that it is straight forward to derive estimates consistent with the one-year horizon, where there also is some freedom methodology wise dependent on the data available. We will consider methods 1 to 5, but not method 6 since it is just a numerical method to compute an approximation of the total loss *given the parameters* rather than a method to derive estimates of the parameters in the respective methods.

We now look at estimates of uncertainty with the presented methods applied on datasets from the company If P&C Insurance. All datasets have been anonymized in respect of what portfolio they represent, and are only characterized as being either short-tailed or long-tailed. We will concentrate our discussion around the most natural quantity to consider in regards of premium risk, which is standard deviation per premium. Methods 1, 4 and 5 results in standard deviations per premium which are a function of the volume for the coming year, and in those cases the volume from the last observation has been used in the tables of results. Three portfolios are analyzed:

- Portfolio 1, which is a relatively large short-tailed portfolio with roughly 100 000 claims per year.
- Portfolio 2, which is a relatively small long-tailed portfolio with roughly 3 000 claims per year.
- Portfolio 3, which is a relatively small short-tailed portfolio with roughly 500 claims per year.

We start with looking at the results when the methods are applied on Portfolio 1, see the results in table 4.1. Yearly data from 1993 to 2010 was used. Note in the table that the reason for the total SD in method 5 not being the sum of the random and the systematic SD is due to the way that the variance structure is specified, see the effect in formula (3.34).

Portfolio 1	Random SD/P	Systematic SD/P	Total SD/P
Method 1			6,1%
Method 2			6,6%
Method 3			6,6%
Method 4	1,5%		1,5%
Method 5	1,5%	6,8%	6,9%

($\lambda = 100\ 000$, $\nu = 5,5$)

Table 4.1: Resulting SD per premium for the methods, applied on Portfolio 1.

Here we see that the estimates from method 1,2,3 and 5 seem to be relatively close to each other. The outlier is clearly method 4. Since method 1,2 & 3 are loss ratio based, they do not really provide any insight into why the estimate from method 4 is low compared to the other methods. However, method 5 provides this insight by the way that the estimate is constructed. Note that method 4, the compound Poisson method, consists only of random variation resulting from the fact that the portfolio is of limited size. Since the portfolio in this case is fairly large, with a frequency of 100 000 claims / year, the random risk will be relatively low and it can be seen that the systematic risk is the driver of volatility in this case.

This example clearly outlines the difference between using a loss ratio method and a more explicit method, where the latter is preferable since it provides more insight into the nature of the total volatility estimate. We also see that the difference between the linear and the quadratic variance structure, method 1 and 2 respectively, is relatively low when having the Normal distribution assumption. Going from Normal to Lognormal distribution with a quadratic variance structure, i.e. going from method 2 to 3, we see that the difference is not significant. As mentioned earlier, we have assumed in table 4.1 that the volume for the coming year is equal to the latest observation. Having a different assumption will result in different effects depending on the method used. For an illustration of this, see figure 4.1, where the premium is normalized to 1 as the ‘base case’ equal to the premium from last year. Note that methods 2 and 3 coincide in this particular example.

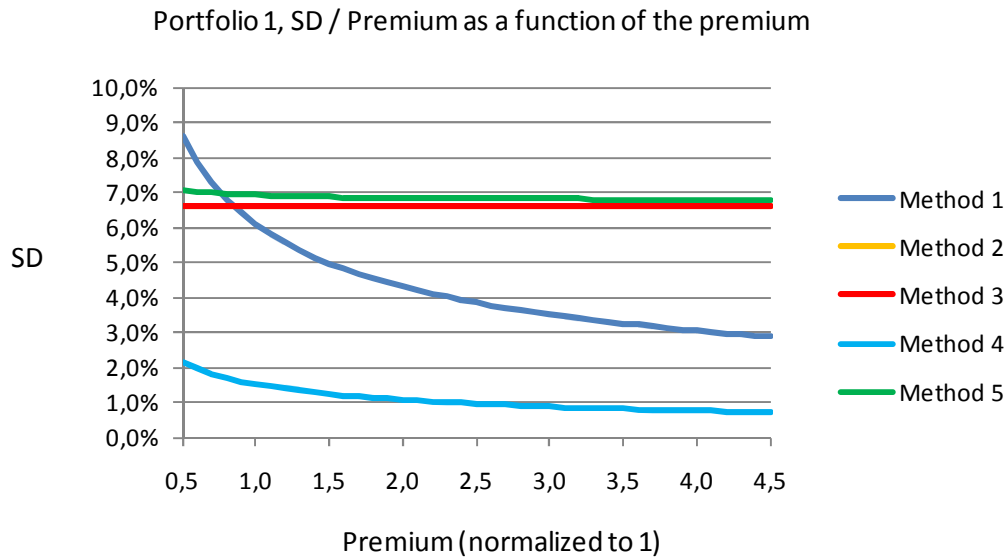


Figure 4.1: Standard deviation per premium as a function of the premium (methods 2 & 3 coincide).

Here we see that, as implied by the variance structures, the standard deviation per premium is not a function of the premium for method 2 and 3. Method 1 and 4 however has a strong premium dependency, implied by their linear variance structures. Method 5, which can be seen as a combination of the linear and the quadratic variance structures, has a small but non-zero premium dependency. The small dependency is simply motivated by the fact that the systematic risk rather than the random risk is driving the overall volatility for this portfolio.

We now look at the more long-tailed Portfolio 2, which compared to Portfolio 1 has a lower frequency. Yearly data from 1994 to 2010 has been used, for more recent year, Chain Ladder has been used to estimate the ultimate frequencies needed in the estimation of systematic risk in method 5. The results are found in table 4.2.

Portfolio 2	Random SD/P	Systematic SD/P	Total SD/P
Method 1			10,8%
Method 2			12,8%
Method 3			12,1%
Method 4	5,1%		5,1%
Method 5	5,1%	11,2%	12,3%

($\lambda = 3\ 000$, $\nu = 3,5$)

Table 4.2: Resulting SD per premium for the methods, applied on Portfolio 2.

This portfolio has a significantly larger random risk, but still the systematic risk is the largest driver of volatility. The difference between methods 1, 2 and 3 are larger in this case, but the differences are not very significant consider the limited number of observations (please see chapter 4.4 for some conclusions around this). Again we see that method 4 underestimates the volatility as it considers only the random risk and excludes the systematic risk. We see also again that the estimate derived from method 5 seem to be in line with the direct loss ratio methods. We now plot the total standard deviation per premium as a function of the premium, see figure 4.2.

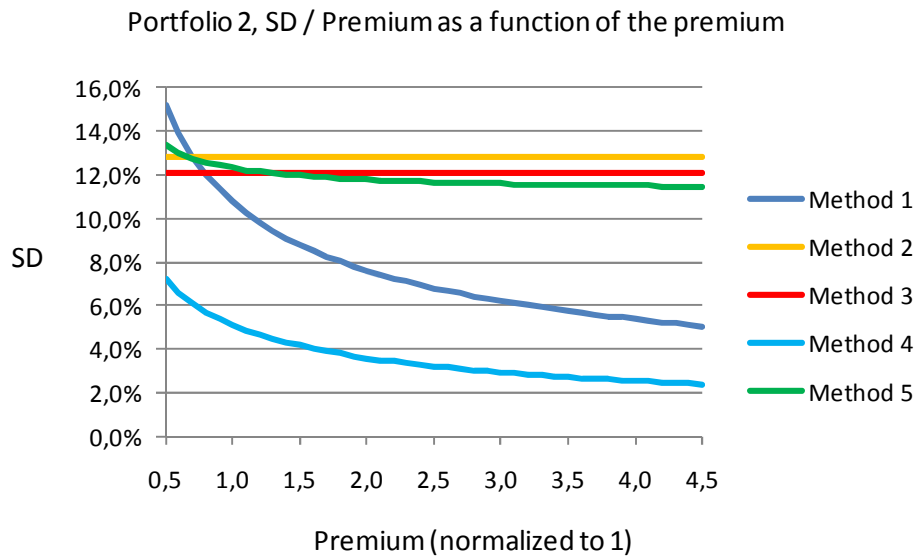


Figure 4.2: Standard deviation per premium as a function of the premium.

Since the systematic risk is driving a large proportion of the overall volatility, the plots are principally similar for Portfolio 1 and Portfolio 2. One difference worth mentioning is that the premium dependency in method 5 is now stronger, since the random risk, measured in percent of the systematic risk, is now larger than for Portfolio 1.

We now look at a short tailed portfolio which a relatively small frequency of 500 claims per year, deonted by Portfolio 3. Yearly observations from 1994 to 2010 were used, and the results are summarized in table 4.3.

Portfolio 3	Random SD/P	Systematic SD/P	Total SD/P
Method 1			28,0%
Method 2			30,9%
Method 3			29,0%
Method 4	22,2%		22,2%
Method 5	22,2%	9,2%	24,0%

($\lambda = 500, \nu = 6,0$)

Table 4.3: Resulting SD per premium for the methods, applied on Portfolio 3.

Also for this portfolio, the loss ratio based methods produce estimates which are in a similar range. Due to the size of this portfolio, it seems that the overall volatility is driven by the random risk rather than the systematic risk. This makes the difference between method 4 and method 5 smaller than for the other portfolios, which is a natural consequence. The estimate produced by method 5 is in this case smaller than the estimate from the loss ratio methods, which could possibly be an effect of the absence of a parameter error for the severity (see chapter 3.9 for a discussion around this possible extension). As for the other portfolios, we look at standard deviation per premium as a function of the premium, in figure 4.3.

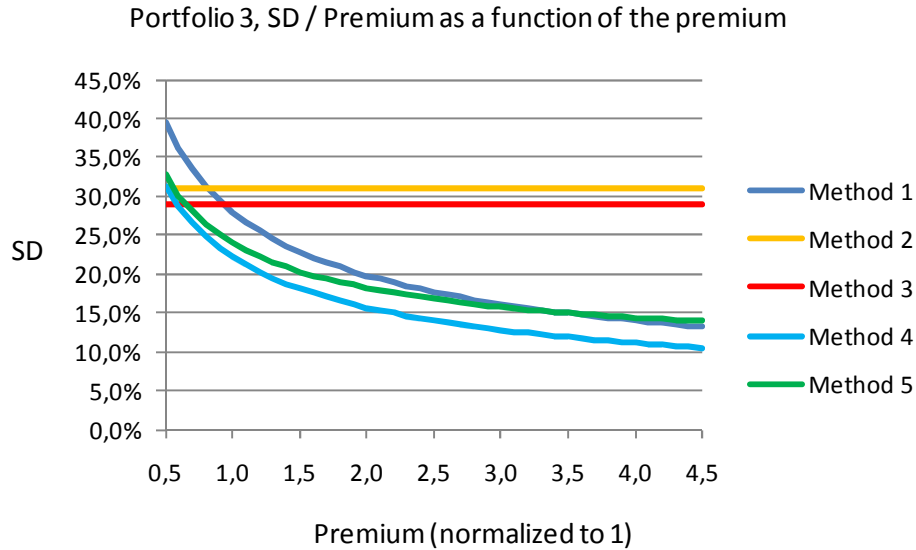


Figure 4.3: Standard deviation per premium as a function of the premium.

In this case method 1, 4 and 5 has almost the same principal behavior, since method 5 is to a large extent driven by the random risk in this case. When the premium increase (or decrease) becomes large we can see that there is a difference between method 5 and the

methods with a linear variance structure, which is explained by the fact that the variance structure for method 5 is a combination of the linear and the quadratic variance structures.

As an overall conclusion of this subchapter we have seen that the loss ratio based methods produce more or less similar estimates and that going from a Normal to a Lognormal assumption does not affect the estimates to large degree. The choice between these three methods is rather related to the variance structure which is considered appropriate. A lesson learned is that the quadratic variance structure is only appropriate when we have a portfolio where the volatility is driven by the systematic risk, which typically is the case for very large portfolios. On the other side of the scale, the linear variance structure is only appropriate for smaller portfolios, where the volatility is driven by random risk. When it comes to the loss ratio methods, the choice of variance structure is only of real significance when the volume of the future period differs significantly from the historical ones, or when the historical premium volume has changed significantly over time.

By this reasoning, method 4, the compound Poisson model, is only appropriate for very small portfolios and will in other cases underestimate the volatility. The estimates produced by method 5, the compound Poisson model with a parameter error for the frequency, produces estimates which are in the same order of magnitude as the observed loss ratio volatility measured using the loss ratio methods. It is however preferable over the other methods since it provides insight into the drivers of the volatility, as well as being superior when it comes to the principal behavior of the standard deviation per premium as a function of the premium, which makes it suitable to use both for small, medium-size and large portfolios.

4.4 On estimation errors

Since parameters are estimated on relatively few data points, it is important to have control over estimation errors to both be able to determine the number of data points needed for desired accuracy, as well as to have the possibility to include an extra capital charge or similar in a solvency model to compensate for possible estimation errors.

We note that we had as distributions assumptions for the total loss ratio in methods 1, 2 and 3 the Normal and the Lognormal distributions. Methods 4 and 5 where distribution free, requiring a prior view on the overall distribution. The Panjer recursion method presented as method 6 could be used to estimate the overall loss distribution given the frequency distribution (which is Poisson, Binomial and Negative Binomial) and the severity distributions (which is an arbitrary distribution with support not including any parts of the negative side of the real line). As has been touched upon, the Normal or Lognormal distribution is commonly used for the overall loss (see for instance chapter 3.5). Even when using a recursive method, the overall loss distribution usually turns out to be close to one of these two distributions. Thus it is sufficient to look at the possible estimation errors with

Normal and Lognormal as overall loss distributions. Of course, different estimators might have a different level of estimation error despite estimating the same quantity, but since we are throughout this thesis using MVUE's this is not an issue. To see why, recall that the Cramér-Rao bound states that for any unknown parameter θ , the variance of any unbiased estimator $\hat{\theta}$ thereof will be bounded by (see Lindgren 1993)

$$\text{Var}(\hat{\theta}) \geq 1/I(\theta) \tag{4.1}$$

where $I(\theta)$ is the Fisher information. Since MVUE's are efficient, equality holds (see Lindgren 1993) and thus the variance of the estimator will only inherit its variance from the underlying distribution (through the Fisher information) rather than the estimator as such.

As a consequence, one method to quantify the uncertainty in estimators, as a function of the number of observations N , is to simulate N independent and identically distributed outcomes from the total distribution M times and in each simulation compute the standard deviation using standard estimators (which are MVUE's). By varying the number of observations N and doing this for both the Normal and the Lognormal distribution we can get a feeling of the uncertainty of the estimates. Analytical derivation of the estimation error of the standard deviation is possible at least for the Normal distribution (see Kenney & Keeping 1951), but leads in the general case of N observations to cumbersome calculus if one is interested in quantiles and not only the standard deviation of the estimator of the standard deviation. Quantiles is of course needed to be able to derive confidence intervals.

By proceeding according to the numerical description above we arrive at table 4.4 for the Normal distribution (based on 2000 simulations in all tables), by simulating from a Normal distribution with a standard deviation of 5%.

Normal distribution, 5% standard deviation						
Number of observations	5	10	15	20	25	50
Average	4,7%	4,9%	4,9%	4,9%	4,9%	5,0%
Standard deviation	1,7%	1,2%	0,9%	0,8%	0,7%	0,5%
5% Percentile	2,1%	3,1%	3,4%	3,6%	3,7%	4,1%
10% Percentile	2,6%	3,4%	3,7%	3,9%	4,0%	4,3%
25% Percentile	3,4%	4,1%	4,3%	4,4%	4,5%	4,6%
75% Percentile	5,8%	5,6%	5,5%	5,4%	5,4%	5,3%
90% Percentile	6,8%	6,3%	6,1%	6,0%	5,9%	5,6%
95% Percentile	7,6%	6,8%	6,5%	6,3%	6,2%	5,8%

Figure 4.4: Estimation errors for the SD of a Normal distribution with SD of 5%.

It is of course not obvious what to consider reasonable estimation errors, but we see that using 5 points we have relatively wide two-sided 90% confidence interval, indicating that

we need to use more observations. With 5 datapoints we actually see some bias in the estimator. Going to 10 or 15 we get significantly higher accuracy, but above that the gain of using more observations becomes smaller.

We now perform the same analysis but with an underlying Lognormal distribution instead, with the same 5% of standard deviation. Results are found in figure 4.5.

Lognormal distribution, 5% standard deviation						
Number of observations	5	10	15	20	25	50
Average	4,7%	4,8%	4,9%	4,9%	4,9%	5,0%
Standard deviation	1,7%	1,2%	1,0%	0,8%	0,7%	0,5%
5% Percentile	2,1%	3,0%	3,4%	3,7%	3,8%	4,1%
10% Percentile	2,6%	3,4%	3,7%	3,9%	4,0%	4,3%
25% Percentile	3,4%	4,0%	4,2%	4,3%	4,5%	4,6%
75% Percentile	5,8%	5,6%	5,5%	5,5%	5,4%	5,3%
90% Percentile	6,9%	6,4%	6,2%	6,0%	5,9%	5,6%
95% Percentile	7,6%	6,9%	6,6%	6,3%	6,2%	5,9%

Figure 4.5: Estimation errors for the SD of a Lognormal distribution with SD of 10%.

Comparing figure 4.4 and figure 4.5 we see no significant difference, only slightly higher estimation errors for the Lognormal but that is expected since it has a heavier tail. We now look at the same setup but with the Normal distribution and a 10% standard deviation, see figure 4.6.

Normal distribution, 10% standard deviation						
Number of observations	5	10	15	20	25	50
Average	9,3%	9,7%	9,8%	9,9%	9,9%	9,9%
Standard deviation	3,4%	2,3%	1,9%	1,6%	1,5%	1,0%
5% Percentile	4,1%	6,2%	6,9%	7,3%	7,5%	8,3%
10% Percentile	5,2%	6,9%	7,4%	7,8%	8,0%	8,6%
25% Percentile	6,9%	8,1%	8,5%	8,8%	8,9%	9,3%
75% Percentile	11,6%	11,2%	11,0%	10,9%	10,8%	10,6%
90% Percentile	13,7%	12,7%	12,2%	11,9%	11,8%	11,2%
95% Percentile	15,3%	13,5%	12,9%	12,6%	12,4%	11,6%

Figure 4.6: Estimation errors for the SD of a Normal distribution with SD of 10%.

The estimation errors does not seem to increase significantly in relative terms, and again it seems that at least 10 – 15 observations are needed for a reasonably small confidence intervals. We perform the same analysis using the Lognormal distribution, again with 10% standard deviation, see figure 4.7.

Lognormal distribution, 10% standard deviation						
Number of observations	5	10	15	20	25	50
Average	9,3%	9,7%	9,8%	9,9%	9,9%	9,9%
Standard deviation	3,5%	2,4%	2,0%	1,7%	1,5%	1,0%
5% Percentile	4,1%	5,9%	6,8%	7,3%	7,6%	8,3%
10% Percentile	5,1%	6,7%	7,3%	7,8%	8,0%	8,6%
25% Percentile	6,8%	8,0%	8,4%	8,6%	8,8%	9,2%
75% Percentile	11,6%	11,3%	11,1%	10,9%	10,8%	10,7%
90% Percentile	13,8%	12,8%	12,4%	12,1%	11,8%	11,3%
95% Percentile	15,3%	13,8%	13,3%	12,8%	12,5%	11,8%

Figure 4.7: Estimation errors for the SD of a Lognormal distribution with SD of 10%.

No significant differences this time either, and once again Lognormal gives slightly higher estimation errors. We now look at the Normal distribution with 20% standard deviation, in figure 4.8.

Normal distribution, 20% standard deviation						
Number of observations	5	10	15	20	25	50
Average	18,6%	19,5%	19,7%	19,7%	19,8%	19,9%
Standard deviation	6,8%	4,7%	3,7%	3,2%	2,9%	2,0%
5% Percentile	8,3%	12,4%	13,8%	14,6%	15,0%	16,5%
10% Percentile	10,3%	13,7%	14,8%	15,6%	16,1%	17,3%
25% Percentile	13,8%	16,2%	17,0%	17,6%	17,8%	18,6%
75% Percentile	23,2%	22,4%	22,1%	21,8%	21,6%	21,2%
90% Percentile	27,3%	25,4%	24,5%	23,9%	23,6%	22,5%
95% Percentile	30,6%	27,0%	25,9%	25,2%	24,8%	23,3%

Figure 4.8: Estimation errors for the SD of a Normal distribution with SD of 20%.

In relative terms, we see no significant increase in estimation error. Last but not least we look at the Lognormal using the same standard deviation, in figure 4.9.

Lognormal distribution, 20% standard deviation						
Number of observations	5	10	15	20	25	50
Average	18,6%	19,3%	19,5%	19,7%	19,7%	19,9%
Standard deviation	7,3%	5,2%	4,3%	3,7%	3,3%	2,3%
5% Percentile	8,1%	11,6%	13,1%	14,2%	14,8%	16,2%
10% Percentile	9,9%	13,0%	14,4%	15,3%	15,7%	17,0%
25% Percentile	13,3%	15,6%	16,5%	17,1%	17,4%	18,3%
75% Percentile	23,0%	22,7%	22,1%	21,9%	21,8%	21,3%
90% Percentile	28,0%	25,9%	25,0%	24,5%	24,2%	22,9%
95% Percentile	31,3%	28,5%	27,2%	26,0%	25,4%	23,9%

Figure 4.9: Estimation errors for the SD of a Lognormal distribution with SD of 20%.

The skewness of the Lognormal distribution is now more obvious, and the estimation errors surely are higher for the Lognormal than for the Normal.

Overall, we see that estimation errors are significant, and that one needs at least 10 to 15 observations to put some credibility on the estimated figures. A higher standard deviation does not seem to increase the estimation error in relative terms for the Normal distribution, but for the Lognormal as the higher standard deviations makes the distribution more skewed.

5. Conclusions

In this chapter we draw some general conclusions around the methods presented, and we make suggestions for potential future work.

5.1 Overall conclusions

We have in this thesis presented and discussed a number of different methods for estimating premium risk. They can be divided into two different groups; method 1 to 3 concentrate on estimating the total loss ratio uncertainty directly by considering loss ratio outcomes, while the other methods suggests analytical models for the loss ratio based on the underlying frequency and severity distributions. While the loss ratio based methods are favorable from a back testing perspective as they by nature are consistent with historical observations, they fall short in providing insight in the estimated volatility. The more explicit methods are much more preferable in this respect. This is in particular important in situations when the forecasting exposure deviates significantly from the historical one, or when we have a prior view of a future portfolio having different risk characteristics than the historical one.

When it comes to the variance structures implied by the different methods, if one does not have a strong view around either a linear or a quadratic structure the ideas of method 5 provides a way of having a variance structure that is a combination of these structures. Considering the time perspective issue, we conclude that there are two main ways to achieve estimates consistent with the chosen perspective. The first is to derive ultimate estimates and then convert them to limited time by using ideas consistent with the reserving methods used. The other method is to arrive at the correct estimates directly by either using the most present ultimate estimates of all ingoing data or by using the ultimate estimates that was available at the end of each accident year.

Overall, we suggest using methods 4 to 6 for practical implementations, since they as argued above have a lot of favorable properties. Method 5, the compound Poisson model with frequency parameter risk, is particularly interesting since due to the resulting variance structure, and it can be used together with method 6 to achieve a numerical estimate of the overall loss distribution. The more simple method 4, the compound Poisson model without parameter risk, could on the other hand be used directly in cases where the portfolio is so small that the pure random risk is the main driver of the overall volatility. The loss ratio based methods however are also valuable in the sense that they provide a valuable benchmark for the other methods, to compare estimated total volatility figures to the historical ones.

5.2 Suggestions for future work

A few suggestions of possible extensions of the presented parameter estimation methods were discussed in chapter 3.9. Of course, also the general setup for premium risk in this thesis can be subject to improvement:

- It was assumed, given a certain amount of exposure, that volatility in earned premiums operating expenses was small compared to volatility in losses. This might not be true for certain portfolios, in which case methods for estimating volatility in these profit & loss elements need to be considered.
- One-year contracts are only considered in this thesis, which is the typical case for within non-life insurance. Exceptions definitely exist, in which case the uncertainty in the cash-flow valued premium reserve for future periods also needs to be considered in a one-year risk setup (the part that is not earned during the year), if one wants to have models consistent with a Solvency II valued balance sheet for multi-year contracts.
- As within reserving, it might be a good idea to consider introducing credibility weighted estimates of the total volatility based on estimates obtained from different methods. For instance number of observations used could be used as a basis for credibility weights, if it differs between methods.
- Independence assumptions between claims and between severity and frequency are in many cases a basis for the methods used. The later assumption could be questioned due to various reasons; one being that raised frequency due to bad weather might lead to more costly type of claims. Assuming a non-zero correlation and considering the effects on the overall volatility is a natural extension.

These are a few natural examples, of course other assumptions made can also be questioned and lead to possible extensions.

6. References

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