ON MOMENTS OF A POLYTOPE

DMITRII V. PASECHNIK, BORIS SHAPIRO, AND MICHAEL SHAPIRO

Abstract. We show that a multivariate generating function of appropriately normalized moments of an arbitrary d-dimensional (compact) polytope $\mathcal{P} \subset \mathbb{R}^d$ w.r.t. an arbitrary homogeneous polynomial weight function is a rational function whose denominator is the product of linear forms dual to the vertices of $\mathcal{P}$ raised to the power equal to the degree of the weight. The used normalization of moments is connected with an integral transform closely elated to the Fantappiè transform of the weight.

1. Introduction

The main motivation for the present paper comes from a new efficient algorithm suggested in [4] which recovers an arbitrary convex polytope from the information about its finitely many axial moments. (Leaving an interested reader to explore new features of Google Scholar we do not attempt in this short note to give an overview of the vast field of the classical inverse moment problem going back to H. Poincaré.) The main result of the present paper is a simple formula for the multivariate generating function of all moments of an arbitrary polytope w.r.t. to an arbitrary homogeneous polynomial weight function which generalizes the formulas for the axial moments of polytopes found independently by M. Brion, J. Lawrence, A. Khovanskii-A. Pukhlikov, and A. Barvinok in the late 1980's and early 1990's.

Notation. Let $\mathcal{P} \subset \mathbb{R}^d$ denote an arbitrary compact d-dimensional polytope not necessarily convex or connected. (We assume that $\mathbb{R}^d$ is endowed with the standard scalar product $\langle .,. \rangle$ and with fixed orthonormal coordinates $(x_1, ..., x_d)$.) Let $\rho(x_1, ..., x_d)$ be an arbitrary non-trivial homogeneous polynomial of some degree δ. Given a multiindex $I = (i_1, ..., i_d)$, we write the monomial $x_1^{i_1} \dots x_d^{i_d}$ as $\mathbf{x}^I$. Given I, define the *moment* $m_I^\rho(\mathcal{P})$ of $\mathcal{P}$ w.r.t. the weight function ρ as

$$m_I^\rho(\mathcal{P}) := \int_{\mathcal{P}} x_1^{i_1} x_2^{i_2} ... x_d^{i_d} \rho(x_1, ..., x_d) dx_1 dx_2 ... dx_d = \int_{\mathcal{P}} \mathbf{x}^I \rho(\mathbf{x}) d\mathbf{x}. \tag{1.1}$$

Define the *normalized moment generating function* of the polytope $\mathcal{P}$ w.r.t. ρ by

$$F_{\mathcal{P}}^\rho(u_1, ..., u_d) = F_{\mathcal{P}}^\rho(\mathbf{u}) = \sum_{I=(i_1,...,i_d),\ |I|=n \geq 0} \frac{(n+d+\delta)!}{i_1! \cdots i_d!} m_I^\rho(\mathcal{P}) \mathbf{u}^I.$$

Note that $F_{\mathcal{P}}^\rho(\mathbf{u})$ admits the following, related to the Fantappiè transform, integral representation (a proof is given at the end of Section 2; see also Remark 3).

$$F_{\mathcal{P}}^\rho(\mathbf{u}) = d! \int_{\mathcal{P}} \frac{\rho(\mathbf{x}) dx_1 \dots dx_d}{(1 - \langle \mathbf{x}, \mathbf{u} \rangle)^{d+1}}. \tag{1.2}$$

Remark 1. The definitions of $m_I^\rho(\mathcal{P})$ and $F_{\mathcal{P}}^\rho(\mathbf{u})$ as well as relation (1.2) remain valid for arbitrary compact sets $\mathcal{P} \subset \mathbb{R}^d$. They can be also extended to a larger class of measurable sets but we do not need this extension for the purposes of the present paper.

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Let $\mathcal{V} = (\mathbf{v}_1, ..., \mathbf{v}_N)$ be the set of all vertices of $\mathcal{P}$. Assuming that $\mathcal{P}$ is a simple polytope, consider for each $\mathbf{v} \in \mathcal{V}$ a fixed set of nonzero vectors, parallel to the edges of $\mathcal{P}$ on $\mathbf{v}$, and denote these edge vectors by $w_1(\mathbf{v}), \dots, w_d(\mathbf{v})$. The polyhedral cone $K_\mathbf{v}$ coinciding with the non-negative real span of these edges at $\mathbf{v}$ is called *the tangent cone* at $\mathbf{v}$. For each $K_\mathbf{v}$ define $|\det K_\mathbf{v}| = |\det(w_1(\mathbf{v}), \dots, w_d(\mathbf{v}))|$ to be the volume of the parallelepiped formed by $w_1(\mathbf{v}), \dots, w_d(\mathbf{v})$.

The first result of this paper is as follows.

Theorem 1. For an arbitrary simple convex polytope $\mathcal{P}$ and the constant weight $\rho \equiv 1$ one has

$$F^1_\mathcal{P}(\mathbf{u}) = (-1)^d \sum_{\mathbf{v}\in\mathcal{V}} \frac{\langle \mathbf{v}, \mathbf{u}\rangle^d |\det K_\mathbf{v}|}{\prod_{j=1}^{d} \langle w_j(\mathbf{v}), \mathbf{u}\rangle} \cdot \frac{1}{1 - \langle \mathbf{v}, \mathbf{u}\rangle} \tag{1.3}$$

$$= (-1)^d \sum_{\mathbf{v}\in\mathcal{V}} \frac{|\det K_\mathbf{v}|}{\prod_{j=1}^{d} \langle w_j(\mathbf{v}), \mathbf{u}\rangle} \cdot \frac{1}{1 - \langle \mathbf{v}, \mathbf{u}\rangle}. \tag{1.4}$$

In particular, if $\mathcal{P} = \Delta$ is a simplex one gets the following.

Corollary 1.

$$F^1_\Delta(\mathbf{u}) = \frac{d!\mathrm{Vol}(\Delta)}{\prod_{\mathbf{v}\in\mathcal{V}} (1 - \langle \mathbf{v}, \mathbf{u}\rangle)} \tag{1.5}$$

Given the normalized moment generating function $F^1_\mathcal{P}(\mathbf{u})$ of an arbitrary polytope $\mathcal{P}$ for the weight $\rho \equiv 1$ one can easily obtain the normalized moment generating function for an arbitrary homogeneous weight ρ. Namely, the following statement holds.

Theorem 2. One has

$$F^\rho_\mathcal{P}(u_1, ..., u_d) = \rho\left(\frac{\partial}{\partial u_1}, ..., \frac{\partial}{\partial u_d}\right) \circ F^1_\mathcal{P}(u_1, ..., u_d), \tag{1.6}$$

where $\circ$ denotes the application of a differential operator to a function.

Remark 2. Relation (1.6) remains valid for arbitrary compact $\mathcal{P} \subset \mathbb{R}^d$.

Notice that an arbitrary polytope $\mathcal{P}$ admits a triangulation which only uses the existing vertices of $\mathcal{P}$, see e.g. [2, Theorem 3.1]. Applying Corollary 1 to such a triangulation we get the following.

Corollary 2. The normalized moment generating function $F^\rho_\mathcal{P}(\mathbf{u})$ of an arbitrary polytope $\mathcal{P}$ w.r.t. an arbitrary homogeneous weight ρ of degree δ is a rational function with denominator

$$\prod_{\mathbf{v}\in\mathcal{V}} (1 - \langle \mathbf{v}, \mathbf{u}\rangle)^\delta.$$

Example. Let Δ be a triangle in $\mathbb{R}^2$ with vertices $v_1 = (1, 1)$, $v_2 = (2, 5)$ and $v_3 = (3, 2)$. Its normalized moment generating function w.r.t. $\rho \equiv 1$ equals

$$F^1_\Delta(u, v) = \frac{7}{(1 - u - v)(1 - 2u - 5v)(1 - 3u - 2v)}.$$

Its Taylor expansion about the origin up to the terms of degree 7 is given by

$$7 + 42u + 56v + 175u^2 + 455uv + 329v^2 + 630u^3 + 2387u^2v + 3367uv^2 + 1750v^3 + 2107u^4 + 10318u^3v$$
$$+ 21217u^2v^2 + 21546uv^3 + 8967v^4 + 6762u^5 + 40082u^4v + 106526u^3v^2 + 157976u^2v^3 + 128772uv^4$$
$$+45276v^5 + 21175u^6 + 145845u^5v + 468895u^4v^2 + 900123u^3v^3 + 10744451u^2v^4 + 741993uv^5 + 227269v^6,$$

which implies that

$$m_{00}=\frac{7}{2}, m_{10}=7, m_{0,1}=\frac{28}{3}, m_{2,0}=\frac{175}{12}, m_{11}=\frac{455}{24}, m_{0,2}=\frac{329}{12}, m_{30}=\frac{63}{2}, m_{21}=\frac{2387}{60},$$
$$m_{12}=\frac{3591}{20}, m_{03}=\frac{175}{2}, m_{40}=\frac{2107}{30}, m_{31}=\frac{5159}{60}, m_{22}=\frac{21217}{180}, m_{13}=\frac{3591}{20}, m_{04}=\frac{2989}{10},$$
$$m_{50}=161, m_{41}=\frac{2863}{15}, m_{32}=\frac{7609}{30}, m_{23}=\frac{5642}{15}, m_{14}=\frac{3066}{5}, m_{05}=1078, m_{60}=\frac{3025}{8},$$
$$m_{51}=\frac{6945}{16}, m_{42}=\frac{13397}{24}, m_{33}=\frac{128589}{160}, m_{24}=\frac{153493}{120}, m_{15}=\frac{35333}{16}, m_{06}=\frac{32467}{8}.$$

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2. Proofs

Following Brion-Lawrence-Khovanskii-Pukhlikov-Barvinok, see [1, 2, 4, 5], define for each vector $\mathbf{z}\in\mathbb{R}^d$ the jth axial moment of $\mathcal{P}$ w.r.t. $\mathbf{z}$ as

$$\mu_j(z)=\int_{\mathcal{P}}\langle\mathbf{x},\mathbf{z}\rangle^j d\mathbf{x}.$$

We will use the following important statement.

Theorem 3. One has the following relation

$$\mu_j(\mathbf{z})=\frac{(-1)^d j!}{(j+d)!}\sum_{v\in\mathcal{V}}\langle v,\mathbf{z}\rangle^{j+d}D_v(\mathbf{z}), \tag{2.1}$$

where $D_v(\mathbf{z}):=\frac{|\det K_v|}{\prod_{j=1}^d\langle w_j(v),\mathbf{z}\rangle}$, and $\mathbf{z}$ is an arbitrary vector for which the product $\prod_{j=1}^d\langle w_j(v),\mathbf{z}\rangle$ does not vanish. Moreover, the following identities hold:

$$\sum_{v\in\mathcal{V}}\langle v,\mathbf{z}\rangle^j D_v(\mathbf{z})=0,\quad \text{for each } j=0,\dots,d-1. \tag{2.2}$$

Proof of Theorem 1. Note that in view of relations (2.2) the right-hand side of (1.3) can be rewritten as (1.4). Indeed, writing $(1-\langle\mathbf{v},\mathbf{u}\rangle)^{-1}=\sum_j\langle\mathbf{v},\mathbf{u}\rangle^j$ and expanding (1.3) w.r.t. to jth powers of $\langle\mathbf{v},\mathbf{u}\rangle$, we see that (2.2) implies that for $j<d$ the sum of all terms $\langle\mathbf{v},\mathbf{u}\rangle^j$ vanishes.

To prove (1.3) consider the generating function

$$\Phi_{\mathbf{z}}(u)=\sum_{j=0}^{\infty}\frac{(j+d)!}{j!}\mu_j(\mathbf{z})u^j.$$

Formula (2.1) implies that $\Phi_{\mathbf{z}}(u)$ is rational. Indeed,

$$\Phi_{\mathbf{z}}(u)=\sum_{j=0}^{\infty}(-1)^d\sum_{\mathbf{v}\in\mathcal{V}}\langle v,\mathbf{z}\rangle^{j+d}\frac{|\det K_{\mathbf{v}}|u^j}{\prod_{k=1}^d\langle w_k(\mathbf{v}),\mathbf{z}\rangle}=$$
$$=(-1)^d\sum_{\mathbf{v}\in\mathcal{V}}\frac{\langle\mathbf{v},\mathbf{z}\rangle^d|\det K_{\mathbf{v}}|}{\prod_{k=1}^d\langle w_k(\mathbf{v}),\mathbf{z}\rangle}\sum_{j=0}^{\infty}\langle\mathbf{v},\mathbf{z}\rangle^j u^j=\sum_{\mathbf{v}\in\mathcal{V}}\frac{\langle\mathbf{v},\mathbf{u}\rangle^d|\det K_{\mathbf{v}}|}{\prod_{k=1}^d\langle w_k(\mathbf{v}),\mathbf{u}\rangle}\cdot\frac{(-1)^d}{1-\langle\mathbf{v},\mathbf{u}\rangle},$$

where $\mathbf{u}=u\mathbf{z}$. On the other hand, recalling the multinomial coefficient formula $\binom{|J|}{J}=\frac{|J|!}{j_1!\dots j_d!}$ for a multiindex $J=(j_1,\dots,j_d)\vdash|J|$, one gets

$$\int_{\mathcal{P}}\langle\mathbf{x},\mathbf{z}\rangle^j d\mathbf{x}=\int_{\mathcal{P}}\left(\sum_{i=1}^d x_i z_i\right)^j d\mathbf{x}=\sum_{J\vdash j}\binom{j}{J}\mathbf{z}^J\int_{\mathcal{P}}\mathbf{x}^J d\mathbf{x}=\sum_{J\vdash j}\binom{j}{J}\mathbf{z}^J m_J^1(\mathcal{P}).$$

Therefore,

$$F^1_{\mathcal{P}}(u_1,...,u_d) = \sum_{(j_1,\dots,j_d)\vdash j} \frac{(j+d)!}{j_1!\cdots j_d!} m_{j_1,\dots,j_d} u_1^{j_1}\cdots u_d^{j_d} =$$

$$= \sum_{J:=(j_1,\dots,j_d)\vdash j} \frac{(j+d)!}{j!}\binom{j}{J} m_J(u\mathbf{z})^J = \sum_{j=0}^{\infty}\frac{(j+d)!}{j!}\mu_j(\mathbf{z})u^j = \Phi_{\mathbf{z}}(u).$$

Formula (1.3) follows. □

Proof of Corollary 1. Note that in the case of a simplex $\Delta = (v_0, v_1, \dots, v_d)$ one has that the value $|\det K_{v_i}|$ for any vertex v_i is the same and equals $d!\mathrm{Vol}(\Delta)$. The right-hand side of (1.3) becomes

$$(-1)^d d!\mathrm{Vol}(\Delta)\sum_{i=0}^{d}\frac{\langle v_i,\mathbf{u}\rangle^d}{\prod_{j=1}^d\langle w_j(v_i),\mathbf{u}\rangle}\cdot\frac{1}{1-\langle v_i,\mathbf{u}\rangle} =$$

$$= (-1)^d d!\mathrm{Vol}(\Delta)\sum_{i=0}^{d}\frac{\langle v_i,\mathbf{u}\rangle^d}{\prod_{j\neq i}\langle v_j - v_i,\mathbf{u}\rangle}\cdot\frac{1}{1-\langle v_i,\mathbf{u}\rangle} =$$

$$= (-1)^d d!\mathrm{Vol}(\Delta)\sum_{i=0}^{d}\frac{\zeta_i^d}{\prod_{j\neq i}(\zeta_j-\zeta_i)}\cdot\frac{1}{1-\zeta_i},$$

where $\zeta_i = \langle v_i,\mathbf{u}\rangle$. Computing the common denominator of the latter expression we obtain

$$F^1_{\Delta}(\mathbf{u}) = \frac{(-1)^d d!\mathrm{Vol}(\Delta)}{\prod_{\mathbf{v}\in\mathcal{V}}(1-\langle\mathbf{v},\mathbf{u}\rangle)}\frac{\sum_{i=0}^d(-1)^i\zeta_i^d\prod\limits_{k>l,k\neq i,l\neq i}(\zeta_k-\zeta_l)}{\prod_{s>t}(\zeta_s-\zeta_t)}.$$

To complete the proof notice that

$$\sum_{i=0}^{d}(-1)^i\zeta_i^d\prod_{k>l,k\neq i,l\neq i}(\zeta_k-\zeta_l) = \prod_{s>t}(\zeta_s-\zeta_t) = \det\begin{pmatrix}1 & 1 & \dots & 1\\ \zeta_0 & \zeta_1 & \dots & \zeta_d\\ \zeta_0^2 & \zeta_1^2 & \dots & \zeta_d^2\\ \dots & & & \\ \zeta_0^d & \zeta_1^d & \dots & \zeta_d^d\end{pmatrix}.$$

Indeed, the middle expression is the usual expansion of the Vandermonde determinant, while the left-hand side is the expansion of the same determinant w.r.t. the last row. □

Proof of Theorem 2. To settle the general case assume first that $\rho(x_1,...,x_d) = \mathbf{x}^K = x_1^{k_1}\cdots x_d^{k_d}$ is a monomial and consider

$$\rho\left(\frac{\partial}{\partial\mathbf{u}}\right)\circ F_{\mathcal{P}}(\mathbf{u}) = \frac{\partial^{|K|}}{\partial u_1^{k_1}\cdots\partial u_d^{k_d}}\circ\sum_{I=(i_1,\dots,i_d)\geq 0}\frac{(|I|+d)!}{i_1!\cdots i_d!}m_I(\mathcal{P})\mathbf{u}^I.$$

One gets

$$\frac{\partial^{|K|}}{\partial\mathbf{u}^K}\circ F_{\mathcal{P}}(\mathbf{u}) = \sum_{I=(i_1,\dots,i_d)\geq 0}\frac{(|I|+|K|+d)!}{\prod\limits_{j=1}^{d}(i_j+k_j)!}\prod_{j=1}^{d}\frac{(i_j+k_j)!}{i_j!}m_{I+K}(\mathcal{P})\mathbf{u}^I =$$

$$= \sum_{I=(i_1,\dots,i_d)\geq 0}\frac{(|I|+d+|K|)!}{\prod_{j=1}^d(i_j)!}m_I^K(\mathcal{P}).$$

Observe that the normalizing coefficients in front of the moments in the latter expression depend only on I and $|K|$ but not on particular entries of K. Therefore, for an arbitrary homogeneous ρ of degree δ one gets by additivity

$$\rho\left(\frac{\partial}{\partial \mathbf{u}}\right) \circ F_{\mathcal{P}}(\mathbf{u}) = \sum_{I=(i_1,\ldots,i_d)\geq 0} \frac{(|I|+d+\delta)!}{\prod_{j=1}^{d}(i_j)!} m_I^{\rho}(\mathcal{P}). \qquad \square$$

Proof of (1.2). We are going to use the following identity, which holds for any formal d-variate power series and any d-variate polynomial g

$$g\left(u_1\frac{\partial}{\partial u_1},\ldots,u_d\frac{\partial}{\partial u_d}\right) \circ \sum_I a_I \mathbf{x}^I \mathbf{u}^I = \sum_I a_I g(I) \mathbf{x}^I \mathbf{u}^I. \tag{2.3}$$

Set $g(\mathbf{z}) := \prod_{\ell=1}^{d}\left(\sum_{k=1}^{d} z_k + \ell\right)$, and $g\left(\mathbf{u}\frac{\partial}{\partial \mathbf{u}}\right) := g\left(u_1\frac{\partial}{\partial u_1},\ldots,u_d\frac{\partial}{\partial u_d}\right)$. Using (2.3) and $\sum_{I\geq 0}\binom{|I|}{I}\mathbf{x}^I\mathbf{u}^I = (1-\langle \mathbf{x},\mathbf{u}\rangle)^{-1}$, one obtains

$$F_{\mathcal{P}}^{\rho}(\mathbf{u}) = \sum_{I\geq 0} g(I)\binom{|I|}{I} m_I^{\rho}(\mathcal{P})\mathbf{u}^I = \int_{\mathcal{P}} \sum_{I\geq 0} g(I)\binom{|I|}{I}\mathbf{x}^I\mathbf{u}^I\rho(\mathbf{x})d\mathbf{x} = \int_{\mathcal{P}} g\left(\mathbf{u}\frac{\partial}{\partial \mathbf{u}}\right)$$

$$\circ \sum_{I\geq 0}\binom{|I|}{I}\mathbf{x}^I\mathbf{u}^I\rho(\mathbf{x})d\mathbf{x} = \int_{\mathcal{P}} g\left(\mathbf{u}\frac{\partial}{\partial \mathbf{u}}\right) \circ \frac{\rho(\mathbf{x})d\mathbf{x}}{1-\langle \mathbf{x},\mathbf{u}\rangle} = \int_{\mathcal{P}} \frac{d!\ \rho(\mathbf{x})d\mathbf{x}}{(1-\langle \mathbf{x},\mathbf{u}\rangle)^{d+1}},$$

where in the final derivation we repeatedly made use of the identity

$$\left(\sum_k u_k\frac{\partial}{\partial u_k} + \ell\right) \circ (1-\langle \mathbf{x},\mathbf{u}\rangle)^{-\ell} = \ell(1-\langle \mathbf{x},\mathbf{u}\rangle)^{-\ell-1}. \qquad \square$$

Remark 3. Another point of view on (1.2) is that it is the result of the application of the differential operator $g\left(\mathbf{u}\frac{\partial}{\partial \mathbf{u}}\right)$ to the Fantappiè transform (see e.g. [6]) $\int_{\mathcal{P}} \frac{\rho(\mathbf{x})d\mathbf{x}}{1-\langle \mathbf{x},\mathbf{u}\rangle}$ of the measure ρ on $\mathcal{P}$.

3. Remarks and open problems

Remark 4. A weaker form of Corollary 2 (i.e. the rationality of $F_{\mathcal{P}}^{\rho}(\mathbf{u})$, but without the claim on the particular shape of the denominator) can be derived directly from (1.2) by using Stokes formula, along the lines of [1, Lemma 1].

Remark 5. An alternative approach to (1.3) is by deriving it from (1.2) directly, using *Gram-Brianchon identity* (3.1), cf. e.g. [2]. This identity relates the characteristic function $[\mathcal{P}] : \mathbb{R}^d \to \mathbb{R}$ (i.e. $[\mathcal{P}](\mathbf{x}) = 1$ if $\mathbf{x} \in \mathcal{P}$ and $= 0$ otherwise) of $\mathcal{P}$ with the characteristic functions of the tangent cones $K_{\mathcal{F}}$ of its nonempty faces $\mathcal{F} \subseteq \mathcal{P}$.

$$[\mathcal{P}] = \sum_{\mathcal{F}\subseteq\mathcal{P}} (-1)^{\dim(\mathcal{F})}[K_{\mathcal{F}}]. \tag{3.1}$$

This is also how (2.1)-like indentities are usually derived (except that one works with Laplace transform). As we are going to multiply both sides of (3.1) by $d!(1-\langle \mathbf{x},\mathbf{u}\rangle)^{-d-1}$ and integrate, we need to justify that the corresponding integrals exist—we show the latter for the case of d even. (The case of d odd is set aside as Problem 1.) Indeed, $K_{\mathcal{F}}$ are not compact, and these improper integrals may have a singularity at $\Theta := \{\mathbf{x} \in \mathbb{R}^d \mid 1 = \langle \mathbf{x},\mathbf{u}\rangle\}$. The latter is of codimension 2, provided that $\mathbf{u} \in \mathbb{C}^d$ are generic. There is no loss in generality in assuming $\Theta \cap \mathcal{P} = \emptyset$. `(needs an explanation???)`

As (3.1) is an identity on characteristic functions, it remains valid when we intersect each set involved with a fixed subset of $\mathbb{R}^d$. Let $D_\Omega \subset \mathbb{R}^d$ be such a subset, namely, the ball D_Ω of radius Ω centered at a point $\theta \in \Theta$. Assume that Ω is suffciently big, so that $\mathcal{P} \subset D_\Omega$. Then the following hols.

$$[\mathcal{P}] = [\mathcal{P} \cap D_\Omega] = \sum_{\mathcal{F} \subseteq \mathcal{P}} (-1)^{\dim(\mathcal{F})} [K_{\mathcal{F}} \cap D_\Omega]. \tag{3.2}$$

Multiplying both sides of (3.2) by $d!(1 - \langle \mathbf{x}, \mathbf{u} \rangle)^{-d-1}$ and integrating, one obtains an expansion of (1.2) as a signed sum of integrals of the form

$$I_\Omega(\mathcal{F}, \mathbf{u}) = \int_{K_{\mathcal{F}} \cap D_\Omega} \frac{d!\; d\mathbf{x}}{(1 - \langle \mathbf{x}, \mathbf{u} \rangle)^{d+1}}.$$

We need to argue that this is a legal operation, as integrals in question are singular, and need not exist, generally speaking.

First of all, observe that $I_\Omega(\mathcal{P}, \mathbf{u}) \equiv 0$. Indeed, here the integration is over the whole D_Ω; doing the change of variables $\mathbf{x} \mapsto \mathbf{x} + \theta$ homogenizes the denominator of the integrand, and shifts the center of D_Ω to the origin. Then the change of variables $\mathbf{x} \mapsto -\mathbf{x}$ preserves D_Ω. It follows that $I_\Omega(\mathcal{P}, \mathbf{u}) = -I_\Omega(\mathcal{P}, \mathbf{u})$, as d is even and the degree of the denominator of the integrand is odd. Thus also $I(\mathcal{P}, \mathbf{u}) := \lim_{\Omega \to \infty} I_\Omega(\mathcal{P}, \mathbf{u}) \equiv 0$.

In the remaining cases the domain of integration $K_{\mathcal{F}} \cap D_\Omega$ is contained in a halfspace, specified by a facet of $\mathcal{P}$. Assume that Θ lies outside $K_{\mathcal{F}} \cap D_\Omega$. Then $I_\Omega(\mathcal{F}, \mathbf{u})$ converges absolutely as $\Omega \to \infty$. Thus $I(\mathcal{F}, \mathbf{u}) := \lim_{\Omega \to \infty} I_\Omega(\mathcal{F}, \mathbf{u})$ is well-defined. Next, we argue that this determines the corresponding analytic function of $\mathbf{u}$, by analytic continuation. `Does this make sense? I am no big expert on this...`

Now we claim that $I(\mathcal{F}, \mathbf{u}) \equiv 0$ whenever $K_{\mathcal{F}}$ contains a line. Indeed, we can change the variables affinely so that the inner integral has the form $\int_{-\infty}^{\infty} \frac{dx_1}{(C - u_1 x_1)^{d+1}}$. The latter is identically 0 when d is even, hence the claim. `here probably more care is needed - working with finite Omega etc?`

Finally, we are left with the case $\mathcal{F} = v \in \mathcal{V}$. Then K_v is a simplicial cone, and an affine change of variables that maps K_v onto the positive orthant allows one to compute these integrals directly to obtain (1.3).

Problem 1. Find an explanation of (1.3) along the lines of Remark 5 for odd d.

Problem 2. Find an appropriate version of Theorem 1 applicable to non-simple and/or non-convex polytopes.

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School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, 637371 Singapore
E-mail address: `dima@ntu.edu.sg`

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
E-mail address: `shapiro@math.su.se`

Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA
E-mail address: `mshapiro@math.msu.edu`