Cloaking Devices and Invisibility

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joint with

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Matti Lassas
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Integral Geometry and Tomography

In Honor of Jan Boman

Stockholm, August, 2008

Cloaking Devices, circa 2006

May: U. Leonhardt: dimension 2 via conformal mapping

J. Pendry, D. Schurig and D. Smith: via singular transformations

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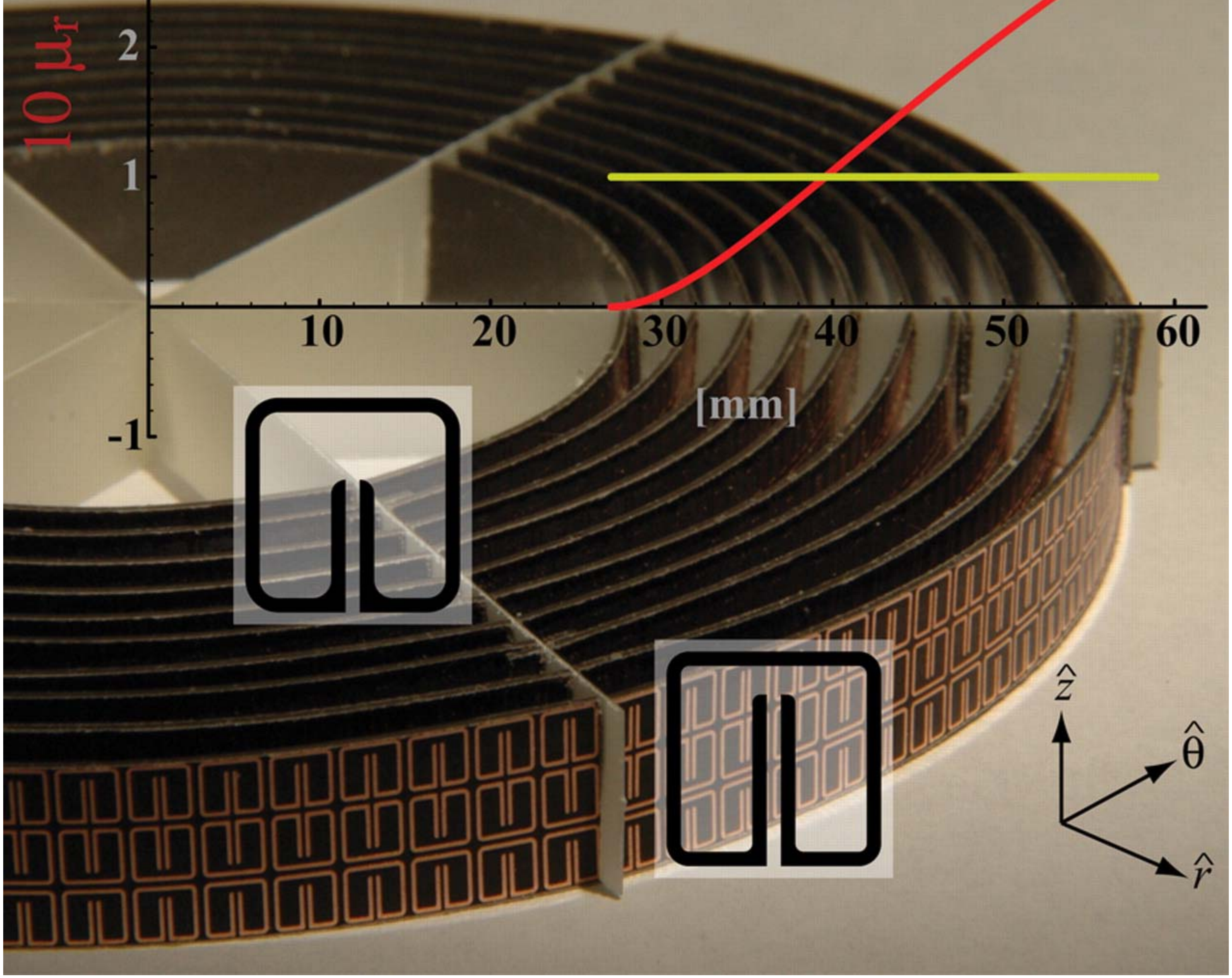
Oct.: Schurig, Mock, Justice, Cummer, Pendry, Starr and Smith:
Physical experiment - Cloaking a copper disc from microwaves.

Metamaterials: arrays of composites having
prescribed variable electromagnetic (EM) material parameters.

Now becoming available at optical wavelengths

N.B. Resonance based $\Longrightarrow$ Monochromatic/ narrow band:
Need to be fabricated and assembled for particular frequency



Schurig, et al., *Science*, **314**, 977 (2006)

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(3) What other artificial wave phenomena can one produce?

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(3) **Electromagnetic wormholes**: invisible optical tunnels $\hookrightarrow$ Change the topology of space *vis-a-vis* EM wave propagation

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Invisibility at frequency $k > 0$ (vector) $\iff$ full unpolarized EM $\iff$
counterexamples to inverse problems for **Maxwell's** equations

Calderón's problem and EIT (Electrical impedance tomography)

- Imaging using electrostatics

- Interior of a region $\Omega \subset \mathbb{R}^n, n = 2, 3$, filled with matter with electrical conductivity $\sigma(x)$ (Ω = human body, airplane wing,...)

- Place voltage distribution on the boundary $\partial\Omega$, inducing potential $u(x)$ throughout Ω

- Measure induced current flow. Ohm's Law $\Longrightarrow$

$$I = \sigma \cdot \frac{\partial u}{\partial \nu}$$

- $u(x)$ satisfies the **conductivity** equation,

$$\nabla \cdot (\sigma \nabla) u(x) = 0 \text{ on } \Omega,$$

$$u|_{\partial\Omega} = f = \text{ prescribed voltage}$$

N.B. $\sigma(x)$ can be a **scalar**- or symmetric **matrix**-valued function (**isotropic** or **anisotropic** conductivity, resp.)

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• Standard assumptions

$$\sigma \in L^\infty, \quad 0 < c_1 I \leq \sigma \leq c_2 I$$

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Q. (Calderón, 1980): Does Λ_σ determine σ?

I.e., is $\sigma(x)$ “visible” to electrostatic measurements?

A.: Yes in many cases.

Isotropic: $n \geq 3$: Sylvester-Uhlmann, . . .
$n = 2$: Nachman, . . . , Astala-Päivärinta

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Anisotropic: NOT SO FAST...

Observation of Luc Tartar $\sim$ 1982:

$\sigma(x)$ **transforms as a contravariant two-tensor**

$\Longrightarrow$ *infinite dimensional* loss of uniqueness

$n(x)$, $\epsilon(x)$, $\mu(x), \ldots$ transform in the *same way* as $\sigma(x)$

Basis for "**transformation optics**" (Ward-Pendry, ... $\geq$ 1996).

- If $F : \Omega \longrightarrow \Omega$ is a C^∞ smooth transformation, then

$\sigma(x)$ **pushes forward** to give a new conductivity, $\tilde{\sigma} = F_*\sigma$,

$$(F_*\sigma)^{jk}(y) = \frac{1}{\det[\frac{\partial F^j}{\partial x^k}(x)]} \sum_{p,q=1}^{n} \frac{\partial F^j}{\partial x^p}(x) \frac{\partial F^k}{\partial x^q}(x) \sigma^{pq}(x)$$

with the RHS evaluated at $x = F^{-1}(y)$.

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• If F is a diffeomorphism, then

$$\nabla(\tilde{\sigma} \cdot \nabla)\tilde{u} = 0 \iff \nabla(\sigma \cdot \nabla)u = 0,$$

where $u(x) = \tilde{u}(F(x))$. Thus, if F fixes $\partial\Omega$, then

$$\Lambda_{\tilde{\sigma}} = \Lambda_\sigma$$

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Best uniqueness result can hope for is:

"Theorem" If $\Lambda_{\tilde{\sigma}} = \Lambda_\sigma$, then there is an $F : \Omega \longrightarrow \Omega$ such that $\tilde{\sigma} = F_* \sigma$.

$n = 2$: True: Sylvester,..., Astala-Lassas-Päivärinta: $\sigma \in L^\infty$

$n \geq 3$: True for $\sigma \in C^\omega$; even $\sigma \in C^\infty$ major open problem.

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BUT: All of the above is under the assumption that σ is bounded above and below.

SO: *Cloaking has to violate this.*

Counterexamples to uniqueness for the Calderón problem
Lassas, Taylor, Uhlmann (2003) ; G., Lassas, Uhlmann (2003)

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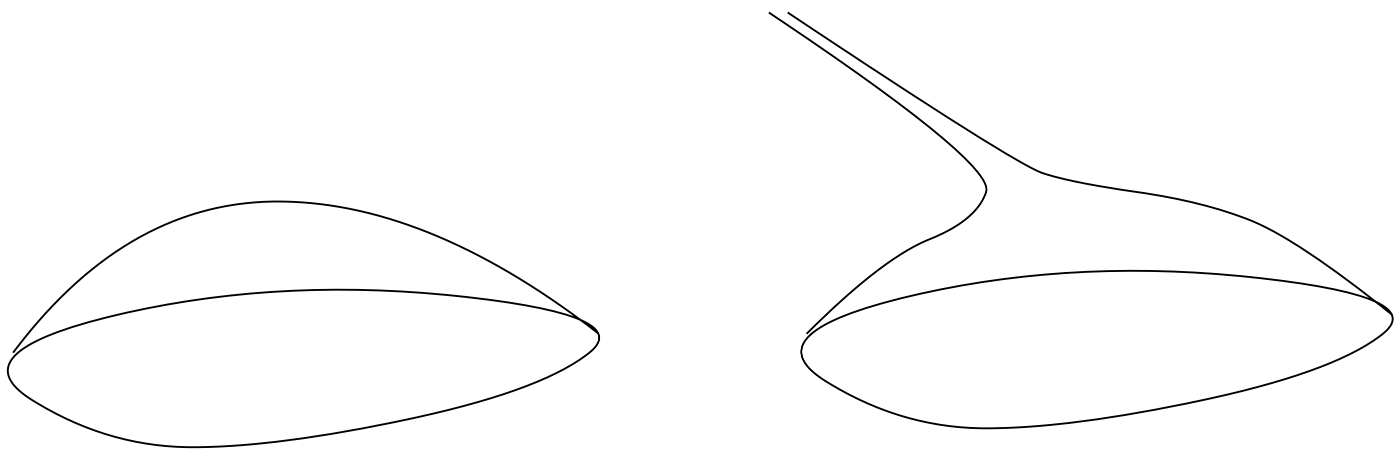
N.B. Conductivity equation no longer uniformly elliptic.

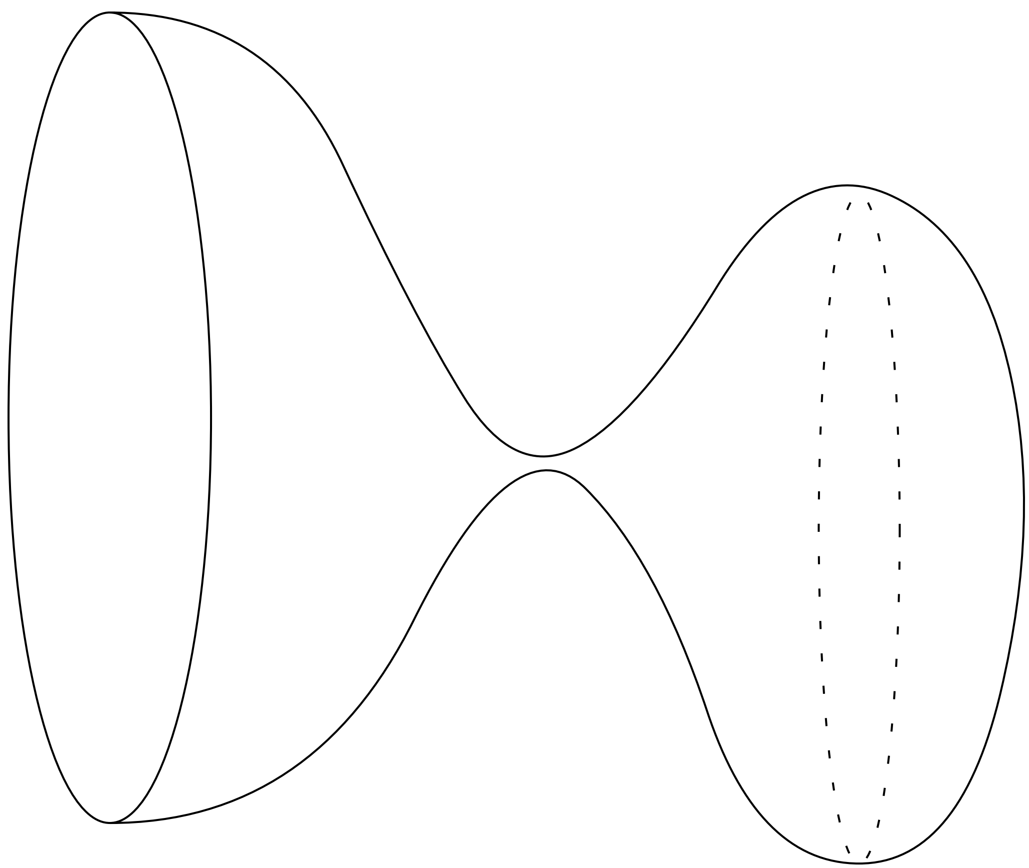
Need to replace the D2N map with the set of **Cauchy data**:

$$\left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial\nu}|_{\partial\Omega} \right) : u \text{ ranges over a space of solutions } \right\}$$

$\leftrightarrow$ "Boundary measurements" rather than scattering data.



Lassas, Taylor and Uhlmann



Greenleaf, Lassas and Uhlmann

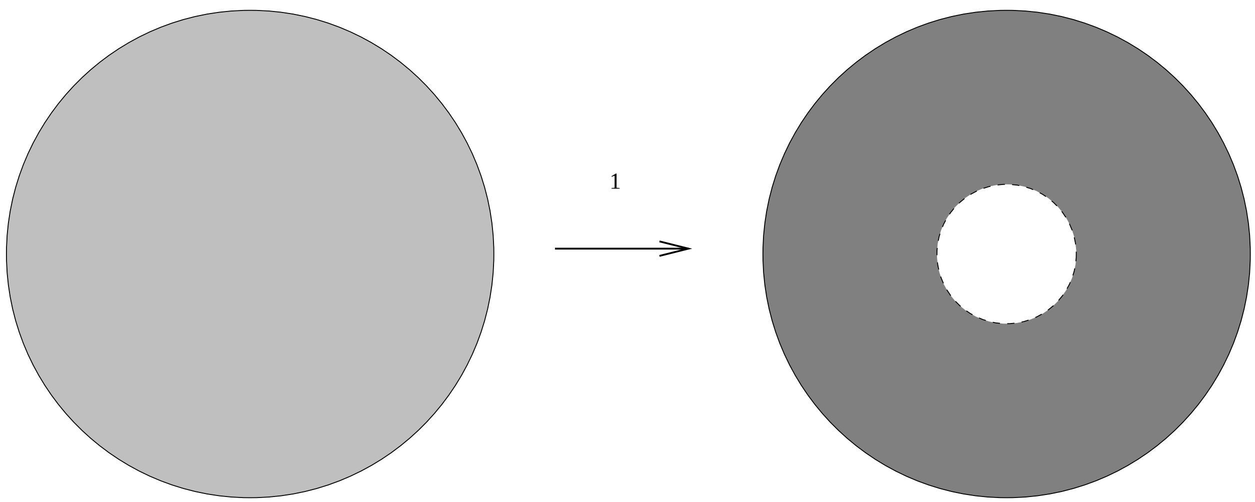
“Anisotropic conductivities that cannot be detected by EIT”

- $\Omega = B(0,2) \subset \mathbb{R}^3$, $\partial\Omega = S(0,2)$
- $D = B(0,1)$ region to be cloaked

“Anisotropic conductivities that cannot be detected by EIT”

- $\Omega = B(0,2) \subset \mathbb{R}^3$, $\partial\Omega = S(0,2)$
- $D = B(0,1)$ region to be cloaked
- Start with homogeneous, isotropic conductivity σ on Ω
- Blow up a point, say $0 \in \Omega \hookrightarrow$ singular, anisotropic $\tilde{\sigma}_1$ on $\Omega \setminus D$.

  One eigenvalue is $\sim (r-1)^2$, the other two $\in [c_1, c_2]$.
- Fill in the “hole” with any nonsingular conductivity $\tilde{\sigma}_2$ on D
- $\tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2)$ form a conductivity on Ω,
  singular at ∂D^+, smooth at ∂D^- (**“single coating”**).



Singular transformation F - “Blowing up a point”

- Singular transformation $F : \Omega \setminus \{0\} \longrightarrow \Omega \setminus \overline{D}$,

$$F(x) = \left(1 + \frac{|x|}{2}\right) \frac{x}{|x|}$$

- Show that the Dirichlet problem for $\nabla \cdot (\tilde{\sigma} \nabla u)$ on Ω has a unique bounded L^∞ (weak) solution.

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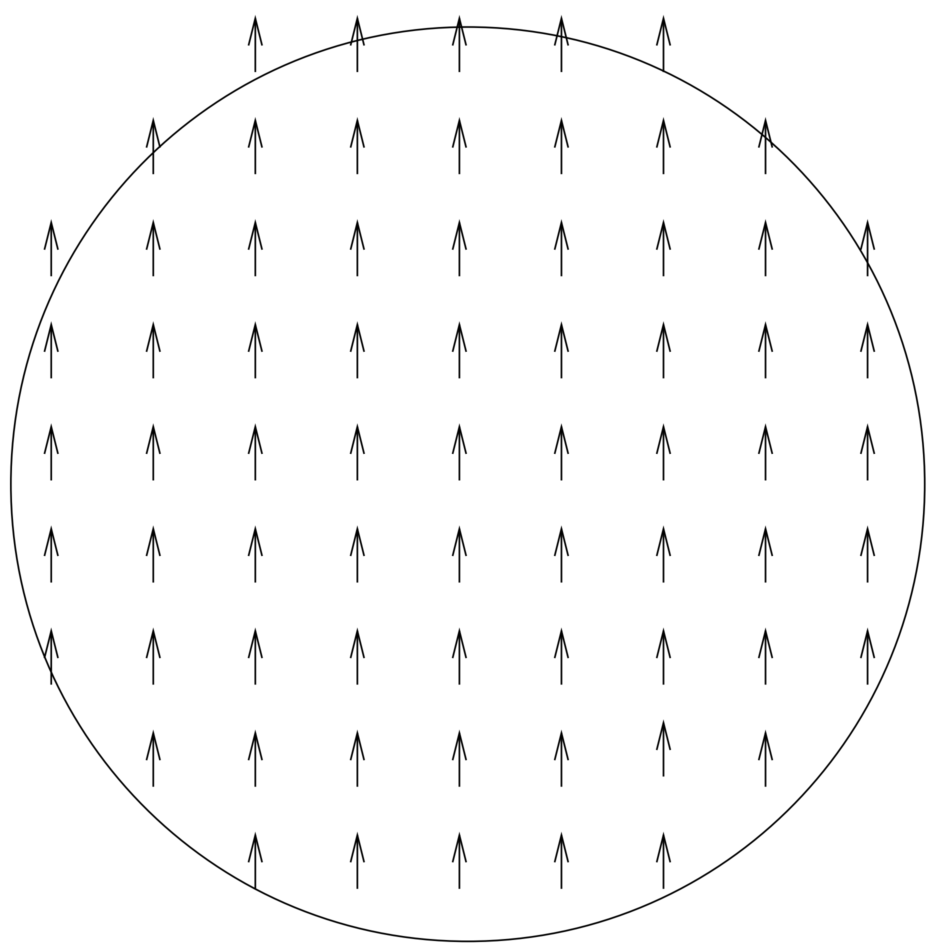
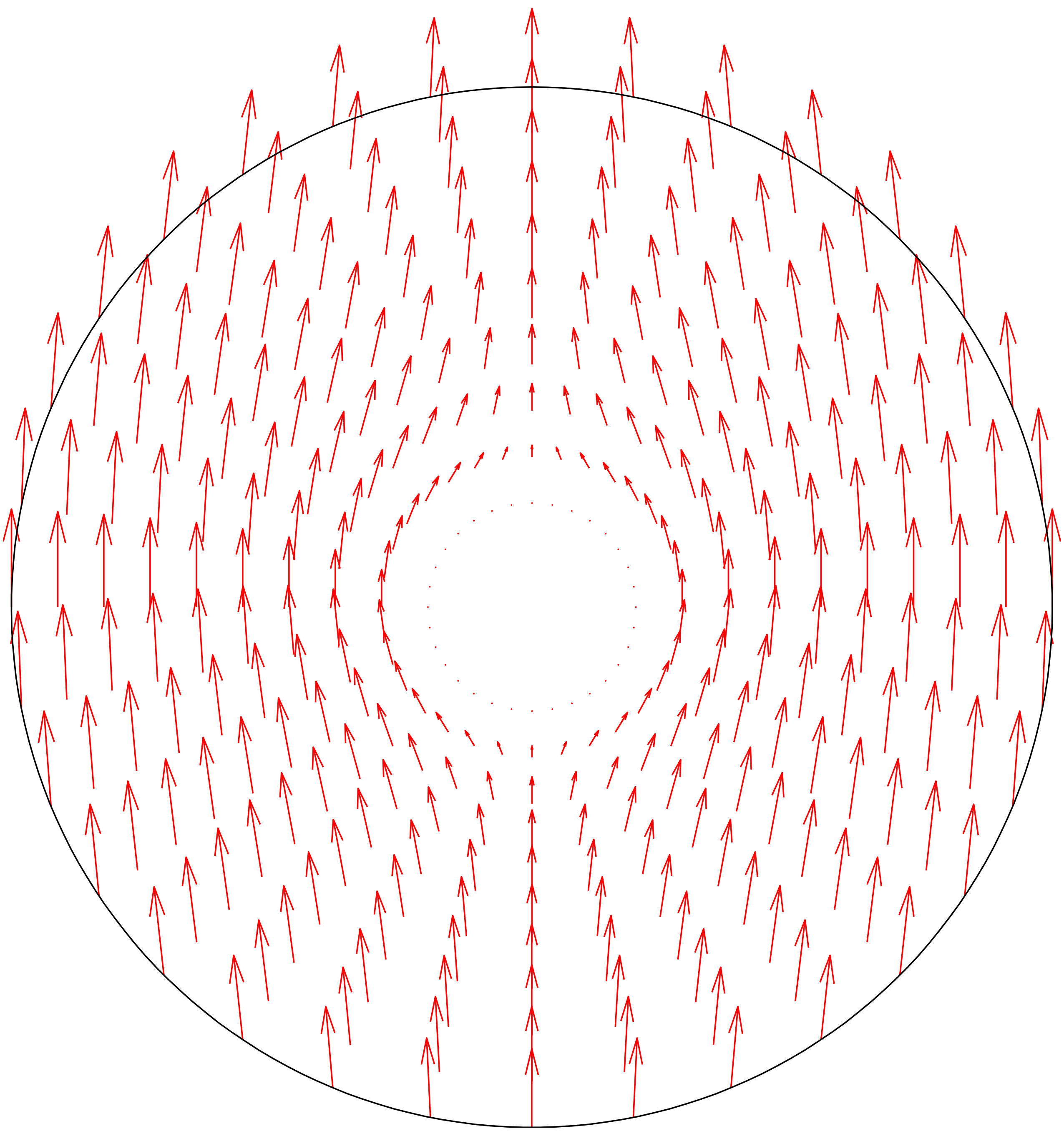
- $\implies$ $\tilde{\sigma}$ and σ have the same Cauchy data

- The material in D is invisible to EIT.

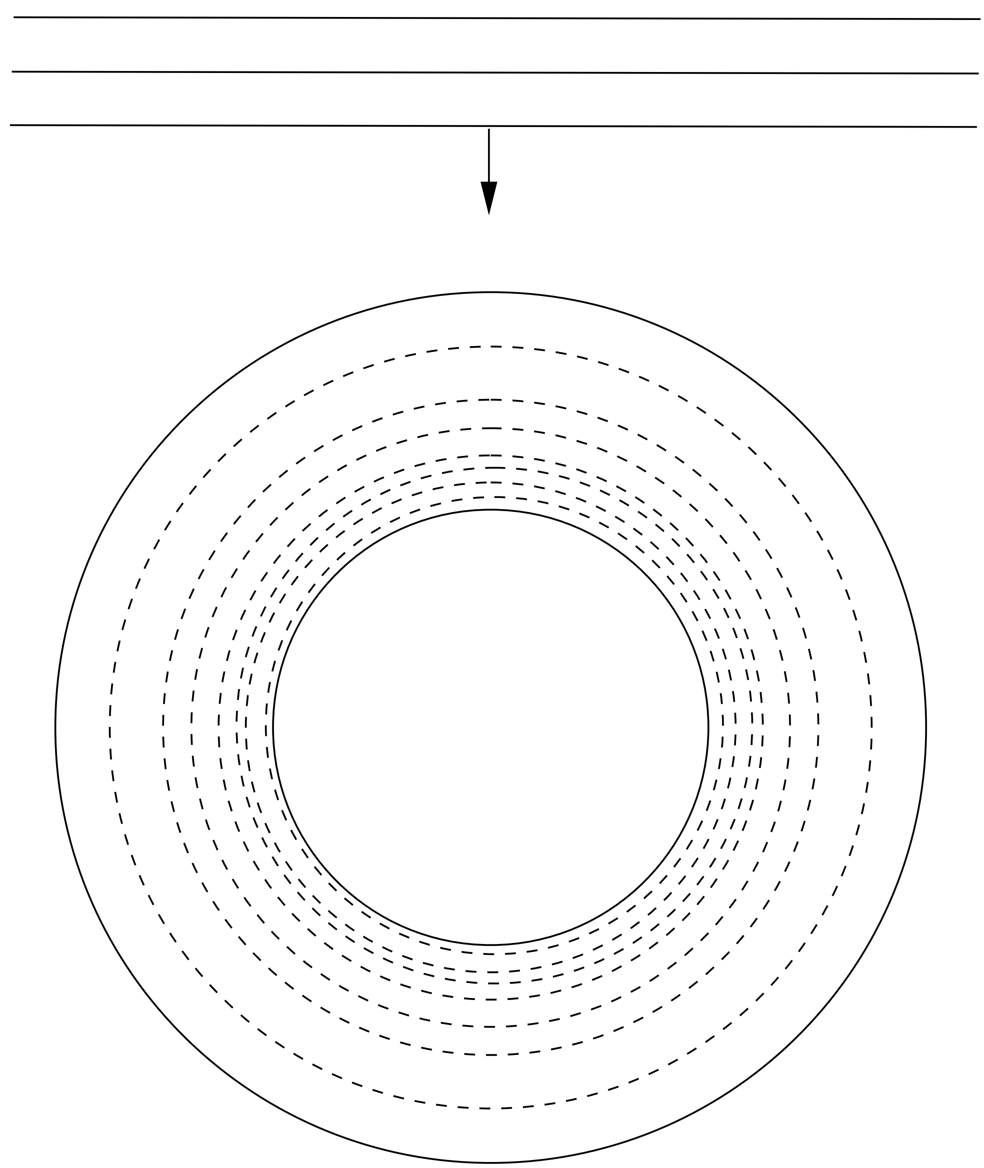
  ($\tilde{\sigma}_2$ can be any smooth nonsingular conductivity.)

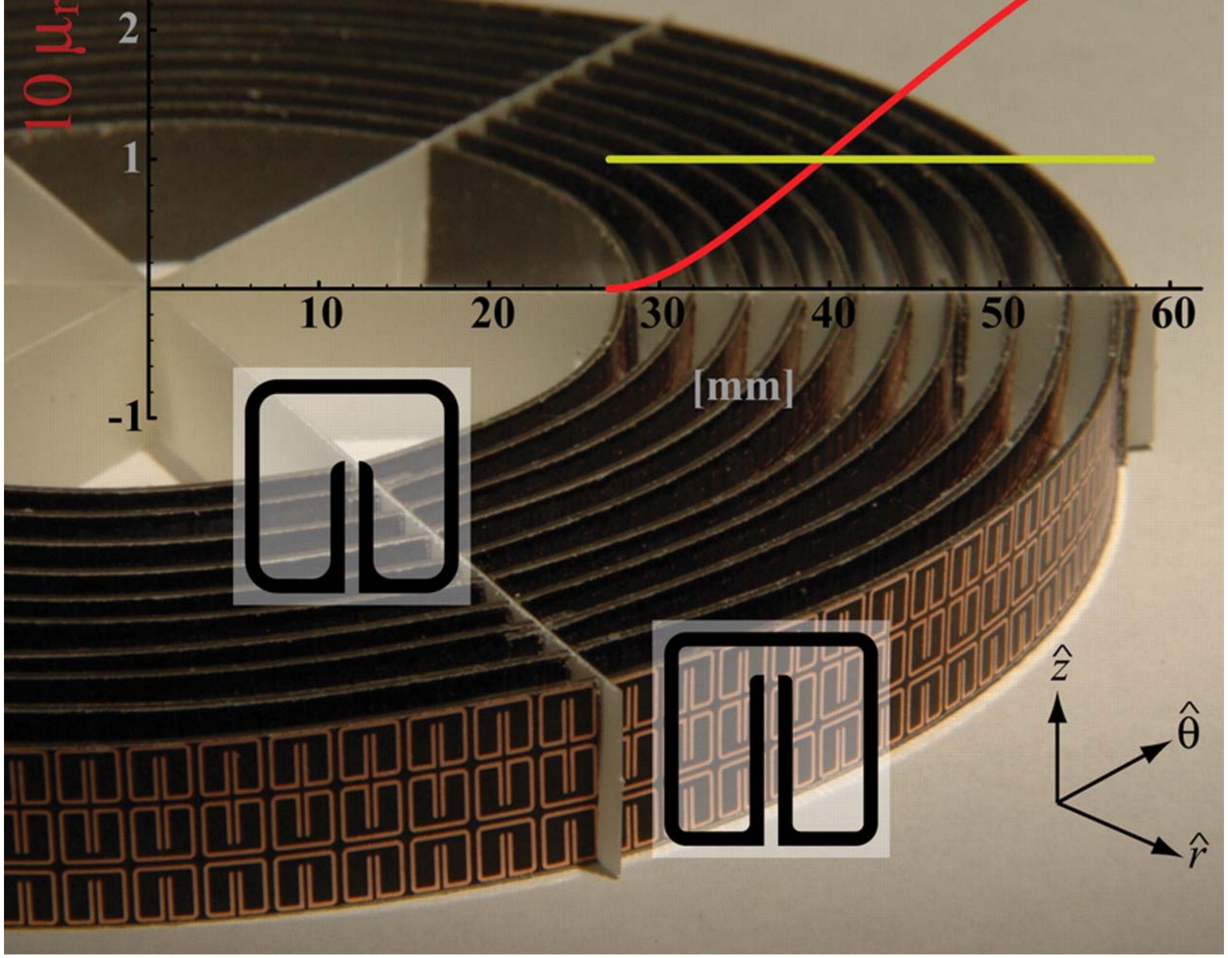




- This is the **same** construction that Pendry, Schurig and Smith (2006) used for Maxwell's equations.

- Electric **permittivity** $\epsilon(x)$ and magnetic **permeability** $\mu(x)$ transform in same way as conductivity $\sigma(x)$





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  (1) Specify material parameters (ϵ, μ) or active objects inside D.
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Q: Can cloaking be made rigorous?

Rigorous treatment $\Longrightarrow$ Hidden boundary conditions

- Need to understand fields on **all** of Ω, not just $\Omega \setminus \overline{D}$
- Singular equations $\Longrightarrow$ need to use weak solutions
- Appropriate notion ($k > 0$): **finite energy, distributional solutions**
- **Prove:** FE solutions on Ω automatically satisfy certain boundary conditions at the cloaking surface ∂D
- **Bonus:** this has real world implications for improving the effectiveness of cloaking.

Physics on a Riemannian manifold

- In dimensions $n \geq 3$,

  Conductivity tensors σ (and $\epsilon, \mu, \dots$) $\leftrightarrow$ Riemannian metrics g:

$$\sigma^{jk} = |g|^{1/2} g^{jk} \leftrightarrow \quad g^{jk} = |\sigma|^{2/(n-2)} \sigma^{jk}$$

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- Certain cases of Maxwell's equtions can be expressed in terms of *one* metric
- Permittivity $\epsilon(x)$ and permeability $\mu(x)$ assumed conformal to g
- Holds for **isotropic media** and their **pushforwards**.

Helmholtz equation at positive frequency

- $(\Delta_g + k^2)u(x) = \rho(x)$ on Ω, with source/sink ρ
- Under diffeomorphism F,

$$g \longrightarrow \tilde{g}, \qquad \tilde{u} \leftrightarrow u = \tilde{u}(F(x))$$

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- [GKLU] (2006) show that for FE distributional solutions,
this correspondance extends to singular cloaking transformation F

- Cloaking "works" for active objects for Helmholtz: scalar optics, acoustics, QM

- FE solution **must** satisfy Neumann condition

$$\frac{\partial u}{\partial \nu}|_{\partial D^-} = 0$$

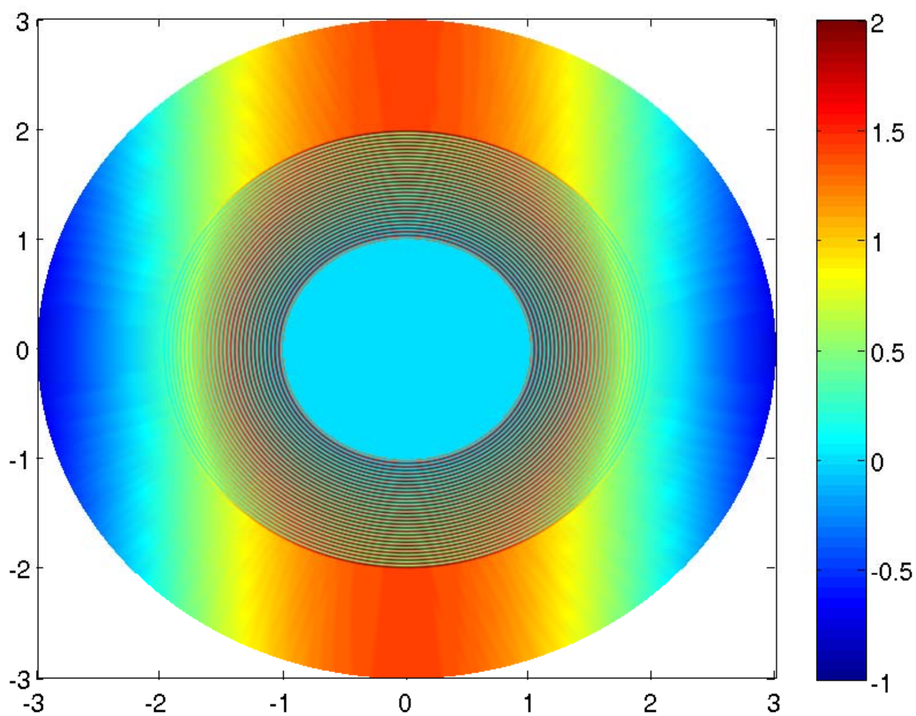
I.e., the cloaking structure on the exterior $\Omega \setminus D$
makes ∂D^- appear to be **perfectly reflecting** (sound hard)
from the inside of D. $\longrightarrow$ "Virtual surface"

Cloaking vs. trapped states

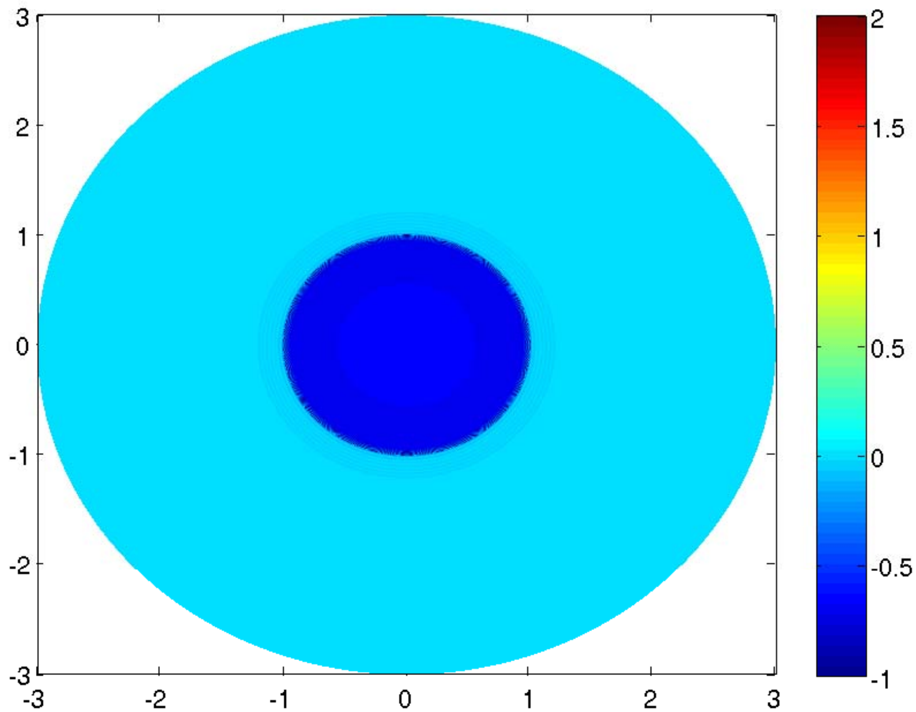
- If $-k^2$ is **not** a Neumann eigenvalue of $(D, \tilde{g}_2)$, waves cannot penetrate ∂D, and $u \equiv 0$ on D: *cloaking works as advertised*

- If $-k^2$ **is** an eigenvalue, then $\exists$ solutions $u \equiv 0$ on $\Omega \setminus D$

  and $=$ a Neumann eigenfunction on D: *trapped states*



Wave passing cloak ($-k^2$ not an eigenvalue)



Trapped state ($-k^2$ an eigenvalue)

Maxwell's equations

$$\nabla \times E = ik\mu(x)H, \quad \nabla \times H = -ik\epsilon(x)E + J$$

- E, H 1-forms; J internal current, a 2-form.

- Permittivity $\epsilon(x)$, permeability $\mu(x)$ 2-tensor fields.

- Can be rewritten in terms of analysis w.r.t a Riemannian metric g

- Under a diffeomorphism F, push $g \longrightarrow \tilde{g}$, $\epsilon \longrightarrow \tilde{\epsilon}$, $\mu \longrightarrow \tilde{\mu}$;
solutions $(\tilde{E}, \tilde{H}) \leftrightarrow (E, H)$

- Again, for FE solutions this extends to singular F

- BUT...

- Hidden boundary conditions at ∂D are **now**

$$\nu \times \tilde{E}|_{\partial D^-} = \nu \times \tilde{H}|_{\partial D^-} = 0$$

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- $J = 0$ is O.K., single coating cloaks **passive** objects

  But: highly unstable, cf. Zhang, et al., and Ruan, et al.

- Similarly for cloaking an infinite cylinder.

- Weder: different notion of solution (von Neumann self. adj. extensions)

Can this be fixed? Yes:

(1) Insert physical surface (lining) at ∂D to kill angular components of $\tilde{E}, \tilde{H}$:

SHS (soft-and-hard surface)

or

(2) The **"double coating"**

Make the metric $\tilde{g}$ be singular from both sides of ∂D
$\Longleftrightarrow$ covering both inside and outside of ∂D
with appropriately matched metamaterials.

Approximate cloaking

Impossible to build device with **ideal** material parameters $\tilde{\epsilon}, \tilde{\mu}$.

Q. What happens to invisibility for a physically realistic cloaking device?

- Extreme values of ideal $\tilde{\epsilon}, \tilde{\mu}$ can't be attained

- Can only discretely sample smoothly varying ideal parameters

See also Ruan, Yan, Neff and Qiu (2007)

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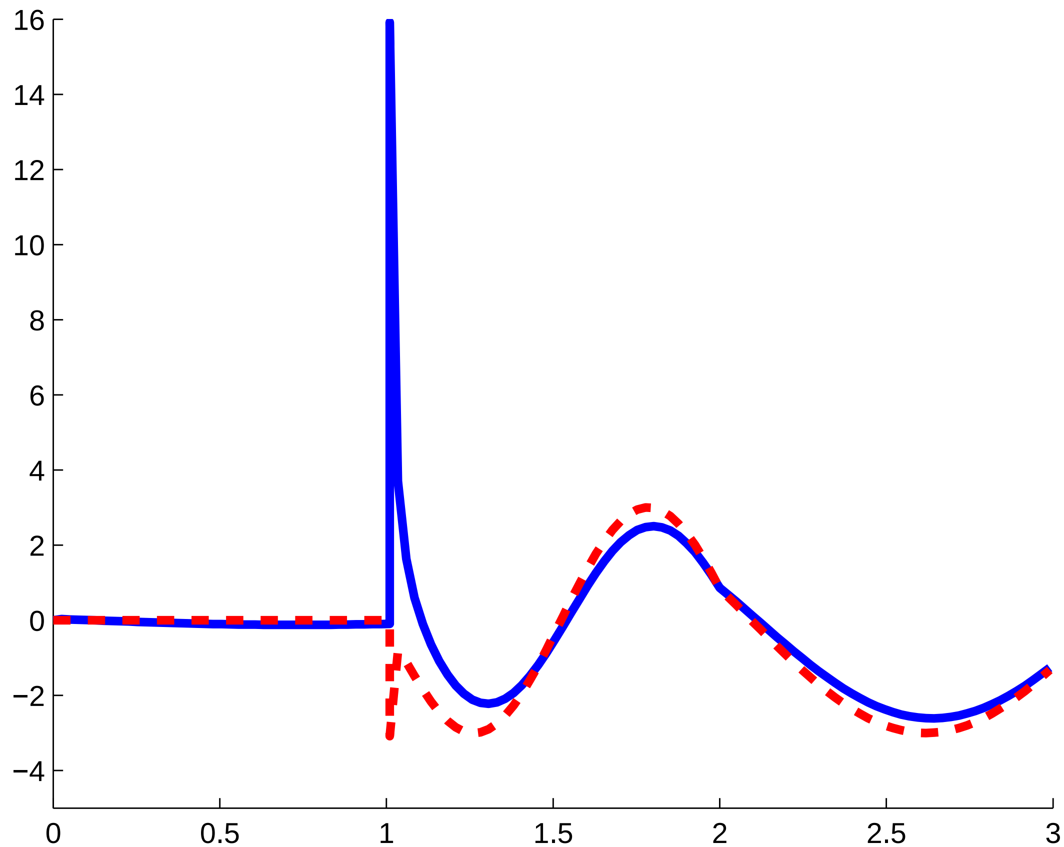
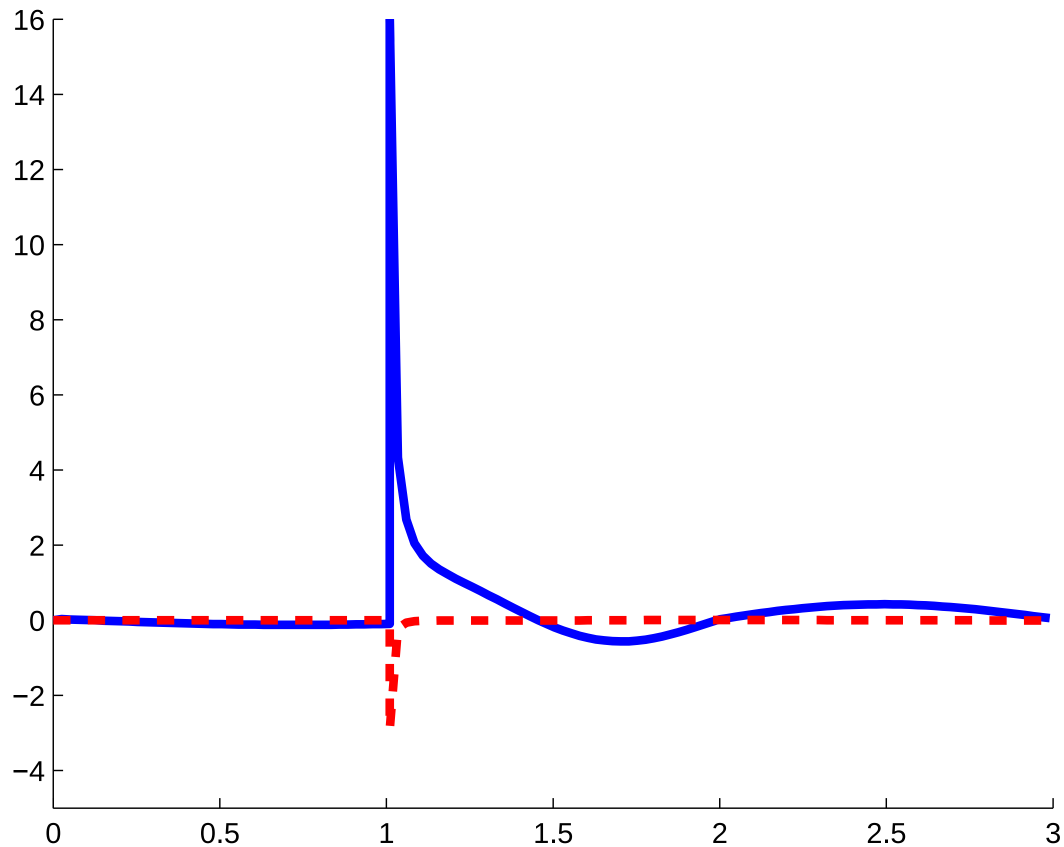
Soft-and-Hard Surface lining

Q. How does including SHS improve approximate cloaking?



Total fields: **(Blue)** without SHS, **(Red)** with SHS



Scattered fields: **(Blue)** without SHS, **(Red)** with SHS

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- So: **difficult** to actually build with current technology
- Cloaking; field rotator; beam splitter;...

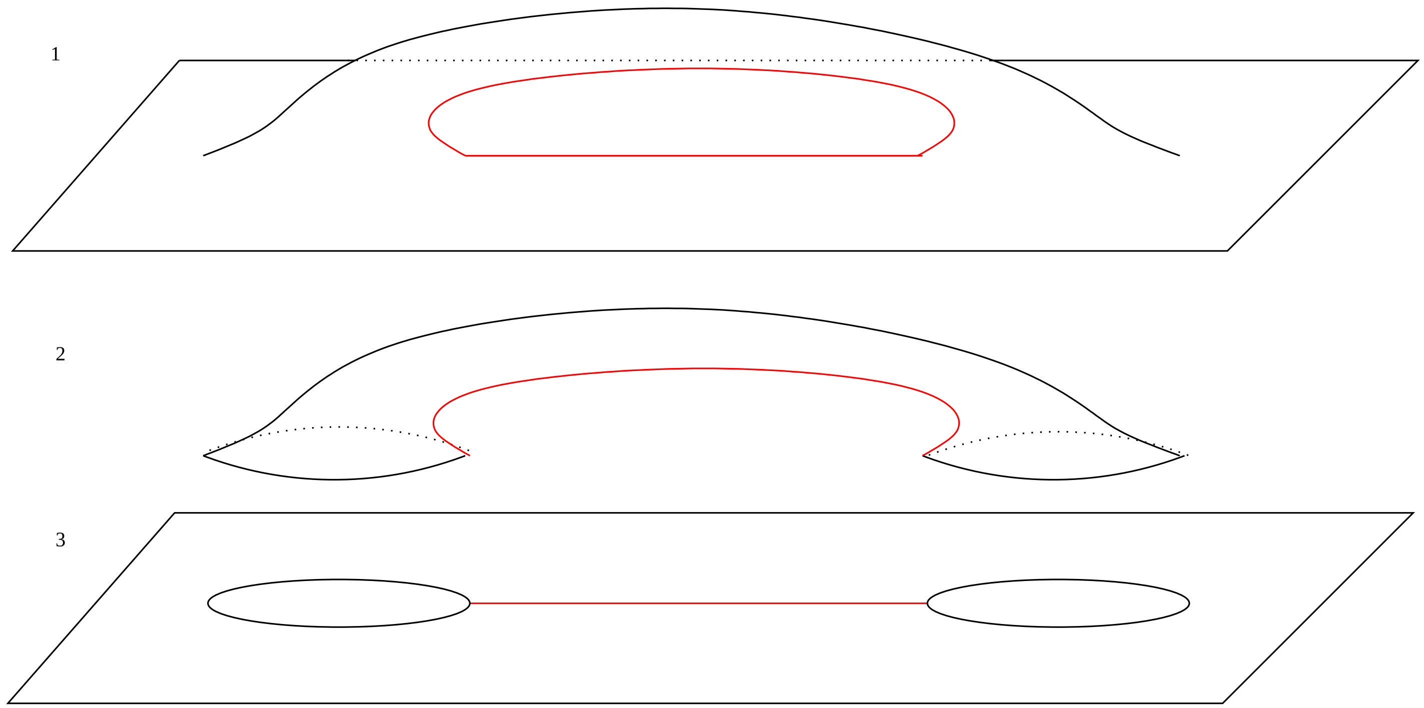
Electromagnetic wormholes [GKLU,2007]

- Invisible tunnels (optical cables, waveguides)

- Change the topology of space *vis-a-vis* EM wave propagation

- Based on "blowing up a curve" rather than "blowing up a point"

- Inside of wormhole can be varied to get different effects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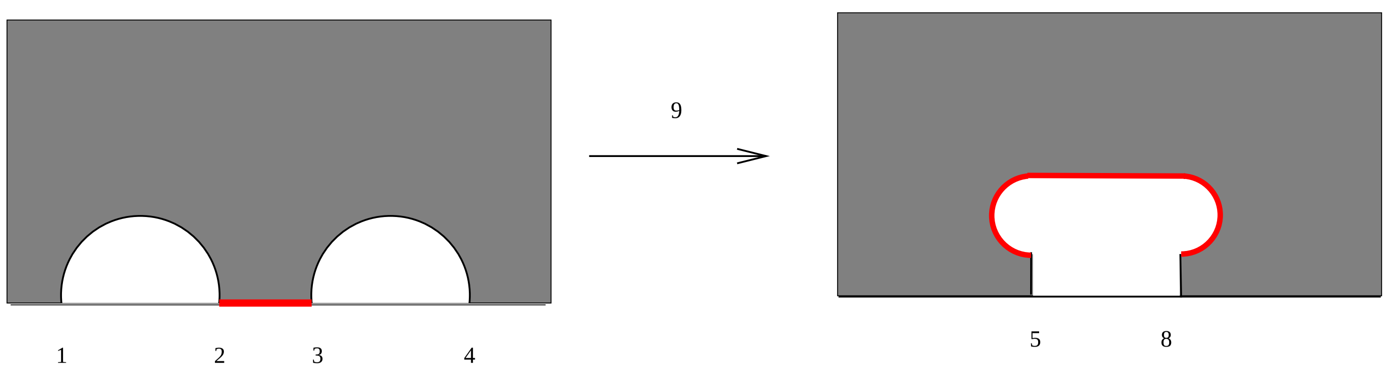
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