Microlocal Methods in Tensor Tomography

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Based on a joint work with GUNTHER UHLMANN

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Tensor Tomography: Main Problem

Let (M, g) be a compact Riemannian manifold with boundary.

Main Problem

Recover a tensor field f_{ij} from the geodesic X-ray transform

$$I_g f(\gamma) = \int f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\, dt$$

known for all (or some) max geodesics γ in M.

One can ask this question for tensor fields of any order m, including 0 (functions).

If $m > 0$, you cannot. For any *potential field dv*, where $v|_{\partial M} = 0$, one has

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If $m > 0$, you cannot. For any *potential field dv*, where $v|_{\partial M} = 0$, one has

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What is dv? If $m = 1$ (f is an 1-form), then v is a function and d is the usual differential. For general m, d is the symmetrized covariant derivative. For example, if $m = 2$, one has

$$(dv)_{ij} = \frac{1}{2}(\nabla_j v_i + \nabla_i v_j).$$

Then $I(dv) = 0$ just follows from the Fundamental Theorem of Calculus.

The natural conjecture is that this is the only obstruction to uniqueness (for some class of manifolds). We call this property *s-injectivity*.

If $m = 0$ (f is a function), this is just injectivity. By the way, each function α has the same ray transform as αg_{ij}, where g is the metric, so any result for 2-tensors implies a result for functions as well.

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Some conditions are clearly needed. On the sphere, for example, each odd function integrates to 0. One can build examples of M with boundary based on that.

One class of manifolds, where no obvious counter-examples exist, are *simple manifolds*.

Definition 1

(M, g) is called simple, if

- $\forall (x, y) \in M \times M$, $\exists$ unique minimizing geodesic connecting x, y, smoothly depending on x, y.
- ∂M is strictly convex

Conjecture 1

If (M, g) simple, then I is s-injective.

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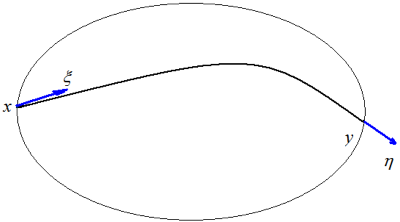
The Lens Rigidity Problem (The Inverse Kinematic Problem)

Define the scattering relation σ and the length (travel time) function ℓ:



$$\sigma : (x, \xi) \to (y, \eta), \quad \ell(x, \xi) \to [0, \infty].$$

Diffeomorphisms preserving ∂M pointwise do not change σ, ℓ!

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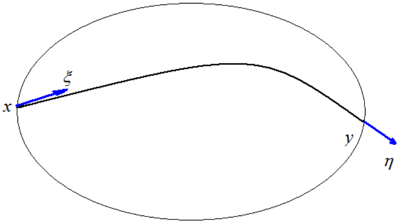

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Boundary Rigidity

Let $\rho(x,y)$ be the distance function on $M(,g)$.

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Does $\rho|_{\partial M\times\partial M}$, determine uniquely g, up to an isometry?

If (M,g) is simple, lens rigidity is equivalent to boundary rigidity. For general manifolds, the lens rigidity is the right question to study.

It turns out that linearizing any of those two problems, we arrive at the problem of inverting If modulo potential tensors. Potential tensors linearize the non-uniqueness due to diffeomorphisms.

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Back to the linear problem

If $m = 0, 1$ (f is a function/1-form), then I is (s)-injective on simple manifolds (Mukhometov; Mukhometov & Romanov, Bernstein & Gerver).

If $m \geq 2$, this is still an open problem if $n \geq 3$. In 2D, solved by SHARAFUTDINOV. The non-linear problem was solved (in 2D) before that by PESTOV & UHLMANN.

Energy Estimates

SHARAFUTDINOV, PESTOV: Under an explicit upper bound on the curvature (implying simplicity), I is s-injective with a (non-sharp) stability estimate. DAIRBEKOV: a bit larger class of simple metrics. The energy method goes back to the original idea of MUKHOMETOV but it is a very non-trivial implementation of it on tensors.

Microlocal Approach

S&UHLMANN: Study I^*I as a ΨDO, and get the most of it. The operator I is an FIO by itself, and this is also used in the analysis. If g is real analytic, use analytic microlocal analysis.

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Solenoidal-potential decomposition (after Sharafutdinov)

Every tensor admits an orthogonal decomposition into a *solenoidal* part f^s and a *potential* part dv,

$$f = f^s + dv, \quad v|_{\partial M} = 0.$$

where $\delta f^s = 0$.

Here the symmetric differential dv is given by $[dv]_{ij} = (\nabla_i v_j + \nabla_j v_i)/2$, and the divergence δ is given by: $[\delta f]_i = g^{jk} \nabla_k f_{ij}$. We have $I(dv) = 0$.

To do this, we solve the elliptic boundary value problem

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Results for simple metrics

Theorem 2 (S.&Uhlmann, Results for simple metrics)

- *s-injectivity for real analytic simple metrics*
- *true also for metrics close enough to real analytic simple metrics*
- *(Recent) Moreover, if I_g is s-injective for some simple g, there is a stability estimate of elliptic type:*

$$\|f^s\|_{L^2(M)} \le C\|N_g f\|_{H^1(M_1)}, \qquad (1)$$

where $M \subset\subset M_1$, $N_g = I_g^* I_g$; and C can be chosen uniform under small perturbations of g.

- *As a result, we get s-injectivity and stability for generic simple metrcs*

Note that (1) is sharp, because N is a ΨDO of order -1.

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Idea of the Proof

The main point is that the linear problem behaves like an elliptic one. First, $N = I^*I$ is a ΨDO of order -1. That however works in an open set, and we chose that set to be the interior of M_1. Then N is elliptic on solenoidal tensors, but those are solenoidal tensors in M_1, not in M!

One can construct explicitly a parametrix to N in M_1, more precisely, we "recover" $f^s_{M_1}$, the solenoidal projection of f extended as zero.

We can always assume $f = f^s$. Next, we compare f^s and $f^s_{M_1}$. They differ by some dw, that is known in $M_1 \setminus M$ (up to compact terms) from the parametrix. Then we recover $w|_{\partial M}$ that helps us find f^s.

After that, one gets a Fredholm equation of the kind

$$(Id + K_g)f = h,$$

where K_g is compact (of order -1) and depends continuously on g.

If -1 is not an eigenvalue of K_g (happens when I_g is s-injective), then there is an estimate.

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One can construct explicitly a parametrix to N in M_1, more precisely, we "recover" $f^s_{M_1}$, the solenoidal projection of f extended as zero.

We can always assume $f = f^s$. Next, we compare f^s and $f^s_{M_1}$. They differ by some dw, that is known in $M_1 \setminus M$ (up to compact terms) from the parametrix. Then we recover $w|_{\partial M}$ that helps us find f^s.

After that, one gets a Fredholm equation of the kind

$$(Id + K_g)f = h,$$

where K_g is compact (of order -1) and depends continuously on g.

If -1 is not an eigenvalue of K_g (happens when I_g is s-injective), then there is an estimate.

S-injectivity for analytic simple metrics

Using analytic ΨDOs. The uniqueness proof for analytic g is based on analytic microlocal analysis. We show that $N = I^*I$ is an analytic ΨDO. If $If = 0$, then $Nf = 0$ near M. Remember, N is not elliptic, but restricted to solenoidal tensors, it is. Then f^s has to be analytic up to the boundary. Next, we show that all derivatives at ∂M vanish by using different arguments. Therefore, $f^s = 0$.

This is actually an oversimplification of what we are doing. We have to work in M_1 first, that gives us a different solenoidal projection $f^s_{M_1}$.

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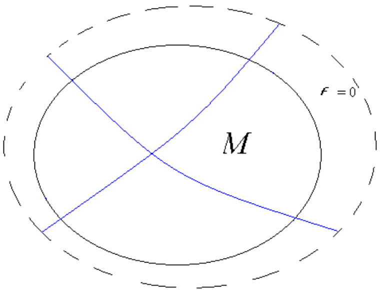
As an easy example, here is how one can prove this theorem for integrals of functions. Note that this is a partial case: if $f(x)$ is a function, not a tensor, then $f(x)g_{ij}$ is a tensor, and

$$\int f(\gamma)g_{ij}\dot\gamma^i\dot\gamma^j\,dt = \int f(\gamma)\,dt.$$



Extend f as zero outside M. Then Nf is still zero because integrals outside M are zero. Now, $Nf = 0$ implies that the extended f is analytic. Since $f = 0$ outside M, then $f = 0$.

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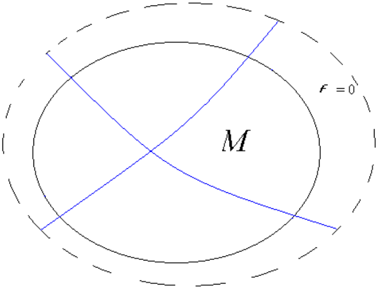

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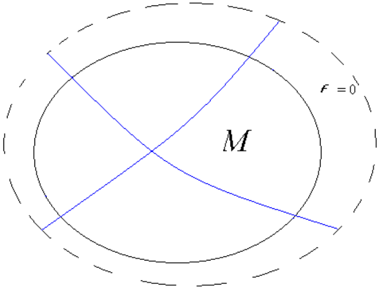

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Results for non-simple manifolds:

We study more general manifolds than the simple ones.

- M does not need to be diffeomorphic to a ball (but some topological restrictions are still needed)
- ∂M does not need to be convex
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Main Condition:

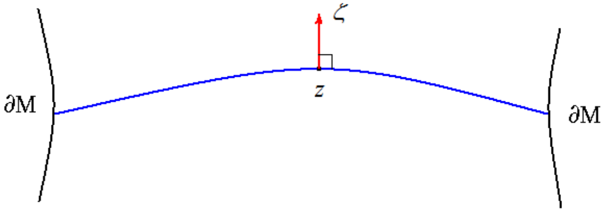
We study $I = I_{\mathcal{D}}$ restricted to $\gamma_{x,\xi}$, where $(x,\xi) \in \mathcal{D} \subset \partial(SM)$. Here $\mathcal{D}$ is chosen so that the conormal bundle of the geodesics issued from $\mathcal{D}$ covers T^*M, and those geodesics have no conjugate points. Such $\mathcal{D}$ are called *complete*.

Definition 3

We say that $\mathcal{D}$ is **complete** for the metric g, if for any $(z,\zeta) \in T^*M$ there exists a maximal in M, finite length geodesic $\gamma : [0,l] \to M$ through z, normal to ζ, such that

- γ belongs to our data (issued from $\mathcal{D}$);
- there are no conjugate points on γ.

We call g **regular**, if a complete set $\mathcal{D}$ exists, i.e., if the maximal $\mathcal{D}$ is complete.



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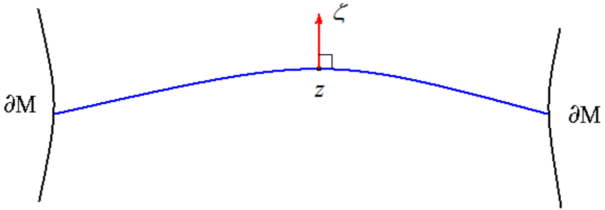
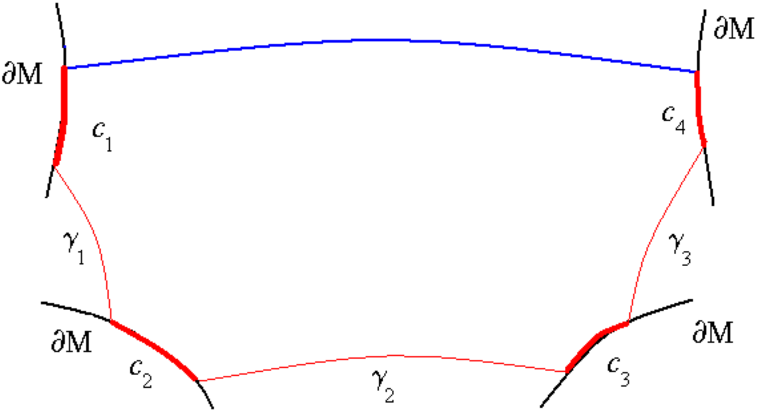
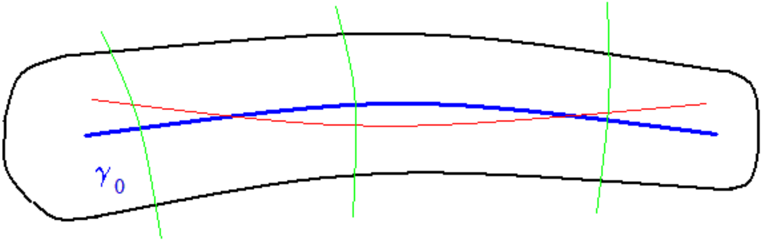


Topological Condition (T): Any path in M connecting two boundary points is homotopic to a polygon $c_1 \cup \gamma_1 \cup c_2 \cup \gamma_2 \cup \cdots \cup \gamma_k \cup c_{k+1}$ with the properties that for any j,
(i) c_j is a path on ∂M;
(ii) $\gamma_j : [0, l_j] \to M$ is a geodesic lying in M^{int} with the exception of its endpoints and is transversal to ∂M at both ends; moreover,
$(\gamma_j(0), \dot\gamma_j(0)) \in \mathcal{D}$.



Example 1: A cylinder around an arbitrary geodesic

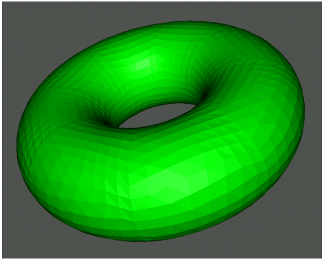
γ_0: a finite length geodesic segment on a Riemannian manifold, conjugate points are allowed.
M: a "cylinder" around γ_0, close enough to it.



One can study the scattering relation only for geodesics almost perpendicular to γ_0, there are no conjugate points on them.

Example 2: The interior of a perturbed torus

$M = S^1 \times \{x_1^2 + x_2^2 \leq 1\}$, with g close to the flat one:



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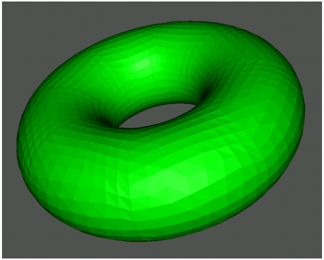

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More generally, one can consider a tubular neighborhood of any periodic geodesic on any Riemannian manifold.

Even more generally, we can study $M \times N$, where M is simple, and N is arbitrary; and study σ for all geodesics over fixed points of N, and all those close to them. A small enough perturbation of this manifold satisfies our assumptions, and can have a terrible topology and all kinds of trapping rays and conjugate points.

The examples above are of that type.

Theorem 4 (S-injectivity for analytic g)

Let g be real analytic. Let $\mathcal{D}$ be open and complete. Then $I_{\mathcal{D}}$ is s-injective.

Theorem 5 (s-injectivity $\Rightarrow$ stability)

Let $\mathcal{D}$ be open and complete. Then s-injectivity of $I_{g,\mathcal{D}}$ implies a locally uniform stability estimate.

In other words, injectivity implies stability!

Theorem 6 (generic s-injectivity)

Let $\mathcal{D}$ be open and complete for g in an open set $\mathcal{G}$ of regular metrics. Then there exists an open dense subset $\mathcal{G}_s$ of $\mathcal{G}$ (in the C^k topology, $k \gg 2$), so that $I_{g,\mathcal{D}}$ is s-injective for $g \in \mathcal{G}_s$.

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The analytic microlocal arguments in this case

The analytic arguments in this case are much more delicate. First, I^*I may not be a ΨDO at all because we may have conjugate points or trapped geodesics. One can consider

$$N_\chi = I^*\chi I,$$

where χ cuts near some subset of geodesics without conjugate points; for example, near a single one γ_0. The C^∞ ΨDO calculus immediately implies

$$\chi If = 0 \quad \Longrightarrow \quad \mathrm{WF}(f) \cap N^*(\gamma_0) = \emptyset$$

(assuming that f is a function at the moment) because N_χ is elliptic at conormal directions to γ_0. Equivalently, χI is an elliptic FIO there.

If g is analytic, and we want to use analytic ΨDOs, we have a major problem: χ destroys the analyticity. In the analytic ΨDO theory, only certain cut-offs, denoted by $g^R(\xi)$ in Treves' book, are allowed. Here, I is an FIO, and the cut-off is of the type $\chi(x, y, \theta)$, $\theta = \theta(x, y, \xi)$. There is no such theory developed yet.

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Instead of trying to use the analytic ΨDO calculus, we use the complex stationary phase method, following Sjöstrand. We get

Lemma 7

γ_0: *non-trapping, without conjugate points. If* $If = 0$ *near* γ_0, *then*

$$WF_{\mathrm{A}}(f^s) \cap N^*(\gamma_0) = \emptyset.$$

This is a non-trivial results even for functions.

Here is an informal version of those 3 theorems: Under the microlocal and the topological condition, we have generic s-injectivity for all metrics satisfying those conditions (ncluding the real analytic ones), and a locally uniform stability estimate.

Moreover, the problem is Fredholm (therefore: finite dimensional and smooth kernel).

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Consider the weighted X-ray transform of *functions* over a general family of curves Γ:

$$If(\gamma) = \int w(\gamma(t), \dot\gamma(t)) f(\gamma(t))\, dt, \quad \gamma \in \Gamma.$$

On can assume that Γ are the solutions of a Newton-type equation

$$\ddot{x} = G(x, \dot{x})$$

with a generator G. (For example, $G = 0$ gives us lines).

Theorem 8 (Frigyik, S & Uhlmann)

I is injective for generic regular (G, w), including real analytic ones. There is a stability estimate.

Here, G is called regular, if the corresponding curves have no "conjugate points" on $\operatorname{supp} w$, and their conormal bundle (on $\operatorname{supp} w$) covers T^*M. This is the same microlocal condition that we had before, and in particular, we can have a subset of "geodesics".

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Magnetic Systems

On (M, g), consider an one form α, and the Hamiltonian

$$H = \frac{1}{2}(\xi + \alpha)_g^2.$$

The corresponding characteristics on the energy level $H = 1/2$ are called unit speed magnetic geodesics. They describe the trajectories of a charged particle in a magnetic field.

The lens rigidity is formulated in a similar way. The boundary rigidity is formulated in terms of the *action* $A(x, y)$, on $\partial M \times \partial M$, not the boundary distance function $\rho(x, y)$. The action $A(x, y)$ is defined by

$$A(x, y) = T(x, y) - \int_{\gamma_{[x,y]}} \alpha,$$

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$$I\phi(\gamma) = \int_\gamma \phi(\gamma, \dot\gamma)\, dt$$

for functions $\phi(x, \xi)$ that are quadratic in ξ:

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A support theorem for tensors

Theorem 9 (S.& Krishnan)

(M, g) simple analytic, K closed geodesically convex subset. If for a symmetric 2-tensor field f we have that $If(\gamma) = 0$ for each geodesic γ not intersecting K, then there exists an 1-form v such that $f = dv$ in $M \setminus K$, and $v = 0$ on ∂M.

True for functions (Venky Krishnan) and 1-forms as well, and the proof if simpler.

Working with f^s is not what we should do now. It is not true in general that $f^s = 0$ in $M \setminus K$. After adding some dv, one can always assume that

$$f_{ni} = 0, \quad \forall i$$

near a fixed geodesic γ_0, in special coordinates, where $\gamma_0 = (0, \dots, 0, t)$. Now, I (or $N = I^*I$) is not elliptic on such tensors, but it is elliptic for ξ with $\xi_n \neq 0$ (not conormal to γ_0).

There are two main difficulties here: The representation above is local only, and N is not elliptic for $\xi_n = 0$. We show however that it is hypoelliptic.

A support theorem for tensors

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One of the ingredients of the proof is a refined version of the analytic microlocal argument discussed above:

Lemma 10

γ_0: non-trapping, without conjugate points. Given $(x_0,\xi^0)\in N^*\gamma_0$, assume that $(x_0,\xi^0)\not\in WF_{\rm A}(\delta f)$, and that $If=0$ near γ_0. Then

$$(x_0,\xi^0)\not\in WF_{\rm A}(f).$$

Another ingredient is the Sato-Kawai-Kawashita Theorem: Let f be supported on one side of a hypersurface S, and $x_0\in S$, $\xi^0\perp S$ at x_0. Assume that f is analytic at (x_0,ξ^0). Then $f=0$ near x_0.

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Modeling Optical Molecular Imaging

The radiative transport equation in Ω is given by

$$\theta \cdot \nabla_x u(x,\theta) + \sigma(x,\theta)u(x,\theta) - \int_{S^{n-1}} k(x,\theta,\theta')u(x,\theta')\,d\theta' = f(x), \quad u|_{\partial_- S\Omega} = 0,$$

where σ is the *absorption* and k is the *collision kernel*. The source term f is assumed to depend on x only. Here, $\partial_- S\Omega$ consists of $x \in \partial\Omega$ and θ pointing inwards.

The boundary measurements are modeled by

$$Xf(x,\theta) = u|_{\partial_+ S\Omega}, \quad (x,\theta) \in \partial_+ S\Omega,$$

where $u(x,\theta)$ is a solution of the transport equation, and $\partial_+ S\Omega$ denotes the points $x \in \partial\Omega$ with direction θ pointing outwards.

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Direct Problem

Given f (and σ, k), find Xf.

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Clearly, it is a linear problem.

Let $\sigma = k = 0$ first.

Then X is just the X-ray transform:

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where

$$E(x,\theta) = \exp\left(-\int_0^\infty \sigma(x+s\theta,\theta)\,ds\right).$$

If $\sigma = \sigma(x)$, then we get the attenuated X-ray transform, that we know how to invert.

Without assuming that any one is zero, BAL and TAMASAN proved injectivity when $k = k(x, \theta\cdot\theta')$, and k is small enough in a suitable norm. The main idea there is to treat k as a perturbation; then X is a perturbation of the attenuated X-ray transform.

Also, results by SHARAFUTDINOV on Riemannian manifolds, smallness conditions on the curvature k and σ.

Our goal is to consider this problem for general (σ, k).

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We need assumptions, even for solvability of the direct problem! Assuming $|k| \ll 1$ is enough. Also, $\int k(\cdot,\theta,\cdot)d\theta < \sigma$ suffices. Those conditions prevent a "nuclear explosion", i.e., the corresponding time-dependent dynamics is bounded. They are not necessary conditions though.

Theorem 11

(a) The direct problem is uniquely solvable for a dense open set of pairs (σ, k) in C^2, even for $f = f(x,\theta)$.
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Fix $\Omega_1 \supset\supset \Omega$. Define X_1 as X but in Ω_1.

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For (σ, k) in an open and dense set of pairs in

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$$\|f\|_{L^2(\Omega)} \le C\|X_1^* X_1 f\|_{H^1(\Omega_1)}, \quad \forall f \in L^2(\Omega),$$

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