Mathematical research during the first decades of the University of Stockholm

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1 Introduction

The University of Stockholm was founded in 1878 as an institution of higher learning. The original Swedish name was "Stockholms Högskola", changed in 1960 to "Stockholms universitet". It was, during its first decades, a privately endowed institution but partially supported by the city of Stockholm. It was founded as a complement, perhaps also as a competitor to the two state universities at Uppsala and Lund. During the century of its existence the University of Stockholm has gradually lost its special profile, becoming a state university like the others. It was very small at the start, with only a faculty of mathematics and natural sciences possessing a very limited number of chairs. One of the first chairs was in mathematics.

Within the young University mathematical research activities of a high standard developed rapidly. The scientific work was largely molded and directed by a single person, Gösta Mittag-Leffler, but the most notable research contributions were made by some of his many students. Mathematical activity at the University during the period Mittag-Leffler was professor, which is to say 1881-1911, was of great significance for the development of mathematical research both within Sweden and internationally. Thus, in connection with the centennial celebration of the founding of the University it is fitting to look back over this important activity and the most interesting of the University's scientifically engaged mathematicians. In order to present events in Stockholm against the proper background I will begin with a description of the situation within the country as a whole as far as regards research within pure mathematics.

2 Mathematics in Sweden

As is well known, Swedish research has long and distinguished traditions within many of the natural sciences. As early as the 18th or beginning of the 19th century Sweden occupied a prominent position internationally in many fields of research — it is sufficient to recall the names Linnaeus, Celsius, Scheele, Torbern Bergman and Berzelius. A mathematician sometimes mentioned in this context is *Samuel Klingenstierna* (1698-1765), distinguished Uppsala professor of mathematics and physics. But his most important scientific contributions were made in physics. As a pure mathematician he was certainly both learned and able, in, among other areas, geometry, as is evidenced by his posthumous manuscripts [19], but his published works were few and he had no influence on the development of mathematics internationally.

It is in fact not until the 1870s that one finds significant Swedish research results in mathematics. The primary reason for the country's mathematics research remaining undeveloped and provincial for such an extended period must quite simply have been that Sweden lacked mathematicians who, independently or through the establishment of international contacts, were able to elevate their research to an international level. Here it may be of interest to make a comparison with Norway, where preconditions for mathematical research presumably approximated those in Sweden. There, however three significant mathematicians appeared before anything began to develop in Sweden: the celebrated Niels Henrik Abel (1802-1829), the group theorist Ludwig Sylow (1832-1918) and Sophus Lie (1842-1899) who during the 1860s began his pioneering research on transformation groups.

The activity of these famed Norwegians was for various reasons more international than Scandinavian in its orientation, and they had hardly any direct contact with Sweden. The work of Sophus Lie was, however, the primary source of inspiration for the very first noteworthy Swedish mathematician, Lund professor *Albert Viktor Bäcklund* (1845-1922) who during the 1870s and 1880s published results within geometry and theory of partial differential equations [18] which drew international attention. Bäcklund's mathematical activities were subsequently inspirational for the second of Lund's important mathematicians of the 19 century, *Anders Wiman* (1865-1959), especially in his early work in geometry [5] . Wiman, who during his time at Lund was also known for his work in algebra, left at the turn of the century for Uppsala. By that time Bäcklund had largely abandoned his work in mathematics to concentrate on his activities as professor of mechanics and physics. Mathematical research in Lund declined and did not revive until Marcel Riesz arrived to be professor in 1926 [7].

Towards the end of the 19th century, then, Lund demonstrated significant mathematical activity. Uppsala, on the other hand, offers little of interest. Professor of Mathematics there at the middle of the century was *Carl Johan Malmsten* (1814-1886). He was responsible for a certain degree of modernization and stimulation of mathematics in Uppsala. But conditions stagnated when he left Uppsala at the close of the 1850s (he was first a cabinet minister, later the governor of the province of Skaraborg). However, he did not lose his interest in mathematics and has a certain importance to us because of his support for Mittag-Leffler during the first years of the University of Stockholm.

None of Malmsten's Uppsala students were responsible for any especially remarkable research results, but two of them are of interest in the present context. They are *Hjalmar Holmgren* (1822-1885), who was appointed professor at the Technological Institute of Stockholm (from 1887 the Institute of Technology), and *Göran Dillner* (1832-1906) who is worthy of mention here as having been Mittag-Leffler's instructor. Perhaps this would be the point to mention that Mittag-Leffler himself accomplished little research of note during his time as a student in Uppsala. Uppsala was not in fact to become a name on the mathematics research map until the turn of the century, and then first of all through *Erik Holmgren* (1872-1943), son of Hjalmar Holmgren. Erik Holmgren's famed uniqueness theorem for partial differential equations dates from 1901. Anders Wiman arrived from Lund during this same year, which set the stage for fruitful research activities in Uppsala in coming years.

It is perhaps remarkable that mathematical research at Uppsala remained at a low level throughout the 1880s and 90s. One supposes that the intensive and high-quality activity during that time at the University of Stockholm should have been stimulating and encouraging. But matters of prestige and personal differences obstructed possible contacts and hindered mathematics at Uppsala from benefitting from events in Stockholm.

In Stockholm there was no organized mathematical research before the founding of the University in 1878. There was the Academy of Sciences, but its activity within mathematics was limited to publishing, and distribution of minor awards. At the Institute of Technology mathematics was merely an area of instruction — research in pure mathematics at Swedish institutes of technology was not actually initiated until the 1950s. Nor were there individuals conducting private mathematical studies of any higher calibre. During the University's first three years, 1878-81, mathematics and mechanics were coordinated, instruction being conducted by Hjalmar Holmgren in addition to his duties as professor at the Institute of Technology. However, actual research activities within mathematics were not initiated until 1881. At that time the University gained its first two professors, and one of them was *Gösta*

Mittag-Leffler (1846-1927), in pure mathematics. It is obvious that we must devote a special place to him in this survey.

3 Mittag-Leffler

Gösta Leffler, who later changed his family name to Mittag-Leffler, grew up in Stockholm and studied at the University of Uppsala, where he received instruction primarily from Dillner. Dillner had started a journal for elementary mathematics and physics which was published in four volumes during the years 1868-74. In a series of essays in the journal he presented Cauchy's theory of analytic functions, and Mittag-Leffler's doctoral dissertation (1872) was a development of Dillner's memoirs concerning the argument principle and its applications. The dissertation is not very impressive, even if one makes allowances for prevalent conditions, i.e., the arid mathematical climate of Uppsala at that time. More promising is a note of 1873 in which Mittag-Leffler proves Cauchy's integral theorem in a fashion reminiscent of Goursat's famed proof of a few years later. Decisive for Mittag-Leffler's development was the journey abroad he undertook as recipient of a scholarship (1873-76) and which he himself described 50 years later in [16]. He traveled first to Paris, where he heard Hermite lecture on elliptic functions and became acquainted with the leading mathematicians of the city. Hermite is reputed to have asked Mittag-Leffler, "Why have you not travelled to Berlin and Weierstrass? He is beyond comparison the foremost of us all". It is relevant that at this point Mittag-Leffler had not even heard of Weierstrass. He took Hermite's advice and visited Weierstrass in Berlin towards the end of the year 1874.

Karl Weierstrass (1815-1897) was one of the truly great mathematicians of the 19th century. His goal was to complete the theory of elliptic functions as developed by Abel and Jacobi. He was successful in his task and in so doing placed both the real and complex analysis on solid foundations.

Weierstrass had been a secondary school teacher until he was more than 40 years old. He was now 60, esteemed by a smaller group but still not generally known among contemporary mathematicians. He published little and was usually satisfied to present his pioneering results through his lectures. Since he was at the same time generous with his ideas and problems, his pupils had extraordinary possibilities to contribute to the development of the new theory of functions.

Mittag-Leffler did not miss this opportunity. He quickly learned his master's mathematics and became one of his most devoted spokesmen. Close contact with Weierstrass and his ideas soon bore fruit in the form of the celebrated Mittag-Leffler theorem on the existence of a meromorphic function with arbitrarily prescribed poles. The road to a general and elegantly proved theorem was, however, a long one. In 1876 Mittag-Leffler published a complicated and only partially correct proof for a special case. In the years to follow the proof was corrected and the theorem gradually generalized, but the proof remained unnecessarily complicated. Weierstrass was the first, in 1880, to give the simple proof which is found in today's textbooks. A work by Mittag-Leffler which is more impressive in certain respects is his review of the development of the theory of elliptic functions (1876). Weierstrass is much in evidence here also, but Mittag-Leffler has the opportunity to document his thorough comprehension of the theory and his exceptional pedagogic and stylistic gifts. This work was certainly a major contributing factor to Mittag-Leffler being appointed professor in Helsinki in 1877, and to his being called four years later to the chair at the University of Stockholm.

Mittag-Leffler rapidly achieved a high scientific standing within Sweden and internationally. But his reputation had less to do with his abilities as a researcher than with other factors. His own research was not highly creative nor trail-blazing and was clearly dependent on Weierstrass. But it is interesting to note that Mittag-Leffler's most impressive scientific work was done at the turn of the century, which is to say after the death of his mentor. Naturally this does not mean his ideas could not have arisen from earlier contacts with the great teacher. What we refer to is a series of longer works in which Mittag-Leffler developed the theory of what is now termed the Mittag-Leffler star, the largest star-shaped domain into which the sum of a power series can be continued analytically. Among other things he summed power series by introducing the factors $\Gamma(n\alpha+1)^{-1}$, where α is positive and tends to 0. Mittag-Leffler studied especially the entire function $E_\alpha(z)$ which is obtained if one applies this procedure to the power series expansion of $(1-z)^{-1}$, something which aroused the interest of Wiman, who was now in Uppsala. It is likely that this was the stimulus for Wiman's work within function theory, especially his valuable researches on the minimum modulus of entire functions [5].

Mittag-Leffler's scientific production was very comprehensive. In addition to what has been mentioned, there were a large number of lesser works e.g. in the theory of linear differential equations and also historical works including a brilliant biography of Abel.

Mittag-Leffler's fame within the mathematical world, however, derived above all from his activities as an organizer and disseminator of mathematical ideas. In this respect his most important contribution was the founding of the journal *Acta Mathematica* in 1882, the year after he had come to the University. It is worth noting in this context that ever since the Franco-Prussian

War (1870-71) there had been great tension in the scientific relationship of the two mathematical powers, France and Germany. The new internationally oriented journal, published by an individual with close ties in both camps, contributed actively to breaking the deadlock.

Acta Mathematica got off to a splendid start. The project had as patron the scientifically-minded King Oscar II, and was supported morally and economically by Hermite, Malmsten and other sponsors. From the first the mathematical content was of the very highest calibre, partly through Mittag-Leffler's excellent contacts with internationally leading mathematicians, partly because of his almost intuitive ability to judge mathematical work and distinguish its essential qualities. Consequently Acta Mathematica was from the beginning one of the leading — and decidedly the most international — of the mathematical journals, and occupies that leading position even today. Among those prominent mathematicians who were granted space in Acta Mathematica in those first years may be noted Georg Cantor, the controversial and bitterly attacked set theorist, whose greatness was recognized by Mittag-Leffler, and Henri Poincaré, perhaps the most distinguished and versatile mathematician of his day. The great prestige enjoyed by Mittag-Leffler as publisher of Acta Mathematica was given expression on his 50th birthday, when he received a congratulatory address bearing the signatures of four hundred mathematicians from all over the world.

Another aspect of Mittag-Leffler's activities in the international sphere was his involvement with mathematical congresses. He was among the mathematicians who signed the invitations to the first international congress of 1897 in Zurich. He was constantly one of the vice presidents at following congresses up until the last he attended, in Toronto 1924, when he was elected honorary president. As regards cooperation in mathematics in Scandinavia, it was a significant gesture on his part to allow the editorial board of Acta Mathematica to be composed of Scandinavian mathematicians, and Mittag-Leffler was solely responsible for taking the initiative for the Scandinavian mathematician congresses, the first of which was held in Stockholm in 1909.

When attempting to form a picture of Mittag-Leffler's activities at the University of Stockholm it is necessary to bear in mind that the University at that time had a completely different orientation than the Universities of Lund and Uppsala. In Stockholm the emphasis was on independent studies and research unrestrained by examinations and rigid course requirements [1]. The school did not gain the right to grant degrees until 1905. Before that time students who wished to be examined had to travel to Uppsala or Lund, and this was also the case for anyone who wished to qualify for the doctorate. Discussion concerning the extent to which Stockholm met the needs of society was vehement and protracted, and Mittag-Leffler was consistently at the head of those within the school who claimed the University should be free of examinations and research-oriented.

Mittag-Leffler was a skillful lecturer with an uncommon ability to enthuse his pupils towards mathematical research, and he supported them in every way possible, including an early introduction to significant research problems — he was never at a loss for such problems, thanks to his rich international contacts and his feeling for what was important in mathematics. His role as advisor to young mathematicians can hardly be overestimated, and will be apparent from the description below of his many students.

Mittag-Leffler was wealthy — he had married the daughter of a prosperous Finnish general — and he had good relationships within influential circles, including the royal court. His social position, together with his scientific reputation and his skill at performing different roles, such as writer and speaker, but also less openly as an often ruthless intriguer, made him a very powerful individual. But through his various maneuvers he became a controversial figure and had many critics, enviers and enemies. Alfred Nobel disliked his struggle for power at the University, and this may have had an influence on the formulation of the Nobel Prize regulations. It is also noteworthy in this context that none of the mathematics professors from Uppsala or Lund were among those signing the 50th birthday message to Mittag-Leffler. Not even his old teacher Dillner, now extraordinary professor at Uppsala, acknowledged the occasion. But in Dillner's case this is perhaps understandable; in 1883 he had in a letter levelled accusations of plagiarism at Mittag-Leffler, asserting that the theorem on meromorphic functions had been taken from Dillner's memoirs in his own journal (mentioned above). The charges were completely unfounded, but Dillner, who obviously did not understand the new function theory, was probably never convinced of his mistake.

Torsten Carleman is responsible for the following sketch characterizing Mittag-Leffler [2]:

"Mittag-Leffler could hardly be considered an unworldly sort of scientist. Certainly he was a great thinker within his chosen field, but he also showed a lively interest in all the events of his time, whether politics, economics, science, literature, art or fundamental human questions. He sought and won wealth and influence. When he had set himself a goal he could with great skill and speed and perhaps also a certain ruthlessness bring to bear all those forces that might forward his designs. In most cases he was indeed successful. He was a man for action and struggle and like all individuals so constituted he acquired devoted friends as well as bitter enemies."

To these observations we can add the following, written by Harald Cramér [3]:

"Mittag-Leffler's personality and accomplishments have been the object of widely different judgements. Enthusiastic, combative and energetic he was surely possessed of many qualities, both lesser and greater. It is often difficult to distinguish to what extent he was driven by a personal desire for honors and power, to what extent by an objective concern for his science and his university. But one of his truly great qualities was undoubtedly his ability to gather about himself a group of gifted pupils, to fill them with enthusiasm for mathematical research and actively support them, even if these students were eventually in not a few cases to greatly surpass their teacher in scientific creativity."

Carleman mentioned Mittag-Leffler's interest in politics. This was actively demonstrated in conjunction with the famous Farmer's March and government crisis (*borggårdskrisen*) of 1914, when he and others including the famed explorer Sven Hedin publicly expressed their support of the monarchy. In a speech delivered to students in Stockholm he attacked the recently resigned prime minister Karl Staaf, whom he indirectly accused of treason. Charges were brought and Mittag-Leffler was fined 400 crowns, but the Supreme Court reversed the decision on the grounds of a technical error in the indictment proceedings.

The picture of Mittag-Leffler would be incomplete without mention of the palatial villa he had built at Djursholm and the enormous mathematical library he assembled there, the largest private mathematical library of his day. The library was open to his students, which was naturally of great significance for research at the University. Mittag-Leffler also had a summer residence in Tällberg, with a magnificent view of Lake Siljan (it is presently the main building of Hotel Dalecarlia). The Villa and library at Djursholm were transferred upon Mittag-Leffler's death to "Professor and Mrs. Mittag-Leffler's Mathematical Foundation". Mittag-Leffler's wish that the foundation should be an active mathematical research institute could not, for financial reasons, be realized until 40 years later, and then primarily due to Lennart Carleson's personal engagement in the matter. The Mittag-Leffler Institute is now an active and important center for research within pure mathematics.

An individual as interesting as Mittag-Leffler should be a suitable object for an extended biography; at present, however, there are only shorter essays available, e.g. [2, 4, 10 and 17]. One of the most readable is an obituary by G. H. Hardy [10]. I will cite his closing words:

"Mittag-Leffler played a very great part in the history of mathematics. He had exceptional opportunities, and exactly the blend of qualities required to take advantage of them to the full. There have been greater mathematicians during the last fifty years, but no one who has done in his way more for mathematics".

4 Kovalevski

Another real celebrity during the early years of the University was *Sonja Kovalevski* (1850-1891). The literature concerning her is plentiful, partly reviewed in [13] and [23]; consequently the treatment here will be more abbreviated than her significance would otherwise warrant.

She grew up in Moscow in an upper-class environment, and at the age of 17 underwent a pro forma marriage in order to leave home and educate herself in Germany. After time spent in Heidelberg she came to Weierstrass in Berlin, becoming his private pupil, since the University of Berlin did not accept female students. Her doctoral dissertation (1874) comprises three works, and one of them contains the famed Cauchy-Kovalevski theorem, which states that an analytic partial differential equation in normal form has locally a unique analytic solution. In 1874, before Mittag-Leffler arrived in Berlin, Sonja Kovalevski returned to Russia, devoting herself to her family and literary activity. In the 1880s she returned to mathematics. A visit with Weierstrass resulted in a larger work, "On the transmission of light in a crystalline medium" printed in 1884. This proved to be an unsuccessful work, for the young Italian Vito Volterra demonstrated that the results were incorrect due to a mistake in the derivation. Mittag-Leffler later tried to minimize her responsibility for the essay, claiming that it was almost a case of teamwork between her and Weierstrass (which is not evident from what was printed), and that Weierstrass had been so occupied with other tasks it had not been possible for him to do his best [15, p. 193].

However, before this work was published Sonja Kovalevski had come to Stockholm. Mittag-Leffler, already during the time he had been in Helsinki, had intended to try to arrange a position for this favorite pupil of Weierstrass. At Stockholm the possibilities were greater and in 1884 he was successful in having her appointed extraordinary professor, a remarkable accomplishment on the part of Mittag-Leffler considering the status of women at that time in scientific circles and the fact that her doctoral dissertation was at this point her only published scientific work.

This richly gifted and evidently fascinating Russian woman naturally attracted the greatest attention in Stockholm. This was not only true at the University, where her lectures were greatly valued, but also in society and in literary circles, into which she was introduced by Mittag-Leffler's sister, the author Anna Charlotta Edgren. For Sonja Kovalevski Stockholm seemed a remote outpost, and she seized every possible opportunity to travel to Paris or to Weierstrass in Germany. In 1888 she completed a significant work on the mathematical theory of the rotation of a solid body around a fixed point, a contribution which was rewarded by the French Academy of Science with the distinguished Bordin Prize and which also facilitated Mittag-Leffler's action to have her given a permanent professorship in 1889. However, a little more than a year later she died of complications following an attack of epidemic influenza.

Sonja Kovalevski's arrival in Sweden was followed by intensive discussions as to her scientific qualifications and her dependence on Weierstrass. In Uppsala the professors were extremely hostile and tore down the announcement of her lectures. She was soon informed of the attitude in Uppsala and the generally more positive atmosphere at the University of Stockholm. Soon after her arrival in 1883 she wrote the following to a fellow author in Russia [13]:

"I must tell you that these two universities, Stockholm and Uppsala, are situated only one-and-a-half hour's distance apart, but Uppsala, one of the oldest in Europe, represents the most orthodox, tradition-bound and intolerant aspects of science, while all of the young, fresh and progressive forces are joined at the University of Stockholm."

The famous author August Strindberg was of course among those who had great difficulty in accepting a prominent female scientist. The evidence of this includes the following letter written in 1886 to the library official and mathematician Gustaf Eneström:

> "Honored Amanuensis
>
> Since I have been given the task of writing for a French journal regarding Mrs. Kovalevski but have not the ability to judge her mathematical work, I am taking the liberty of requesting your statement as regards her ability. It would be especially useful for me to know in what relationship her dissertation stands to Professor Weierstrass, and if her mathematical merit is in any proportion to her great scientific reputation.
>
> Further, I would like to learn more about the appointment. Was Leffler's salary divided? And were two professors needed?
>
> With the hope that you will honor me with a frank answer, and with a promise of discretion,
>
> August Strindberg"

Strindberg did not receive any answer, nor can we answer the question of Sonja Kovalevski's independence visa-vis her teacher. In any case it is obvious that she was a gifted and important researcher and she most certainly contributed to drawing the attention of the mathematical world to the small University of Stockholm.

5 The years of 1881-1891

In the remaining pages I shall describe the foremost among those young mathematicians who received their training at the University during the period Mittag-Leffler was professor. Practically all of them can be considered to have been his pupils; even though the University had two professors of mathematics after 1884 and the group of established mathematicians in Stockholm was eventually a large one, it was always Mittag-Leffler who took charge of the young mathematicians and directed them during the first years of their studies.

Among the earliest of Mittag-Leffler's pupils at the University were a small group of Finnish students who had begun their research studies already during Mittag-Leffler's four years in Helsinki and had followed their teacher to Stockholm. Foremost among them was *Hjalmar Mellin* (1854-1933) [14], who stayed at Stockholm for a few terms. His doctoral dissertation treated algebraic functions, and he later devoted himself primarily to different special functions within complex function theory such as the gamma function, hypergeometric functions, the zeta-function and in conjunction with this he studied the concept which is now named after him, the Mellin transform. He was eventually appointed professor at the Helsinki University of Technology.

Mittag-Leffler was professor at Stockholm for three decades, but it was especially during his first decade that mathematics attracted a truly distinguished group of gifted young students. The most prominent of these were Edvard Phragmén, Ivar Bendixson, Helge von Koch and Ivar Fredholm. All developed into internationally renowned researchers and all remained at the University as professors, the three first in mathematics, Fredholm in mechanics and mathematical physics.

Edvard Phragmén (1863-1937) grew up in Örebro, and began his studies in Uppsala in 1882, but transferred in the following year to Stockholm. His first work, four small essays on different subjects including function theory and Cantor's set topology, had already been published by 1883-84. Mittag-Leffler early saw Phragmén's great promise. In 1888 he was appointed editorial secretary of Acta Mathematica and as such immediately made a remarkable contribution. What had happened was as follows. Mittag-Leffler had persuaded King Oscar II to offer a mathematics prize to the individual giving the best solution to one of four prize problems. One of the prizes was awarded to Poincaré for an essay on the three-body problem. When the work was to be published in Acta Mathematica and Phragmén set about checking details in the manuscript, he detected certain unclear points which were then corrected — or so it was believed. When the published journal had been sent to a few subscribers, however, all further distribution was stopped and all released copies recalled. The reason was that Phragmén's examination had directly or indirectly led to the discovery of a serious mistake after publication. Revision caused a delay of a year, and Poincaré expressed his gratitude to Phragmén in a postscript.

Of the greatest value for the editorial work of Acta Mathematica was Phragmén's uncommon ability to analyze and critically examine, and his astuteness in this respect was also evidenced e.g. in the group of works he published in 1890-92 which treat a variety of subjects, often starting with an analysis of older theories. He speaks as critic in such lines as the following which introduce a work in which an argument of Schwarz is revised and corrected: "Most mathematicians would probably not have any objection to this proof. However, there is one point which I find dissatisfying, ..."

But Phragmén's activities as researcher were not concerned purely with matters of editing and criticising the work of others. He also opened up new paths, especially in a work from 1904 in which Liouville's theorem on bounded entire functions is extended to entire functions bounded outside a sector and majorized by $\exp(|z|^\alpha)$ within. Phragmén's proof was special and did not admit of generalizations; however the Finnish mathematician Ernst Lindelöf saw the general nature of the theorem, and their teamwork, one of the earliest and most significant in Scandinavian mathematics, led to the fundamental Phragmén-Lindelöf principle dating from 1908. Another pioneering work of Phragmén was presented at the Scandinavian Congress of 1913. Here the problem is more special, lying within the theory of uniform convergence of trigonometric series. But Phragmén was far ahead of his time, as has been certified by the expert Antoni Zygmund, who rediscovered parts of the theory 26 years later, unaware of Phragmén's paper.

In 1892 Phragmén succeeded Sonja Kovalevski as professor at the University, but left the post in 1903 to become head of the national insurance inspection and several years later executive in a private insurance company. The interesting biography [3] gives us a good idea of Phragmén's character: he is clear, logically objective, perhaps somewhat cool and occasionally sarcastic in his statements. The picture is less complete concerning his teaching at the University and his reasons for resigning from the professorship.

A biographic depiction of *Ivar Bendixson* (1861-1935) is found in [23]. He studied in Uppsala for a few years and enrolled at the University of Stockholm in 1882. In the following year his first mathematical work was published, several minor results relating to Cantor's set theory, including the well-known Cantor-Bendixson theorem that a closed point set can be represented as the union of a perfect and a denumerable set.

His most important work, however, lies within differential equation theory and dates from around the turn of the century. Poincaré had earlier presented a theory for ordinary differential equations regarding the behaviour of solutions in the neighborhood of singularities Bendixson developed and extended Poincarés results. The Poincaré-Bendixson theory is still considered to be an extremely useful part of the theory of ordinary differential equations and is included as a fundamental part of the theory of so-called autonomous systems.

Bendixson was extra lecturer at the University and the Institute of Technology during the 1890s. In 1900 he was appointed professor at the Institute of Technology and in 1905 he succeeded Phragmén as professor at the University. He was a very skilful lecturer and of superior stylistic ability. But he abandoned active research relatively early [12] and became otherwise occupied, especially by his work as university rector, a position he held from 1911 until he resigned from the professorship in 1927. He was wealthy, and was one of the most prominent spokesmen for the liberal Swedish political party.

It is possible to read about *Helge von Koch* (1870-1924) in [4] and [23]. He was 17 when he began his studies at Stockholm and defended his dissertation when only 22 years old. The starting point for his paper was a work by Poincaré treating the so-called Hill's equation in which the determinant concept had been generalized to matrices of infinite order. In his doctoral dissertation and in a number of following essays von Koch built up a theory of infinite determinants with applications to systems of infinitely many equations.

von Koch's thesis defense was the object of a little verse in the mathematics students' skit in Uppsala a few years later [3]:

And if a poor Psittacus
with borrowed feathers came
student of the great
Mittacus he left, plucked clean, in shame.

It may be assumed that Psittacus (parrot) is a play on Koch (French *coq*—rooster) and that his feathers were considered to have been borrowed from Poincaré. In Uppsala, then, the defense was seen as a defeat for von Koch and nobody imagined that the dissertation would be of considerable importance in the future, as a step in the development of functional analysis and as source of inspiration for Fredholm's integral equation theory.

von Koch was also very productive in other areas of mathematics such as differential equations and function theory where he worked with the Mittag-Leffler star, and not least, analytic number theory. One of his results on Riemann's zeta function was singled out by Hilbert in his celebrated speech at the Paris Congress of 1900.

von Koch succeeded Bendixson as professor at the Institute of Technology in 1905, moving to the University in 1911 to replace Mittag-Leffler. According to [12] his lectures were more formal and elegantly polished than those of Bendixson.

A thorough and well-written biographic sketch of *Ivar Fredholm* (1866-1927) is to be found in [21]. Like Bendixson and von Koch, Fredholm grew up in Stockholm. He began studies in Uppsala and came to Stockholm in 1888 where he mainly studied mathematics, but also physics. He was perhaps somewhat in the shadow of Phragmén, Bendixson and von Koch, four years his junior, all of whom had had an earlier and more spectacular start. It is indeed doubtful whether he had considered a research career — during the 90s he worked at an insurance agency and underwent a probationary year as instructor at the teacher's college. His doctoral dissertation (1898) treats boundary value problems of the theory of elasticity.

He spent the spring of 1899 in Paris together with Ernst Lindelöf and attended lectures of Poincaré, Picard and Hadamard. It was probably at this time that he comprehended the fundamental relations later formulated in his celebrated alternative theorem for integral equations of what we now term Fredholm's type. The starting point was work of Neumann, Poincaré and Volterra, and important information had also been obtained from von Koch's determinant theory. Fredholm's results were soon widely known. In Göttingen Erik Holmgren presented the work to the great David Hilbert, and Fredholm's theory was of decisive significance for Hilbert's own research. Fredholm's short essay of 1900 has had an extremely strong influence on the development of integral equation theory and functional analysis, and it is hardly an exaggeration to claim that his work is the most wellknown and most important Swedish contribution to the development of mathematics. Remarkable in this connection is that before this work Fredholm was almost totally unknown internationally, and that he worked out his theory completely independently. Fredholm had many ideas and manuscript drafts on the further development of his theory, but only a small portion of these were published; other researchers such as Hilbert and Picard rediscovered and were given credit for many of his results.

With his strong concern for physics Fredholm was never interested in a professorship in pure mathematics. After the turn of the century he worked in insurance, where he made valuable contributions within insurance mathematics. In 1906 he was appointed professor of rational mechanics and mathematical physics at the University.

Through his integral equation theory Fredholm became a celebrity in the scientific world. In 1908 he was the recipient of the distinguished Poncelet Prize and he received many other honors. However he seems to have been indifferent to public acclaim. He was modest and retiring, restrained in the matter of publishing his results, and not exactly a brilliant lecturer. Generally speaking he seems to have been quite the opposite of Mittag-Leffler, and indeed their important contributions within mathematics were of completely different natures.

A few other mathematicians who began their studies at the University during the 1880s deserve mention. One of them is *Gustaf Kobb* (1863-1934) (see [23]). His doctoral dissertation dealt with algebraic functions and he had also done valuable work within the calculus of variations, where he generalized results of Weierstrass from one to two dimensions with the aid of results of Picard concerning partial differential equations. But then as now there were few research posts for mathematicians and he found the competition overwhelming. He therefore oriented himself towards mechanics and became professor of mathematics and mechanics at the Institute of Technology in 1912. He was an active liberal politician and served as an M.P.

Tension between Uppsala and Stockholm was partly due to political differences. The University of Stockholm was a center of liberal ideas and many of the young mathematicians took a clearly radical stance. Sonja Kovalevski had correspondence with Russian nihilists (after her death Mittag-Leffler saved these letters from falling into the hands of the Russian embassy). Bendixson and Kobb were radical, at least in the eyes of the conservative Uppsala. This was important at the turn of the century for example, when it was declared that the person to succeed to the extraordinary professorship after Dillner would have to be qualified in algebra or geometry. This was done partly to favor an Uppsala docent (who died, however, before the appointment), partly to shut out the Stockholm scholars Bendixson and Kobb.

Another capable researcher was *Frans de Brun* (1866-1930), whose doctoral dissertation was a development of Weierstrass' theory on algebraic functions. He had also other work in function theory. de Brun too oriented himself towards mechanics; he gained the docent title at Uppsala, and later became a teacher at a gymnasium in Stockholm.

Gustaf Cassel (1866-1945) defended his dissertation (1894) on the use of automorphic functions in the study of linear differential linear equations. After some years as a mathematics teacher at a gymnasium he went over to economics, in which field he eventually became a leading international figure, and professor at the University from 1904.

It is interesting that after 1909 the University had another mathematically highly qualified professor in a field even more remote from mathematics; this was *Carl Delin* (1865-1934) professor of law history and Roman law (!). He had in fact been docent in mathematics at Lund (1893-1900) before taking up a career in law.

Here follow a few summary comments on mathematics at the University during the 80s. The students were offered rather advanced lectures by the professors, dealing primarily with complex function theory and applications within e.g. analytic number theory and differential equations, and the mathematical seminar met every other week. There seem to have been about ten participants per term. There was also an assistant's position, the holder of which was given the task of preparing new students so that they might soon follow the professors' instruction. There was no actual basic training as such, rather it was assumed that newly enrolled students had the knowledge necessary for higher studies. To judge from [22] the level of scientific activity was praiseworthy, even during the first decade, which is shown in part through the considerable number of scientific publications. In the list of scientific works may be noted a short essay by Nanny Lagerborg on the movement of a body around a fixed point, evoking the image of a young girl following closely in Sonja Kovalevski's footsteps.

6 The years 1891-1901

Before turning to the leading Swedish figures who joined the mathematics group at the University in the 90s we will pause for a moment to consider the earlier mentioned Finnish mathematician *Ernst Lindelöf* (1870-1946). He was a frequent guest at the University during the 90s and consequently should be mentioned in this context. He was the son of Lorenz Leonard Lindelöf, who had been Mittag-Leffler's predecessor at Helsinki. Ernst Lindelöf became eventually a leading function theorist [4] and, as professor at Helsinki, was an outstanding teacher, founder of the still flourishing Finnish school of function theory (his students included the Nevanlinna brothers and P.J. Myrberg). It may be assumed that Ernst Lindelöf's interest in function theory was stimulated in part by Mittag-Leffler's first pupil, Mellin (who was the most prominent mathematician in Helsinki during Lindelöf's student years there), in part by the mathematicians at Stockholm. Thus one can, with some justification claim that the special interest of the Finnish mathematical school in function theory ultimately can be traced to Mittag-Leffler.

Among the new mathematicians of the 90s, three names are especially interesting: Severin Wigert, Erik Stridsberg and Hakon Grönwall.

Severin Wigert (1871-1941) was primarily interested in analytic number theory and function theory. He is best known for a theorem on entire functions which is found in textbooks on the subject. Wigert was an assistant at the University at the turn of the century and stayed at the University as a researcher. However he never held any permanent position, supporting himself by means of his own private fortune. He belonged to that group of researchers who, in line with the aims of the University, never bothered to obtain the doctorate; Phragmén and Bendixson acted similarly. The latter two were instead awarded honorary doctorates by Uppsala in 1907 after Wiman and Holmgren had been appointed to professorships there and the tension between the two schools had abated. Wigert was also made honorary doctor by Uppsala in 1927.

Erik Stridsberg (1871-1950) (see [8]) defended a dissertation on transcendental numbers (1909). He became an insurance mathematician but retained his interest in pure mathematics, publishing work both in algebra and analysis. Stridsberg is a good illustration of the close connection between mathematics and the insurance companies in those days. Malmsten and Mittag-Leffler were pioneers in establishing this cooperation, followed by Phragmén, Fredholm, Stridsberg, and many others. The benefits were mutual: as for the mathematicians, the insurance companies meant employment, economic support for congresses, etc.

Reference [11] describes *Hakon Grönwall* (1877-1932) and his meandering career within the sciences. He began studies at Uppsala at the age of 16, soon transferring to Stockholm where by 1898 (that is, by the age of 21) he had published 10 mathematical papers and had received a doctorate from Uppsala with highest marks. The reason for his leaving Stockholm and Sweden, probably already in the following year, was, according to his biographer Einar Hille, this: the authorities, sadly lacking in sense of humor, treated a student's prank in a way which hurt his youthful pride and his strong love of liberty.

Grönwall subsequently began a new career; he attended the Technical University of Berlin and from 1902 was a practising engineer.

From 1904 he was in the U.S., now under the name T.H. Gronwall, returning suddenly in 1911 to his old science. A number of papers from 1912 establish him as a first-class mathematician. During the remaining twenty years of his life his interests oscillated between mathematics and more applied research. His working intensity was great, as was his mathematical versatility: his work touches on, among other things, analytic number theory, Fourier series, function theory, differential equations; and especially in physical chemistry he is known as a very important contributor.

Here, following [22], are some general glimpses of Stockholm mathematics during the 90s. The direction taken by research was unchanged from the previous decade, and advanced lectures and seminars were still held as before, but it appears that the seminar met to an increasing degree in the professor's residences rather than at the University. During the 90s an annual series of elementary lectures on analysis was instituted. A teaching position was established for the purpose, and was held by Bendixson in 1892-99. Towards the end of this period, at any rate, about twenty students attended his lectures.

An important event was a three month lecture series by the visitor Paul Painlevé in 1895 on analytic ordinary differential equations. The Frenchman Painlevé was a leading researcher in this area — he later became a prominent politician. Oscar II had offered the invitation to Painlevé, and he was himself present at one of the lectures, where he was overheard helping his chief of staff by giving him a (correct) definition of the modulus of a complex number.

7 The years 1901-1911

Painlevé's lectures were mimeographed and were the starting point of a dissertation produced ten years later by a young Stockholm mathematician. This was *Johannes Malmquist* (1882-1952) Mittag-Leffler's foremost pupil during his last decade as professor [20]. Malmquist's subject area lay near Bendixson's, but in this case too the teacher was Mittag-Leffler. Malmquist's doctoral dissertation was in the area of analytic ordinary differential equations, and he dealt with the same subject in his later work. Most celebrated of his many- results is a theorem of 1913 that a differential equation $w' = R(z, w)$, where R is rational, must be a Riccatti equation if it has a uniform and nonrational solution in its domain of existence. Malmquist was awarded his doctorate in 1909 at the University's first conferment ceremony; in 1913 he was appointed professor at the Institute of Technology. For many years he was editorial secretary of Acta Mathematica.

Nils Zeilon (1886-1959), a student principally of Fredholm, was another of the noted mathematicians to begin studies during this decade. Zeilon worked in partial differential equations, in an area bordering on mathematical physics, where his interests included problems of hydrodynamics. In 1926 he became professor of mathematics at Lund.

Certain changes in mathematics instruction took place during the first decade of the 20th century. The teaching position in elementary analysis was withdrawn when Bendixson became professor at the Institute of Technology, and the lectures were thereafter delivered by the professors, first by Phragmén and later by his successor Bendixson. Phragmén lectured to a small group, while Bendixson attracted an audience of around 30. The number of those taking advanced courses grew during this period from 15 to 20. A prominent foreigner visited the University also during this decade: in the spring of 1906 Vito Volterra held a series of lectures attended by an audience of 30.

8 Eneström

Before concluding this presentation with a few words about developments in the years immediately after Mittag-Leffler left his professorship, we should first touch on the activities of *Gustaf Eneström* (1852-1923) who deserves to be mentioned in this context even though he had no direct ties to the University. He became a *filosofie kandidat* (B.S.) at Uppsala and then moved to Stockholm, where he held various library positions while also carrying out tasks in statistics and insurance mathematics. His purely mathematical education was relatively limited, but he is remembered for a theorem in pure mathematics, the so-called Eneström-Kakeya theorem on the roots of a special type of polynomial, a theorem he arrived at in 1893 while working on an assignment in insurance mathematics. His main research interest, however, was the bibliography and history of mathematics, where thanks to his scrupulous and critical researches and enormous productivity he must be considered among the foremost.

When Acta Mathematica started in 1882 Eneström was given certain secretarial duties, primarily to edit Bibliotheca Mathematica, a bibliography of newly arrived literature which was to be issued as an appendix to Acta. Eneström also included historical notes in this appendix. In 1888 there occurred a break between Eneström and Mittag-Leffler, perhaps quite simply because Mittag-Leffler wanted Phragmén as secretary for Acta Mathematica. From this point Eneström published Bibliotheca Mathematica on his own, with great scientific contributions and also great personal economic sacrifice. The journal was produced through 1914, however during the last years only with the support of a German publishing house.

Despite the fact that Mittag-Leffler was himself a bibliophile and interested in mathematical history he does not seem to have shown much respect for Eneström's scientific activities. As is asserted in Eneström's biography [9], enthusiasm for Weierstrass's function theory on the part of Mittag-Leffler and fascination with the possibilities of application which were revealed did not leave much enthusiasm over for the historical aspects of the subject. Eneström became isolated in Sweden. He did not fit into the overall pattern, but for our purposes here at any rate, his activities were part of Stockholm mathematics during the time of Mittag-Leffler.

9 Concluding words

In 1911 Mittag-Leffler left his professorship at the University after having dominated mathematical activities in Stockholm for 30 years, and Helge von Koch returned to the University as Mittag-Leffler's successor. The year 1911 is an important one for several reasons: Bendixson began his long term as rector and consequently abandoned mathematical research almost entirely, Malmquist was temporarily appointed as professor at the Institute of Technology and far off in the U.S. Grönwall had returned to the science of his youth. But perhaps the most significant event of 1911 was that a young Hungarian mathematician, *Marcel Riesz* (1886-1969), at Mittag-Leffler's initiative, joined the ranks of the Stockholm mathematicians [7]. The situation at the University at this time has been described in [12] by *Einar Hille* (b. 1894), who was himself enrolled at the University in 1911. It is interesting to read that it was in fact the 68 year old Mittag-Leffler who gave Hille his first research problem, that von Koch and (especially) Bendixson were more passive, and Riesz was in reality the one responsible for initiating Hille into the path which he later was to pursue so successfully. Also *Harald Cramér* (b. 1893), famed researcher in probability and mathematical statistics, who from the beginning worked in pure mathematics, was a student of Riesz. Indeed, it was Riesz who now took over Mittag-Leffler's role, stimulating and leading the mathematically gifted young students of the University.

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