Rigidity of broken geodesics and inverse problems for radiative transfer equation

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in collaboration with

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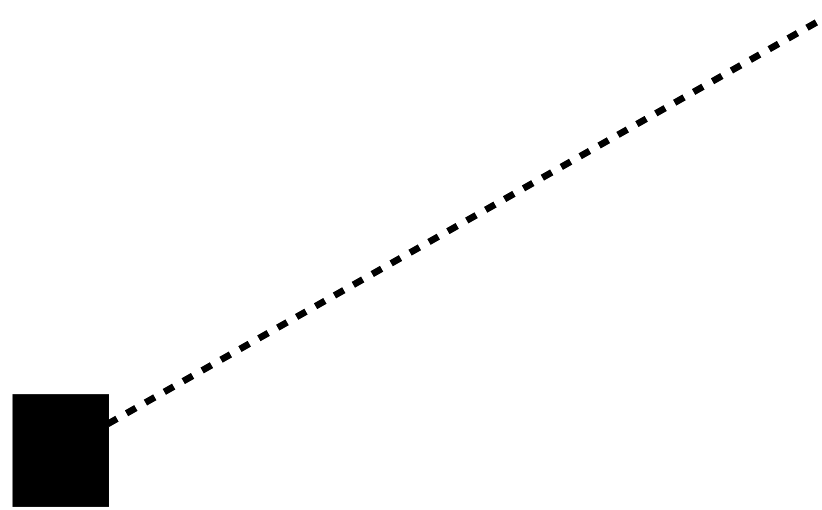
At Greenwich Royal Observatory, a green laser points exactly north along the prime meridian line.
Where the laser ray is exactly located?



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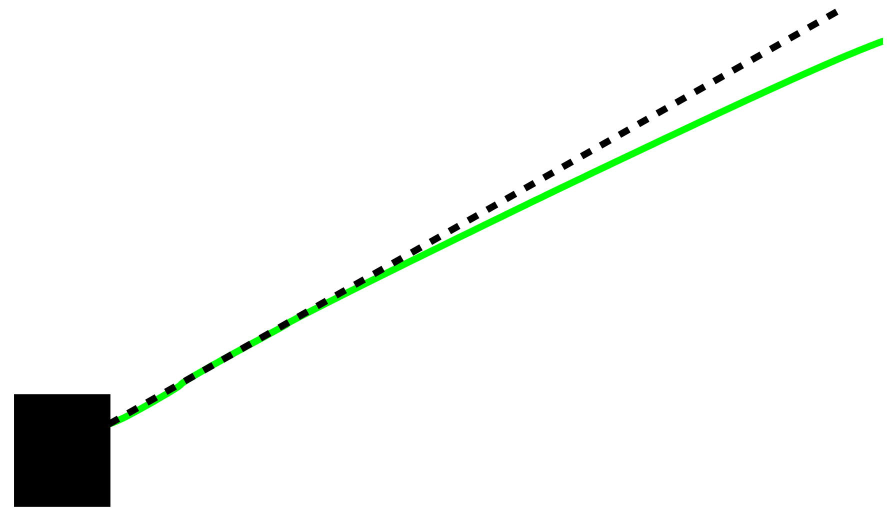

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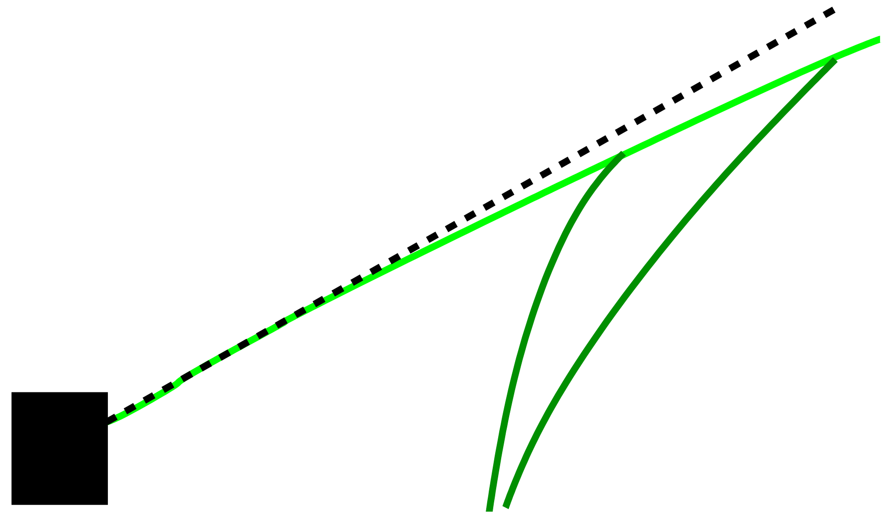

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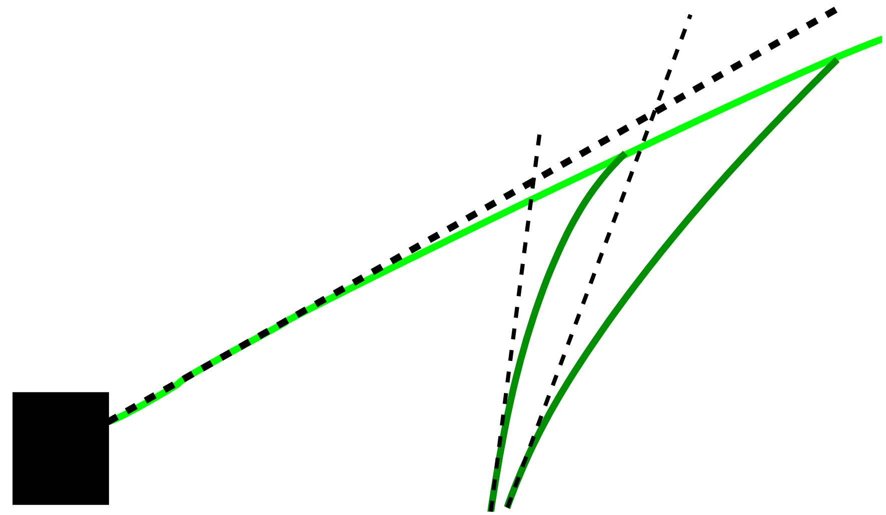
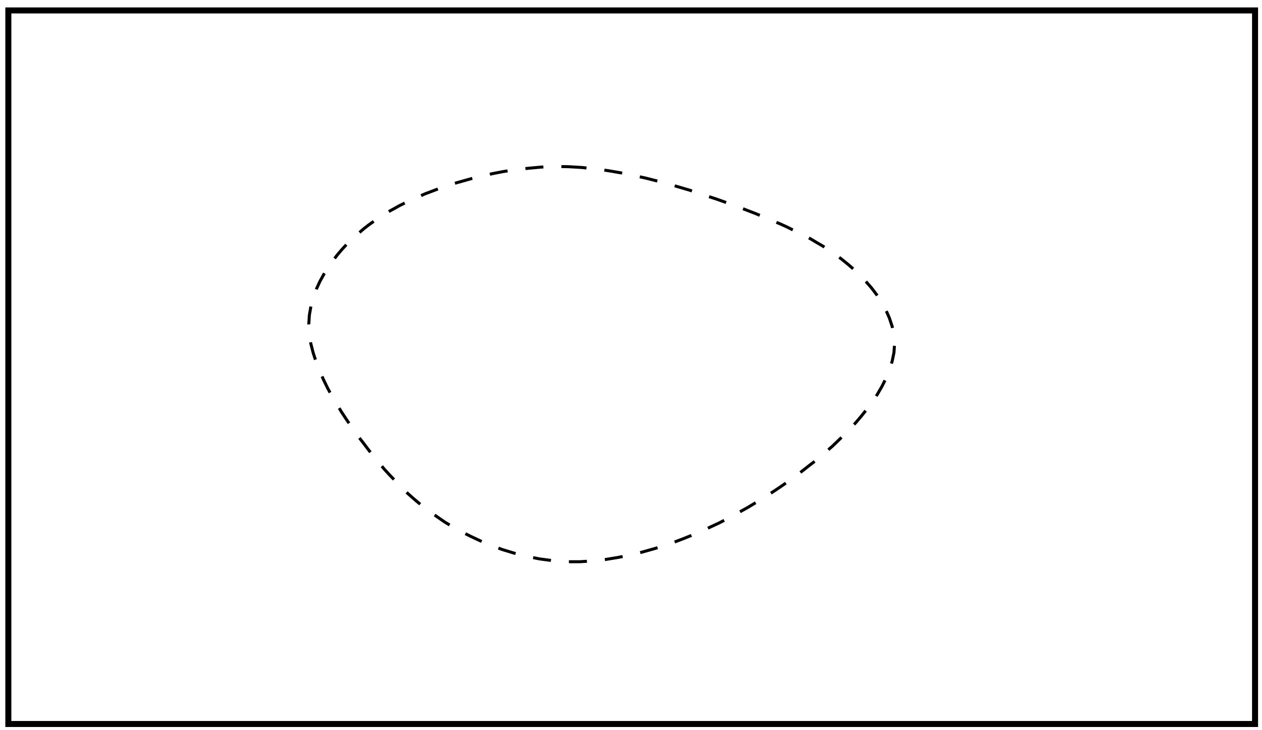


We consider radiative transfer equation with a varying light speed.
Let (N, g) be a complete Riemannian manifold and $M \subset N$ compact. Assume that g is known outside M
We send photons outside M that scatter inside M and measure the photons outside M.



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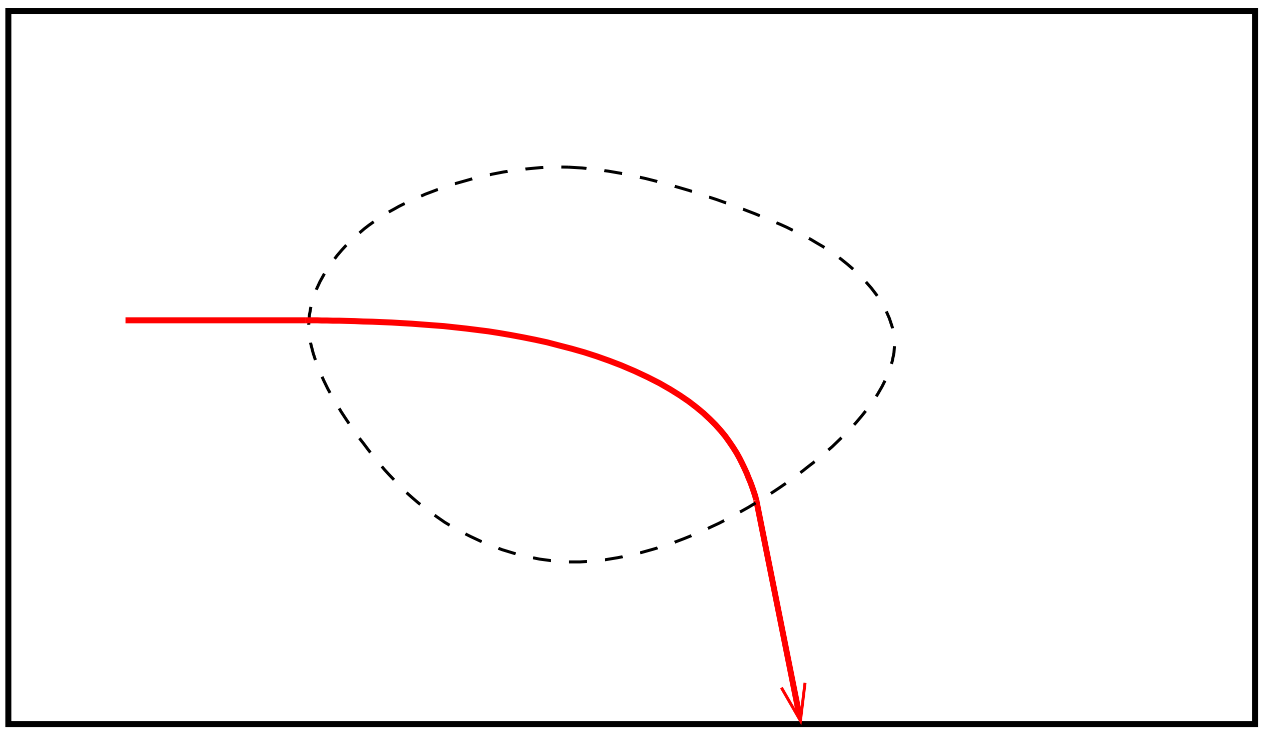

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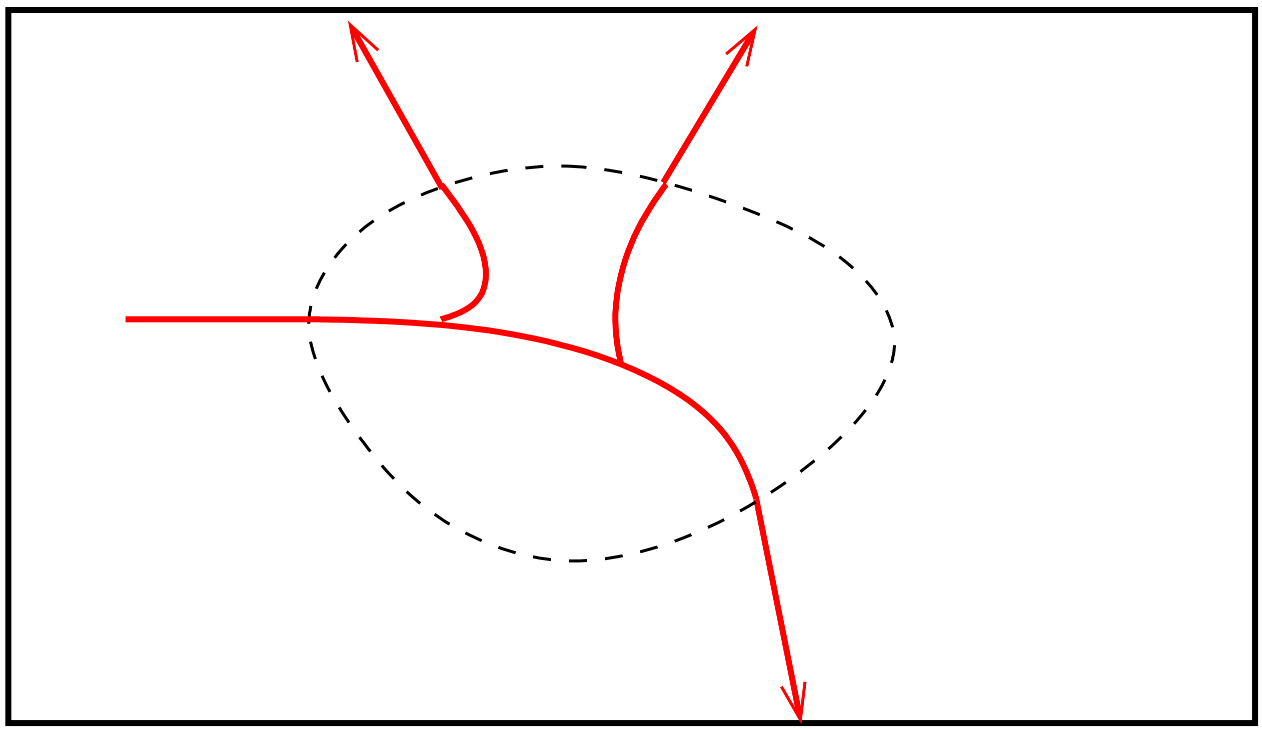

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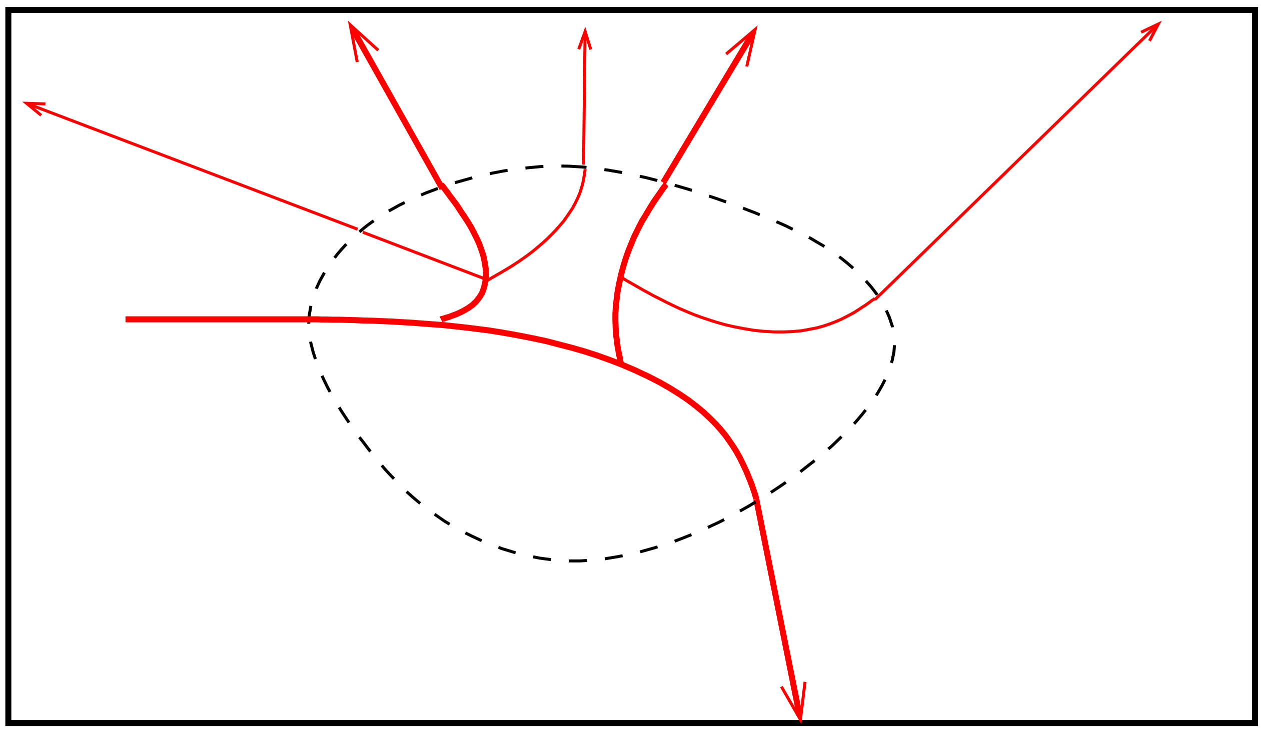
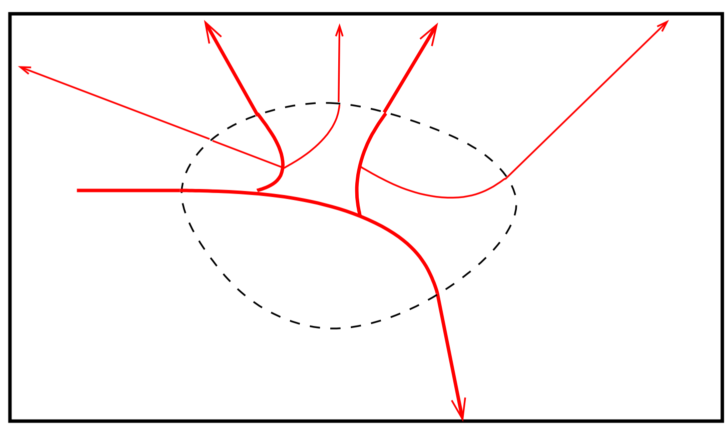


Radiative transfer equation. Consider

$$(Hu)(t,x,\xi) + \sigma(x,\xi)u(t,x,\xi) - (Ku)(t,x,\xi) = 0$$
$$u(t,x,\xi)|_{t=0} = w(x,\xi).$$

Here $t \in \mathbb{R}_+$ and $(x,\xi) \in SN = \{(x,\xi) \in TN : \ \|\xi\|_g = 1\}$.
H is the geodesic flow on the sphere bundle $SN \times \mathbb{R}$,

$$Hu(t,x,\xi) = \frac{\partial u}{\partial t} + \xi^i \frac{\partial u}{\partial x^i} - \xi^i \xi^j \Gamma^k_{ij}(x) \frac{\partial u}{\partial \xi^k},$$
$$Ku(t,x,\xi) = \int_{S_x N} K(x,\xi,\xi')u(t,x,\xi')\, dS_g(\xi').$$

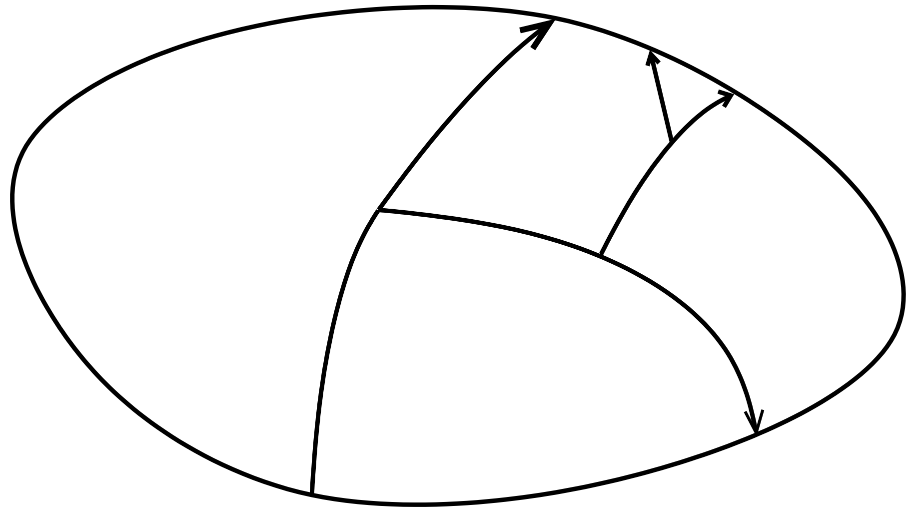


Previous results on radiative transfer the problem:

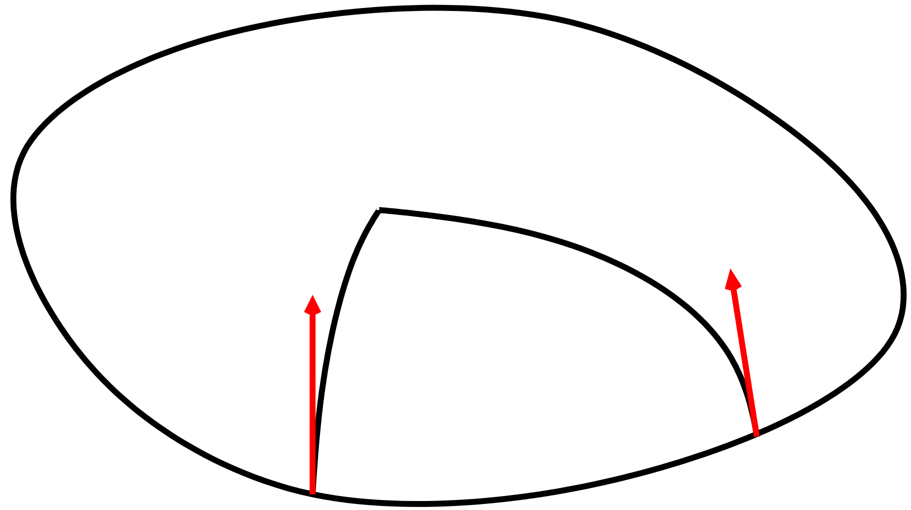
- Choulli-Stefanov
- McDowall
- Arridge

Determination of a non-trapping metric using travel times

- Mukhomethov, Romanov
- Michel
- Gromov
- Croke, Otal
- Sharafutdinov
- Pestov-Uhlmann
- Stefanov-Uhlmann



Let us consider single scattering in M.



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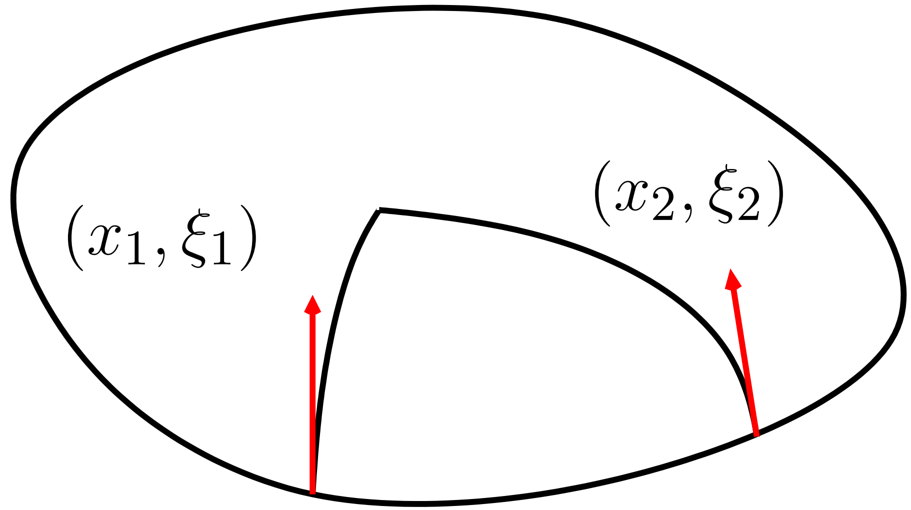


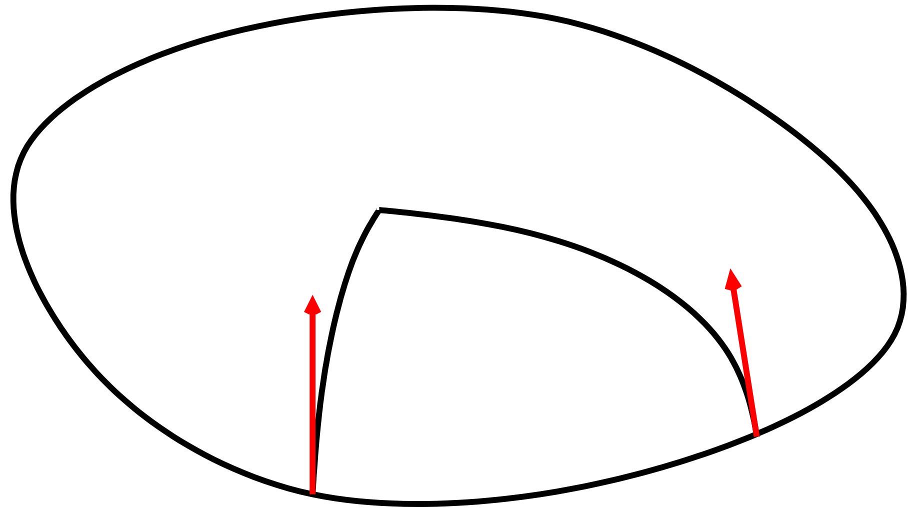
Denote by $\gamma_{x,\xi}$ be a geodesic with $\gamma_{x,\xi}(0) = x$, $\partial_t \gamma_{x,\xi}(0) = \xi$. Let $x_1, x_2 \in \partial M$, $\xi_1 \in S_{x_1}M$, $\xi_2 \in S_{x_2}M$. We say that (x_1, ξ_1), (x_2, ξ_2) and time t are in broken scattering relation if

$$\gamma_{x_1,\xi_1}(s_1) = \gamma_{x_2,\xi_2}(s_2), \quad \text{and } t = s_1 + s_2,$$

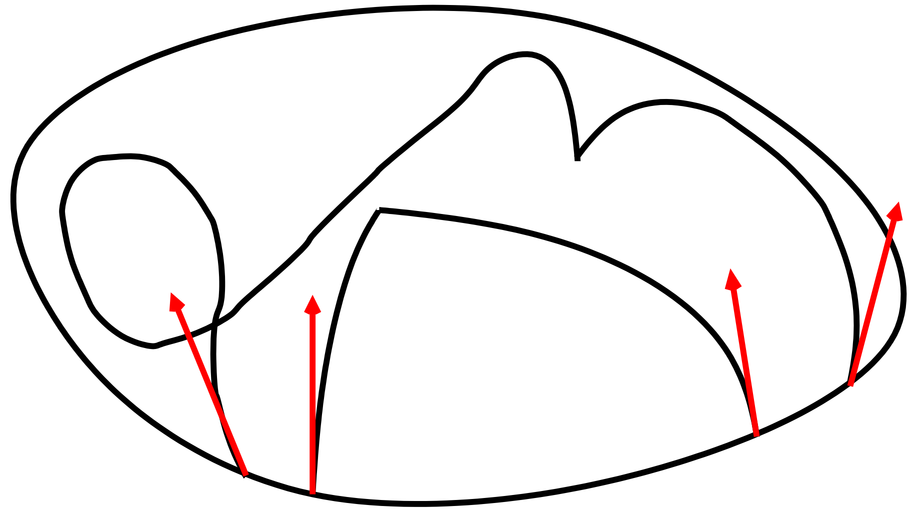
Then we denote

$$\big((x_1, \xi_1), (x_2, \xi_2), t\big) \in \mathcal{B}.$$

Then there is a broken geodesic from (x_1, ξ_1) to $(x_2, -\xi_2)$.

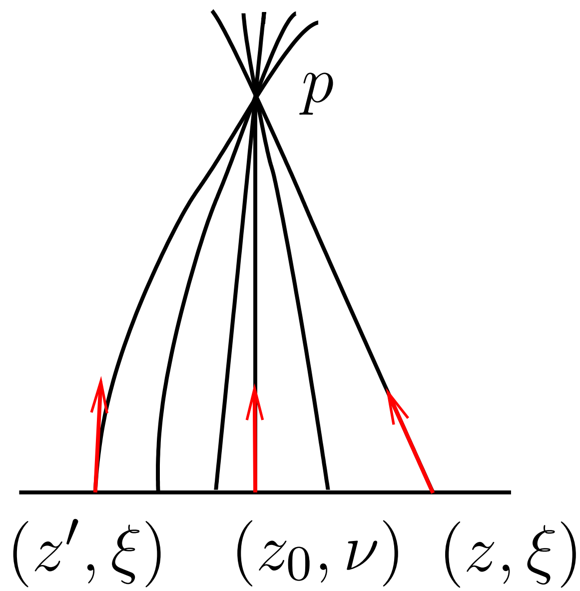


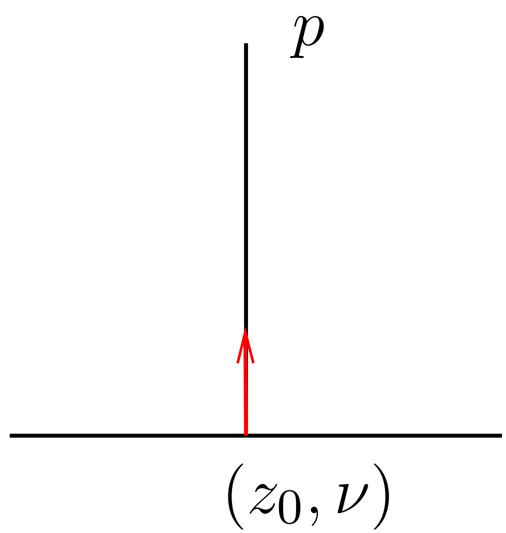
Theorem 1 *Let* (M, g) *be a compact Riemannian manifold with a non-empty boundary of dimension* $n \geq 3$*. Then* ∂M *and the broken scattering relation* $\mathcal{B}$ *determine the manifold* (M, g) *up to an isometry.*



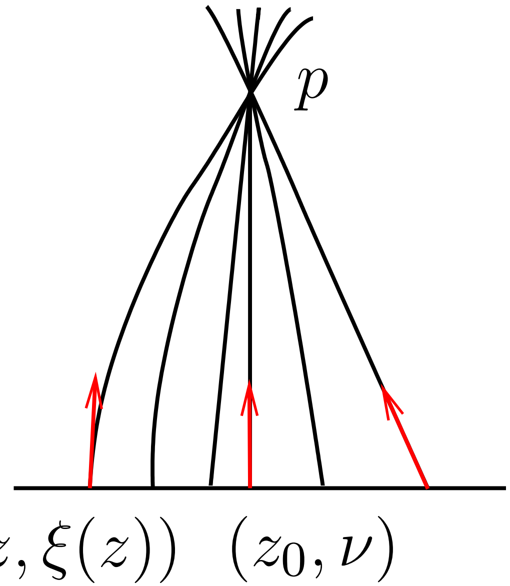
Theorem 2 *Let* (M, g) *be a compact Riemannian manifold with a non-empty boundary of dimension* $n \geq 3$*. Then* ∂M *and the broken scattering relation* $\mathcal{B}$ *determine the manifold* (M, g) *up to an isometry.*

Idea of the proof. Using boundary data we want to recognise when a family of geodesics intersect at the same point.





Let $z_0 \in \partial M$, $U \subset \partial M$ its neighborhood and $p = \gamma_{z_0,\nu}(t_0)$.



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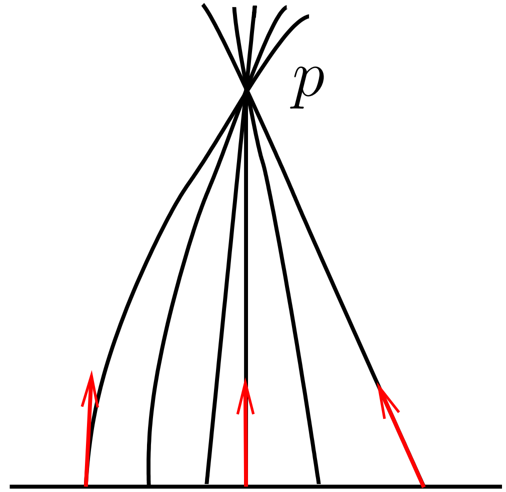
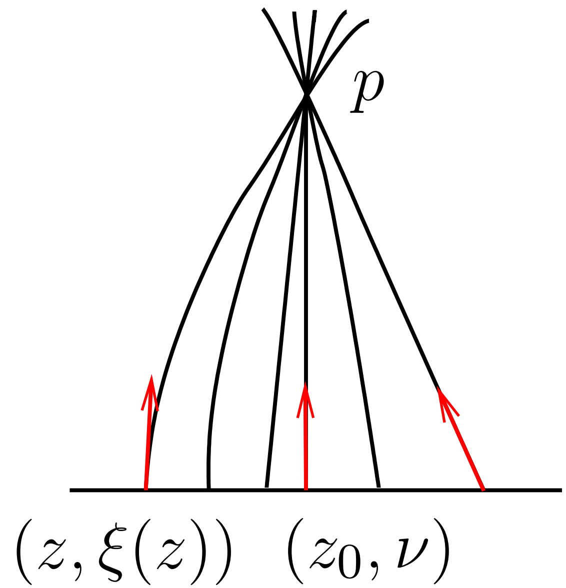


If all geodesics $\gamma_{z,\xi(z)}$ intersect at p and $t(z) = \text{dist}\,(z, p)$, then

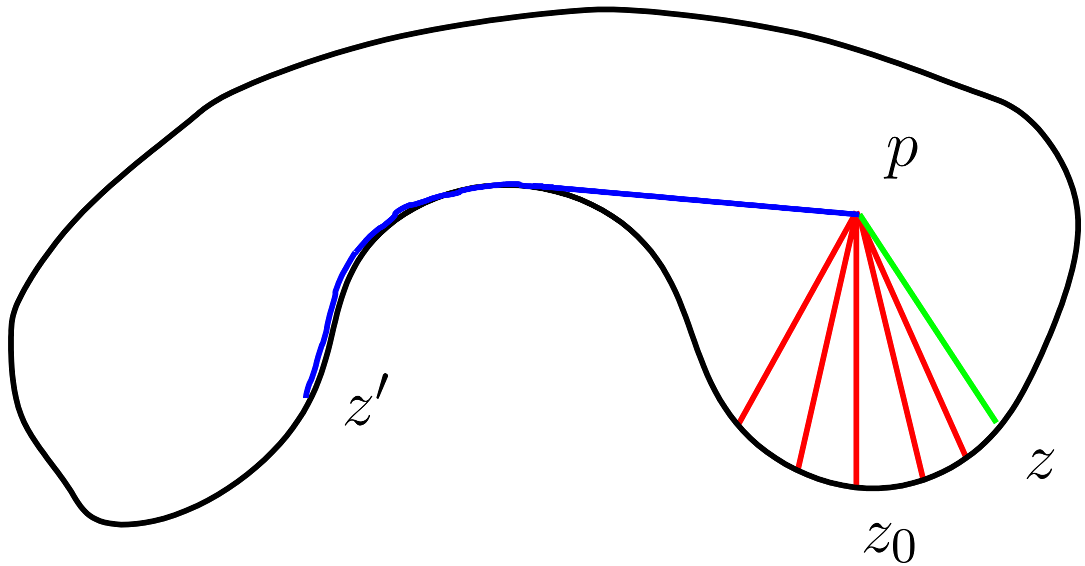
$$\big((z, \xi(z)), (z', \xi(z')),\, t(z) + t(z')\big) \in \mathcal{B}, \quad z, z' \in U \subset \partial M, \tag{1}$$

$$\xi(z_0) = \nu, \quad t(z_0) = t_0, \quad dt(z)|_{z_0} = 0. \tag{2}$$

Definition 1 $(U, \xi(\,\cdot\,), t(\,\cdot\,)\,)$ *is a family of focusing directions for z_0 and t_0 if (1) and (2) are valid.*



Lemma 1 *Let* $(U, \xi(\,\cdot\,), t(\,\cdot\,))$ *be a family of focusing directions for* $z_0 \in \partial M$ *and* $t_0 < \tau(z_0)$ *where* $\tau(z_0)$ *is a critical distance determined by the boundary data. Then all geodesics* $\gamma_{z,\xi(z)},\ z \in U$ *intersect at the point* p *and* $t(z) =$ *dist*(z, p)*.*



Lemma 2 *Let* $(U, \xi(\,\cdot\,), t(\,\cdot\,))$ *be a family of focusing directions for* $z_0 \in \partial M$ *and* $t_0 < \tau(z_0)$*. The broken scattering relation* $\mathcal{B}$ *determines function*

$$z \mapsto \textit{dist}\,(z, p), \quad z \in \partial M,\ p = \gamma_{z_0,\nu}(t_0).$$

Boundary distance functions.

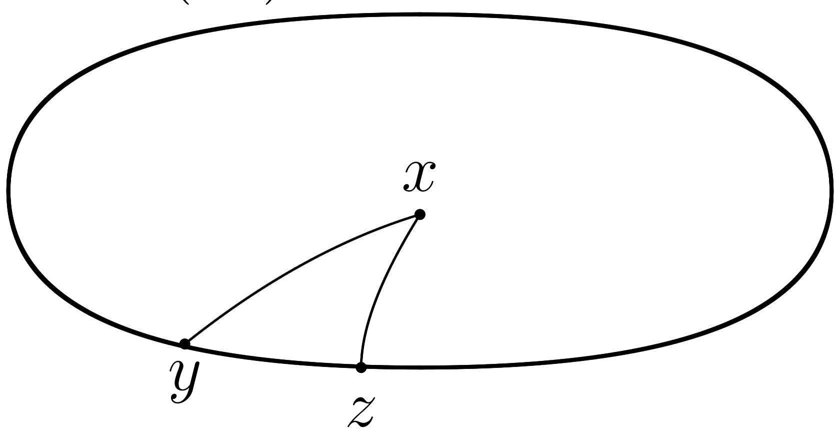
For $x \in M$ define

$$r_x(z) = \text{dist}\,(x,z),\ \ z \in \partial M.$$

Let

$$R : M \to C(\partial M), \quad R(x) = r_x.$$

Next we consider $R(M)$ as a submanifold on $C(\partial M)$.



In the Belishev-Kurylev-Tataru method the boundary distance functions are used to solve hyperbolic inverse problems.

Lemma 3 (Kurylev) *The set $R(M)$ has a Riemannian manifold structure which is isometric to (M, g).*

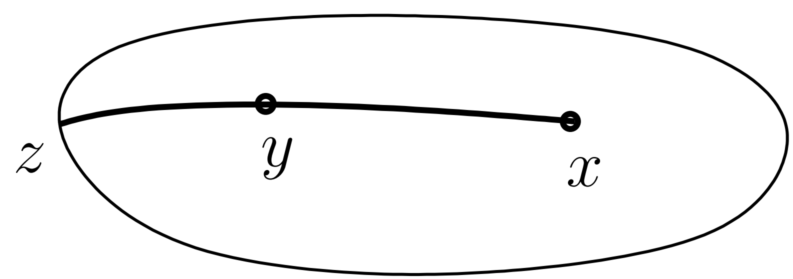
Example: Assume that (M, g) is compact and all geodesics are the shortest paths between their endpoints.
By triangular inequality we have

$$\|r_x - r_y\|_{C(\partial M)} \leq \text{dist}\,(x, y), \quad x, y \in M.$$

For any $x, y \in M$ the geodesic from x to y hits later to $z \in \partial M$ and

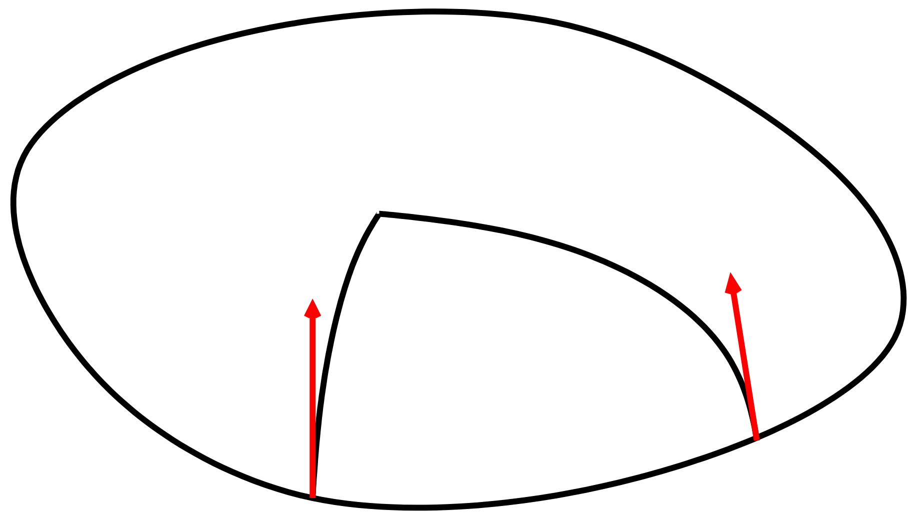
$$\|r_x - r_y\|_{C(\partial M)} \geq |r_x(z) - r_y(z)| = \text{dist}\,(x, y)$$

Then (M, d) is isometric to $(R(M), \|\cdot\|_\infty)$.



Lemma 4 (Kurylev) *The set $R(M)$ has a Riemannian manifold structure which is isometric to (M, g).*

The broken scattering relation $\mathcal{B}$ determines the boundary distance functions $R(M)$ and thus (M, g) upto an isometry.



Inverse problem or radiative transfer equation.

Consider the equation

$$(Hu)(t,x,\xi) + \sigma(x,\xi)u(t,x,\xi) - (Ku)(t,x,\xi) = 0,$$
$$u(t,x,\xi)\big|_{t=0} = w(x,\xi).$$

on a complete and simple Riemannian manifold (N,g).

$$Hu(t,x,\xi) = \frac{\partial u}{\partial t} + \xi^i \frac{\partial u}{\partial x^i} - \xi^i \xi^j \Gamma^k_{ij}(x) \frac{\partial u}{\partial \xi^k},$$
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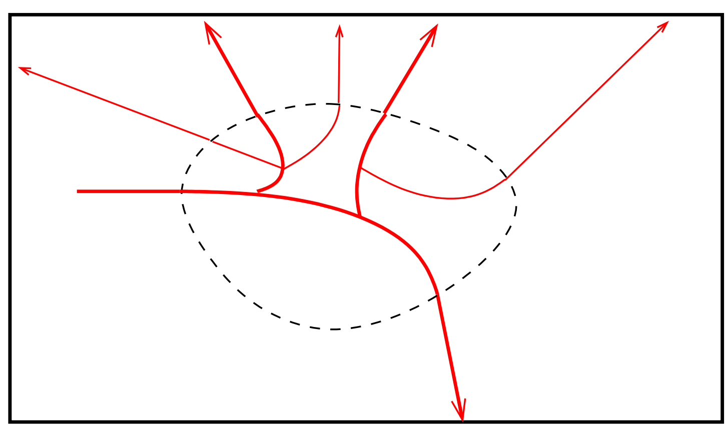
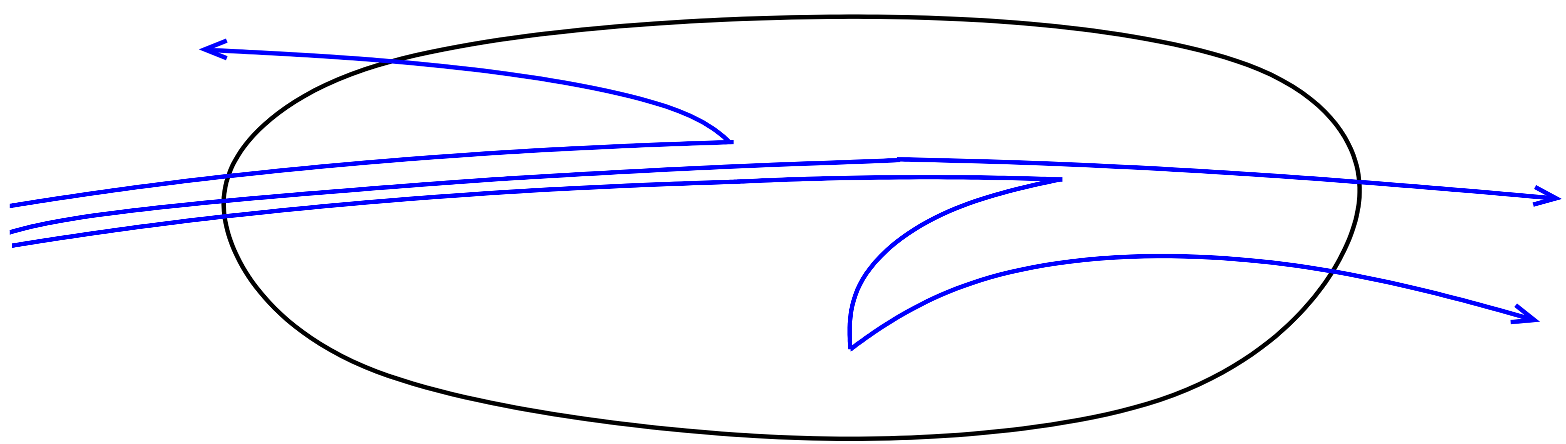


Let $M \subset N$ be compact, $U = N \setminus M$. Assume that we are given the measurement map

$$A : C_0^\infty(SU) \to C^\infty(\mathbb{R}_+ \times SU), \quad A(u|_{t=0}) = u|_{\mathbb{R}_+ \times SU}.$$

Theorem 3 *Let N be a complete simple manifold of dimension $n \geq 3$, $M \subset N$ be compact and strictly convex. Assume that $K \in C_0^\infty(SM \dot{\times} SM)$ and $K(x,\xi,\xi') > 0$ for all $x \in M^{\rm int}$.*
Then $U = N \setminus M$ and the measurement map A determine uniquely (M, g).



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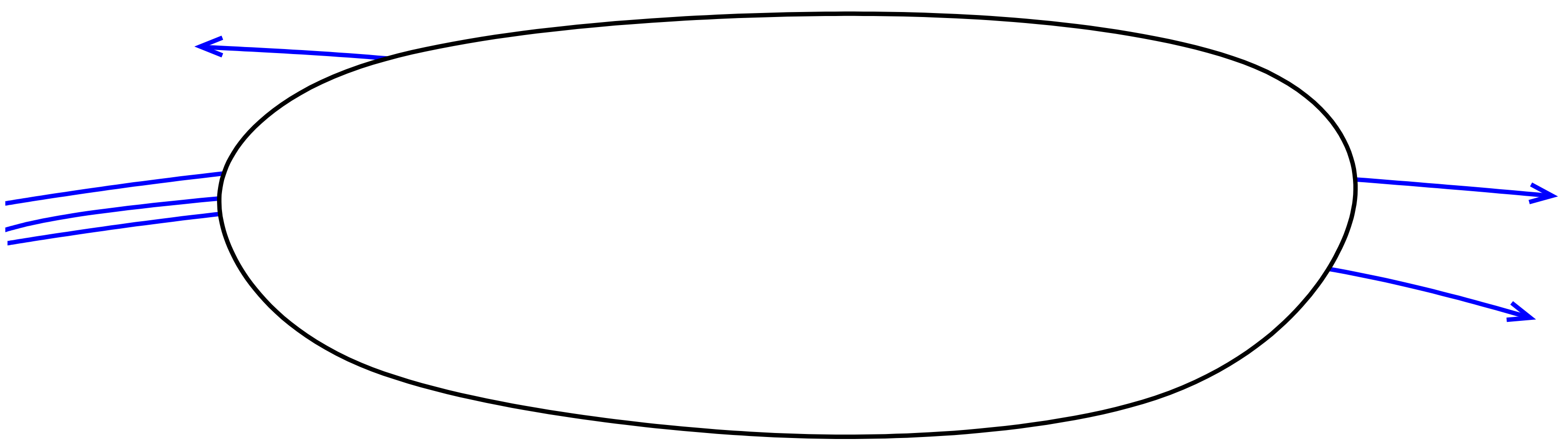
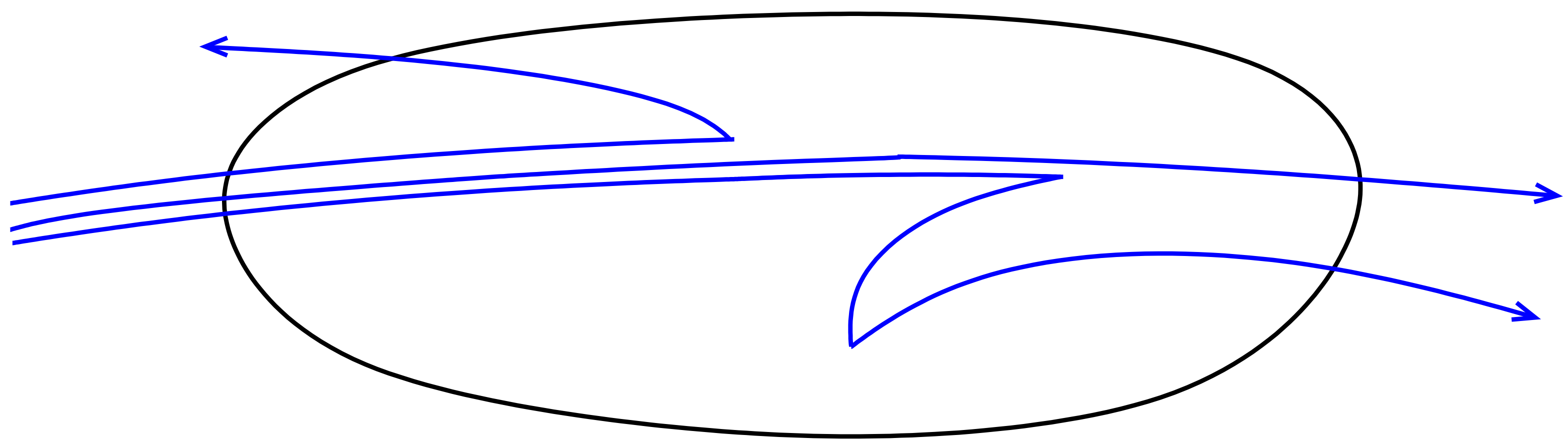


Idea of solution: Consider the Born series

$$u = u_0 + u_1 + u_2 + u_3 + \ldots, \quad u_{j+1} = (H + \sigma)^{-1} K u_j.$$

Using Melrose-Uhlmann-Greenleaf calculus of conormal distributions, we can show that the ballistic photons u_0 and the single scattering photons u_1 dominate in the Born series.



Consider $X = \mathbb{R}^n$ with coordinates $x = (x', x'') \in \mathbb{R}^d \times \mathbb{R}^{n-d}$. Denote

$$S = \{x' = 0\}, \quad \Lambda = N^* S.$$

We say that $u \in \mathcal{D}'(X)$ is a Lagrangian distribution associated with Λ and denote $u \in I^m(X; \Lambda)$, if

$$u(x) = \int_{\mathbb{R}^d} e^{ix' \cdot \theta} a(x, \theta)\, d\theta, \quad a(x, \theta) \in S^{m+n/4-d/2}(X \times \mathbb{R}^d \setminus 0).$$

Note that $WF(u) \subset \Lambda$.
If $\Lambda_1, \Lambda_2 \subset T^* X$ are two cleanly intersecting Lagrangian manifolds, we can define

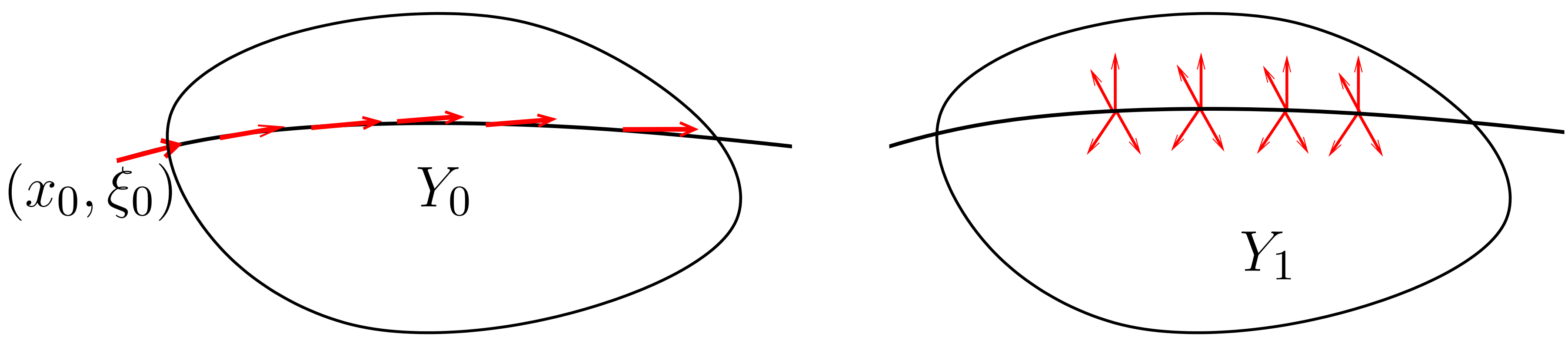
$$u \in I^{p,l}(X; \Lambda_1, \Lambda_2), \quad WF(u) \subset \Lambda_1 \cup \Lambda_2.$$

In the following we use $X = SN$.

Let $\gamma_0 = \gamma_{x_0,\xi_0}$ be geodesic starting from (x_0, ξ_0) and

$$Y_0 = \{(\gamma_0(t), \partial_t \gamma_0(t)) \in SN : \; t \in \mathbb{R}\},$$
$$Y_1 = \{(x, \xi) \in SN : \; x \in \gamma_0(\mathbb{R})\}$$

and define $\Lambda_0 = N^* Y_0$ and $\Lambda_1 = N^* Y_1$.



Let u be solution with the initial data $u|_{t=0} = \delta_{x_0,\xi_0}$.
Let $\widehat{u}(k, x, \xi) = (\mathcal{L}u(\cdot\,, x, \xi))(k)$ be the Laplace transform of u in time t. Then

$$(P + \sigma + k)\widehat{u} - K\widehat{u} = w_0 \quad \text{in } (x, \xi) \in SN,$$

where $w_0(x, \xi) = \delta_{(x_0,\xi_0)}(x, \xi)$ and

$$Pv(x, \xi) = \xi^j \frac{\partial v}{\partial x^j}(x, \xi) - \xi^l \xi^j \Gamma^m_{lj}(x) \frac{\partial v}{\partial \xi^m}(x, \xi).$$

The operator $P + \sigma + k$ has a right inverse

$$\widehat{Q}_k : C_0^\infty(SN) \to C^\infty(SN).$$

Mapping properties of $\widehat{Q}_k$ and the parametrix Q of H are known by Melrose-Uhlmann-Greenleaf calculus.

We can write $K = K_1 K_2$,

$$K_j f(x,\xi) = \int_{S_x N} K_j(x,\xi,\xi') f(x,\xi')\, dS_g(\xi'), \quad j = 1,2$$

where $K_j(x,\xi,\xi') \in C_0^\infty(SN \dot{\times} SN)$.
The terms in the Born series can be written as

$$\begin{aligned} \widehat{u}_j(k) &= \widehat{Q}_k (K \widehat{Q}_k)^{j-1} K \widehat{u}_0(k) \\ &= \widehat{Q}_k K_1 G^{j-1} K_2 \widehat{u}_0(k), \quad j \geq 1, \end{aligned}$$

where for fixed k the operator

$$G = K_2 \widehat{Q}_k K_1$$

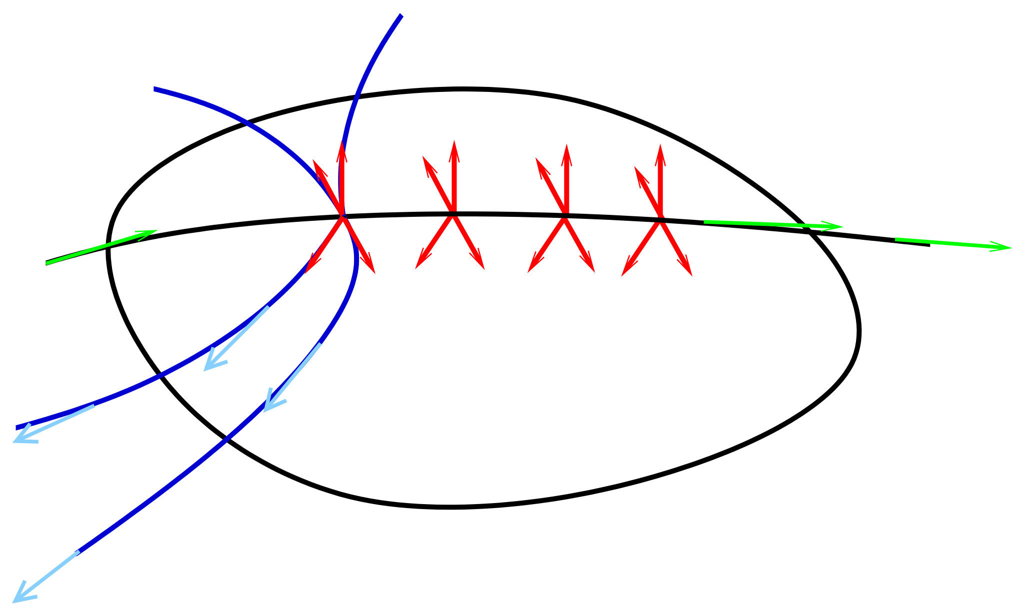
is pseudodifferential operator of order (-1), that is, an operator increasing smoothness by one.

Lemma 5 *We have*

$$\widehat{u}_0(k,x,\xi) = c_0(x,k)\delta_{Y_0}(x,\xi) \in I^{r_0}(SN;\Lambda_0), \quad r_0 = (2n-3)/4,$$
$$\widehat{u}_j(k) \in I^{r_j,-\frac{1}{2}}(SN;\Lambda_1,\Lambda_2), \;\; r_j = -j + \frac{1}{4} + \epsilon, \;\; \epsilon > 0, j \geq 1$$

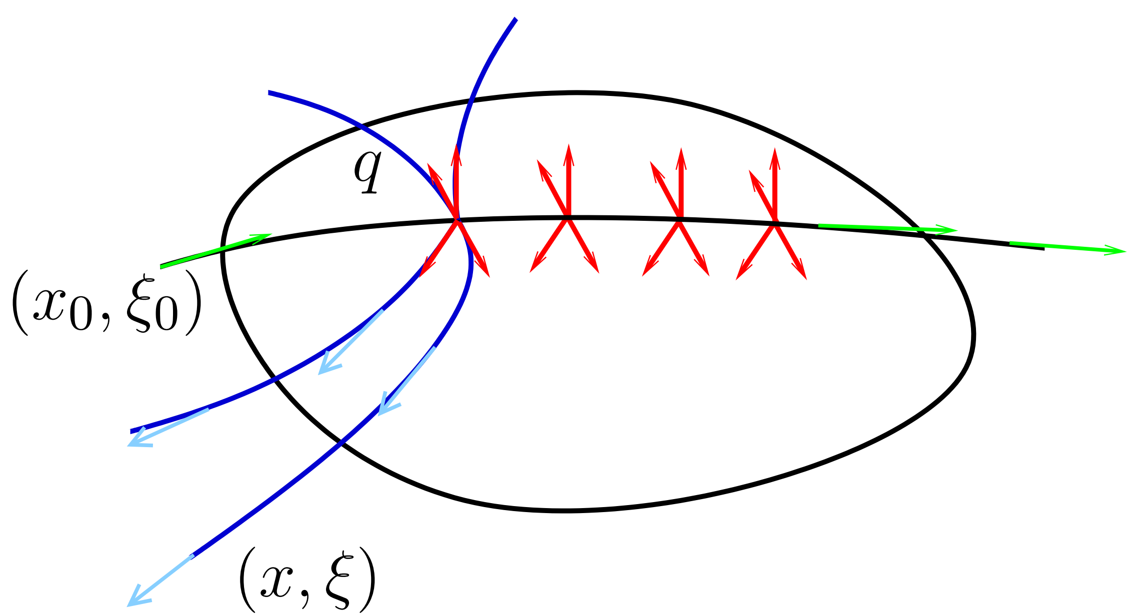
where Λ_2 *is the flow-out of* Λ_1 *in char* (P^{-1})*.*

Thus with a fixed k the solution $\widehat{u}(x,\xi,k)$ determines the singularities of $\widehat{u}_1(x,\xi,k)$.



Singularities of $\widehat{u}(x, \xi, k)$ determine all points (x, ξ), $x \notin M$, such that there is a broken geodesic from (x_0, ξ_0) to (x, ξ) with a breaking point in M^{int}. Let $q = \gamma_0(s)$ be the breaking point.
As $k \to \infty$, the principal symbol of $\widehat{u}(k)$ near $\Lambda_2 \setminus \Lambda_1$ has the asymptotics

$$a^p(x, \theta; k) = e^{k(dist\,(x_0,q)+dist\,(q,x))}(c_1(x, \theta) + \mathcal{O}(k^{-1}))$$



Thus the singularities of $\widehat{u}(x,\xi,k)$ determine all points (x,ξ), $x \notin M$, such that there is a broken geodesic from (x_0,ξ_0) to (x,ξ) with a breaking point in $M^{\rm int}$ and the function

$$\text{dist}\,(x_0,q) + \text{dist}\,(q,x), \quad q = \gamma_0(s_1).$$

Thus the singularities of $\widehat{u}(x,\xi,k)$ determine the broken scattering relation $\mathcal{B}$ that further determines (M,g) upto an isometry.