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Dynamic Models of Private Lending Business

Xiaofang Li

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106 91 Stockholm

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Abstract

In this paper, we consider the problem of risk measure for private lending portfolio of a commercial bank against the credit risk by utilizing the ruin theory from Insurance. We build a portfolio model, and extrapolate from this model to three models regarding different situations concerned. Our main objective is to determine the probability distributions of the risk processes and Ruin probabilities using Central Limit Theory and Monte Carlo Simulation technology, furthermore, investigate the loan conditions that make sure the Ruin probability in accordance with the risk appetite of the bank.

*Postal address: Mathematical Statistics, Stockholm University, SE-106 91, Sweden. E-mail: luckyfirst0812@hotmail.com . Supervisor: Håkan Andersson.

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1 Introduction

Private lending is one of the most important business for banks and other financial institutions. A large part of the net interest income of the bank has its origin in the lending activities. The business is however associated with substantial financial risk-in particular credit risk and interest rate risk. It is very important for the bank to define an adequate risk appetite and to hold proper capital buffers to ensure that the business will survive also under stressed economic conditions.

In the simplest set up, the bank lends some amount of money to a client. An amortization period is determined together with a number of payment dates during that period. At each payment date, a fraction of the loan is amortized. Also, the client pays an interest consisting of market interest rate, an administration fee, a risk premium to cover for credit losses, plus a margin for the bank. If the client fulfils all payments then bank will be able to repay the investors that have provided the loan amount in the first place, and will still have a surplus from the deal. On the other hand, if the client defaults, and if no collateral has been provided, then the bank might incur a substantial loss on the deal. The client interest rates should thus be carefully set so that the business provides a steady inflow of cash in the long run.

In this work a simple portfolio model of the transaction between the bank and its clients is designed with the main purpose of investigating the distributional properties of the result generated by the business during a given time period

Pricing methodologies for portfolio credit risk have of course been studied extensively. Risk sensitive pricing is closely related to the concept of capital allocation. Given a certain risk horizon, a high quantile of the portfolio loan loss distribution is estimated. Then it is determined how much a given loan A contributes to this loss quantile. This calculation provided a guideline pricing future loans that are similar to loan A in all respects. In the long run, all deals of the portfolio will be properly priced. A drawback of this approach is that loan losses and interest revenues are treated separately, even though in reality they are highly dependent. In contrast, in the present work amortization and interest payments are integrated into the risk calculation in the sense that the future cash flows of the receive leg (from the point of view of the bank) are removed at the exact time of default of the client. We believe that this approach gives a more consistent picture of the risk of the portfolio.

2. Financial Background

2.1 Lending Business

Depository institutions include commercial banks, savings and loan associations (S&Ls), saving banks, and credit unions. All of them are financial intermediaries that accept deposits. These deposits represent the liabilities of those deposit-accepting institutions. With the funds raised by deposits and other funding sources, depository institutions both make direct loan to various entities and invest in securities. They generate income through two sources: the loans they make and the securities they purchase, and fee income.

In order to make their various businesses work normally, banks have to raise the basic funds. They usually have there sources of funds: deposits, non-deposit borrowing, common stock and retained earnings. Banks are highly leveraged financial institutions, meaning that most of their capital come from borrowing, namely the deposits and non-deposit borrowing. Non-deposit borrowing is borrowing from the Federal Reserves through the discount window facility, borrowing reserves in the federal funds markets, and borrowing by the issuance of instruments in the money and bond markets. There are principal three types of deposit accounts: (1) demand deposits (checking accounts), which pay no interest and can be withdrawn upon demand; (2) savings deposits, paying interest, typically below market interest rates, do not have specific maturity, and usually can be withdrawn upon demand; (3) time deposit (certificates of deposit), have a fixed maturity date and pay either a fixed or floating interest rate., some of them can be sold in the open market prior to their maturity if the depositors need money.

Commercial banks offer various services among financial system, which can be primarily classified into three parts: individual banking; institutional banking and global banking. Different banks certainly are more active in certain of these services than others. For example, money center banks[1] are more active in global banking. Institutional banking is lending to institutions such as non-financial corporations, financial corporations and governments, gaining interest and fee income. Global banking is a kind of similar work as investment banking company, covering a broad of activities including corporate financing, capital market and foreign exchange products and services, generating principally fee income as opposed to interest income. We primarily focus on individual banking in our paper.

Individual banking consists of consumer lending, residential mortgage lending, consumer installment loans, credit card financing, automobile and boat financing, brokerage services, student loans, and individual-oriented financial investment services such as personal trust and investment services. Banks gain interest and fee income.

2.2 Credit Risk

We categorize broadly the risks faced by financial institutions as following:

[1]Money center bank is a bank that raises most of its funds from the domestic and international money markets and that relies very little on deposits or depositors

- Market risk - the risk of value depreciation of portfolio, investment portfolio or trading portfolio due to the change in value of the market risk factors (the four standard market risk factors are stock prices, interest rates, foreign exchange rates, and commodity prices) including equity risk, interest rate risk, currency risk and commodity risk.
- Credit risk - the risk of the changes of default or of reductions in market value cause by changes in the credit quality of issuers or counterparties.
- Liquidity risk - the risk that the expenses of adjusting financial positions will increase substantially or that a firm will lose access to financing.
- Operation risk - the risk due to fraud, systems failures, trading errors and lots of many other internal organizational risks.

The latest notable increased attention on credit risk can be traced in part to the concerns of regulatory agencies and investors regarding the risk exposures of financial institutions though their large positions in Over-the-Counter derivatives and to the fast developing markets for credit-sensitive instruments that admit financial institutions and investors to trade these risks.

In financial system, default occurs when debtor fail to take their legal obligations according to the debt contacts, e.g. have not made a scheduled payment, or have violated a loan covenant of the debt contract, no matter they are unwilling or unable to pay their debts. It can happen with all kinds of debt obligations encompass bonds, mortgages, loans, and promissory notes.

Definition Credit risk is the risk that borrowers default on obligations to the banks or bonds issuers default on obligations to the lender, so it is also called as default risk. We principally focus on the default on credit risk for loan portfolio of bank, namely, consumer credit risk.

Consumer credit risk is the risk of that customers default on repayment for consumer credits products, such as mortgage, unsecured personal loan, credit card, overdraft etc. And personal loan is our researching object.

Banks are financial intermediaries that originate loans and consequently take the credit risk when borrowers default on the obligations of loans. Banks undertake relatively high credit risk as a result of two resources: (1) bank loans usually concentrate on specific regions and industries, hence limit the effect of reducing credit risk through diversification; (2) credit risk is the primary risk in a loan. With the changing of risk-free interest rates, most of commercial loans are designed at floating interest rate, as a result, the changes of default-free interest rates make rarely risks for commercial banks. However, the credit risk premiums are fixed when the loan contracts were signed, banks suffer losses as the loan benefits are not enough to cover the higher risk caused by the increased credit risk premiums.

Credit risk is driven by both unsystematic and systematic components. Unsystematic credit risk contains the probability of a borrower's default owing to circumstances that are essentially unique to the individual, whereas systematic credit risk can be recognized as the probability of a borrower's default caused by more general economic fundamentals. Banks increasingly realize the necessity of measuring and managing the credit risk of the loans they have originated on a portfolio basis as well as a loan-by-loan basis due to the fact that only the aggregate credit exposure is the relevant factor for the future solvency of a bank.

Credit risk is closely related to potential return of an investment, the gains on the lending business correlate strongly to their perceived credit risk. The higher the perceived credit risk, the higher the rate of interest investors (bank) will demand for lending their capital. Credit risks are calculated on the base of the borrowers' overall ability to repay, such as the borrowers' collateral assets, revenue-generating ability and taxing authority and so forth.

2.3 Credit Analysis

The audited financial statements of a large firm might be analyzed when it issues or has issued bonds. Before approving a commercial loan, a bank will analyze the financial statements of a small business, referring to either case, whether the business is large or small. In another words, a bank need to check the creditworthiness of a business or organization, this action is referred to as credit analysis.

Credit analysis includes a wide variety of financial analysis techniques such as the creation of projections, ratio and trend analysis and a detailed analysis of cash flows. Besides, it also involves an examination of collateral and other sources of repayment as well as credit history and management ability. Furthermore, before making or renewing a commercial loan, a bank will emphasize the cash flow of the borrower. A typical measurement of repayment ability is the debt service coverage ratio. A credit analyst in a bank measures the cash flow generated by a business excluding depreciation and any other non-cash or extraordinary expenses before interest expense. A typical measurement of repayment ability is the debt service coverage ratio, which divides this cash flow amount by the debt service that will be required to accord with, both principal and interest payments on all loans.

Credit analysis and risk management are far from exact sciences of quantitative methods, derivatives, finance and accounting, Uncovering how credit analysis proceeds through evaluating financial ratios, cash flows, and the firm's objectives compared to its industry peers, using both conceptual and numerical examples, analyzing the interaction among credit risk and other types of risk.

2.4 Credit Risk Management

Financial risk management is the practice of creating economic value in a company by means of financial instruments to manage exposure to risk, particularly credit risk and market risk.

Similar to general risk management, financial risk management requires to investigating their sources, measuring them, and schemes to control them. Financial risk management can be qualitative as well as quantitative. As a specialization of risk management, financial risk management concentrates on when and how to use financial instruments to manage efficiently exposures to risk.

As a set of activities, risk management by a financial institution may involve:

- Measuring the extent and sources of exposure;
- Charging each position a cost of interest rate appropriate to its risk;
- Allocating scarce risk capital to traders and profit centers;
- Proving information on financial integrity of the firm to outside parties, such as investors, rating agencies and regulators;
- Evaluating the performance of profit centers according to the risks taken to achieve profits;
- Mitigating risk by means of various means and policies.

At the mean time, an important objective applying specifically to credit risk management is assigning and enforcing counterparty default exposure limits in view of the important market imperfections, adverse selection and moral hazard in credit markets.

Measuring and managing credit risk are among the major challenges for any lender. Commercial banks have traditionally used standardization of loan examination and diversifications of loan borrowers to mitigate credit risk, they currently use credit derivatives[2] to tailor their credit exposure[3]. Broadly speaking, they shed credit risk via credit derivatives and other means of credit risk transfer, such as securitizations, to shed risk in several areas of their credit portfolio, including large corporate loans, loans to small firms, and counterparty credit risk on over-the-counter (OTC) derivatives.

Risk management and analysis were done qualitatively in the past, but now with the advent of powerful computing software, quantitative risk management and analysis can be done quickly and effortlessly.

2.4.1 Risk Measure

A risk measure is used to determine the aggregation of cash requirement in reserve in order to make the risks acceptable taken by financial institutions, such as banks and insurance companies, in accord with the regulator.

The senior managers of banks and other financial institutions, along with their research and development staffs feel obliged to develop policies and systems to measure credit risk. Key elements of their credit risk pricing and risk-measurement systems encompass: the sources of risk factors to be examined and their joint probability distribution, methodologies for measuring changes in credit quality and default over a host of counterparties.

The different risk measure methods are not equally attractive within a firm or from the perspective of aggregation of risk measures across positions or trading units. To select a proper risk measure, we need to judge about whether it is closely related to the vital

[2]A Credit Derivative is a securitized derivative whose value is derived from the credit risk on an underlying bond, loan or any other financial asset. For example, a bank concerned that one of its customers may not be able to repay a loan can protect itself against loss by transferring the credit risk to another party while keeping the loan on its books.
[3]Credit Exposure measures the outstanding obligation of the debtor.

economic costs of financial risk; easily communicated to, and understood by; estimable at a reasonable cost and under a reasonable tolerance; meaningfully aggregated from individual entities or desks into an overall risk measure for the corporations. In calculating the credit risk, we expect trade-offs from these criteria.

Well-known examples of risk measures are Value-at-Risk (VaR), expected shortfall, the price of market value insurance and volatility. In recent years the attention has turned towards coherent[4] risk measurement. Besides, in order to better understand the nature of their exposures to credit risk, risk managers have explored several complementary measures of credit risk including market value of default loss and exposure.

2.4.2 Risk Modeling

Risk modeling applies formal econometric techniques into the determination of the aggregate risk in a financial portfolio and is one of many subtasks in the broader area of financial modeling.

Risk modeling utilizes a great many of techniques including market risk, Monte Carlo, Historical Simulation, or Extreme Value Theory to analyze a portfolio and make forecasts of the possible losses that would be incurred due to various risks. Such risks are typically categorized into credit risk, liquidity risk, interest rate risk, and operational risk categories.

Many large financial intermediary institutions use risk modeling to help portfolio managers assess the quantity of capital reserves need to retain, and to assist their purchases and sales of a variety of financial assets.

Formal risk modeling is required under the Basel II accord by the various national depository institution regulators, referring to all the major international banking institutions.

2.4.3 Capital Adequacy

The capital requirement is a bank regulation, which sets a framework on how much capital banks and depository institutions must hold against the financial distress owning to variety of risks. In order to calculate the total measure of assets, each asset is multiplied by a risk weighting factor that, in principle, represents the credit quality of the asset.

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[4]Artzer et al.(1999) analyze desirable propertied for a portfolio risk measure. These authors call risk measure m(.) coherent if it fulfills the following four axioms,

1. Subadditivity: For any portfolio payoffs X and Y, $m(X+Y) \leqq m(X) + m(Y)$.
2. Homogeneity: For any number
3. Monotonicity: $m(X) \leqq m(Y)$ if $X \leqq Y$.
4. Risk-free condition: $m(X + k) = m(X) - k$, for any constant k.

The Basel Committee on Banking Supervision sets the standard for each country's banking capital requirements. In 1988, the Committee organized a capital measurement system-Basel Accord, and this framework is now being replaced by a new and significantly more complex capital adequacy framework commonly referred to as Basel II which was initially published in June 2004

Basel II aims to form an international standard that banking regulators can use when creating regulations about the amount of capital requirement that banks need to reserve to protect against the types of financial and operational risks banks face. Advocates of Basel II believe that such an international standard can help prevent the international financial system from the types of problems. For accomplishing this, Basel II sets up severe risk and capital management requirements in order to ensure a bank surviving the risk the bank exposes itself to through its lending and investment businesses. Above all, these rules mean that the greater risk the bank is exposed, the more the amount of capital the bank needs to handle in reserve to safeguard its solvency and overall economic stability. Each national regulator normally calculates bank capital in a fairly different way, designed to accord with the common requirements within their individual national legal framework.

2.4.4 Amortization of Premium

While paying just the interest each period will lead to a low outflow of cash each month, the debtor might not save enough to pay the principal. Charges made against the interest received on a debt in order to offset a premium paid for the debt. Thus, with each periodic payment, a debtor is paying back interest as well as part of his or her premium. This leads to higher periodic payments than in the case when only interest is paid out.

Amortizing the premium each period reduces the credit risk of the debt, since the creditor gets some part of the principal each time period, as opposed to allowing a debtor to forfeit on all of it at the maturity of the loan; and makes debt management easier, especially when the principal is large. Amortization of premium is a common feature in cases when a person or company takes on a large amount of debt at one time, such as a mortgage.

2.5 Objectives of Research

To mitigate the impact of default risk, lenders often charge rates of return that correspond to the debtor's level of default risk. The higher the risk, the higher the required return and vice versa.

Our aim objective is to study the law of large numbers and the central limit theorem for this quantity as time tends to infinity. In cases where it is impossible to arrive at closed analytical formulas, we have to resort to Monte Carlo simulation. Based on these results, a proper risk premium should be chosen to make sure that the bank’s ruin probability (i.e. the probability that the bank loses all its initial capital) is in accordance with the risk appetite[5] formulated by the management. In this sense, the problem is

[5]Risk appetite is frequently used throughout the risk management community. Risk appetite is the amount of risk exposure or potential adverse impact from an event that

very much related to ruin problems in the theory of insurance mathematics.

the organization is willing to accept/retain at the organizational level. When the risk appetite threshold has been breached, risk management treatments and business controls are performed to bring back the exposure level within the accepted range.

3 Model Descriptions

The portfolio model is designed as follows. We work in continuous time, measured in years. Today is $t=0$. In this simple setup we assume that all market interest rates are zero, this assumption could of course be weakened. Deals arrive at the portfolio according to a Poisson process X(t) with rate λ. The loan size of deal i is denoted by M_i and the time to maturity is denoted by L_i. Default of deal i happen at time Z_i. During the life- time of the contract the client i amortizes at rate M_i / L_i and pays a risk premium $M_i r_i$ per time unit. Further, a given client i defaults on its payment obligations on constant rate μ_i (default probability[6]).

The assumptions in the model are:

(1) No collaterals have been taken by the bank, hence in case default all future cash flows between the client and the bank are removed;

(2) The loan size process (M_i), Poisson process X and the default arrival process (Z_i) are mutually independent;

(3) $X=(X(t))_{t\geq 0}$ is a counting process on $[0,\infty)$: X(t) is the number of the deals which occurred by time t;

(4) Initial capital the bank holds is a constant, W(0).

It is clear that the effective time that client i remains in the system, T_i, follows a truncated Exponential distribution:

$$T_i = \min\left(L_i, Z_i\right) \qquad \text{where} \quad Z_i \sim Exp\left(\mu_i\right).$$

It follows that the total result generated by the clients that have arrived between 0 and t is given by

$$\sum_{i=1}^{X(t)} \{-M_i + (\frac{M_i}{L_i} + M_i r_i) T_i\}, \quad t \geq 0$$

The surplus or risk process of the bank generated by all the deals that have arrived between 0 and t is

$$W(t) = W(0) + \sum_{i=1}^{X(t)} \{(\frac{M_i}{L_i} + M_i r_i) T_i - M_i\}, \quad t \geq 0 \tag{3.1}$$

[6]Default Probability is the degree of likelihood that the borrower of a loan or debt defaults on his or her obligations. When the borrower is unable to pay, they are then said to be in default of the debt, the lenders of the debt have legal avenues to attempt obtaining at least partial repayment. Generally speaking, the higher the default probability a lender estimates a borrower to have, the higher the interest rate the lender will charge the borrower as compensation for bearing higher default risk. It is measured by year, and has the value $\text{DP} = 1 - e^{-\mu} \approx 1 - (1-\mu) = \mu$ if μ is small.

From the model we know that the process $\left(X(t)\right)_{t\geq 0}$ is a homogeneous Poisson process with intensity λ, $X(t) \sim Po(\lambda t)$ and deals arrive at the random instants of time $0 < N_1 < N_2 < .....$ Then by the theorem A.1, we have representation

$$X(t) = \#\{i \geq 1 : N_i \leq t\}, t \geq 0, \tag{3.2}$$

where

$$N_n = Y_1 + Y_2 + ... + Y_n, n \geq 1, \tag{3.3}$$

and the $Y_1 = N_1, Y_2 = N_2 - N_1, ...Y_i = N_i - N_{i-1}$ are iid exponentially distributed with $EY_1 = 1/\lambda > 0$. (N_n) and (Y_n) are referred to as the sequences of the arrival time and inter-arrival times of the homogeneous Poisson process X respectively.

The above model can be restricted and modified, in several ways. Hence follows some suggestions.

Model A: Homogeneous loans

The loan size, the time to maturity, the premium and the default rate are assumed to be constants, and identical for all clients.
The model becomes

$$W(t) = W(0) + \sum_{i=1}^{X(t)} \{(\frac{M}{L} + Mr)T_i - M\},$$

Where $T_i = \min\left(L, Z_i\right), Z_i \sim Exp(\mu)$.

Model B: Default rate depends on premium

The loan size, the time to maturity and the premium are assumed to be constants, and identical for all clients. The default rate is assumed to be an affine function of the premium; $\mu = \mu(r) = \alpha + \beta r$. This assumption is motivated by the obvious fact that a heavy payment burden leads of financially stressed clients**.**

We consequently have the following model:

$$W(t) = W(0) + \sum_{i=1}^{X(t)} \{(\frac{M}{L} + Mr)T_i - M\},$$

where $T_i = \min\left(L, Z_i\right)$, $Z_i \sim Exp(\alpha + \beta r)$.

Model C: Default rate depends on macroeconomics

The loan size, the time to maturity and the premium are assumed to be constants, and identical for all clients. The default rate is allowed to be influenced by macroeconomic conditions. To model such behavior, define $\mu = \mu(t)$ to be a continuous-time Markov Chain with states μ_L and μ_H, $\mu_L < \mu_H$, that changes state according to prescribed transition intensities. Obviously, in this setup we need to model explicitly how the portfolio changes with time.

The model becomes

$$W(t) = W(0) + \sum_{i=1}^{X(t)} \{(\frac{M}{L} + Mr)T_i - M\},$$

where, $T_i = \min\left(L, Z_i\right)$, $Z_i \sim Exp\left(\mu\left(t_i\right)\right)$, t_i denotes the arrival time of the i th client.

4 Methods of Analysis

In actuarial risk management it is an important issue to estimate the performance of the portfolio. Ruin theory[7] is used by actuaries in order to follow the insurer's surplus and ruin probability which can be explained as the probability of insurer's surplus drops bellow a specified lower bond such as minus initial capital.

With the model description above in Section 3, it is clear that our theme is closely related to the ruin theory in Non-life Insurance Mathematics, see [7].

The quantity W(t) is nothing but the bank's balance at a given time t, and the process $W = (W(t))_{t\geq 0}$ describes the cashflow in the portfolio over time. The $M_i/L_i + M_i r_i$ represent the income obtained from the i th client, and $\sum_{i=1}^{X(t)} \left(M_i/L_i + M_i r_i\right) T_i$ describes the whole inflow of the capital into the business by time t; M_i represent the capital quantity borrowed by the i th client, $\sum_{i=1}^{X(t)} M_i$ describes the outflow of capital due to loan lending occurred in $[0,t]$. It is obvious that the process decrease at those arrival time points $0 < N_1 < N_2 < .....$, reducing by size M_i at arrival time N_i of the i th client. It turns out that values of W(t) are possible negative if there is a sufficiently large lending size M_i which makes W below zero. We call the event that W ever falls below the threshold A ($A \geq 0$) RUIN. That means:

$$Ruin = \{W(t) < A for some t > 0\},$$

then we have

$$Ruin = \bigcup_{t\geq 0}\{W(t) < A\} = \left\{\inf_{t\geq 0} W(t) < A\right\}.$$

The probability of ruin is then given by

$$\psi(W(0)) = P\left(Ruin \mid W(0)\right) = P\left(\inf_{t\geq 0} W(t) < A\right), \tag{4.1}$$

which denotes the probability of ultimate ruin the bank will face during time $[0,t]$.

In examining the nature of the risk associated with a credit portfolio of a commercial bank, it is often interest to assess how the portfolio may be expected to perform over an extended period of time. One approach concerns the use of ruin theory. We use Ruin Probability as risk measure in our paper, specifically, ruin is said to occur if the

[7]Ruin theory is a branch of actuarial science that researches an insurer's vulnerability to insolvency on base of mathematical modeling of the insurer's surplus. The theory permits the derivation and calculation of many ruin-related measures and quantities including the probability of ultimate ruin, the distribution of an insurer's surplus immediately prior to ruin, the deficit at the time of ruin and so forth.

bank's surplus reaches a specified lower bound, theoretically zero, but in practice and in our paper, we set minus of the initial capital $W_{\min}$ as the threshold, that means

$$\psi(W(0)) = P\left(Ruin \mid W(0)\right) = P\left(\inf_{t\geq 0} W(t) < W_{\min}\right).$$

Besides this, we are also interested in the probability of the process $\left(W(t)\right)_{t\geq 0}$ less than $W_{\min}$ at fixed time point t.

Consequently, our main tasks are obtaining the following two probabilities by means of asymptotic calculation or Monte Carlo simulation:

- $\psi_1 = P\left(W(t) < W_{\min}\right)$ (4.2)

which means the probability of W(s) drops below the minus of the initial capital at time t. It comes in hand when the bank needs to check the ruin probability at a fixed time point.

- $\psi_2 = P\left(\inf_{0<s<t} W(s) < W_{\min}\right)$ (4.3)

which means the probability of W(s) value ever drops below the minus of the initial capital during the time interval $[0,t]$, it is a useful measure for the bank to monitor the risk level during one period; a dynamic and functional indication for the bank to set proper premium rate and loan conditions to make sure the low ruin probability in accord with the bank's risk appetite.

We achieve above objective, calculating the two ruin probabilities ψ_1 and ψ_2 by two methods: (1) Approximation to the Distribution of Risk Process W(t) using the Central Limit Theorem; (2) Approximation to the Distribution of Risk Process W(t) By Monte Carlo Techniques.

4.1 Method of Central Limit Theorem

The model is as described above, see Section 3. By the Law of Large Number and Central Limit Theorem, see theorem D.2, if $\mathrm{var}(T_1) < \infty$ and $\mathrm{var}(W_1) < \infty$, we have

$$\begin{aligned}&\sup_{x\in\mathbb{R}}\left|P\left(\frac{W(t)-EW(t)}{\sqrt{\mathrm{var}\left(W(t)\right)}}\leq x\right)-\Phi(x)\right| \\ &= \sup_{y\in\mathbb{R}}\left|P\left(W(t)\leq y\right)-\Phi\left(\left(y-EW(t)\right)\Big/\sqrt{\mathrm{var}\left(W(t)\right)}\right)\right|\to 0\end{aligned}, \quad (4.1.1)$$

where the Φ is the distribution function of the standard normal distribution N(0,1). As in classical statistics, where one is interested in the construction of asymptotic confidence bands for estimators and in hypothesis testing, one could take this central limit theorem as justification for replacing the distribution of W(t) by the normal distribution with mean EW(t) and variance var(W(t)): for large t, we have

$$P\left(W(t) \le y\right) \approx \Phi\left(\left(y - EW(t)\right) \Big/ \sqrt{\operatorname{var}\left(W(t)\right)}\right) \qquad (4.1.2)$$

We can use (4.1.2) to determine the probability ψ_1 in (4.2) for Model A and B.

4.2 Method of Monte Carlo Simulation

It is clear by the construction of ruin probability in (4.3) that this probability distribution is not easily evaluated since one have to study a very complicated functional of a sophisticated random process. One way out of this situation is to use the power and memory of modern computers to approximate the distributions of W(t).

Monte Carlo methods are on the base of the analogy between probability and volume. The mathematics of measure formalizes the intuitive notion of probability, associating an event with a set of outcomes and defining the probability of the event to be its volume or measure with respect to that of a universe of possible outcomes. Monte Carlo uses this identity in reverse, calculating the volume of a set by means of regarding the volume as a probability. The law of large numbers ensures that this estimate converges to the real value as the number of observations increases. The central limit theorem provides information about the likely magnitude of the error in the estimate after a finite number of draws.

When you build a model with a spreadsheet like Excel, you have a few equations with a certain number of input parameters, producing a set of outputs. This type of model is usually deterministic, meaning that you get the same results no matter how many times you re-calculate. And Monte Carlo simulation is a method for iteratively evaluating a deterministic model using sets of random numbers as inputs. By this way, you are essentially turning the deterministic model into a stochastic model. This method is often used when the model is complex, nonlinear, or involves more than just a couple uncertain parameters.

Monte Carlo simulation is a computerized mathematical technique that allows people to explain risk in quantitative analysis and decision making. It performs risk analysis by considering random sampling of probability distribution functions as model inputs to produce a large number of possible outcomes instead of a few discrete scenarios, each time using a different set of random values. Depending upon the number of uncertainties and the ranges specified for them, a Monte Carlo simulation could involve thousands of recalculations before it is complete. Monte Carlo simulation furnishes the decision –maker with a range of possible outcomes and the probabilities they will occur for any choice of action. In this way, Monte Carlo simulation provides a much more comprehensive view of what may happen, telling you that could happen as well as how likely it is to happen.

For example, if we knew the distribution of $X(t)$ and iid W_i, the arrival Poisson process and the income from the i th client respectively, we could simulate an iid sample $X_1,...,X_m$ from the distribution of $X(t)$. Then we could draw iid samples

$$W_1^{(1)},\dots,W_{X_1}{}^{(1)},\dots,W_1^{(m)},\dots,W_{X_m}{}^{(m)}$$

from the distribution of W_1 and calculate iid copies of $W(t)$:

$$W^1 = W(0)+\sum_{i=1}^{X_1} W_i^{(1)},\dots,W^m = W(0)+\sum_{i=1}^{X_m} W_i^{(m)}.$$

The probability $P(W(t)\in A)$ for some Borel set A could be approximated by virtue of the strong LLN:

$$\widehat{P}_m = \frac{1}{m}\sum_{i=1}^{m} I_A\left(W^i\right) \xrightarrow{a.s.} P\left(W(t)\in A\right) = p = 1-q \quad \text{as} \quad m\to\infty. \tag{4.2.1}$$

Note that $m\widehat{P}_m \sim Bin\left(m,p\right)$. The approximation of p by the relative frequencies $\widehat{P}_m$ of the event A is called crude Monte Carlo simulation.

5 Results

5.1 Model A: Homogeneous Loans

$$W(t) = W(0) + \sum_{i=1}^{X(t)} \{(\frac{M}{L} + Mr)T_i - M\},$$

where $T_i = \min(L, Z_i)$, $Z_i \sim Exp(\mu)$, $i.i.d.$

The (4.2) could be approximated by the following formulas:

$$\psi_1 = P(W(t) \le W_{\min}) \approx \Phi\left((W_{\min} - EW(t)) / \sqrt{\mathrm{var}(W(t))}\right) \qquad (5.1.1)$$

where

$$EW(t) = W(0) + \lambda t\left[-M + \left(M/L + Mr\right)\left(1 - e^{-\mu L}\right)/\mu\right], \qquad (5.1.2)$$

$$\begin{aligned} Var(W(t)) &= \lambda t \cdot (M/L + Mr)^2 \left(1 - 2\mu L e^{-\mu L} - e^{-2\mu L}\right)/\mu^2 \\ &+ \lambda t \cdot \left[-M + \left(M/L + Mr\right)\left(1 - e^{-\mu L}\right)/\mu\right]^2. \end{aligned} \qquad (5.1.3)$$

Φ denotes the Standard Normal Distribution function, which means $(W_{\min} - EW(t)) / \sqrt{\mathrm{var}(W(t))} \sim N(0,1)$. The proof is given in Section 5.5.

By construction of the risk process W, ruin can occur only at the clients arrival times $t = N_n$ for some $n \ge 1$, consequently, we can express ruin as

$$Ruin = \left\{\inf_{t \ge o} W(t) < W_{\min}\right\} = \left\{\inf_{n \ge 1} W(N_n) < W_{\min}\right\} \qquad (5.1.4)$$

We have fact which will be very useful later that

$$X(N_n) = \#\{i \ge 1 : N_i \le N_n\} = n \quad a.s. \qquad (5.1.5)$$

As a result, the (4.3) is calculated by means of Monte Carlo simulation as following algorithm:

(1) Draw an iid sample $X^{(1)}, X^{(2)}, ..., X^{(m)}$ from the Poisson distribution $Po(\lambda t)$;
(2) Draw iid samples $Z_1, Z_2, ..., Z_{X^{(1)}}$ from the Exponential distribution $Exp(\mu)$;
(3) Get corresponding iid random variables $T_1, T_2, ..., T_{X^{(1)}}$ by means of $T_i = \min(L, Z_i)$;
(4) Calculate iid copies of $W(t)$:

$$W\left(N_1\right)=W(0)+\left\{\left(\frac{M}{L}+Mr\right)T_1-M\right\},$$

$$W\left(N_2\right)=W(0)+\left\{\left(\frac{M}{L}+Mr\right)T_1-M\right\}+\left\{\left(\frac{M}{L}+Mr\right)T_2-M\right\},$$

$$\ldots,$$

$$W\left(N_{X^{(1)}}\right)=W(0)+\sum_{i=1}^{X^{(1)}}\left\{\left(\frac{M}{L}+Mr\right)T_i-M\right\};$$

(5) Get $\inf\limits_{0<i\le X^{(1)}} W\left(N_i\right)$ on the base of step (4);

(6) Repeat steps (2)-(5) m times, obtaining iid

$$\inf_{0<i\le X^{(1)}} W\left(N_i\right),\ \inf_{0<i\le X^{(2)}} W\left(N_i\right),\ldots,\ \inf_{0<i\le X^{(m)}} W\left(N_i\right).$$

From (4.2.1), we get the approximation values for ruin probability ψ_2 in (4.3) by (5.1.5) and by virtue of the strong Law of Large Numbers as:

$$\psi_2\approx\widehat{P_m}=\frac{1}{m}\sum_{j=1}^{m}I_{(-\infty,W_{\min})}\left(\inf_{0<i\le X^{(j)}} W\left(N_i\right)\right). \tag{5.1.6}$$

Note that in step (4), we used the fact (5.1.5) we mentioned before.

5.2 Model B: Default Rate Depends on Premium

$$W(t)=W(0)+\sum_{i=1}^{X(t)}\{(\frac{M}{L}+Mr)T_i-M\} \tag{5.2.1}$$

where $T_i=\min\left(L,Z_i\right),\ Z_i\sim Exp\left(\alpha+\beta r\right),\ i.i.d.$

The same as Model A, the (4.2) could be approximated by LLN and Central Limit Theorem as the following formulas:

$$\psi_1=P(W(t)\le W_{\min})\approx\Phi\left((W_{\min}-EW(t))/\sqrt{\mathrm{var}(W(t))}\right) \tag{5.2.2}$$

where, similarly to the Model A, we set $\mu=\alpha+\beta r$, then we can get

$$\begin{aligned}EW(t)&=W(0)+EX(t)\cdot EW_1\\&=W(0)+\lambda t\left[-M+\left(M/L+Mr\right)\left(1-e^{-(\alpha+\beta r)L}\right)\big/\left(\alpha+\beta r\right)\right],\end{aligned} \tag{5.2.3}$$

$$
\begin{aligned}
Var(W(t)) &= EX(t)\cdot Var(W_1)+Var(X(t))\cdot(EW_1)^2 \\
&= \lambda t\cdot(M/L+Mr)^2\left[1-2(\alpha+\beta r)Le^{-(\alpha+\beta r)L}-e^{-2(\alpha+\beta r)L}\right]\Big/(\alpha+\beta r)^2 \;. \\
&+\lambda t\cdot\left[-M+(M/L+Mr)\left(1-e^{-(\alpha+\beta r)L}\right)\Big/(\alpha+\beta r)\right]^2
\end{aligned}
\tag{5.2.4}
$$

Note W_i represents the net income from the ith client, $W_i=(M/L+Mr)T_i-M$. Φ denotes the Standard Normal Distribution function, which means $(W_{\min}-EW(t))/\sqrt{\mathrm{var}(W(t))}\sim N(0,1)$. The proof is given in Section 5.5.

From the model B (5.2.1), it is clear that the only difference of Model B refers to the default intensity, μ is changed into $\alpha+\beta r$ to indicate the relationship between premium rate and default rate that the higher the premium is set, the larger probability people will default on their obligations, which is practical in real businesses of banks and worthy of interest.

Consequently, the only distinction of Monte Carlo simulation algorithm for Model B from Model A lies in the step (2). To Model B, for step (2) in Section 5.1 when we generate samples for Z_i from Exponential distribution, we substitute

(2)' Draw iid samples $Z_1,Z_2,\ldots,Z_{X^{(1)}}$ from the Exponential distribution $Exp(\alpha+\beta r)$.

The approximation values of ruin probability ψ_2 in (4.3) for Model B are the same as the one of Model A represented in (5.1.6). By (5.1.5) and the strong LLN we have

$$
\psi_2\approx\widehat{P_m}=\frac{1}{m}\sum_{j=1}^{m}I_{(-\infty,W_{\min})}\left(\inf_{0<i\le X^{(j)}}W(N_i)\right). \tag{5.2.5}
$$

5.3 Model C: Default Rate Depends On Economics

The model becomes

$$
W(t)=W(0)+\sum_{i=1}^{X(t)}\{(\frac{M}{L}+Mr)T_i-M\}, \tag{5.3.1}
$$

where, $T_i=\min(L,Z_i)$, $Z_i\sim Exp(\mu(t_i))$.

In this model, $\mu(t_i)$ is a continuous Markov Chain with states μ_L and μ_H, $\mu_L<\mu_H$, that changes state according to prescribed intensities. μ_L denotes the default rate of the borrower at "good" economical environment whereas μ_H denotes the default rate of the borrower at economics depression time.

The place where get changed in Model C from previous two models is also the default intensity, constant μ is replaced by a continuous-time Markov Chain $\mu(t)$, stressing the situation that the default rate could possibly be influenced by macroeconomic conditions. People tend to abide by their loan contracts with the bank at “good” economic environment as opposed to be apt to default on their obligations during economic depression years (“bad” time). It is interesting for the bank to investigate how the portfolio changes with respect to time through this model.

Unlike the previous models, we need to utilize Monte Carlo Simulations for calculating the ruin probabilities ψ_1 and ψ_2.

From the model description in Section 3, we know that the deals (borrowers) arrive according to a homogeneous Poisson distribution $X = (X(t))_{t\geq 0}$ with intensity λ, $X(t) \sim Po(\lambda t)$ and arrival times $0 \leq N_1 \leq N_2 \leq \ldots$ Then by Theorem A.2 in Appendix A, it is demonstrated that given the number of arrivals of a homogeneous Poisson process in the interval $[0,t]$, $X(t) = X_1$, their arrival times $N_1, N_2, \ldots N_{X_1}$ constitute the points of a uniform ordered sample in $[0,t]$.

The Markov Chain $\{\mu(t), t \geq 0\}$ has two states μ_L and μ_H, for the sake of convenience, we assume the economics is “good” at initial time $t = 0$, in another words, the Markov Chain starts from state μ_L, $\mu(0) = \mu_L$. According to the Markov Chain theory, see Theorem B 2.1 in Appendix B, we know the waiting time S before the chain transit to next state is Exponential distributed with a intensity. Here, the intensity with which the chain move to μ_H from μ_L is q_{HL}, and the waiting time of the chain jumps to μ_L from μ_H is Exponential distributed with intensity q_{LH}, as a result, the time pointes at which the Markov Chain (economics environment) jumps are $0 < S_1 < S_1 + S_2 < \ldots$. Then it is cleat that the default rate of those borrowers that arrive at time $t \in [0, S_1]$ is μ_L, the default rate of those borrowers that arrive at time $t \in (S_1, S_1 + S_2]$ is μ_H etc.

The algorithm for simulating ψ_1 and ψ_2 is prescribed as:

(1) Initialize the Markov Chain $\{\mu(t), t \geq 0\}$ to μ_L and time $t = 0$;
(2) Draw iid samples $X^{(1)}, X^{(2)}, \ldots, X^{(m)}$ from the Poisson distribution $Po(\lambda t)$;
(3) Draw iid samples $S_1, S_3, S_5 \ldots$ from $Exp(1/q_{LH})$, $S_2, S_4, S_6, \ldots$ from another distribution $Exp(1/q_{HL})$ respectively;
(4) Simulate iid order samples $N_1 < N_2 < \ldots < N_{X^{(1)}}$ from Uniform distribution $U[0,t]$;
(5) Draw Z_i from $Exp(\mu_L)$ if $N_i \in [0, S_1] \bigcup \left(\sum_{i=1}^{2} S_i, \sum_{i=1}^{3} S_i \right] \bigcup \left(\sum_{i=1}^{4} S_i, \sum_{i=1}^{5} S_i \right] \bigcup \ldots$, else, draw Z_i from $Exp(\mu_H)$, $i = 1, 2, \ldots X^{(1)}$;

(6) We get $T_1, T_2, ..., T_{X^{(1)}}$ through $T_i = \min(L, Z_i)$;

(7) Calculate the observation value of this simulation

$$W^{(1)} = W(0) + \sum_{i=1}^{X^{(1)}} \left\{ \left(\frac{M}{L} + Mr \right) T_i - M \right\};$$

(8) Repeat steps (3)-(7) m times, we can get the m iid observations values $W^{(1)}, W^{(2)}, ..., W^{(m)}$ of the risk process $W(t)$ at time t.

Differ from previous two models which use deterministic methods to calculate the ruin probability at time t ψ_1, For model C, we utilize Monte Carlo simulation method, by (4.2.1) and strong LLN, we have

$$\psi_1 \approx \widehat{P}_m = \frac{1}{m} \sum_{j=1}^{m} I_{(-\infty, W_{\min})} \left(W^{(j)}(t) \right) \tag{5.3.2}$$

The algorithm of Monte Carlo simulation for ψ_2 is presented as following:

(1) Initialize the Markov Chain $\{\mu(t), t \geq 0\}$ to μ_L and time $t = 0$;

(2) Draw iid samples $X^{(1)}, X^{(2)}, ..., X^{(m)}$ from the Poisson distribution $Po(\lambda t)$;

(3) Draw iid samples $S_1, S_3, S_5 ...$ from $Exp(1/q_{LH})$, $S_2, S_4, S_6, ...$ from another distribution $Exp(1/q_{HL})$ respectively;

(4) Simulate iid order samples

$$N_1 < N_2 < ... < N_{X^{(1)}}$$

from Uniform distribution $U[0, t]$;

(5) Draw Z_i from $Exp(\mu_L)$ if

$$N_i \in [0, S_1] \bigcup \left(\sum_{i=1}^{2} S_i, \sum_{i=1}^{3} S_i \right] \bigcup \left(\sum_{i=1}^{4} S_i, \sum_{i=1}^{5} S_i \right] \bigcup ...,$$

else,

draw it from $Exp(\mu_H)$, $i = 1, 2, ... X^{(1)}$;

(6) We get $T_1 < T_2 < ... < T_{X^{(1)}}$ in terms of $T_i = \min(L, Z_i)$;

(7) Calculate the iid copies of $W(t)$:

$$W(N_1) = W(0) + \left\{ \left(\frac{M}{L} + Mr \right) T_1 - M \right\},$$

$$W(N_2) = W(0) + \left\{ \left(\frac{M}{L} + Mr \right) T_1 - M \right\} + \left\{ \left(\frac{M}{L} + Mr \right) T_2 - M \right\},$$

$$\ldots,$$

$$W\left(N_{X^{(1)}}\right) = W(0) + \sum_{i=1}^{X^{(1)}} \left\{ \left(\frac{M}{L} + Mr \right) T_i - M \right\};$$

(8) Get $\inf_{0<i\leq X^{(1)}} W\left(N_i\right)$ on the base of step(7);

(9) Repeat steps (3)-(8) m times, obtaining iid observations

$$\inf_{0<i\leq X^{(1)}} W\left(N_i\right), \inf_{0<i\leq X^{(2)}} W\left(N_i\right), \ldots, \inf_{0<i\leq X^{(m)}} W\left(N_i\right).$$

The same as the calculation of approximation values for ruin probability ψ_2 in (4.3), for Model A and B, for model C, we have

$$\psi_2 \approx \widehat{P_m} = \frac{1}{m} \sum_{j=1}^{m} I_{(-\infty, W_{\min})} \left(\inf_{0<i\leq X^{(j)}} W\left(N_i\right) \right). \tag{5.3.3}$$

Note that in step (7), we used the fact (5.1.5) we mentioned before.

5.4 Numerical Cases

In this Section, we substitute real numbers for the parameters in Model A and C to perform the two methods, Central Limit Theorem Approximation and Monte Carlo Simulation, to achieve our objectives-calculating ruin probabilities ψ_1 and ψ_2.

To see how the initial capital quantity and interest rate exert influence on the ruin probability, we calculate ψ_1 and ψ_2 individually under different levels of W(0) and r for model A and C respectively. We give the outputs, compare ψ_1 and ψ_2 during same model and between different models, explore the reasons, dig out the regular rules underlying those numbers and at last, we give meaningful advices for seeking of specified low ruin probability.

We measure the surplus W(t) at $t = 10$ years; the clients arrival according to the Poisson process with intensity $\lambda = 1000$, which means there are 1000 deals come per year, $X(10) \sim Po(10*1000)$; every client borrow money by quantity $M = 5$ (*1000 EUR) and the time period for every client to pay all loan back is 2 years $L = 2$; the capital threshold $W_{\min} = 1000$ (*1000 EUR).

For Model A, the default rate μ is identical for every client, we assign it with $\mu = 0.1$, then the time point Z_i of the ith client default is Exponentially distributed with intensity 0.1, $Z_i \sim Exp(0.1)$. From the Section 5.1 we know ψ_1 is obtained by utilizing deterministic approximation of Central Limit Theorem, see (5.1.1)-(5.1.3). The results are given in following Table 5.4.1.

W(0) / r	1100	1200	1300	1400
5%	65.4%	35.15%	12.33%	2.64%
5.25%	8.7%	1.65%	0.18%	0.01%
5.5%	0.1%	0.01%	0	0
5.75%	0	0	0	0

Table 5.4.1 ψ_1 of Model A with different levels of W(0) and r

The ψ_2 of Model A are obtained by using Monte Carlo simulation, for seeking of accuracy, as close as possible to the real value, we simulate 1000 trajectories of process W(t), $t=10$. In out paper, we use MATLAB 7.0 to do the Monte Carlo simulation, outputs are given in following Table 5.4.2.

W(0) / r	1100	1200	1300	1400
5%	81.3%	46.2%	15.1%	4.7%
5.25%	23.3%	4.4%	0.4%	0.2%
5.5%	3.1%	0.3%	0	0
5.75%	0.5%	0	0	0

Table 5.4.2 ψ_2 of Model A with different levels of W(0) and r

The substituting numbers of the parameters of Model C are the same as Model A except the default rate. The default rate in Model C is not a constant but a Markov Chain. We assign those clients that arrive at "good" times with default rate $\mu_L = 0.08$, those clients that arrive at "bad" times with default rate $\mu_H = 0.12$; the jump rate of the Markov Chain μ from "good" times to "bad" times is $q_{LH} = 0.2$, waiting time before transiting to "good" times from "bad" times has the same intensity $q_{HL} = 0.2$. Monte Carlo simulation is used for calculating both ψ_1 and ψ_2 utilizing MATLAB 7.0, and the same as Model A, we also simulate 1000 trajectories of process W(t) at $t=10$ years. Outputs of are given in following Table 5.4.3 and 5.4.4 respectively.

W(0) / r	1100	1200	1300	1400	1500
5%	40.1%	30%	28.3%	24.1%	16.9%
5.25%	26%	19.1%	16.9%	11%	8.1%
5.5%	16%	8.9%	8.8%	6.5%	2.7%
5.75%	9%	4.4%	3.2%	1.6%	0.4%
6%	2.5%	0.4%	0.4%	0.2%	0

Table 5.4.3 ψ_1 of Model C with different levels of W(0) and r

W(0) / r	1100	1200	1300	1400	1500
5%	48.5%	37.8%	35.5%	28.7%	21.5%
5.25%	35.3%	26.5%	22.5%	15.1%	10.6%

5.5%	26.8%	12.8%	11.3%	9.1%	3.8%
5.75%	16.2%	7.6%	5.3%	2.7%	0.7%
6%	6.7%	1.7%	0.8%	0.3%	0

Table 5.4.4 ψ_2 of Model C with different levels of W(0) and r

After observing Tables 5.4.1-5.4.4, we have following results:

(1) Besides observing the definitions as (4.2) and (4.3), it is also proved numerically here that for both Model A and C, ψ_2 is not less than ψ_1.
(2) When $r \geq 5.25\%$ ψ_1 of Model C is higher than the ψ_1 of Model A. That is because W of Model C has higher Variance than Model A, which makes it more possible for Model C that a trajectory ends up below the threshold $W_{\min}$. Similarly, ψ_2 of Model C is higher than corresponding ruin probability of Model C when $r \geq 5.25\%$.
(3) When $r = 5\%$, it is a opposite situation from (2), ψ_1 and ψ_2 of Model A are higher than corresponding ψ_1 and ψ_2 of Model C. Actually, when $r = 5\%$, we have "negative drift", the bank is continuously losing money. Furthermore, recall that W of Model A has a smaller variance than Model C, in other words, if we start too close to $W_{\min}$, the ruin probability of Model A may very well be "higher" than Model C.
(4) If the bank aims to control the ruin probability below 1% during one period, for example in our paper $t \in [0,10]$, allowing for the influence from Micro-economics, the risk manager will choose ψ_2 as the indication. So from the Table 5.4.4, it is clear that r = 5.75%, $W(0)$ = 1500 and r = 6%, $W(0)$ = 1300 meet its requirement.

When we prescribed ψ_1 for Model B in Section 5.2, we used the Expectation Value of W(t), see (5.2.3), from which we have

$$EW_1 = -M + (M/L + Mr)\left(1 - e^{-(\alpha+\beta r)L}\right)\Big/(\alpha + \beta r),$$

which denotes the mean value of income from one client. In our paper, we primarily focus on how r and β influence this value. EW_1 acts much more sensitive with respect to the different value of β than r, so it is impossible for us to present one table like (5.4.1)-(5.4.4) to show how C performs with different interest rate. We use MATLAB to implement it, calculating different values of EW_1 with a huge number of r under several distinct values of β and plotting them. In our paper, we just give consequences after our observation.

(1) When $\beta = 0$, the model B becomes the Model A;
(2) When $0 < \beta \leq 1$, the path of EW_1 strictly increasing with respect to r;
(3) $1 < \beta \leq 2$, EW_1 takes values according to a parabola open towards down, increases to the top, then goes down;

(4) $\beta > 2$, the values of EW_1 strictly decreases;

(5) When $\beta \geq 1.4$, no matter how high the interest rate we set, the values of EW_1 are negative, the bank will definitely lose money.

5.5 Calculation of $EW(t)$ and $VarW(t)$

$$E(Z|Z<L)P(Z<L)=\int_0^L x\cdot \mu e^{-\mu x}dx=-\int_0^L xd\left(e^{-\mu x}\right)$$
$$=-\left(x\cdot e^{-\mu x}|_0^L-\int_0^L e^{-\mu x}dx\right)=-\left(L\cdot e^{-\mu L}+\frac{1}{\mu}\cdot e^{-\mu x}|_0^L\right)$$
$$=-\left(e^{-\mu L}-1\right)/\mu-Le^{-\mu L}$$
$$=1/\mu-\left(1/\mu+L\right)e^{-\mu L}$$

$$ET=E\left(E(T|Z)\right)=E(T|Z>L)\cdot P(Z>L)+E(T|Z<L)\cdot P(Z<L)$$
$$=EL\cdot P(Z>L)+E(Z|Z<L)\cdot P(Z<L)$$
$$=L\left[1-P(Z<L)\right]+1/\mu-\left(1/\mu+L\right)e^{-\mu L}$$
$$=L\cdot e^{-\mu L}+1/\mu-\left(1/\mu+L\right)e^{-\mu L}$$
$$=\left(1-e^{-\mu L}\right)/\mu$$

$$EW_1=E\left[-M+\left(M/L+Mr\right)\right]=-M+\left(M/L+Mr\right)ET$$
$$=-M+\left(M/L+Mr\right)\left(1-e^{-\mu L}\right)/\mu$$

$$EW(t)=W(0)+EX(t)\cdot EW_1$$
$$=W(0)+\lambda t\left[-M+\left(M/L+Mr\right)\left(1-e^{-\mu L}\right)/\mu\right]$$

$$E\left(Z^2|Z<L\right)P(Z<L)=\int_0^L x^2\cdot \mu e^{-\mu x}dx=-\int_0^L x^2de^{-\mu x}$$
$$=-\left(x^2\cdot e^{-\mu x}|_0^L-\int_0^L e^{-\mu x}dx^2\right)=2\int_0^L x\cdot e^{-\mu x}dx-L^2e^{-\mu L}$$
$$=-2\frac{1}{\mu}\int_0^L xde^{-\mu x}-L^2e^{-\mu L}=2\left[1/\mu-\left(1/\mu+L\right)e^{-\mu L}\right]/\mu-L^2e^{-\mu L}$$
$$=\left(-L^2-2L/\mu-2/\mu^2\right)e^{-\mu L}+2/\mu^2$$

$$
\begin{aligned}
ET^2 &= E\left(E\left(T^2 \mid Z\right)\right) = E\left(T^2 \mid Z > L\right)\cdot P\left(Z > L\right) + E\left(T^2 \mid Z < L\right)\cdot P\left(Z < L\right) \\
&= EL^2 \cdot P\left(Z > L\right) + E\left(Z^2 \mid Z < L\right)\cdot P\left(Z < L\right) \\
&= L^2 \cdot e^{-\mu L} + \left(-L^2 - 2\,L/\mu - 2/\mu^2\right) e^{-\mu L} + 2/\mu^2 \\
&= -2\left(L + 1/\mu\right) e^{-\mu L}/\mu + 2/\mu^2
\end{aligned}
$$

$$
\begin{aligned}
VarT &= ET^2 - \left(ET\right)^2 = -2\left(L + 1/\mu\right) e^{-\mu L}/\mu + 2/\mu^2 - \left[\left(1 - e^{-\mu L}\right)/\mu\right]^2 \\
&= \left(1 - 2\mu L e^{-L/\mu} - e^{-2L/\mu}\right)/\mu^2
\end{aligned}
$$

$$
\begin{aligned}
Var\left(W_1\right) &= \left(M/L + Mr\right)^2 \cdot Var\left(T\right) \\
&= \left(M/L + Mr\right)^2 \left(1 - 2\mu L e^{-\mu L} - e^{-2\mu L}\right)/\mu^2
\end{aligned}
$$

$$
\begin{aligned}
Var\left(W\left(t\right)\right) &= EX\left(t\right)\cdot Var\left(W_1\right) + Var\left(X\left(t\right)\right)\cdot\left(EW_1\right)^2 \\
&= \lambda t\cdot\left(M/L + Mr\right)^2 \left(1 - 2\mu L e^{-\mu L} - e^{-2\mu L}\right)/\mu^2 \\
&+ \lambda t\cdot\left[-M + \left(M/L + Mr\right)\left(1 - e^{-\mu L}\right)/\mu\right]^2
\end{aligned}
$$

When we set $\mu = \alpha + \beta r$, we get

$$
\begin{aligned}
EW\left(t\right) &= W\left(0\right) + EX\left(t\right)\cdot EW_1 \\
&= W\left(0\right) + \lambda t\left[-M + \left(M/L + Mr\right)\left(1 - e^{-(\alpha+\beta r)L}\right)/\left(\alpha + \beta r\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left(W\left(t\right)\right) &= EX\left(t\right)\cdot Var\left(W_1\right) + Var\left(X\left(t\right)\right)\cdot\left(EW_1\right)^2 \\
&= \lambda t\cdot\left(M/L + Mr\right)^2 \left[1 - 2\left(\alpha + \beta r\right) L e^{-(\alpha+\beta r)L} - e^{-2(\alpha+\beta r)L}\right]/\left(\alpha + \beta r\right)^2 . \\
&+ \lambda t\cdot\left[-M + \left(M/L + Mr\right)\left(1 - e^{-(\alpha+\beta r)L}\right)/\left(\alpha + \beta r\right)\right]^2
\end{aligned}
$$

6 Summary and Conclusions

Credit risk become more and more important in modern society, exerting great influence on all trades and professions such as Finance, Insurance, Real Estate, Energy, Education, Manufacture, Healthcare, transportation and so on. This leads to an accelerating demands of credit risk manage for privates as well as companies.

Our research object is a commercial bank. The bank borrow money from the residents and market, then lend them to the clients including individuals, families, firms or do investment to earn the price differences. The credit risk is an essential risk which the bank has to confront and address, that explains why risk management department of a bank becomes crucial, not only supplying risk management service to external customers but also controlling internal risks. Except for the historical management experience and traditional methods, the quantitative analysis is playing a effective and common role in modern credit risk management.

In our paper, we built three dynamic portfolio models to investigate how the capital flow is affected by the credit risk with respect to the private lending business of the bank. By utilizing the ruin theory in Insurance, we theoretically described how to calculate two ruin probabilities by two methods, Central Limit Theorem approximation and Monte Carlo simulation. Furthermore, we numerically investigated how different values of initial capital and interest rate influence the two ruin probabilities respectively. As a result, besides using ψ_1 and ψ_2 as the risk measure for the credit risk exposure of the bank, they also can be utilized to determine proper interest rate, how much capital the bank should hold (initial capital) and even proper value of β, to ensure its business survive or make sure those ruin probabilities in accordance with the risk appetite formulated by the management, avoiding sharp declining amount of clients due to too high interest rate or inacceptable loss on account of too low one.

It would be also interesting to investigate the loan conditions of real-life unsecured consumption loans offered by different financial institutions, with the purpose to find out whether the effective interest rates taken by the institutions can be motivated from a credit risk point of view.

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Appendix A: Poisson Processes

A Poisson process, named after the French mathematician Siméon-Denis Poisson (1781–1840), is a stochastic process in which events happen continuously and independently of one another. The classical Poisson processes models are the radioactive decay of atoms, telephone calls arriving at a switchboard, page view requests to a website, and rainfall.

The Poisson process is a set $\{N(t): t \geq 0\}$ of random variables, where $N(t)$ refer to the number of events that have occurred between time 0 and t. The number of events that have occurred between time a and time b is given as $N(b) - N(a)$ and is Poisson distributed. Each realization of the process $\{N(t)\}$ is a non-negative integer-valued step function that is non-decreasing, but intuitively, it is usually easier to regard it as a point pattern on $[0, \infty)$, the points in time where the step function jumps, e.g. the points in time where an event occurs.

The Poisson process is a continuous-time process, it is also an example of continuous-time Markov process, i.e. a Poisson process is a pure-birth process, the simplest example of a birth-death process and it is also a point process on the real half-line.

Definition A stochastic process $\{N(t): t \geq 0\}$ is said to be a Poisson process if the following conditions hold:

- The process starts at zero: N(0)=0 a.s.;
- Independent increments: for any t_i , $i = 0, ..., n$ and $n \geq 1$ such that $0 = t_0 < t_1 < ... < t_n$, the increments $N(t_{i-1}, t_i]$, $i = 1, ...n$, are mutually independent ($N(t_{i-1}, t_i]$ describes the the number of events that have occurred between time t_{i-1} and time t_i);
- The exists a non-decreasing right-continuous function μ : $[0, \infty) \to [0, \infty)$ with $\mu(0) = 0$ so that the increments $N(s, t]$ have a Poisson distribution $Pois(\mu(s, t])$, μ is referred to as the mean value function of N;
- Stationary increments: the probability distribution of the number of occurrences counted in any time interval only depends on the length of the interval, $N(s, t] \underline{\underline{d}} N(s + h, t + h] \sim Pois(\lambda(t - s))$.

Consequences of this definition include:

- The probability distribution of $N(t)$ is a Poisson distribution, $N(t) = N(t) - N(0) = N(0, t] \sim Pois(\mu(0, t]) = Pois(\mu(t))$.
- The probability distribution of the waiting time until the next occurrence is an exponential distribution, as illustrated in the following part.

Homogeneous Poisson process**:** The most popular Poisson process corresponds to the case of a linear mean value function μ **:**

$$\mu(t) = \lambda t, t \geq 0, \tag{A.1}$$

for some $\lambda > 0$. A process with such a mean value function is said to be homogeneous, in homogenous otherwise. The quantity λ is the intensity of the homogeneous Poisson process. If $\lambda = 1$, N is called standard homogeneous Poisson process.

For the homogeneous Poisson process, the number of events occurring in time interval $(t, t+\tau]$ follows a Poisson distribution with associated parameter $\lambda\tau$. This representation is given as

$$P[(N(t+\tau) - N(t)) = k] = \frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!}, k = 0,1,\ldots, \quad \text{(A.2)}$$

where N(t+τ)−N(t) is the number of events occurring in the time interval $(t, t+\tau]$.

Just as a Poisson random variable is characterized by its scalar parameter λ, a homogeneous Poisson process is characterized by its rate parameter λ, which represents the expected number of events (or arrivals) that occur per unit time.

Renewal process

For any homogeneous Poisson process with intensity $\lambda > 0$ and arrival times $0 \le T_1 \le T_2 \le \ldots$ we have the representation

$$N(t) = \#\{i \ge 1 : T_i \le t\}, t \ge 0, \quad \text{(A.3)}$$

where

$$T_n = W_1 + \ldots + W_n, n \ge 1, \quad W_i \sim Exp(\lambda) \quad i.i.d \quad \text{(A.4)}$$

Since the random walk (T_n) with non-negative step sizes W_n is also referred to as renewal sequence, a process N with representation (A.3)-(A.4) for a general iid sequence (W_i) is called renewal (counting) process.

Theorem A.1 (The homogeneous Poisson process as a renewal process)

(1) The process N given by (A.3) and (A.4) with an iid exponential $Exp(\lambda)$ sequence (W_i) constitutes a homogeneous Poisson process with intensity $\lambda > 0$.

(2) Let N be a homogeneous Poisson process with intensity λ and arrival times $0 \le T_1 \le T_2 \le \ldots$. Then N has representation (A.3), and (T_i) has representation (A.4) for an iid exponential $Exp(\lambda)$ sequence (W_i).

Theorem A.2 (Order statistic property of the Poisson process)

Consider the Poisson process $N=\left(N(t)\right)_{t\geq 0}$ with continuous a.e. positive intensity function λ and arrival times $0<T_1<T_2<\ldots a.s.$ Then the conditional distribution of $(T_1,\ldots,T_n)$ given $\{N(t)=n\}$ is the distribution of the ordered sample $\left(X_{(1)},\ldots,X_{(n)}\right)$ of an iid sample $X_1,\ldots,X_n$ with common density $\lambda(x)/\mu(t)$, $0<x\leq t$:

$$\left(T_1,\ldots,T_n \mid N(t)=n\right) \underline{\underline{d}} \left(X_{(1)},\ldots,X_{(n)}\right).$$

In other words, the left-hand vector has conditional density

$$f_{T_1,\ldots,T_n}\left(x_1,\ldots,x_n \mid N(t)=n\right)=\frac{n!}{\left(\mu(t)\right)^n}\prod_{i=1}^{n}\lambda(x_i),\quad 0<x_1<\ldots<x_n<t.$$

Consider a homogeneous Poisson process with intensity $\lambda>0$. Then Theorem A.2 yields the joint conditional density of the arrival times T_i:

$$f_{T_1,\ldots,T_n}\left(x_1,\ldots,x_n \mid N(t)=n\right)=n!t^{-n} \tag{A.5}$$

Appendix B: Continuous-time Markov Chains

B1 Markov Chain

Definition

A Markov chain is a sequence of random variables X_1, X_2, X_3, ... with the Markov property: given the present state, the future and past states are independent,

$$\Pr\left(X_{n+1} = x \mid X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\right) = \Pr\left(X_{n+1} = x \mid X_n = x_n\right)$$

The possible values of X_i form a countable set S which is referred to as the state space of the chain.

The probability of moving from state i to state j in n time steps is

$$p_{ij}^{(n)} = \Pr\left(X_n = j \mid X_0 = i\right)$$

B2 Continuous-time Markov Chain

As before we assume that we have a finite or countable state space S, but now the continuous-time Markov chain $X = \left\{X\left(t\right) : t \geq 0\right\}$ which satisfies the Markov property has a continuous time parameter $t \in \left[0, \infty\right)$. Transitions from one state to another can occur at any instant of time.

In continuous-time Markov chain, there are no smallest time steps and hence we cannot talk about one-step transition matrices any more. If we are in state $i \in S$ at time t, then the probability of transition to a different state $j \in S$ at time t + h is

$$\Pr\left\{X(t+h) = j \mid X(t) = i\right\}$$

We are interested in small time steps, i.e. small values of h > 0. We have

$$\lim_{h \to 0} \frac{\Pr\left\{X(t+h) = j \mid X(t) = i\right\}}{h} = q_{ij}$$

We can write this as

$$\Pr\left\{X(t+h) = j \mid X(t) = i\right\} = q_{ij}h + o(h)$$

And o(h) here is a convenient abbreviation for any function with the property that $\lim_{h\to o} o(h)/h = 0$, the function is of smaller order than h. Hence, over a sufficiently small interval of time, the probability of a particular transition between different states is roughly proportional to the duration of that interval.

The Markov property states that at any times $t+h>t>0$, given the whole history of the process up to and including time t, the conditional probability distribution of the process at time $t+h$, depends only on the state of the process at time t. For all $i \neq j$, $0<t_0<t_1<...<t_n<t$, we have

$$\Pr\left\{X(t+h)=j \mid X(t)=i, X(t_i)=x_i\right\} = \Pr\left\{X(t+h)=j \mid X(t)=i\right\} = q_{ij}h+o(h)$$

From this we get, for every $j \in S$,

$$\Pr\left\{X(t+h)=j \mid X(t)=j\right\} = 1-\sum_{i\in S} q_{ji}h+o(h) = 1+q_{jj}h+o(h)$$

If we define $q_{jj} = -\sum_{i\in S} q_{ji}$.

We say that q_{ij} gives the rate at which we try to enter state j when we are in state i, or the jump intensity from i to j.

We can put all the transition probabilities information into a matrix $Q=\left(q_{ij}: i,j\in S\right)$, which contains all the information about the transitions of the Markov chain X. This matrix is called the transition matrix of the Markov chain.

Given the transition matrix Q one can construct the paths of a continuous time Markov chain as follows. Suppose the chain starts in a fixed state $X_0 = i$ for $i \in S$. Let

$$T = \min\left\{t: X_t \neq X_0\right\}$$

be the first jump time of X, we always assume the minimum always exists.

Theorem B2.1 The Markov chain started in $X_0 = i$, the random variables T and X(T) are independent, T is exponentially distributed with rate $q_{ii} = \sum_{j\neq i} q_{ij}$, which means

$$\Pr\left(T>t\right) = e^{-q_{ii}t} \quad \text{for} \quad t \geq 0.$$

Moreover,

$$\Pr\left(X(T)=j\right)=\frac{q_{ij}}{q_{ii}}.$$

And the chain starts afresh at time T. That represents the probability distribution of the waiting time until the first transition is an exponential distribution with rate parameter q_{ii}, and hence continuous-time Markov processes are memoryless processes.

Appendix C: Law of Large Numbers

We intuitively view the probability of a certain outcome as the frequency with which that outcome occurs in the long run when the experiment is repeated a large number of times. And probability is mathematically defined as a value of a distribution function for the random variable representing the experiment. In probability theory, the law of large numbers (LLN) is a theorem which describes the result of implementing the same experiment a large number of times. By the law, the average of the results obtained from a large number of trials should be close to the expected value, and will tend to be closer when more trials are performed.

Theorem C.1 (Chebyshev Inequality)

Let X be a continuous random variable with density function f. Suppose X has a finite expected value $\mu = E(X) < \infty$ and finite variance $\sigma^2 = V(X) < \infty$. Then for any positive number $\varepsilon > 0$ we have

$$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

We omit the proof here. Note that this theorem says nothing if $\sigma^2 = V(X)$ is infinite.

With the Chebyshev Inequality we can state the Law of Large Numbers for continuous case.

Theorem C.2 (Weak Law of Large Numbers) Let $X_1, X_2, ..., X_n$ be independent random variables with a continuous density function f, finite expected value μ, and finite variance σ^2. Let $S_n = X_1 + X_2 + ... + X_n$ be the sum of the X_i. Then for any small number $\varepsilon > 0$ we have

$$\lim_{n \to \infty} P\left(\left| \frac{S_n}{n} - \mu \right| \geq \varepsilon \right) = 0,$$

Or equivalently,

$$\lim_{n \to \infty} P\left(\left| \frac{S_n}{n} - \mu \right| < \varepsilon \right) = 1.$$

Note that this theorem is not necessarily true if σ^2 is infinite.
The Weak Law of Large Numbers says that the average value of n independent trials tends to the expected value as $n \to \infty$, in the precise sense that, given ε, the probability of the differ between average value and the expected value more than σ^2 tends to 0 as $n \to \infty$ (convergence in probability).

Theorem C.3 (Strong Law of Large Numbers) Let $X_1, X_2, ..., X_n$ be an independent random variables with a continuous density function f, finite expected value μ, and finite variance σ^2. Let $S_n = X_1 + X_2 + ... + X_n$ be the sum of the X_i. Then for any small number $\varepsilon > 0$ we have

$$\mathrm{P}\left(\lim_{n\to\infty}\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) = 0\,,$$

which expresses the fact that the sample converges almost surely to the distribution mean.

Appendix D: Central Limit Theorem

In probability theory, the central limit theorem (CLT) states that if S_n is the sum of n mutually independent and identically distributed random variables, then the distribution function of S_n is well-approximated by a certain type of continuous function known as a normal density function, which is given by the formula

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

When $\mu = 0$ and $\sigma = 1$, we call this particular normal density function the standard normal density, denoted by $\phi(x)$:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

The Central Limit Theorem tells us, quite generally, what happens when we have the sum of a large number of independent random variables each of which contributes a small amount to the total. Since real-world quantities are often the balanced sum of many unobserved random events, this theorem provides a partial explanation for the prevalence of the normal probability distribution. The CLT also justifies the approximation of large-sample statistics to the normal distribution in controlled experiments.

The Central Limit Theorem for independent trials process is as follows.

Theorem D.1 (Central Limit Theorem) Let $S_n = X_1 + X_2 + ... + X_n$ be the sum of n independent random variables with common distribution having expected value μ and variance σ^2. Then, for $a < b$,

$$\lim_{n\to\infty} P(a < \frac{S_n - n\mu}{\sqrt{n\sigma^2}} < b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx$$

Depend on the appendix A and C, we give following theorem.

Theorem D.2 (The strong law of large numbers and the central limit theorem in the renewal mode)
Assume the renew model for W.

(1) If the inter-arrival times and the claim sizes X_i have finite expectation, W satisfies the strong law of large numbers:

$$\lim_{n\to\infty} \frac{W(t)}{t} = \lambda E X_1 \quad a.s.$$

(2) If the inter-arrival times and the claim sizes X_i have finite variance, W satisfies the central limit theorem:

$$\sup_{x\in\mathbb{R}}\left|P\left(\frac{W(t)-EW(t)}{\sqrt{\operatorname{var}(W(t))}}\le x\right)-\Phi(x)\right|\to 0,$$

where Φ is the distribution function of the standard nomal $N(0,1)$ distribution.