A Semi-classical inverse problem motivated by passive imaging in seismology

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The simplest semi-classical inverse spectral problem

Let us consider a 1D Schrödinger operator

$$\widehat{H}_\hbar = -\hbar^2 \frac{d^2}{dx^2} + V(x)$$

on some intervall I with self-adjoint boundary conditions and discrete spectrum $\lambda_1(\hbar) < \cdots < \lambda_n(\hbar) < \cdots$.

Does the semi-classical spectrum $\{\lambda_n(\hbar) | n = 1, \cdots\}$ mod $O(\hbar^\infty)$ determines the potential V?

The answer is YES under some genericity assumptions.

Before discussing the main result, I will quickly review some motivations coming from **passive imaging in seismology**.

Topics

1. (Quick presentation of) Passive imaging:

   correlations of noisy fields give the Green function

2. (Quick presentation of) “Classical propagation” in wave guides:

   effective Hamiltonian’s from a spectral problem

3. A semi-classical inverse spectral problem

I. Motivation:

a very short review of
the method of passive imaging in seismology

A. The classical method in seismology

uses waves created by an earthquake or an explosion. These waves propagate inside the earth and propagation times allow to get some knowledge of the earth structure. This method has some intrinsic limitations:

- non seismic areas
- the power generated by explosives is limited!

B. The method of passive imaging (Michel Campillo (LGIT, Grenoble) and co-workers). **The goals: finding the geological structure of the earth crust; real time imaging and volcanoes eruption forecasting.**

1. Recording the **seismic noise** at the stations of a network. *The noisy field at a single point contains no information, but noises at different points are correlated.*

2. Computing the **time correlation functions** of noises recorded during a long time (months).

3. The correlation function $C_{A,B}(\tau)$ of the seismic waves at the points A and B is very similar to the signal observed at the

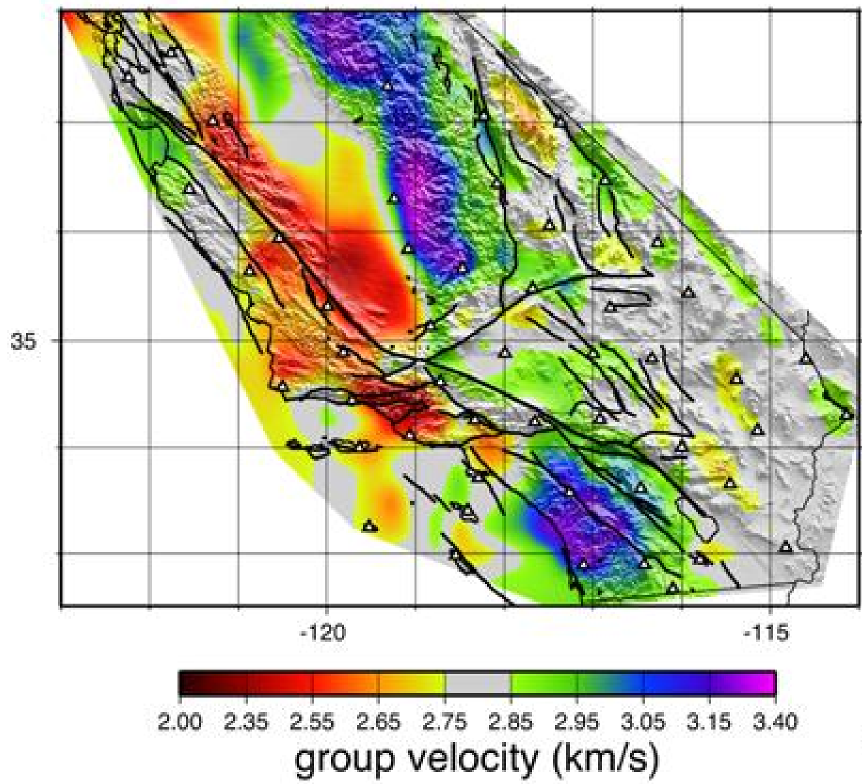
point A when an earthquake occurs at the point B and is propagated during a time τ: the **Green function**.

4. **Surface waves** give the main contribution:

   the time-correlation functions $C_{A,B}(\tau)$ of noisy elastic waves allows to know the velocity maps of surface waves for each frequency (in some windows).

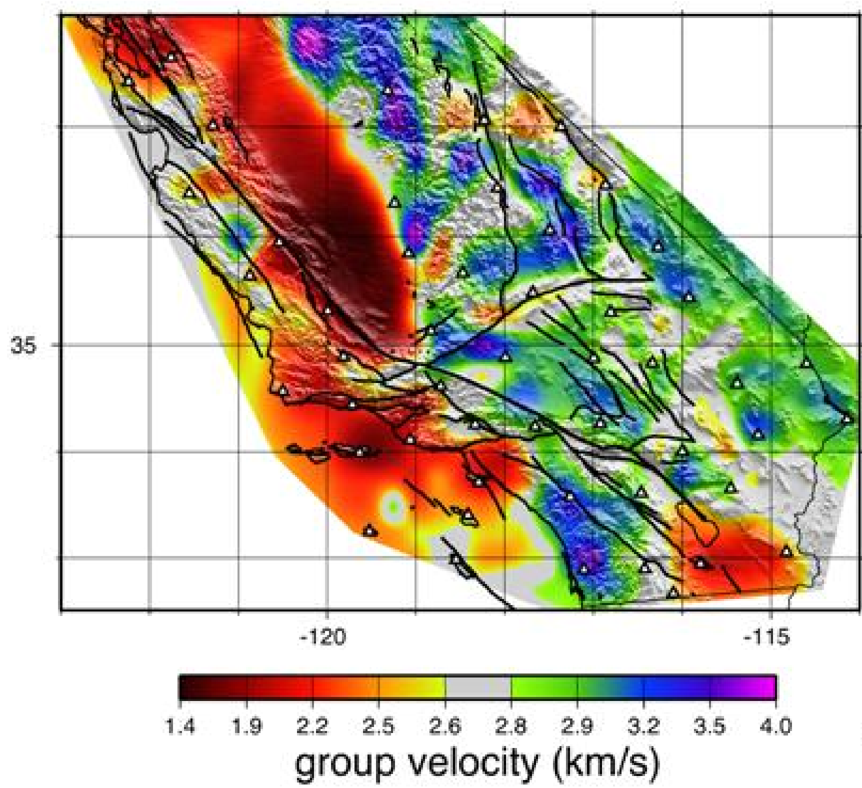
   More precisely, they give the *dispersion relation* (the *effective Hamiltonian*) of surface waves.

High resolution velocity map obtained from noise (Rayleigh 15 s ~ middle crust)



Main geological features

High resolution velocity map obtained from noise (Rayleigh 7.5 s)



Main geological features

C. Passive imaging: a mathematical statement

$$u_t + \widehat{H}u = f \tag{1}$$

- $u = u(x,t)$ the field
- $x \in X$, X a *smooth manifold* of dimension d with a smooth measure $|dx|$
- $\widehat{H}$ the generator of the free dynamics is acting linearly on $L^2(X)$. It satisfies some attenuation property: if we define the semi-group $\Omega(t) = \exp(-t\widehat{H}), t \geq 0$, there exists $k > 0$, so that we have the estimate $\|\Omega(t)\| = 0(e^{-kt})$.

- $f(x,t)$, the source of the noise is a random field assumed to be *stationary in time* and *ergodic.* We will write

$$K(s-s',x,y) := \mathbb{E}\left(f(x,s)\overline{f(y,s')}\right)$$

the *covariance* kernel of f. For simplicity, we will assume that $K(t,x,y) = L(x,y)\delta(t=0)$.

This simple model can be easily generalized to usual wave equation: just write it with vector valued fields as usual.

$$u_{tt} + a(x)u - \Delta u = f$$

$$\mathbf{u} = \begin{pmatrix} u \\ u_t \end{pmatrix}, \mathbf{f} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \widehat{H} = \begin{pmatrix} 0 & -1 \\ \Delta & a \end{pmatrix} ,$$

$$\mathbf{u}_t + \widehat{H}\mathbf{u} = \mathbf{f} \ .$$

The solution of Equation (1) with $f \equiv 0$, $u(t) = \Omega(t)(u(0))$, can be written as

$$(\Omega(t)u)(x) = \int_X P(t,x,y)u(y)|dy| \ .$$

$P(t,x,y)$, the Schwartz kernel of $\Omega(t)$, is called the **propagator**. It satisfies:

$$\int_X P(t,x,y)P(t',y,z)|dy| = P(t+t',x,z) \ .$$

The **causal** solution of Equation (1) is:

$$u(x,t) = \int_0^\infty ds \int_X P(s,x,y) f(y, t-s) |dy| \tag{2}$$

The kernel $Y(s)P(s,x,y)$ is called the **Green function.**

The **correlation** of 2 complex fields $\varphi(t)$ and $\psi(t)$ is defined by:

$$C_{\varphi,\psi}(\tau) := \lim_{T \to +\infty} \frac{1}{T} \int_0^T \varphi(t)\overline{\psi(t-\tau)}dt \ .$$

The correlation of the fields at A and B is then given for $\tau > 0$, by

$$C_{A,B}(\tau) = \int_0^\infty ds \int_{X \times X} |dx||dy| P(s+\tau, A, x) L(x,y) \overline{P}(s, B, y) \quad (3)$$

and $C_{A,B}(-\tau) = \overline{C_{B,A}(\tau)}$.

Some use-full notations:

$[A](x,y)$ is the Schwartz kernel of the operator A.

$\hat{a}$ is the operator of Schwartz kernel $a(x,y)$.

We get the nicer formula:

$$\begin{array}{l} \text{for } \tau > 0,\ C_{A,B}(\tau) = [\Omega(\tau)\Pi](A,B) \\ \text{with } \Pi = \int_0^\infty \Omega(s)\hat{L}\Omega^\star(s)ds \end{array} \qquad (4)$$

Recall that $\hat{L}$ is defined from:

$$L(x,y)\delta(t-t') = \mathbb{E}(f(x,t)\overline{f}(y,t'))\ .$$

For simplicity, we will consider the case where the source is a **white noise**

A white noise of an Hilbert space $(\mathcal{H}, \langle .|. \rangle)$ is a Gaussian random field f whose correlation satisfies: $\mathbb{E}(\langle f|v\rangle \overline{\langle f|w\rangle}) = \langle w|v\rangle$.

If we assume

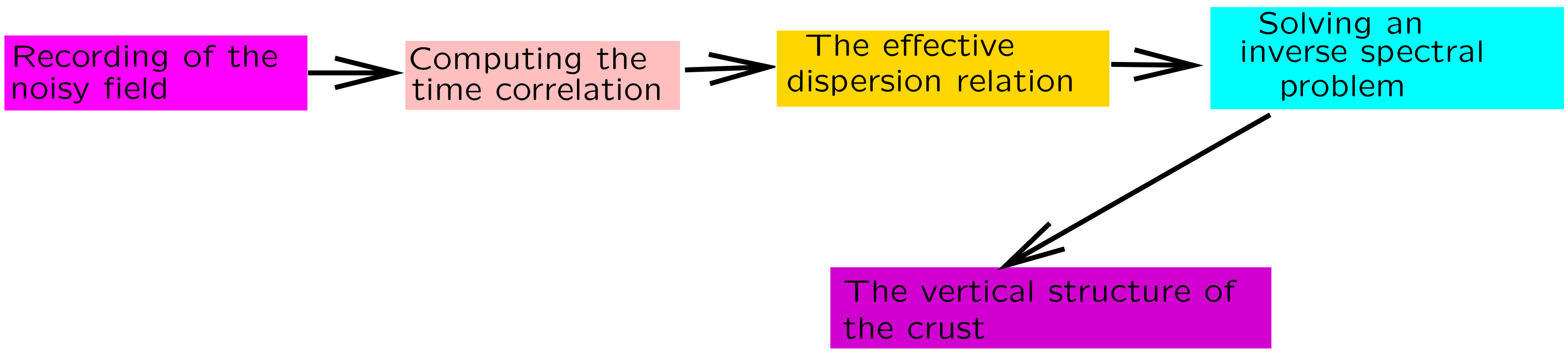
- f a white noise
- $\widehat{H} = \widehat{H}_0 + k$ with $k > 0$ a constant and $\widehat{H}_0$ anti-self-adjoint with unitary propagator P_0,

we get, for $\tau > 0$, an exact formula:

$$C_{A,B}(\tau) = \frac{1}{2k} P_0(\tau, A, B) \ .$$

More general sources can be worked out using semi-classical analysis: the correlation is then given by a FIO with the same canonical relation as the Green function. From this, we can always recover the classical dynamics of surface waves.

The scheme of the reconstruction:



II. Effective Hamiltonian's for waves guides

For seismic waves, the earth crust acts usually as a wave guide because the speed of elastic waves is increasing with the depth. A simple academic model will be the following acoustic (scalar) wave equation in the half space $\{(\mathbf{x}, z) \in \mathbb{R}^2 \times \mathbb{R} \mid z \leq 0\}$:

$$(\star) \begin{cases} u_{tt} - \text{div}\,(\, N(\mathbf{x}, z)\ \text{grad}\, u) = 0 \\ u(t, \mathbf{x}, 0) = 0 \end{cases}$$

Let us assume that

- $N(\mathbf{x}, z) \to +\infty$ as $z \to -\infty$ (large velocity at infinity),
- $N_0(\mathbf{x}) := \inf_{z \leq 0} N(\mathbf{x}, z) > 0$

A. N independent of $\mathbf{x}$

Let $\lambda_j(\xi) > N_0\|\xi\|^2$ be an eigenvalue of

$$L_\xi = -\frac{d}{dz}N(z)\frac{d}{dz} + N(z)\|\xi\|^2$$

(with Dirichlet boundary condition).

Then $u(\mathbf{x}, z, t) := e^{i(\langle \mathbf{x}|\xi\rangle - \omega t)}\varphi_j(z)$ is a solution of Equ. $(\star)$. Using an $\mathbf{x} \leftrightarrow \xi$ Fourier transform, reduces Equ. $(\star)$ to $v_{tt} - \Lambda_j v = 0$ (with $v = v(t, \mathbf{x})$) where Λ_j is a ΨDO of symbol λ_j.

The effective classical dynamics on the boundary is given by the dispersion relation

$$\omega^2 - \lambda_j(\xi) = 0 \; .$$

B. $N = N(\varepsilon\mathbf{x}, z)$ is slowly dependent of $\mathbf{x}$

If $\lambda_j(\varepsilon\mathbf{x}, \xi)$ is an eigenvalue of $L_{\varepsilon\mathbf{x},\xi}$, adiabatic theory allows to show that the effective dynamics is given in the variables

$$(\mathbf{X} = \varepsilon\mathbf{x}, \xi)$$

by the dispersion relation

$$\omega^2 - \lambda_j(\mathbf{X}, \xi) = 0 \ .$$

Conclusion:

The j^{th} mode of the surface waves have a classical dynamics given by the Hamiltonian's $\pm\sqrt{\lambda_j(\mathbf{X},\xi)}$. It leads to the following 2 natural inverse spectral problems:

1. Does the function $\lambda_1(\mathbf{X},\xi)$ determines $N(\mathbf{X},z)$?
   (this was asked by Bernard Helffer)

2. Does the knowledge of all $\lambda_j(\mathbf{X},\xi)$'s for ξ large (the semi-classical spectrum) determine $N(\mathbf{X},z)$?

We will answer the second question for a Schrödinger equation. The first one is maybe more difficult!

III. A semi-classical inverse problem

$$\widehat{H}_\hbar = -\hbar^2 \frac{d^2}{dx^2} + V(x) \ .$$

- $-\infty \leq a < b \leq +\infty$ and $V : I =]a, b[\to \mathbb{R}$ smooth
- $-\infty < \inf V = E_0 < E_\infty := \liminf_{x \to \partial I} V(x)$
- Self-adjoint boundary conditions

The spectrum of $\widehat{H}_\hbar$ is discrete in $]-\infty, E_\infty[$:

$$(E_0 <)\lambda_1(\hbar) < \lambda_2(\hbar) < \cdots < \lambda_n(\hbar) < \cdots (< E_\infty) \ .$$

We will denote by $H = \xi^2 + V(x)$ the classical Hamiltonian.

Can one recover V in the domain $\{V(x) < E_\infty\}$ from the spectra $\sigma(\hat{H}_\hbar) \cap]-\infty, E_\infty[$ modulo $O(\hbar^\infty)^*$?

*These spectra are mod $O(\hbar^\infty)$ independent of the boundary conditions

Theorem 1 Let us assume that V satisfies the generic **conditions (A), (B) and (C) and let $E < E_\infty$. Then V can be explicitly reconstructed in $I_E = \{x | V(x) \leq E\}$ modulo trivial moves (symmetry-translation) from the semi-classical spectrum**

$$\sigma(\widehat{H}_\hbar) \cap]-\infty, E[$$

modulo $o(\hbar^3)$.

Condition (A) : parity defect

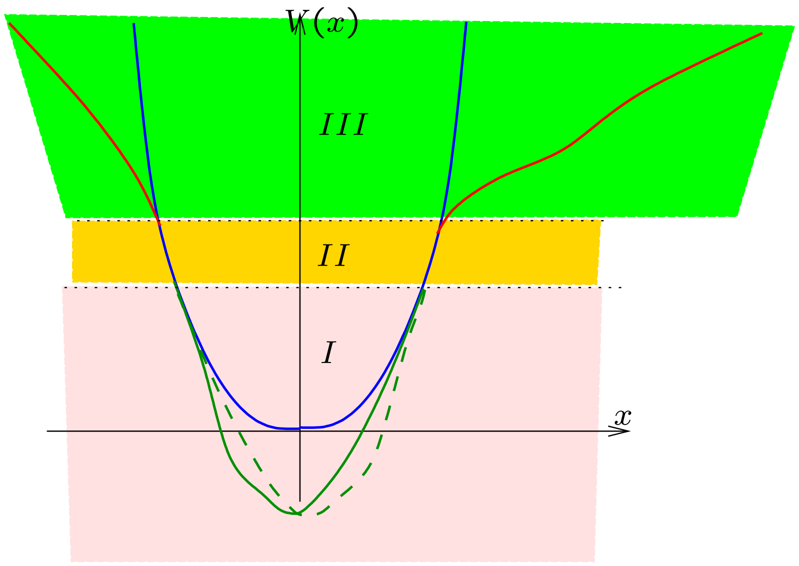
If $x_- < x_+$ satisfy

$$\forall n = 0, 1, \cdots, \; V^{(n)}(x_-) = (-1)^n V^{(n)}(x_+)$$

then V is even w.r. to $x_+ + x_-/2$.

(True if V is analytic)

2 potentials with one well and the same semi-classical spectra



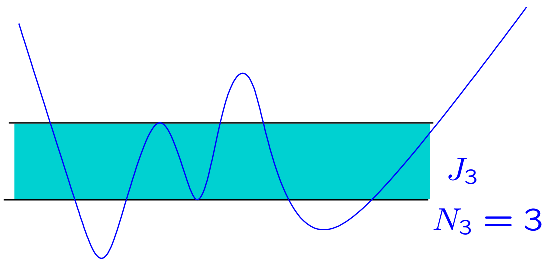
Condition (B) : critical points

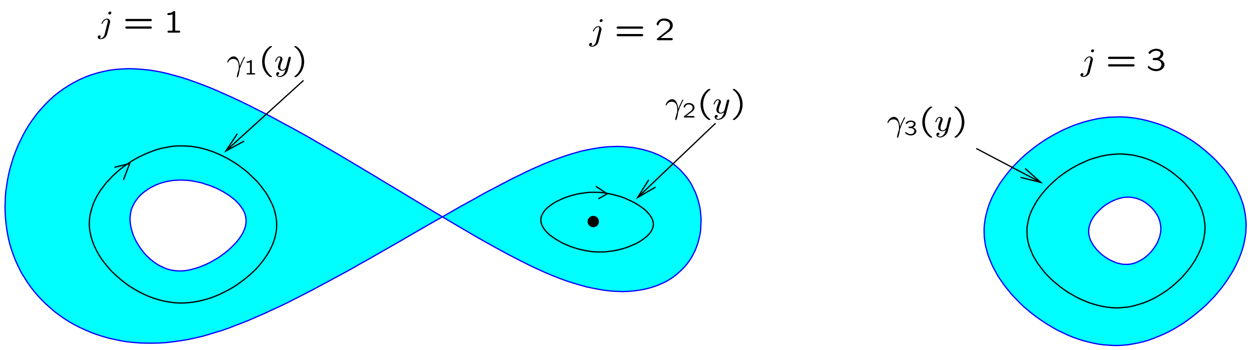
- *Weakly non degenerate:* if $V'(x) = 0$, there exists n with $V^{(n)}(x) \neq 0$ (true if V is Morse).

- *Pairwise distinct critical values.*

Let $E_0 < E_1 < \cdots < E_k < \cdots < E_\infty$ be the critical values and let $x_0,\ x_1, \cdots$ the corresponding critical points. Let $J_k =]E_{k-1}, E_k[$. **The N_k wells of order k** are the connected components of $V(x) < E_k$. The index $j = 1, \cdots, N_k$ will label the wells of order k.

The phase space picture





Condition (C) : separation of the wells

If $T_{j_1}(y)$ and $T_{j_2}(y)$ are the **periods** of 2 wells of order k in $J_k =]E_{k-1}, E_k[$, they are *weakly transverse **, including at the point E_{k-1} if x_{k-1} is ND local minimum of V.

Z: Does not holds for quartic potentials

*2 smooth functions f and g are weakly transverse, if for each x with $f(x) = g(x)$, there exists n with $f^{(n)}(x) \neq g^{(n)}(x)$

Previous works

- Borg-Gelfand-Levitan-Marchenko (50' to 60'): 2 spectra, $\hbar = 1$
- David Gurarie (95'): similar semi-classical results with application to surfaces of revolution, but less precise statements
- Victor Guillemin and YCdV (2007): Taylor expansion at ND critical points with $V'''(x_k) \neq 0$.

Comments:

- Comparison with B-G-L-M:
  - B-G-L-M does not apply in the essentially self-adjoint case
  - asymptotic/exact,
  - part of the spectrum/whole spectrum,
  - explicit reconstruction/quite indirect reconstruction via the spectral function.

- Numerical reconstruction from spectra for 2 values of $\hbar$ (in progress).

The direct problem I: Weyl asymptotic

We have

$$\#\{\lambda_n(\hbar) \leq y\} \sim \frac{1}{2\pi\hbar} A(y)$$

With $A(y) = \text{Area}\left(\xi^2 + V(x) \leq y\right)$. With Assumption (B), the singularities of $A(y)$ are exactly the critical values E_k of V and the singularities determine $V''(x_k)$.

The direct problem II: Bohr-Sommerfeld rules (see YCdV, Ann. Henri Poincaré, 2005)

$y \in J_k$, $j = 1, \cdots N_k$ the wells. The *semi-classical action* of the $j-$th well is a formal power series in $\hbar$:

$$S_\hbar^j(y) \equiv S_0^j(y) + \pi\hbar + \sum_{l=1}^{\infty} \hbar^{2l} S_{2l}^j(y) \text{ mod } O(\hbar^\infty)$$

with

- $S_0^j(y) = \int_{\gamma_j(y)} \xi dx$ is the classical action, $(S_0^j)'(y) = T_j(y) = \int_{\gamma_j(y)} dt$,

- The next term is:

$$S_2^j(y) = -\frac{1}{12}\frac{d}{dy}\int_{\gamma_j(y)} V''|dt|$$

- All terms are of the form

$$S_{2l}^j(y) = \sum_N \left(\frac{d}{dy}\right)^N \int_{\gamma_j(y)} P_{l,N}(V', V'', \cdots)|dt| \ ,$$

with $P_{l,N}$ some universal polynomials.

The B-S rules are

$$S_{\hbar}^{j}(y) \in 2\pi\hbar\mathbb{Z}$$

The semi-classical spectrum in J_k is the union of spectra given by the B-S rules for the N_k wells.

The direct problem III: ΨDO Trace formulas

$f \in C_o^\infty(J_k)$ and $F(y) = -\int_y^{+\infty} f(z)dz$.

$$\text{Trace}F(\hat{H}) \equiv \frac{1}{2\pi\hbar}\left(\int_{T^\star I} F(H)dxd\xi + \hbar^2 \int_{J_k} f(y)\left(\sum_{j=1}^{N_k} S_2^j(y)\right)dy + \hbar^4 \cdots\right)$$

*The proof is a direct application of the calculus of the Weyl symbol of $F(\hat{H})$ using Moyal formula**

In the case of ONE well, this implies that $T(y)$ and $S_2(y)$ are determined by the spectrum modulo $o(\hbar^2)$.

*see Alfonso Gracia-Saz, Ann. IF, 2005, for explicit computations of the Weyl symbol of the function of a ΨDO.

The direct problem IV: Gutzwiller Trace formula

If $D_\hbar(y) = \sum \delta(\lambda_n(\hbar))$, as a *micro-function*[*] in $T^\star J_k$, we have

$$D_\hbar \equiv \frac{1}{2\pi\hbar} \sum_{j=1}^{N_k} \sum_{l\in\mathbb{Z}} D_l^j$$

with

$$D_l^j = (-1)^l e^{ilS_0^j(y)/\hbar} T^j(y)(1 + il\hbar S_2^j(y) + O(\hbar^2)) \ .$$

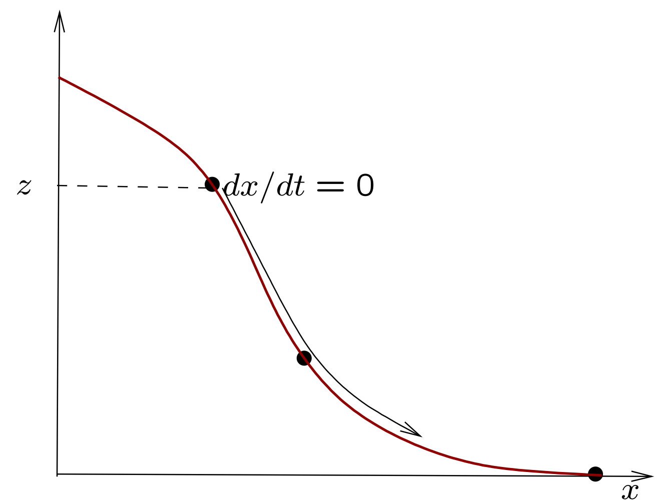
This is proved from the B-S rules via the Poisson summation formula.

[*] i.e. mod $O(\hbar^\infty)$ in the phase space, or both sides are equal modulo $O(\hbar^\infty)$ if we apply a ΨDO whose symbol is compactly supported

Main inputs: direct problem + N. Abel:

"Auflösung eine mechanichen Aufgabe" by Niels Abel (1826)

Abel solved the (classical) inverse "toboggan problem": finding the shape of a toboggan from the arrival times function $\tau(y)$



$H = \xi^2 + V(x)$ with $V :]-\infty, 0] \to \mathbb{R}^+$, smooth, decaying, $V(0) = 0$. Consider the Cauchy problem

$$\begin{cases} \dot{x} = 2\xi, \dot{\xi} = -V'(x) \\ x(0) = y,\ \dot{x}(0) = 0, \end{cases}$$

and define the arrival time $\tau(y)$ by $x(\tau(y)) = 0$.

Abel's Aufgabe: recovering V from the function τ.

$$\tau(y) = \frac{1}{2} \int_0^y \frac{W'(u)du}{\sqrt{y-u}} \ ,$$

with W the inverse function of V.

Solution: If we define

$$\mathcal{A}(f)(y) := \int_0^y \frac{f(x)dx}{\sqrt{y-x}}$$

we have:

$$\mathcal{A} \circ \mathcal{A}(f)(z) := \pi \int_0^z f(x)dx \ .$$

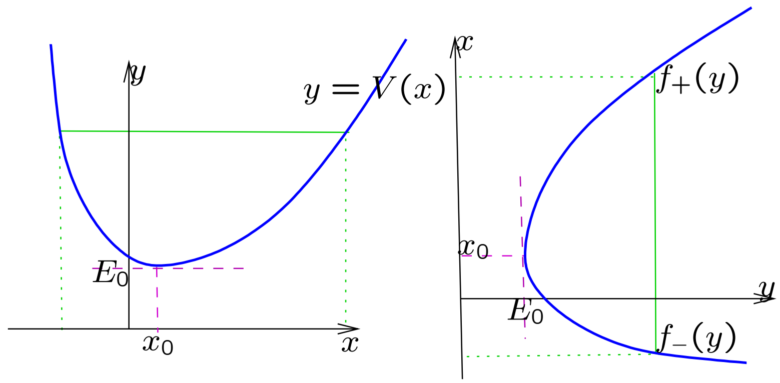
And hence $f = \pi^{-1}(\mathcal{A}^2 f)'$.

An immediate Corollary is that the **period $T(y)$ as a function of the energy** (the "classical spectrum") of an EVEN potential well determines the potential (because $T(y) = 4\tau(y)$).

This is no more true for an arbitrary potential!!

What the main result says is that this is true for the "semi-classical spectrum".

The case of one well: $E = E_1$



Let us define $F_{\pm} = \frac{1}{2}(f_{+} \pm f_{-})$. From the ΨDO trace formula, we recover S_0 and S_2, hence (because we know the limit of $\int_{\gamma(y)} V''(x)dt$ as $y \to E_0$), we know also the integrals

$$I(y) := T(y)/2 = \int_{E_0}^{y} \frac{F'_{-}(u)du}{\sqrt{y-u}}$$

and

$$J(y) := \int_{E_0}^{y} \frac{d}{du} \left(\frac{1}{f'_{+}(u)} - \frac{1}{f'_{-}(u)} \right) \frac{du}{\sqrt{y-u}} \ .$$

Using Abel's result, we recover F'_- and $(F'_+)^2$. Using Assumption A, we recover F_- and F_+ up to sign change. Hence, we recover V up to translation-symmetry.

The case of several wells

Assumption C allows to separate the spectra associated to different wells using the Gutzwiller formula and the:

Lemma 1 *If*

$$\sum_{j=1}^{N} a_j(y) e^{iS_j(y)/\hbar} = o(1)$$

in $L^2(J)$ and, if the functions $T_j(y) = S_j'(y)$ are weakly transverse, all a_j's vanish.

This allows to get the semi-classical actions S_0^j, S_2^j associated to the different wells from the spectra mod $o(\hbar^3)$. The proof is then by induction on k.

Several problems:

- Explicit approximate reconstruction of V from the spectra of $\widehat{H}_{\hbar_1}$ and $\widehat{H}_{\hbar_2}$ with $\hbar_j$ small,

- Extension to matrix potentials,

- What can still be done in dimension ≥ 2 at least in the integrable case?

Thanks for your attention...

More on

http://www-fourier.ujf-grenoble.fr/~ycolver/