Dedicated to Jan Boman's 75-th birthday

Integral Geometry and Tomography

Stockholm University, SWEDEN, August 12-15, 2008

Reconstruction methods in thermoacoustic tomography and all that

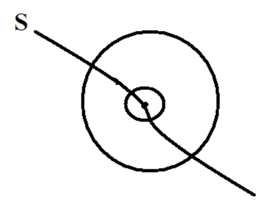
Peter Kuchment
Math. Dept., Texas A&M University, College Station, TX, USA
Joint with L. Kunyansky (U. Arizona),
D. Darrow, Y. Hristova, L. Nguyen (TAMU)

Partially supported by NSF, DNDO, and DOE grants

An outline

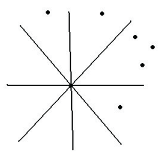
- Pre-history: how it all started.
- Thermoacoustic Tomography: can one hear the heat of a body?
- Some references
- Comparison of three approaches to TAT reconstructions
- Time reversal
- Other hybrid modalities. AET example
- Synthetic focusing

1. **Pre-history: how it all started.**



V. Lin and A. Pinkus

P.K., N. Zobin



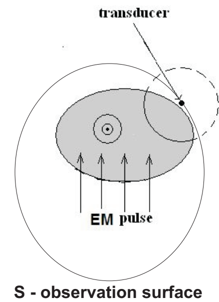
M. Agranovsky and E. T. Quinto

Agranovsky&Berenstein&Kuchment, Agranovsky&Quinto

AND THEN WE ALL GOT TIRED

2. **Thermoacoustic Tomography: Can One Hear the Heat of a Body?**

Hybrid techniques: Thermoacoustic (TAT) and Photoacoustic Tomography (PAT), UMOT, AET, etc.

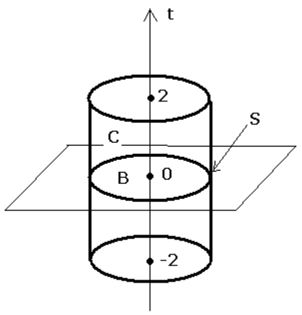


TAT goal: recover EM energy absorption $f(x)$.
Cancerous cells absorb several times more energy than healthy ones $\Rightarrow$ high contrast. Also high resolution of ultrasound.

3. Mathematics of TAT

$$\begin{cases} p_{tt} = c^2(x)\Delta_x p, & t \geq 0, \quad x \in \mathbb{R}^3 \\ p(x,0) = f(x), & p_t(x,0) = 0, \\ p(y,t) = g(y,t) & \text{for } y \in S, \quad t \geq 0. \end{cases}$$

$c(x)$ - speed of sound; $g(y,t)$ - measured data, S - *observation surface*. One needs to recover f from g.



WHY IS THIS PROBLEM SOLVABLE?

When $c = 1$, equivalent to inversion of *restricted spherical mean Radon transform*:

$$R_S f(p,r) = \omega^{-1} \int_{|y-p|=r} f(y)\, d\sigma(y), \quad p \in S, \quad r \geq 0.$$

We will assume that supp f - compact, S closed. There is uniqueness of reconstruction then.

4. **Some related references**

- Surveys
  - M. Agranovsky, P. Kuchment, and L. Kunyansky, On reconstruction formulas and algorithms for the thermoacoustic tomography, to appear in L. H. Wang (Editor) "Photoacoustic imaging and spectroscopy," CRC Press 2009. Available at arXiv.
  - D. Finch and Rakesh, The spherical mean value operator with centers on a sphere, Inverse Problems **23**(2007), No. 6, S37–S50.
  - D. Finch and Rakesh, Recovering a function from its spherical mean values in two and three dimensions, to appear in same volume as [1].

- P. Kuchment and L. Kunyansky, Mathematics of thermoacoustic tomography, European J. Appl. Math., **19** (2008), Issue 02, 191-224

- other
  - Y. Hristova, P. Kuchment, and L. Nguyen, On reconstruction and time reversal in thermoacoustic tomography in homogeneous and non-homogeneous acoustic media, to appear in Inverse Problems.
  - P. Kuchment and L. Kunyansky, in progress
  - Y. Hristova in progress
  - L. Nguyen, in progress

5. **Three types of inversion formulas and procedures**

- **Filtered backprojection (FBP) formulas**

(Finch-Patch-Rakesh '04, Xu-Wang '05, Finch-Haltmeier-Rakesh '06, Kunyansky '06)

$$f(y) = -\frac{1}{8\pi^2}\Delta_y \int\limits_S \frac{g(z,|z-y|)}{|z-y|} dA(z),$$

$$f(y) = -\frac{1}{8\pi^2}\int\limits_S \left(\frac{1}{t}\frac{d^2}{dt^2} g(z,t)\right)\Bigg|_{t=|z-y|} dA(z).$$

Known for **constant speed** and S - **sphere**. Incorrect reconstructions when f is supported (even partially) outside S.

• **Eigenfunction series expansions** (Kunyansky '06, Agranovsky-PK '07)
B - interior of S, $\psi_k(x), \lambda_k^2$ - eigenfunctions and eigenvalues of $A = -c^2(x)\Delta$ in B with Dirichlet conditions on S, E - harmonic extension from S to interior. Speed c-non-trapping.

$$f(x) = (Eg|_{t=0}) - \int_0^\infty (A)^{-\frac{1}{2}} \sin(\tau (A)^{\frac{1}{2}}) E(g_{tt})(x,\tau) d\tau.$$

Series expansion

$$f(x)|_B = \sum_k f_k \psi_k(x),$$
$$f_k = -\lambda_k^{-1} \int_0^\infty \int_S \sin(\lambda_k t) g(x,t) \overline{\frac{\partial \psi_k}{\partial \nu}(x)} dx dt.$$

• **Time reversal** (Finch-Patch-Rakesh '04, Burgholzer et al '07, Hristova-PK-Nguyen '08)

Choose T so large that $p(x,T)$ is small inside S. Solve back in time

$$\begin{cases} p_{tt} = c^2(x)\Delta_x p, & t \geq 0, \quad x \in \mathbb{R}^3 \\ p(x,T) = 0, & p_t(x,T) = 0, \\ p(y,t) = g(y,t) & \text{for } y \in S, \quad t \geq 0. \end{cases}$$

to find at $t = 0$ an approximation for $f(x) = p(x,0)$.
Exact only when dimension $n > 1$ is odd and speed is constant. Works approximately, best in odd dimensions and with non-trapping speed.

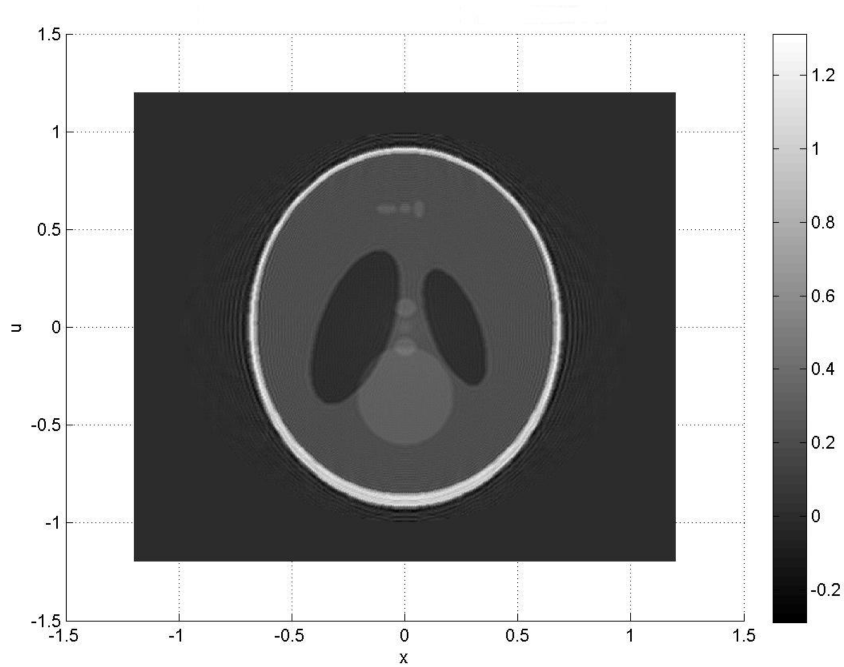
6. Comparison of requirements of the three methods

- FBP methods.
  - Require that the source f to be reconstructed is supported inside the observation surface S. Otherwise reconstructions are incorrect.
  - Work for constant sound speed only.
  - Known only for spheres.
  - Require that the time derivative p_t vanishes at $t = 0$.
  - Can be stably implemented.

- Eigenfunction expansions
  - Do not require the source f to be supported inside S.
  - Require that the solution $p(x,t)$ inside S decays sufficiently fast with time.
  - Theoretically work for variable sound speed.
  - Theoretically work for any observation surface S.
  - Do not require the time derivative p_t to vanish at $t = 0$.
  - Can be very nicely implemented for a constant speed and cubic S. Probably cannot be implemented well numerically for variable speeds and general observation surfaces.

- Time reversal
  - Does not require the source f to be supported inside S.
  - Requires that the solution $p(x,t)$ inside S decays with time.
  - Does not require the time derivative p_t to vanish at $t = 0$.
  - Easily implementable numerically for variable sound speeds and arbitrary observation surfaces.

7. **Exploring the time reversal** Error analysis (Y. Hristova, in preparation).



Time reversal reconstruction of Shepp-Logan phantom

- **Non-trapping condition**

  Hamiltonian system with $H = \frac{c^2(x)}{2}|\xi|^2$:

$$\begin{cases} x'_t = \frac{\partial H}{\partial \xi} = c^2(x)\xi \\ \xi'_t = -\frac{\partial H}{\partial x} = -\frac{1}{2}\nabla\left(c^2(x)\right)|\xi|^2 \\ x|_{t=0} = x_0, \xi|_{t=0} = \xi_0. \end{cases}$$

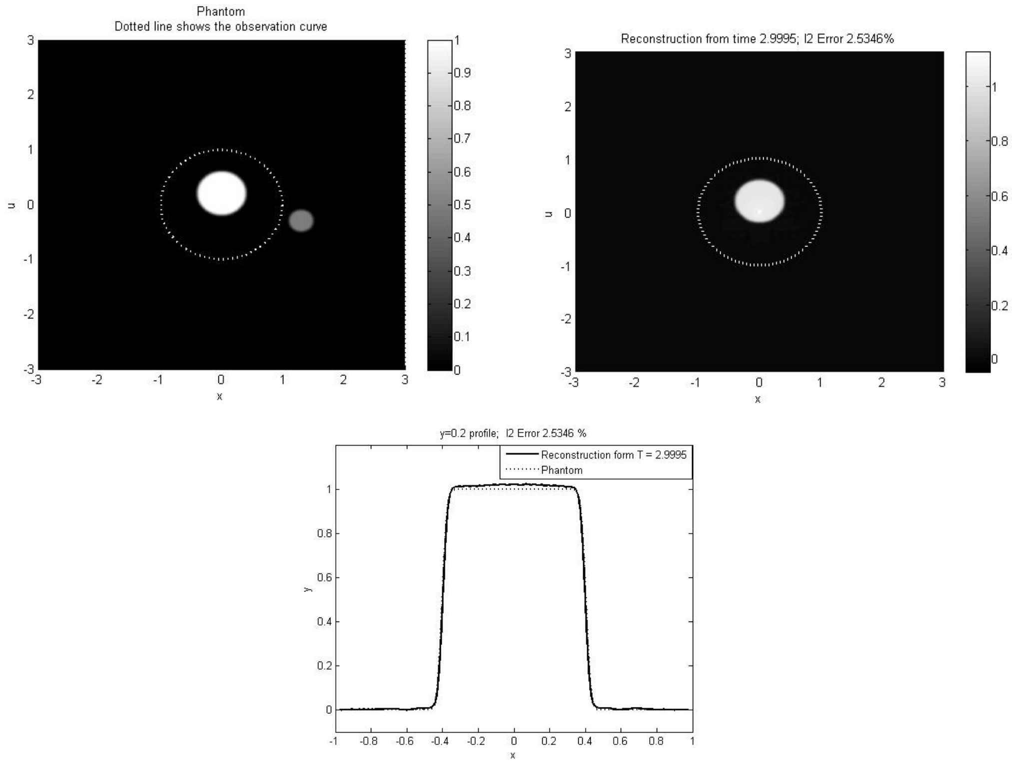
  Solutions - *bicharacteristics*, projections to $\mathbb{R}^n_x$ - *rays*.

  Non-trapping: Rays ($\xi_0 \neq 0$) tend to ∞ when $t \to \infty$.

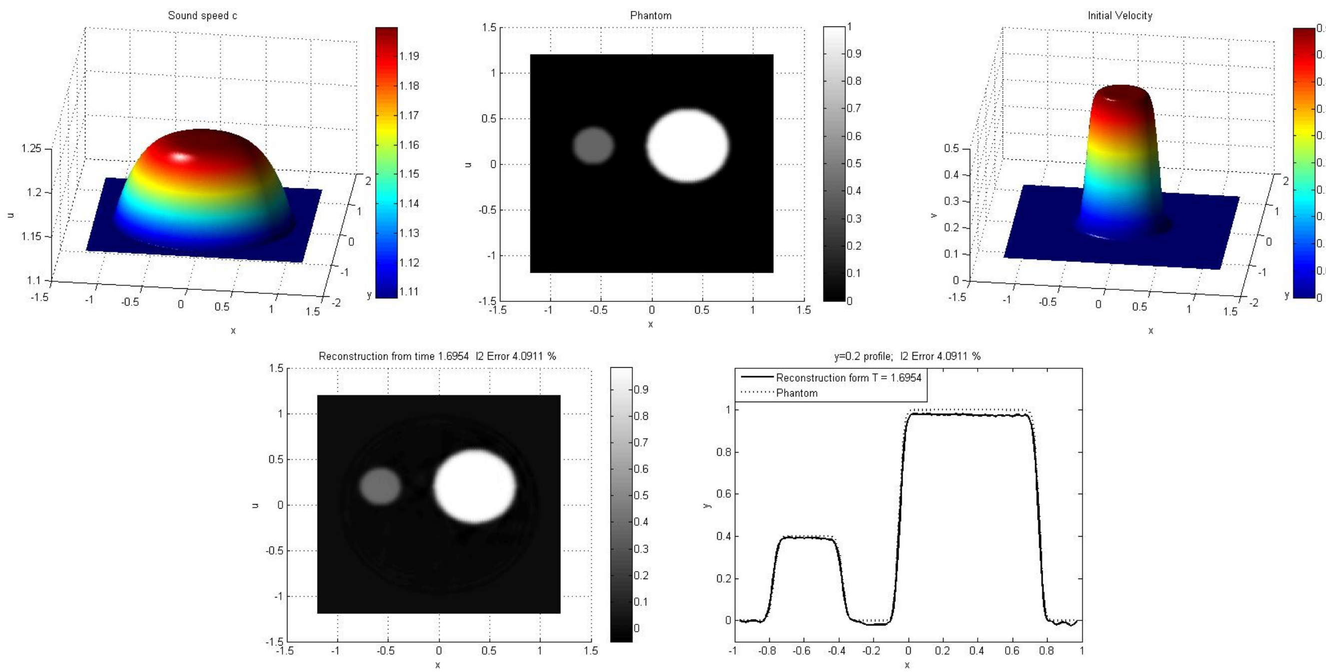
  Non-trapping $\Rightarrow$ decay and eventual smoothing in any compact region.

  "Crater" speed $c(x) = |x|$ for $r_1 < |x| < r_2$ is trapping for $r_1 < |x_0| < r_2, \xi_0 \perp x_0$.

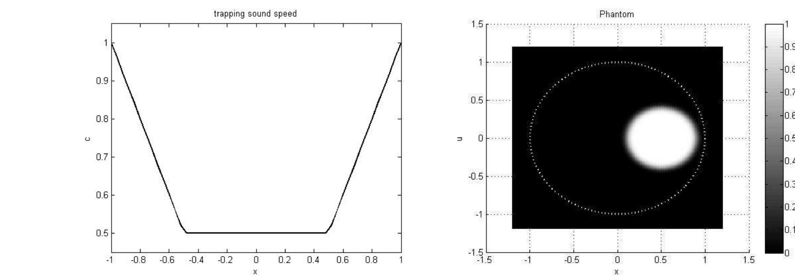
• **Examples of time reversal performance**



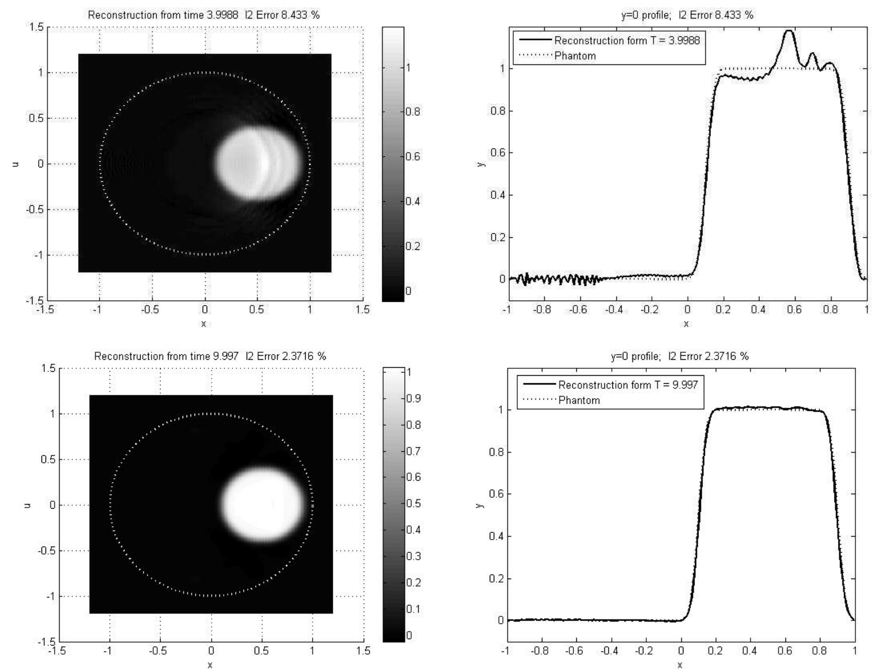
Phantom with a part outside the observation surface and its reconstruction.



Reconstruction with a non-zero initial velocity

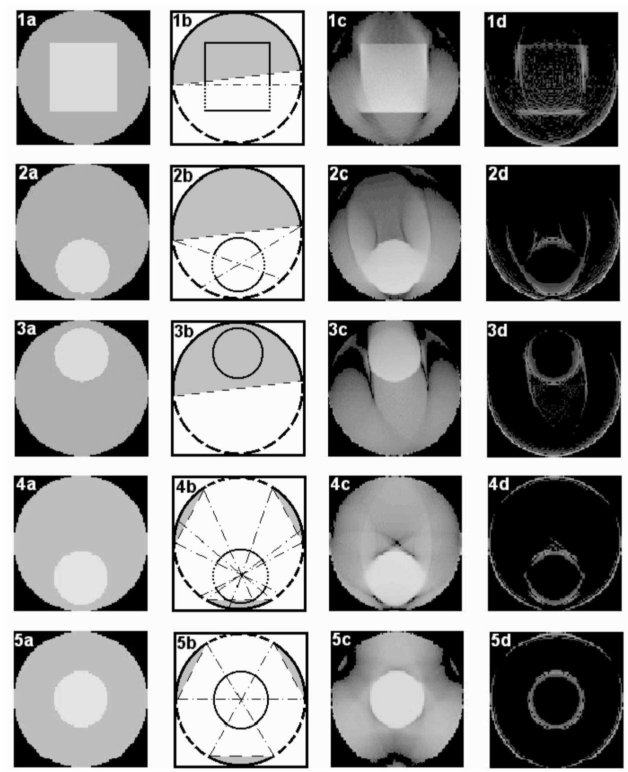


A trapping speed and phantom

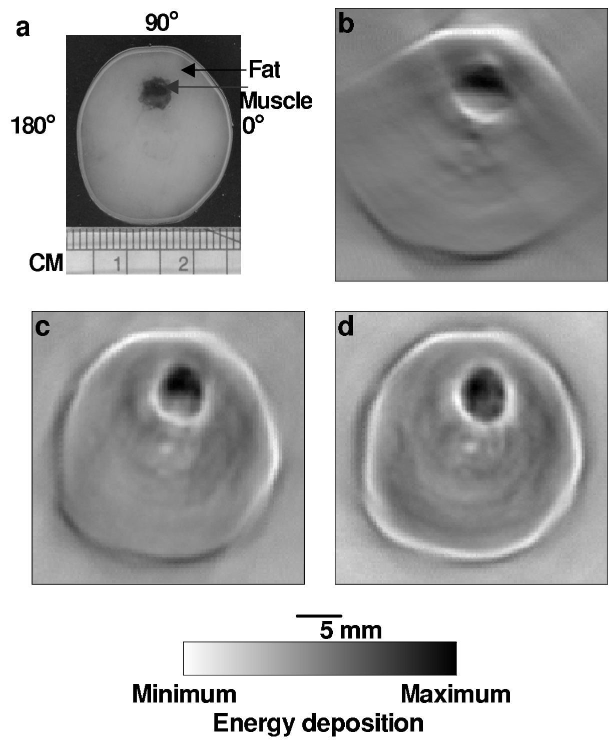


Reconstructions for $T = 4, 10$

- **Limited view due to trapping**
  ♠ Limited view at constant speed (Louis-Quinto '00, Xu-Wang-Ambartsoumian-PK '04)

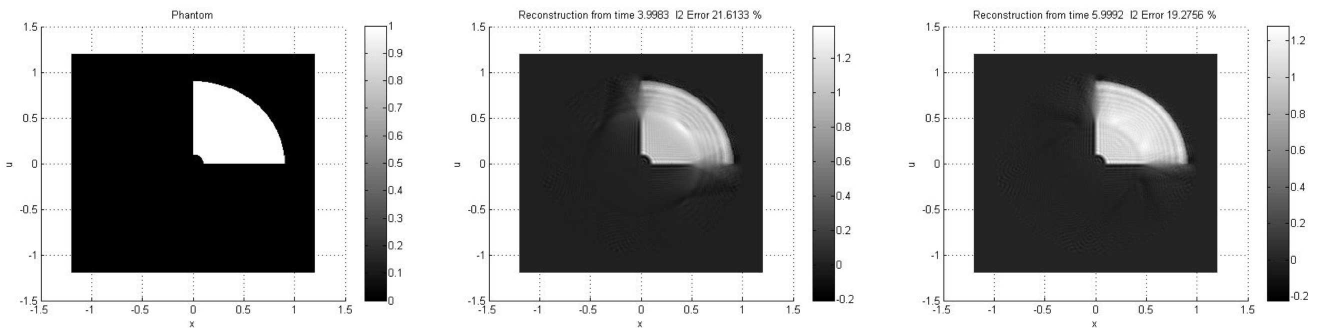


The parts unstable to reconstruct are blurred.



a
90°
180°
0°
Fat
Muscle
CM
1
2
b
c
d
5 mm
Minimum
Maximum
Energy deposition

♠ Variable speed and “full view” (Hristova-PK-Nguyen ’08)

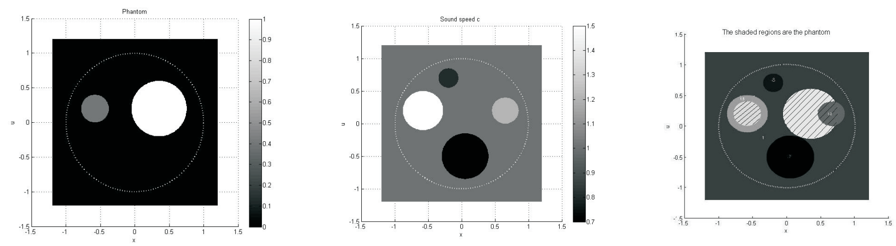


Phantom and time reversal reconstructions ($T = 4, 6$).
“Limited view” blurring due to trapping.

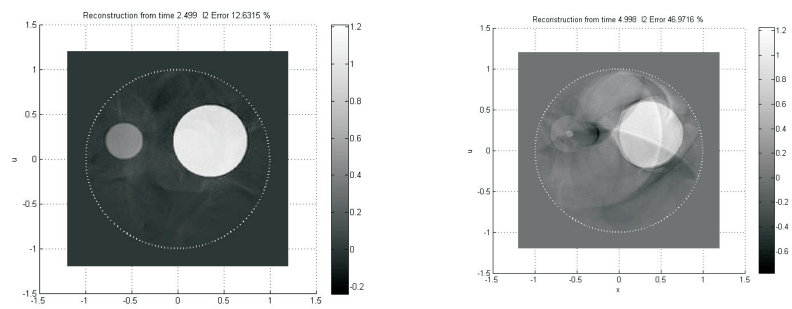
Analysis of singularities (L. Nguyen, in preparation).

- Finding the sound speed

  What if one gets the speed wrong?



Phantom, sound speed, and their overlap.



Reconstructions: correct (left) and average (right) speed.

- **Can one find the speed?**

  Transmission ultrasound tomography before TAT finds the speed approximately (Xu-Wang).

  Q.: Can one find the speed c and the image f from the same TAT data?
  A.: ??

  Successful numerical experiments (Anastasio-Zhang '06).

  If f is supported strictly inside S, a **constant** speed is determined uniquely (PK-Nguyen '08).

  If f is supported inside S, the range conditions (Agranovsky-PK-Quinto '07) **locally** uniquely determine the coefficient α in speed $\alpha c(x)$ (PK-Nguyen '08).

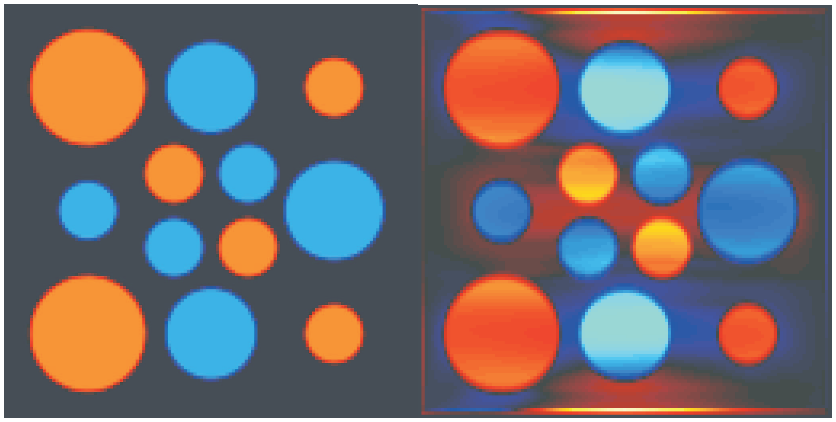
  Transmission eigenvalues results $\Rightarrow$ distinguishing two speeds such that $c_0(x) > c_1(x)$. (Finch '08)

8. **Ultrasound modulated imaging**

In several newly developing types of tomography, a tomographic measurement is accompanied by scanning a focused ultrasound beam, which modifies properties of a medium. It is assumed that observing the changes in the measurements might help reconstruction. An example is the AET (acousto-electric tomography, coined by L. Wang and Y. Xu '04), which can also be called ultrasound modulated EIT. Use focused ultrasound to perturb locally the electric properties of the material. Pick up the signal on the boundary and use it to reconstruct the conductivity σ in

$$\nabla \sigma(x) \nabla u = 0$$

Example



A conductivity phantom and its reconstruction via focused US scanning

Also see H. Ammari, M. Fink, et al, SIAP 2008.

9. **Synthetic focusing** (P. K. and L. Kunyansky, '08)

Problems with focusing ultrasound. Synthetic focusing – gets responses to focused irradiation without focusing. A simplest version:

Images $\psi(y)$ are mapped into data by a forward measurement operator

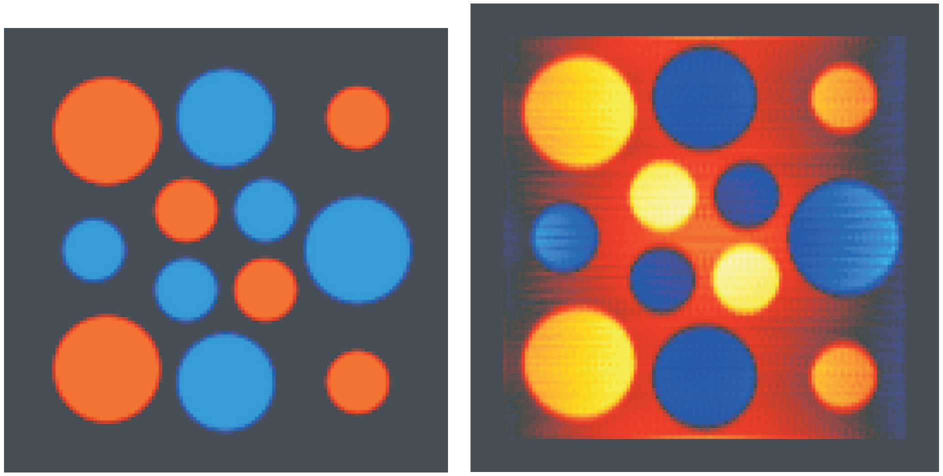
$$\psi(y) \mapsto A(\psi).$$

A localized perturbation $\epsilon\delta_x(y)$. Perturbed signal

$$f(x) := A(\psi + \epsilon\delta_x) - A(\sigma) \approx A'(\psi)\epsilon\delta_x$$

E.g., spherical perturbation $\delta_{p,t}(x) := \epsilon\delta(|x-p|-t))$ and response $g(r,p) := A'(\psi)\delta_{p,t}$. Then $f(x)$ is recovered from $g(p,t)$ by a TAT procedure.

An example of using synthetic focusing in AET



Conductivity phantom and synthetic reconstruction