Section 2: The Project proposal

Aspects of the asymptotic theory of linear ordinary differential equations

I. State-of-the-art and objectives

Investigation of asymptotics of the spectrum for all kinds of spectral problems is a major ingredient in numerous articles in pure and applied mathematics, mathematical and theoretical physics as well as many other areas of natural sciences. At the same time results on the asymptotic behavior of the corresponding sequences of eigenfunctions are incomparably fewer and scattered sporadically in the literature starting with the pioneering Ph.D. thesis of J.D.Birkhoff from 1913. In several recent papers I initiated a further development of this field and discovered its extremely rich connections and plausible applications to several classical branches of mathematics.
Goal: The main purpose of this proposal is to pursue a systematic investigation of the asymptotic distributions of the zero loci of solutions and eigenfunctions to linear ODEs with polynomial coefficients depending on parameter(s).

Objectives:

for the second order linear differential equations:

1) generalize the famous Evgrafov-Fedoryuk-Sibuya's WKB-theory for solutions of the Schrödinger equation with a polynomial potential to the case of a rational potential and clarify the role of quadratic differentials in this theory;

2) obtain upper bounds for the number of finite singular trajectories for rational quadratic differentials in order to get the upper bound for the number of spectral families for the corresponding Schrödinger equations;

3) accomplish the study of the (weak) asymptotics for arbitrary (weakly) converging sequences of Jacobi polynomials;

for higher order linear differential equations:

4) accomplish the description of the asymptotic root distributions for the sequence of the eigenpolynomials of an arbitrary exactly solvable operator and apply it to obtain further results in the classical Bochner-Krall problem in orthogonal polynomials;

5) find connections of the latter results with the structure of Stokes lines for linear differential equations of higher order;

6) obtain general asymptotic results for Heine-Stieltjes resp. Van Vleck polynomials in the general Heine-Stieltjes spectral problem for linear differential equations of arbitrary order including a description of the support of the asymptotic measures;

7) develop local and global theory of meromorphic differentials of order greater than 2 including the appropriate notion of a Jenkins-Strebel differential on a compact Riemann surface;

8) characterize algebraic functions in $\mathbb{CP}^1$ which can be represented almost everywhere as the Cauchy transform of real/positive measures.

Below I briefly describe the state-of-the-art situation for the above subprojects.

Operators of second order:

1-2. Evgrafov-Fedoryuk-Sibuya WKB-theory and quadratic differentials. It is a well-known fact in mathematics and quantum mechanics that for practically all linear differential operators their spectrum and eigenfunctions can not be found explicitly. Still for a wide class of operators one can obtain a rather detailed asymptotic information about the location of their spectrum and the asymptotic distribution of the zeros of their eigenfunctions. This topic is closely related to the classical Nevanlinna theory, the so-called oscillation theory in complex domain, and the WKB-theory, see e.g., [EF], [Fe], [Hil], [Sib], [Wa1], [Wa2] and a more recent [La].

The classical WKB-method (Wentzel-Kramers-Brillouin) was developed on the physics level of rigorousness in the 1920's for the purpose of obtaining approximate solutions of linear second order differential equations and, in particular, of the Schrödinger equation. Similar technique was even earlier suggested by Liouville and Birkhoff. For example, in the simplest case of the Schrödinger equation

$$-y'' + \lambda P(z) y = 0 \tag{1}$$

with a polynomial potential $P(z)$ and spectal parameter λ the main idea of the WKB-method is to compare an appropriate fundamental solution of (1) and its approximate solutions given by:

$$\frac{1}{\sqrt[4]{P(z)}} e^{\pm\lambda \int_{z_0}^{z} \sqrt{P(t)}dt}. \tag{2}$$

One has to show that for large λ in appropriate domains in $\mathbb{C}$ the functions (2) give the leading term of a fundamental solution of (1) which helps, in particular, to get asymptotic information about the spectrum. Although the idea is very basic nevertheless mathematically rigorous global in $\mathbb{C}$ WKB-theory for (1) was developed only in the late 60's by Y. Sibuya in USA and M. Evgrafov-M. Fedoryuk in Russia. They studied non-standard spectral problems looking for the eigenfunctions exponentially decreasing in a pair of non-neighboring Stokes sectors of $P(z)$. Recall that for $P(z) = a_0 z^d + a_1 z^{d-1} + ... + a_d,\ a_0 \neq 0$ of degree d one defines its (open) *Stokes sectors* $S_j, j = 0, ..., d+1$ as given by the condition:

$$S_j = \left\{ z : \left| \arg z - \frac{\phi_0}{d} - \frac{2\pi j}{d+2} \right| < \frac{\pi}{d+2} \right\},$$

where $\phi_0 = \arg a_0$. This set-up includes as a special case the standard boundary value problem for anharmonic oscillators with a real potential and eigenfunctions belonging to $L^2(\mathbb{R})$. M. Fedoryuk was apparently the first one to observe the role of the quadratic differential $\Psi = P(z)dz^2$ in this context. He explained that the maximal domains in which that WKB-method works are the so-called canonical domains in $\mathbb{C}$ obtained as certain unions of the subdomains into which $\mathbb{C}$ is cut by the Stokes lines of Ψ. (For the formal definition of the (global) Stokes line consult [Wa1] or [Fe]. Detailed information about quadratic differentials can be found in [Str].)

As I recently observed in [ShQ] there are much deeper connections between the original Schrödinger equation and the corresponding differential Ψ. In particular, each short geodesic of the canonical singular metric on $\mathbb{C}$ induced by Ψ. i.e. a geodesic connecting the roots of $P(z)$ leads to an infinite series of eigenvalues of the corresponding boundary value problem. Moreover, certain subset of the Stokes line of Ψ serves as the accumulation curve of the zeros of appropriately scaled eigenfunctions, see [GES2].

In mathematical physics the case of polynomial potentials (i.e., anharmonic oscillators) is considered as an important mathematical model but hardly having any relevance in physics. On the other hand, allowing potentials to be rational functions opens many more possibilities but also creates substantial difficulties since solutions of such equations are no longer univalent functions. In case of a rational potential $R(z)$ a natural type of boundary-value problem to consider are the so-called 2-point problems when one looks for the values of the spectral parameter for which there exists a solution regular at two given poles of $R(z)$. No appropriate version of the global Evgrafov-Fedoryuk theory for rational potentials exists at present although one can easy guess that the quadratic differential $R(z)dz^2$ plays an important role as well. Especially, the behavior of its geodesics which can be incomparably more complicated than in the polynomial case is very essential in understanding of the spectral properties of Schrödinger equations under consideration. In particular, the following conjecture seems to be highly plausible.

Conjecture 1. *Each short geodesic of a rational quadratic differential $R(z)dz^2$ (i.e. connecting two singular points of $R(z)$) corresponds to an infinite sequence of eigenvalues for the 2-point boundary* *value problem of the Schrödinger equation $-y'' + \lambda R(z)y = 0$. This sequence of eigenvalues approaches a certain ray in $\mathbb{C}$ whose slope is determined by the considered geodesics.*

Recently A.Eremenko and A.Gabrielov [EG2] made an essential progress in the understanding of the so-called spectral discriminant of Schrödinger equations with polynomial potentials. The spectral discriminant is a real hypersurface in the space of polynomial potentials of a given degree for which the corresponding Schrödinger equation has a solution decreasing in at least two distinct Stokes sectors. One can hope to obtain at least partially similar information in the case of rational potentials due to the fact that one can deform a rational potential into a polynomial one by moving its poles to infinity.

The latter conjecture obviously motivates project 2. Notice that in [ShQ] I found sharp upper and lower bounds for the number of short geodesics of $P(z)dz^2$ in the case of polynomial potentials $P(z)$. Namely, the following result holds.

Theorem 1. *For any polynomial $P(z)$ of degree d the number of short geodesics of the quadratic differential $P(z)dz^2$ can take an arbitrary integer value in between $d-1$ and $\binom{d}{2}$.*

On the other hand, at the moment I do not have even a reasonably motivated conjecture what upper and lower bounds might appear in the case of rational $R(z)$.

3. Asymptotics of sequences of Jacobi polynomials. The study of the (weak) asymptotics of sequences (of the root-counting measures) of orthogonal polynomials goes back to the fundamental treatise [Sz]. (Recall that for a polynomial of degree n its root-counting measure is a discrete probability measure having mass $1/n$ at each of its roots.) Recently serious progress in the case of sequences of Jacobi polynomials was obtained in e.g., [KuMF] and [KuMFO] using the technique of the Riemann-Hilbert problem developed by P. Deift and his collaborators. A number of other authors were studying similar problems for e.g., hypergeometric, Laguerre, Bessel polynomials. Practically all these papers are dealing with the case of real but non-standard parameters for which the corresponding polynomials are no longer orthogonal w.r.t. the standard weights. It is well-known that the study of the (weak) asymptotics of sequences of classical orthogonal polynomials is closely related to the study of the asymptotic behavior of solutions to the Gauss hypergeometric equation and, more generally, Riemann second order equation depending on a spectral parameter λ in a non-standard way. More exactly, to obtain interesting asymptotic results one should instead of the usual spectral problem consider the so-called *homogeneous spectral problem* where the term with the first derivative is multiplied by λ and the dependent variable is multiplied by λ^2. In such approach there is no need to keep additional parameters real. Using potential theory one can also observe that if the weak convergence takes place the accumulation set for the zeros of the considered sequence of Jacobi (or similar) polynomials will coincide with some horizontal trajectories of a quadratic differential of the form $P(z)dz^2/Q^2(z)$ where $P(z)$ and $Q(z)$ are quadratic and coprime polynomials. To my surprise is turned out that neither the homogenized spectral problem for the classical hypergeometric Gauss equation nor the detailed study of the behavior of trajectories of the latter family of (quite simple) quadratic differentials were ever carried out although this will solve the problem of the (weak) asymptotics for sequences of Jacobi and hypergeometric polynomials in complete generality.

Operators of higher order:

4. Exactly solvable operators and Bochner-Krall problem. Much reacher and less studied field opens if one considers the asymptotics of the eigenfunctions for linear differential equations of order exceeding 2. A model case of this situation is the case of arbitrary exactly solvable differential operators. Recall that an operator $T = \sum_{i=0}^{k} Q_i(z)\frac{d^i}{dz^i}$ is called *exactly solvable* if all $Q_i(z)$ are polynomials of degree at most i and there exists at least one i_0 for which $\deg Q_{i_0} = i_0$. One can easily show that any exactly solvable T has a unique eigenpolynomial in each sufficiently large degree and one can pose the following natural question.

Problem 1. Describe the limiting set (measure) to which the sequence of zero loci (root-counting measures) of this sequence of eigenpolynomials converges?

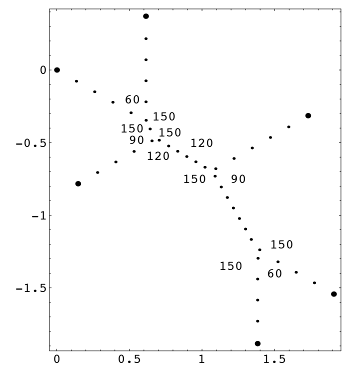

FIGURE 1. Small dots shown the roots of the eigenpolynomial of degree 50 for the operator $T = Q_6(z)\frac{d^6}{dz^6}$ where $Q_6(z)$ is a sextic whose roots are the larger dots. (Numbers on the picture are the angles between the respective edges.)

For the class of the so-called non-degenerate operators characterized by $\deg Q_k = k$ the results are fairly complete, see [BR]. An illustration of the limiting distribution of the roots of the eigenpolynomials for a non-degenerate exactly solvable operator of order 6 is shown on Fig. 1. On the other hand, at the present moment in the important degenerate case there are only conjectures describing the root asymptotics, see [Ber].

Why should one care about Problem 1? To answer let us present a modern reformulation of the classical Bochner-Krall problem from 1929, see [Bo].

Problem 2. Characterize the class of exactly solvable operators T whose sequence of eigenpolynomials are the sequences of orthogonal polynomials with respect to some real weight supported on $\mathbb{R}$?

Bochner himself knew that such T must be of even order and formally self-adjoint. But in spite of numerous efforts the Bochner-Krall problem has only being solved for operators of order 2 and 4, see [EKLW]. Studying the root asymptotics of the sequence of these eigenpolynomials one can easily rule out large classes of exactly solvable T. This approach has been successfully applied in the non-degenerate case, see [BRSh] and [KL], but the degenerate case which requires appropriate rescaling of the eigenfunctions is still widely open. One should mention that in all known cases one finds behind each limiting set (measure) an interesting rational differential whose order equals to the order of the original operator.

Besides the study of the usual sequences of eigenpolynomials for exactly solvable operators one can also consider the sequences of generalized eigenpolynomials which appear in the homogenized spectral problem for such operators, cf. [BBSh]. Conjecturally they also possess very interesting asymptotic root-counting measures discussed in [BBSh] and provide us with a rich family of examples. The study of the homogenized spectral problem for arbitrary exactly solvable operators is a natural extension of a part of project 3 dealing with the particular case of the hypergeometric equation.

5. Stokes lines of equations of higher order. The notion of a (global) Stokes line of a given linear differential equation of order 2 with polynomial or entire coefficients in $\mathbb{C}$ is a classical object of study and elaborated in e.g. [Fe] and [Wa1]-[Wa2]. It is frequently discussed in physics and technical literature in the situations when the Stokes phenomenon takes place, see e.g. [Be] and [BM]. Its local version near the singularities exists for equations of all orders and even for systems of linear differential equations. But the existence of a global Stokes line which is well behaved and responsible for the global asymptotic behavior of the underlying equation is so far known only for order 2. The major problem with equations of higher order is that their local Stokes lines determined near the singularities (called the *turning points* in this area) might intersect at non-singular points creating the so-called *secondary turning points*, see [BNR]. If one naively creates new Stokes lines from these secondary points then they will also intersect at the ternary turning points etc. Generically, these derived turning points and their derived Stokes line will densely cover some domains in $\mathbb{C}$ creating an untracktable situation. It is expected at least for generic linear ode that only finitely many of these derived turning points and Stokes lines are essential for the asymptotic behavior of the solutions to the original equation. A group of Japanese mathematicians from RIMS, Kyoto under the leadership of Professor T.Kawai has made important contributions to this field, see e.g. [AKT1]-[AKT2]. In particular, they have developed certain algorithms allowing to select the essential secondary, ternary etc turning points as well as the corresponding Stokes lines. But general existence and uniqueness results are unavailable at present except for very special cases.

There are general reasons to believe and special cases when this is rigorously proven that the accumulation set for the zeros of generalize eigenvalues constitutes an important part of the global Stokes line of the underlying linear differential equation. Interesting and rather unknown results on the root distributions of eigenfunctions of such operators were obtained in the late 80's by S. Bank, see [Ba1], [Ba2]. But these results were never interpreted from the point of view of the global Stokes line.

Working on this subproject I hope to obtain detailed information about the structure of the global Stokes line in the case of exactly solvable operators/equations which , on one hand, are too special to be covered by the methods a'la Kawai, and, on the other hand, form a large and important for applications class of equations.

6. Heine-Stieltjes theory. Another classical source of (generalized) polynomial eigenfunctions is the theory of the Lamé equation, see e.g. [WW] and [BW]. In the case of the Lamé equation many prominent mathematicians including Heine, Klein, Stieltjes [He], [St] studied polynomial solutions to linear ordinary differential equation as follows. Consider an arbitrary operator $T = \sum_{i=0}^{k} Q_i(z)\frac{d^i}{dz^i}$ with polynomial coefficients. The number $r = \max_{0\leq i\leq k}(\deg Q_i(z) - i)$ is called the *Fuchs index* of T and the operator T is called a *higher Lamé operator* if it satisfies the conditions (i) $r \geq 0$ and (ii) $\deg Q_k(z) = k + r$. Given such an operator T consider the following multiparameter spectral problem: for each positive integer n find a polynomial $V(z)$ of degree at most r such that the equation

$$TS(z) + V(z)S(z) = 0 \tag{3}$$

has a polynomial solution $S(z)$ of degree n. Polynomials $V(z)$ and $S(z)$ are called a *higher Van Vleck* and *higher Heine-Stieltjes polynomial*, respectively. One can show that under some genericity assumptions on T for each n there exists exactly $\binom{n+r}{r}$ pairs (V, S) solving problem (3). Since the number of pairs (V, S) grows with n one needs to study certain natural subsequences of polynomial solutions.

The next localization result recently proven in [Sh] guarantees that there exists an abundance of converging subsequences of (appropriately normalized) Van Vleck polynomials.

Theorem 2. *For any higher Lamé operator T and any $\epsilon > 0$ there exists a positive integer N_ϵ such that the zeros of all its Van Vleck polynomials and Heine-Stieltjes polynomials of degree $n \geq N_\epsilon$ belong to $Conv^\epsilon_{Q_k}$. Here $Conv_{Q_k}$ is the convex hull of all zeros of the leading coefficient Q_k and $Conv^\epsilon_{Q_k}$ is its ϵ-neighborhood in the usual Euclidean distance on $\mathbb{C}$.*

This localization result allows to find an abundance of converging subsequences of normalized Van Vleck polynomials. In a recent joint paper [HoSh] we were able to describe the asymptotic root distribution of the roots of Stieltjes polynomials whose sequence of normalized Van Vleck polynomials converges to a given polynomial $V(x)$. At the same time the situation with the asymptotics of Van Vleck polynomials is a complete mystery except for one special case treated in [ShT], [STT]. Preliminary results and certain ideas lead to the following conjecture.

Let Pol_r denote the space of all monic polynomials of degree r. Take any higher Lamé operator T and for a Van Vleck polynomial $V(z)$ of T denote by $\widetilde{V}(z)$ its monic scalar multiple. For a given positive integer n denote by $\{V_{n,i}(z)\}$ the set of all Van Vleck polynomials $V(z)$ whose Stieltjes polynomials have degree exactly n. (Notice that for sufficiently large n the set $\{V_{n,i}(z)\}$ belongs to

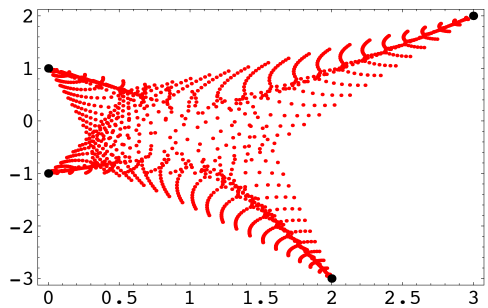

FIGURE 2. The union of zeros of all $\binom{27}{2}$ Stieltjes polynomials of degree 25 for the operator $T = (z^2+1)(z-3I-2)(z+2I-3)\frac{d^3}{dz^3}$.

Pol_r, i.e. each Van Vleck has degree exactly r.) Transform the set $\{V_{n,i}(z)\}$ into a finite probability measure $\sigma_n(T)$ in Pol_r by assigning to each polynomial the finite mass equal to the inverse of the cardinality of $\{V_{n,i}(z)\}$.

Conjecture 2. *For any higher Lamé operator T one has that*

- *the sequence $\sigma_n(T)$ converges weakly to a probability measure $\Sigma(T)$ compactly supported in Pol_r;*
- *the measure $\Sigma(T)$ depends only on the leading monomial $Q_k(z)\frac{d^k}{dz^k}$ of T.*

Preliminary numerical results in this direction show very surprising pictures of the asymptotic root distributions for the Stieltjes polynomials, see Fig. 2. They seems to have a fractal-looking structure which at moment is completely unexplained. And again behind each such limiting set (measure) one finds an interesting rational differential whose order equals to the order of the original equation.

7. Local and global theory of high order differentials. As I mentioned before the theory of quadratic differentials is a well established area of mathematics with numerous applications, see [Str]. On the other hand, although differentials of higher order sometimes appear in e.g., algebraic geometry their general theory is practically non-existing. One of very rare exceptions is a old paper [Str2].

In the study of the asymptotic root distributions for the generalized eigenpolynomials of exactly solvable and higher Lamé operators one observes a systematic appearance of higher order differentials in $\mathbb{CP}^1$ possessing a number of additional properties, see [HoSh]. This fact along with some additional motivation from algebraic geometry justifies the necessity to develop a general theory of such differentials. At the moment I have succeeded in the local study of these differentials, i.e. in obtaining local normal forms of such differentials near their poles or zeros (but not in complete generality). In principle, this can be straightforwardly carried out using the ideas of Strebel's classical treatise [Str].

On the other hand, a much more challenging and rewarding task is to develop an appropriate notion of a Jenkins-Strebel differential, i.e. a differential for which almost all its horizontal trajectories are closed. The necessity of introducing this concept again comes from other part of the present project. Namely, high order differentials appearing in [HoSh] come with a similar structure as Jenkins-Strebel differentials, i.e. with a system of closed curves covering almost all $\mathbb{CP}^1$. Each curve consists of pieces of trajectories and it can switch from one trajectory to the other on sets with peculiar additional properties. Observe that the straight-forward generalization of the above definition of Jenkins-Strebel differentials to higher orders is impossible since (except for trivial degenerate examples coming from quadratic differentials) there are no high order differentials with almost all closed trajectories. In order to get a nontrivial theory one has to allow to switch from one type of trajectories of such a differential to the another on certain switching sets. At the moment I have a rather clear idea how the notion of a Jenkins-Strebel differential should be defined, but there is still a number of technical difficulties to overcome.

8. Global representability of algebraic functions as the Cauchy transform of real measures. The study of local and global properties of the Cauchy transform and the Cauchy-Stieltjes integral was initiated by A. Cauchy and T. Stieltjes in the middle of the 19th century. This development is still continuing at present with dozens of well-known mathematicians contributing to this topic over more than a century. Numerous papers and several books are partially or completely devoted to this area, see e.g. [Bell], [CMR], [Du], [Ga], [GaMa], [Mu], [Za], [Wi]. This area is closely connected with the (logarithmic) potential theory and, especially, its inverse problem and inverse moment problem.

During the last 2 decades the notion of a motherbody of a solid domain or, more generally, of a positive Borel measure was discussed in the geophysics and mathematics literature, see e.g. [Sj], [SaStSha], [Gu], [Zi]. This notion was pioneered in the 60's by a Bulgarian geophysicist D. Zidarov [Zi] and mathematically developed by B. Gustafsson in [Gu]. Although several interesting results about this notion were obtained there seems still to be no consensus even about the definition of a motherbody and no general existence and uniqueness results are known. The intuitive meaning of a motherbody of a given domain/positive measure is the set/positive measure such that a) it creates the same potential as the original domain/measure outside the support of the original domain/measure and b) the support of a motherbody is as small as possible. The usual technique to obtain a motherbody is to apply to the original domain/measure different forms of the so-called inverse balayage introduced initially by H. Poincaré. But at present there are just isolated examples and no general understanding for which domains one can find a motherbody whose support has smaller dimension than the dimension of the ambient space which is the most desirable situation. One can can also substitute the original domain/positive measure by its potential near infinity, then take its complete analytic continuation and ask when there exists a set/measure whose potential coincides with a branch of the latter analytic continuation almost everywhere. This problem is quite closely related to the original motherbody problem.

Although one might think that this topic has very little to do with the main theme of the overall project it turns out that they are closely related in the case of $\mathbb{C}$. Namely, in $\mathbb{C}$ one can associate to a domain/measure both its logarithmic potential and its Cauchy transform (which is the complex potential) and pose the following natural question.

Main problem. Given a germ $f(z) = a_0/z + \sum_{i\geq 2}^{\infty} a_i/z^i,\ a_0 \in \mathbb{R}$ of an algebraic (or, more generally, analytic) function near ∞ is it possible to find a compactly supported in $\mathbb{C}$ real measure whose Cauchy transform coincides with (a branch of) the analytic continuation of $f(z)$ a.e. in $\mathbb{C}$? Additionally, for which $f(z)$ it is possible to find a positive measure with the above properties?

Now if one considers a weakly converging sequence of generalized eigenpolynomials to an exactly solvable or a higher Lamé operator then the sequence of their root-counting measures weakly converges to a probability measure whose Cauchy transform satisfies the symbol equation of the original operator. (The symbol of an operator $T = \sum_{i=0}^{k} Q_i(z)\frac{d^i}{dz^i}$ with polynomial coefficients is the bivariate polynomial $S_T(z,u) = \sum_{i=0}^{k} Q_i(z)u^i$.) In other words, normalized logarithmic derivatives of the original sequence of polynomials usually converge after appropriate scaling and renormalization to a branch of the algebraic function $S_T(z,u) = 0$. This circumstance provides us with large families of examples of algebraic functions for which there is a positive solution to the latter problem. It seems highly plausible that any algebraic function with a branch representable a.e. in $\mathbb{C}$ as the Cauchy transform of a positive measure can be obtained in this way.

Impact of the project

In my opinion the major impact of the suggested project is threefold. The first aim is to clarify the connection between the asymptotics of the (appropriately scaled) eigenfunctions for several classes of linear differential operators and the corresponding differentials whose order equals the order of the operators. (As a suggested intermediate step one should substantially develop the theory of such differentials.)

The second aim it to obtain deeper results about the asymptotic root distributions of exactly solvable operators and apply them to make a serious progress (and hopefully a final solution) of the Bochner-Krall problem in orthogonal polynomials.

The third aim to solve the problem which algebraic functions have branches which be represented almost everywhere as Cauchy transforms of real/positive measures which means a substantial advance in the classical potential theory.

II. Methodology

In this sections I will briefly describe some partial results and possible approaches one can apply to obtain progress in the above subprojects.

1-2. Evgrafov-Fedoryuk-Sibuya WKB-theory and quadratic differentials. We use the same terminology as in the state-of-the-art section. Consider the classical Schrödinger equation

$$-\frac{d^2y}{dz^2}+\lambda^2R(z)y=0, \tag{4}$$

where $R(z)$ is a rational function and λ is a large real parameter. To this equation one usually associates its *global Stokes line* obtained as follows. A turning point of (4) is a zero or pole of $R(z)$; a local Stokes line propagating from a given turning point z_0 is given by the equation $\Re\, S(z,z_0)=0$, where $S(z,z_0)=\int_{z_0}^{z}\sqrt{R(t)}dt$. The union of all these local Stokes lines forms the global Stokes line ST of (4). Already M. Fedoryuk noticed that ST can be also interpreted as the set of all finite singular trajectories (i.e. entering the turning points of $R(z)$) of the quadratic differential $R(z)dz^2$.

If $R(z)=P(z)$ is a polynomial of degree k then the asymptotic behavior of the global Stokes line at ∞ is as follows. Its branches tend to the one of $k+2$ equally spaced with the angle $\frac{2\pi}{k+2}$ rays (called the *Stokes rays*) whose configuration only depends on the leading coefficient of $P(z)$. (Stokes rays are tangents to the different branches of the global Stokes line near the irregular singularity of (4) at ∞.) Fundamental results of Evgrafov-Fedoryuk-Sibuya [Fe, Sib] show that the asymptotic expansion in λ^{-1} of fundamental solutions to (4) exists in each unbounded connected component of $\mathbb{CP}^1\setminus ST$. For polynomial $P(z)$ it is natural to define a certain class of solutions of (4) and its eigenfunctions, see e.g. [Sib]. It is well-known that for any of the $k+2$ (Stokes) sectors restricted by two neighboring Stokes rays and for any fixed value of parameter λ there exists and unique (up to a constant factor) solution of (6) which exponentially decreases in this sector. Such a solution is called *subdominant* in the respective sector. For generic values of λ each subdominant solution increases exponentially in any other sector and it always increases in both neighboring sectors. But if one chooses 2 non-neighboring sectors then for an infinite discrete set of values of λ there exists a solution subdominant in both sectors. These values of λ are called *spectral* and the corresponding solution is called the *eigenfunction*. The main achievement of Evgrafov-Fedoryuk-Sibuya's theory in the modern language is a theorem claiming that to each short geodesic of $P(z)dz^2$ corresponds an infinite sequence of eigenvalues of the latter spectral problem which asymptotically are equally spaced along a certain ray in $\mathbb{C}$ whose slope is defined by the geodesics.

Our main goal here is to obtain an analog of this theory for rational $R(z)$. Although the situation with rational $R(z)$ in (4) is formally very similar to that of a polynomial potential $P(z)$ there are new essential difficulties. In particular, their spectral problem should be substituted by the 2-point boundary value problem. The global behavior of Stokes lines can be rather involved. In particular, they can be dense in some domains in $\mathbb{C}$ etc. The first case to consider is when the latter situation does not occur. Then it is rather straightforward that the WKB-asymptotics holds in each open connected component of the complement to the Stokes line in $\mathbb{CP}^1$. For some values of the argument of λ there will appear parts of the Stokes lines connecting simple poles of $R(z)$. One expects that close to this ray in the space of the spectral parameter one can find an infinite sequence of eigenvalues such that there exists a (multivalued) solution of (4) having a branch analytic at the poles connected by the latter parts of the Stokes line. What happens if the Stokes line is dense in a subdomain in $\mathbb{C}$ is not clear at the moment.

3. Asymptotics of sequences of Jacobi polynomials. Recall that the classical Jacobi polynomial $P_n^{(\alpha,\beta)}$ is given by

$$P_n^{(\alpha,\beta)}(z) = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (z-1)^k (z+1)^{n-k}.$$

The following (unpublished) proposition relates the asymptotic root-counting measure of a sequence of Jacobi polynomials to a certain quadratic differential.

Proposition 1. (i) *If the sequence $\{\mu_n\}$ of the root-counting measures of a sequence $\{p_n(z)\} = \{P_n^{(\alpha_n,\beta_n)}(z)\}$ of Jacobi polynomials weakly converges to a measure μ which is compactly supported in $\mathbb{C}$ and has no point masses then the limits $A = \lim_{n\to\infty} \frac{\alpha_n}{n}$ and $B = \lim_{n\to\infty} \frac{\beta_n}{n}$ exist;*
(ii) *In the above notation if $B \neq 0$ then the Cauchy transform $\mathcal{C}_\mu$ of the limiting measure satisfies almost everywhere in $\mathbb{C}$ the algebraic equation*

$$(1-z^2)\mathcal{C}_\mu^2 - ((A+B)z + A - B)\mathcal{C}_\mu + A + B + 1 = 0. \tag{5}$$

(iii) *If $B \neq 0$ then the support of μ consists of finitely many horizontal trajectories of the quadratic differential*

$$\Psi(z) = -\frac{(A+B+2)^2 z^2 + 2(A^2-B^2)z + (A-B)^2 - 4(A+B+1)}{(1-z)^2} dz^2. \tag{6}$$

The latter quadratic differential Ψ has two (simple) roots and three double poles (including ∞). For any values of A and B it has a singular trajectory connecting its zeros which is always in the support of the limiting root-counting measure. On the other hand, when it also have a closed trajectory it might also belong to the support, see e.g., [KuMF]. The detailed study of the above family of differentials will provide a complete information about the asymptotics of sequences of Jacobi polynomials satisfying the assumptions of Proposition 1.

4. Exactly solvable operators and Bochner-Krall problem. Recall that a differential operator with polynomial coefficients $T = \sum_{i=1}^{k} Q_i(z) \frac{d^i}{dz^i}$ is called *exactly solvable (ES)* if $\deg Q_i \leq i$ and there exists j such that $\deg Q_j = j$. An exactly solvable T is called *non-degenerate* if $\deg Q_k = k$ and *degenerate* otherwise.

As was already mentioned one can easily check that any ES-operator T has a unique polynomial eigenfunction $p_n^T(z)$ of degree n for all sufficiently large n. The main problem of this project is to describe the asymptotic root distribution of the sequence $\{p_n^T(z)\}$. In the non-degenerate case substantial progress in this problem was obtained in [BR]. These results were then applied in [BRS] and [KL] to obtain interesting results in the Bochner-Krall problem. The important remaining case to consider is the case of degenerate ES-operators.

One can show that for an arbitrary degenerate ES-operator the maximal modulus r_n among the roots of $p_n^T(z)$ tends to ∞ when $n \to \infty$. Thus, in order to obtain a converging sequence of root-counting measures one has to scale eigenpolynomials which motivates the problem about how fast does the maximal modulus r_n grow. The following conjecture was formulated in [Ber].

Conjecture 3. *Let $T = \sum_{i=1}^{k} Q_i(z) \frac{d^i}{dz^i}$ be a degenerate exactly-solvable operator of order k and denote by i_0 the largest i for which $\deg Q_i = i$. Then*

$$\lim_{n\to\infty} \frac{r_n}{n^d} = c_T,$$

where $c_T > 0$ is a positive constant and

$$d := \max_{i \in [i_0+1, k]} \left(\frac{i - i_0}{i - \deg Q_i} \right).$$

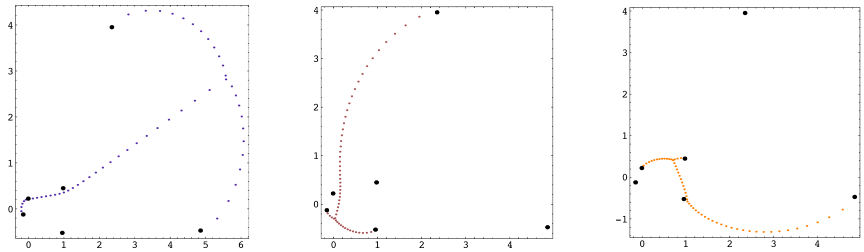

FIGURE 3. Three root-counting measures for the homogenized spectral problem $\left(z^3 - 2Iz^2 + (4+2I)z\right)\frac{d^3y}{dz^3} + \lambda(z^2 + Iz + 2)\frac{d^2y}{dz^2} + \frac{\lambda^2}{5}(z - 2 + I)\frac{dy}{dz} + \lambda^3 y = 0.$ (Larger dots on all pictures are the turning points of the original equation. Notice that tripoid points above are the secondary turning points.)

As an immediate consequence of the latter conjecture we get that the following.

Corollary 1. *If Conjecture 3 holds then the Cauchy transform $C(z)$ of the asymptotic root measure μ of the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$ of an arbitrary exactly-solvable operator T as above satisfies almost everywhere in $\mathbb{C}$ the following algebraic equation:*

$$z^{i_0} C^{i_0}(z) + \sum_{i \in A} \alpha_{i, \deg Q_i} z^{\deg Q_i} C^i(z) = 1.$$

Here A is the set consisting of all i for which the maximum $d := \max_{i \in [i_0+1, k]} \left(\frac{i - i_0}{i - \deg Q_i}\right)$ is attained, i.e. $A = \{i : \frac{i - i_0}{i - \deg Q_i} = d\}$.

The latter statement (if settled) gives a very detailed information on the asymptotic root distribution of the family of scaled eigenpolynomials for an arbitrary degenerate ES-operator. For application to the Bochner-Krall problem the following intermediate question of independent interest can be formulated.

Problem 3. Characterize all ES-operators whose polynomial eigenfunctions have only real roots.

5. Stokes lines for equations of higher order. In [AKT1] the authors developed a certain heuristics how to determined which secondary, ternary etc turning points and resp. local Stokes lines are important for the global asymptotics of generic linear differential equation with polynomial coefficients satisfying a number of additional restrictions. In [AKT2] they tested their procedure in the case of third order equations which allows explicit Mellin-Barnes intergal representation of their solutions. In this case the asymptotic behavior of fundamental systems can be analyzed explicitly and shows a good coincidence with the algorithm suggested in [AKT1].

In [BBSh] we initiated the study of the asymptotic root-counting measures of the homogenized spectral problem for exactly solvable operators. The homogenized deformation of a given linear differential operator (introduced already by J. Birkhoff) is exactly the set-up considered in [AKT1]-[AKT2] in the study of the global Stokes lines. I am convinced that the supports of the limiting root-counting measures from [BBSh] are essential parts of the global Stokes line of exactly solvable equations, see example on Fig. 3. On the other hand, exactly solvable operators is an incomparably broader test class of operators than those whose solutions can be found by explicit integrals a'la Mellin-Barnes. Additional circumstance in favor of this class of operators is that the existence of the underlying positive measures supported on (conjectural) parts of the global Stokes line apparently means that its behavior is not as complicated as it in principle might be. One knows from the experience with second order equations that the global Stokes line has a much simpler structure if it contains fragments connecting pairs of turning points. At the moment there is still much work to be done to understand the properties of supports for asymptotic root-counting measures appearing in [BBSh].

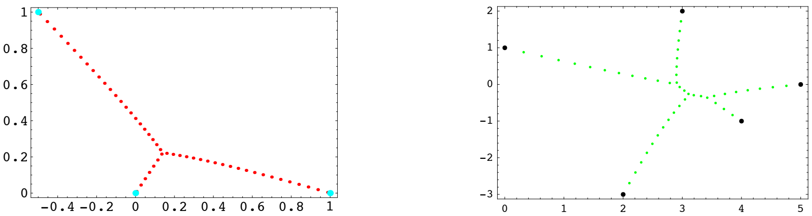

FIGURE 4. Measures $\Sigma(T)$ for the classical Lamé operator $T = Q(z)\frac{d^2}{dz^2} + \frac{1}{2}Q'(z)\frac{d}{dz}$, with $Q(z) = z(z-1)\left(z+\frac{1}{2}-i\right)$ (left) and $T = Q(z)\frac{d^4}{dz^4}$ with $Q(z) = (z-5)(z-I)(z-4+I)(z-2+3I)(z-3-2I)$ (right).

6. Heine-Stieltjes theory. We will use notation from the-state-of-the-art section. Conjecture 2 was proven in [STT] and [MFR] for the classical Lamé equation, i.e. for the second order equations with the Fuchs index r equal to 1. In this case each Van Vleck polynomial $V(z)$ is linear and is uniquely defined by its unique root. Moreover, there exists $n+1$ such polynomials for a given positive integer n. Their root distribution for $n = 50$ is shown on the left part of Fig. 4. The limiting measure $\Sigma(T)$ is the union of three real semi-analytic curves connecting the root of $Q(z)$ with the value of z corresponding to the Lavrentiev's continuum.

For other higher Lamé operators a strategy to follow is to use the conformal welding technique to prove Conjecture 2. In a very recent collaboration with Professor D. Marshall (U Washington) we were able to prove the following (unpublished) result for higher Lamé operators with $r = 1$.

Theorem 3. *For any higher Lamé operator T with Fuchs index $r = 1$ Conjecture 2 holds. Moreover, the support of $\Sigma(T)$ is topologically a tree in $\mathbb{C}$ whose leaves (i.e. hanging vertices) are exactly the roots of the leading coefficient $Q_k(z)$.*

This result is illustrated on the right picture of Fig. 4. The case of $r > 1$ is so far out of reach of our methods but an appropriate modification of conformal welding looks promising. Another important potential research direction for Heine-Stieltjes theory is to study the homogenized version of the standard spectral problem considered above. It will undoubtedly show new interesting features in the appearing sequences of generalized polynomial eigenfunctions.

7. Local and global theory of higher order differentials. By a meromorphic differential Ψ of order k on a (compact) Riemann surface Γ without boundary we mean a meromorphic section of the k-th tensor power $(T^*_{\mathbb{C}}\Gamma)^{\otimes k}$ of the holomorphic cotangent bundle $T^*_{\mathbb{C}}\Gamma$. The zeros and poles of Ψ are called its *singular* points.

Following §6 of Ch.3 in [Str] one can get normal forms for a k-differential near its singular points. Namely, for $\Psi = f(z)(dz)^k$ near the origin set $f(z) = z^n(a_n + a_{n+1}z + \ldots)$, where $a_n \neq 0$ and $n \in \mathbb{Z}$. We can choose a single branch of the k-th root

$$(a_n + a_{n+1}z + \ldots)^{\frac{1}{k}} = b_0 + b_1 z + b_2 z^2 + \ldots,\ b_0 \neq 0.$$

Thus, $\sqrt[k]{f(z)} = z^{\frac{n}{k}}(b_0 + b_1 z + b_2 z^2 + \ldots)$. The next lemma gives an example of the normal form.

Lemma 1. *In case when there is no non-negative integer l such that $\frac{n}{k} + l = -1$ then there exists a local coordinate ζ near the origin such that*

$$\Psi = (dW)^k = \left(\frac{n+k}{k}\right)^k \zeta^n (d\zeta)^k.$$

Moreover ζ is defined uniquely up to a factor $\exp(l\frac{2\pi i}{n+k}),\ l = 0, 1, \ldots, n+k-1.$

Concerning global properties of such differentials notice that main results of [BR] and [HoSh] give a interesting hint which Ψ should be called a Jenkins-Strebel differential of higher order. Our suggestion is as follows.

Definition. We say that a k-differential Ψ given on a compact curve Γ without boundary is *Strebel* if there exists a continuous (potential) function $u : \Gamma \setminus Sing(\Psi) \to \mathbb{R}$ such that
(i) u has a limit (finite or infinite) at each singular point $p \in Sing(\Psi)$;
(ii) u is piecewise harmonic and the 1-current $\mathcal{C}_u := \frac{\partial u}{\partial z} dz$ coincides with one of k branches of $\sqrt[k]{\Psi^*} = i \sqrt[k]{\Psi}$ a.e. on Γ;
(iii) level curves of u are tangent to (one of the) line fields of Ψ a.e. on Γ.

But one has to check a number of properties and, in particular, to prove that the above definition does not enlarge the classical set of quadratic Strebel differentials. The following conjecture gives interesting examples of high order Strebel differentials.

Conjecture 4. *Any rational differential of the form* $\frac{(\sqrt{-1})^k dz^k}{Q_k(z)}$ *where* $Q_k(z)$ *is a arbitrary monic polynomial of degree* k *is Strebel.*

8. Global representability of algebraic functions as the Cauchy transform of real measures. We will discuss some conditions of representability of a branch of an algebraic function in $\mathbb{CP}^1$ almost everywhere as the Cauchy transform of a real measure of total mass 1 compactly supported in $\mathbb{C}$. The obvious necessary condition for that is the existence of a branch at ∞ which behaves asymptotically as $\frac{1}{z}$. Such a branch will be called a *probability branch*. If there exists a real (resp. probability) measure with the above property then we say that the algebraic function *admits a real (resp. probability) motherbody measure*.

Lemma 2. *The curve given by the equation* $P(\mathcal{C}, z) = \sum_{(i,j) \in S(P)} \alpha_{i,j} \mathcal{C}^i z^j = 0$ *has a probability branch at* ∞ *if and only if* $\sum_i \alpha_{i,M(P)-i} = 0$ *where* $M(P) = \min_{(i,j) \in S(P)} i - j$*. In particular, there should be at least two distinct monomials in* $S(P)$ *whose difference of indices equals* $M(P)$*.*

The following fundamental conjecture is motivated by our study of the Heine-Stieltjes problem.

Conjecture 5. *An arbitrary irreducible polynom* $P(\mathcal{C}, z)$ *with a probability branch and* $M(P) = 0$ *admits a probability motherbody measure.*

One can easily check that no real motherbody measure exists unless $M(P) \geq 0$. The case $M(P) = 0$ is covered by the latter conjecture. What about $M(P) > 0$? More exactly, given a finite set S of monomials satisfying the assumptions of Lemma 2 consider the linear space Pol_S of all polynomials $P(\mathcal{C}, z)$ whose Newton polygon is contained in S. What is the (Hausdorff) dimension of the subset $MPol_S \subseteq Pol_S$ of polynomials admitting a motherbody measure?

Our examples seem to confirm the following daring conjecture.

Conjecture 6. *Under the assumption of the latter conjecture the (Hausdorff) codimension of* $MPol_S$ *equals* $M(S)$*.*

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