KAZHDAN–LUSZTIG POLYNOMIALS FOR CERTAIN VARIETIES OF INCOMPLETE FLAGS

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Abstract. We give explicit formulas for the Kazhdan-Lusztig P- and R-polynomials for permutations coming from the variety $F_{1,n-1}$ of incomplete flags consisting of a line and a hyperplane.

1. Introduction

In [KL1] Kazhdan and Lusztig have associated with each Coxeter group W a family of so-called P-polynomials indexed by pairs of elements $x \prec y$ in W (here $\prec$ denotes the Bruhat partial order on W). If W is the Coxeter group of some (semi)simple group G, then $P_{x,y}(q)$ measures the singularity of the Schubert variety $V_y \subset G/B$ near the Schubert cell $\mathcal{C}_x \subset V_y$. In particular, $P_{x,y}(q) = 1$ for all $x \prec y$ if and only if V_y is nonsingular. Explicit calculation of P-polynomials for an arbitrary pair $x \prec y$ is a very hard problem, even for the case $W = S_n$. One of the most advanced results in this direction is a simple combinatorial algorithm for calculation of P-polynomials for Grassmannian permutations (see [LS1]). Several other particular cases are considered in [Br]. In the case $W = S_n$ there exist several special criteria of nonsingularity of V_y, see e.g. [LSe, LSa]. For example, according to [LSa], the Schubert variety $V_y \subset SL_n/B$ is nonsingular if and only if $y = (y_1, ..., y_n)$ avoids the following two types of subsequences:

1) $y_k < y_l < y_i < y_j$, or
2) $y_l < y_j < y_k < y_i$,

for some $1 \leqslant i < j < k < l \leqslant n$. For evident reasons, a subsequence of the first type is denoted by 3412, while that of the second type, by 4231. Apparently, there exist two different basic types of singularities of V_y related to the two permutations above. Very recently, an almost explicit description of any $P_{x,y}$, where y is a vexillary permutation, i.e. 4231-avoiding, was found, [LS2].

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Besides, paper [LSa] contains the following interesting conjecture describing combinatorially the set of all singular pairs $x \prec y$.

Given $y = (y_1, ..., y_n) \in S_n$, let Z be the set of all $\tau' \prec y$ such that either 1 or 2 below holds.

1. There exist $1 \leqslant i < j < k < l \leqslant n$ such that

(a) $y_k < y_l < y_i < y_j$;

(b) let $\tau' = (b_1, ..., b_n)$, then there exist $1 \leqslant i' < j' < k' < l' \leqslant n$ such that $b_{i'} = y_k, b_{j'} = y_i, b_{k'} = y_l, b_{l'} = y_j$;

(c) let τ (resp., y') be the element obtained from y (resp., τ') by replacing y_i, y_j, y_k, y_l by y_k, y_i, y_l, y_j (resp., $b_{i'}, b_{j'}, b_{k'}, b_{l'}$ by $b_{j'}, b_{l'}, b_{i'}, b_{k'}$), then $\tau' \succ \tau$ and $y' \prec y$.

2. There exist $1 \leqslant j < j < k < l \leqslant n$ such that

(a) $y_l < y_j < y_k < y_i$;

(b) let $\tau' = (b_1, ..., b_n)$, then there exist $1 \leqslant i' < j' < k' < l' \leqslant n$ such that $b_{i'} = y_j, b_{j'} = y_l, b_{k'} = y_i, b_{l'} = y_k$;

(c) let τ (resp., y') be the element obtained from y (resp., τ') by replacing y_i, y_j, y_k, y_l by y_j, y_l, y_i, y_k (resp., $b_{i'}, b_{j'}, b_{k'}, b_{l'}$ by $b_{k'}, b_{i'}, b_{l'}, b_{j'}$), then $\tau' \succ \tau$ and $y' \prec y$.

Conjecture ([LSa]). Singular locus of V_y consists of all elements of Z that are maximal in the Bruhat order.

Another family of polynomials defined in [KL1], so called R-polynomials, often helps to calculate P-polynomials (see [KL1, De, Br]). When q is a prime power, $R_{x,y}(q)$ calculates the number of points in the intersection $V_x \cap w_0 V_y$ over GF_q, where w_0 is the longest element in W. R-polynomials also have a transparent geometrical interpretation over $\mathbb{C}$ (see [SSV1, Cu]). Their explicit calculation is, in general, a simpler problem than that for P-polynomials; nevertheless, one encounters here rather complicated combinatorial problems ([De, SSV2, Br]).

In this note we give simple explicit formulas for P- and R-polynomials for two classes of permutations related to incomplete flags consisting of a line and a hyperplane. Occuring permutations admit both types of forbidden subsequences in the simplest form (either only 3412, or only 4231, but not both of them simultaneously) and provide a nice illustration of the two basic types of singularities of Schubert cells in SL_n/B. Moreover, all singular pairs $x \prec y$ are exactly the ones predicted by the above conjecture.

Let us denote by $F_{i_1,...,i_k}$ the variety of all incomplete flags of type $L^{i_1} \subset L^{i_2} \subset \cdots \subset L^{i_k} \subseteq \mathbb{C}^n$. For brevity, the variety $F_{1,2,...,n}$ of complete flags is denoted by F_n. There exists a natural bundle $F_n \to F_{i_1,...,i_k}$ that just drops redundant subspaces. Evidently, the fiber of this bundle is diffeomorphic to $F_{i_1} \times F_{i_2 - i_1} \times \cdots \times F_{n+1-i_k}$. Each complete flag $f \in F_n$ defines a decomposition of $F_{i_1,...,i_k}$ into Schubert cells. This decomposition is consistent with the above bundle, i.e. the inverse image of a Schubert cell in $F_{i_1,...,i_k}$ is the union of some Schubert cells in F_n. It is easy to see that the index set of this union is an interval in the Bruhat order on S_n. Thus, with each $F_{i_1,...,i_k}$ we associate two sets of permutations, namely, the *maximal* and the *minimal* elements of the corresponding intervals. These sets are denoted $\overline{\mathcal{M}}_{i_1,...,i_k}$ and $\underline{\mathcal{M}}_{i_1,...,i_k}$, respectively.

We consider the variety $F_{1,n-1}$; each point of this variety is a flag consisting of a line and a hyperplane. Below we provide explicit expressions for the polynomials $P_{x,y}(q)$ in the cases $y \in \overline{\mathcal{M}}_{1,n-1}$, x arbitrary and $y \in \underline{\mathcal{M}}_{1,n-1}$, x arbitrary. Besides, we present explicit expressions for the polynomials $R_{x,y}(q)$ in the cases $x, y \in \overline{\mathcal{M}}_{1,n-1}$ and $x, y \in \underline{\mathcal{M}}_{1,n-1}$.

2. Results

It is easy to see that permutations in $\overline{\mathcal{M}}_{1,n-1}$ are of the form $(n-1, n-2, \dots, 1, \dots, n, \dots, 3, 2)$, while those in $\underline{\mathcal{M}}_{1,n-1}$ of the form $(2, 3, \dots, 1, \dots, n, \dots, n-2, n-1)$. Recall that $P_{x,y}(q) \equiv P_{x^{-1},y^{-1}}(q)$ (see [Dy]) and $R_{x,y}(q) \equiv R_{x^{-1},y^{-1}}(q)$. Therefore, it is possible to state all the results in terms of inverse permutations, which seems to us more convenient.

Theorem 1. *Let $y = (i, n, n-1, \dots, 1, j)$, $x = (x_1, \dots, x_n)$. Then*

(i) *y is singular if and only if $i > j$ (and thus y contains only forbidden subsequences of type 3412);*

(ii) *a pair $x \prec y$ with y singular is singular if and only if $x_1 < j$ and $x_n > i$;*

(iii) *if a pair $x \prec y$ is singular, then V_y in some neighborhood of $\mathcal{C}_x$ is diffeomorphic to $K \times A$, where K is a cone of real dimension $4(i-j)+2$ and A is an affine space; thus,*

$$P_{x,y}(q) = 1 + q^{i-j}.$$

Theorem 2. *Let $y = (i, 1, 2, \dots, n, j)$, $x = (x_1, \dots, x_n)$. Then*

(i) *y is singular if and only if $i > j+2$ (and thus y contains only forbidden subsequences of type 4231);*

(ii) *a pair $x \prec y$ is singular if and only if there exists a solution of the following equation and two inequalities in z:*

$$z(z+1) = 2\sum_{p=1}^{z} x_p, \qquad j+1 \leqslant z \leqslant i-2. \tag{1}$$

(iii) *if y is singular, then V_y admits a small resolution of singularities and*

$$P_{x,y}(q) = (1+q)^r,$$

where r is the number of solutions to (1).

Observe that in the cases described in Theorems 1 and 2 the conjecture of Lakshmibai–Sandhya holds true.

Theorem 3. *Let $x = (i, n, n-1, \dots, 1, j)$, $y = (k, n, n-1, \dots, 1, l)$. Then:*

1. *If $k + j < n+1$ or $l + i > n+1$, then $R_{x,y}(q) \equiv 0$.*

2. *Let $l + i \leqslant n+1 \leqslant k+j$. Denote by $\Omega_{j,k}$ the segment $[n+1-j, k]$, and by $\Omega_{l,i}$ the segment $[l, n+1-i]$.*

(i) *If $\Omega_{j,k} \cap \Omega_{l,i} = \varnothing$, then*

$$R_{x,y}(q) = (q-1)^{(k+j)-(l+i)}.$$

(ii) *If $\Omega_{j,k} \cap \Omega_{l,i} \neq \varnothing$ and one of $\Omega_{j,k}$ and $\Omega_{l,i}$ degenerates to a point, then*

$$R_{x,y}(q) = (q-1)^{(k+j)-(l+i)-1}.$$

(iii) *If* $\Omega_{j,k} \cap \Omega_{l,i} \neq \varnothing$ *and both* $\Omega_{j,k}$ *and* $\Omega_{l,i}$ *are nondegenerate, then* $R_{x,y}(q) = (q-1)^a(q^2-q+1)^b$, *where*

$$a = |k+i-n-1| + |l+j-n-1| - 1,$$
$$b = \frac{1}{2}\big((k+j)-(l+i)-|k+i-n-1|-|l+j-n-1|\big)-1.$$

Observe that Theorem 3 allows to calculate R-polynomials for $x=(i,1,2,\dots,n,j)$, $y=(k,1,2,\dots,n,l)$, since by [Hu, Ch.7] one has

$$R_{x,y}(q) \equiv R_{(n+1-k,n,n-1,\dots,1,n+1-l),(n+1-i,n,n-1,\dots,1,n+1-j)}(q).$$

3. Proofs

Proof of Theorem 1. Claim (i) follows immediately from the Lakshmibai–Sandhya criterion mentioned in the introduction.

To prove (ii) observe that the natural projection $\pi\colon F_n \to F_{1,n-1}$ has a smooth fiber diffeomorphic to F_{n-2}. Let V be a Schubert cycle in $F_{1,n-1}$, and $\delta = \pi^{-1}(V)$. Then the stalk of the IH sheaf on δ at an arbitrary point x is isomorphic to the stalk of the IH sheaf on V at the point $\pi(x)$, since π is a bundle with a smooth fiber. Therefore, by [KL2], in order to find P-polynomials we have to calculate the local intersection homology for Schubert cycles in $F_{1,n-1}$.

The Schubert cycle $V_y \subset F_{1,n-1}$ corresponding to the permutation $y=(i,n,n-1,\dots,1,j)$ is diffeomorphic to the Schubert cycle $V_{y^{-1}}$; the latter is a subset of $\mathbb{C}P^{n-1}\times\mathbb{C}P^{n-1}$ defined by the following equations:

$$\begin{gathered} p_1 = p_2 = \cdots = p_{j-1} = 0, \qquad q_{i+1} = q_{i+2} = \cdots = q_n = 0, \\ p_j q_j + \cdots + p_i q_i = 0. \end{gathered} \tag{2}$$

Therefore, V_y is diffeomorphic to $K\times A$, where K is a cone of real dimension $4(i-j)+2$ and A is an affine space. Evidently, the singular locus of this variety is an affine subspace $p_1=\cdots=p_i=0$, $q_j=\cdots=q_n=0$. Therefore, if $\pi(x)$ does not belong to this subspace, then $P_{x,y}(q)\equiv 1$. However, x is projected to the above subspace exactly if $x_1<j$ and $x_n>i$. This condition gives the second claim of the theorem.

To complete the proof it remains to study the stalk of the IH sheaf at the singular locus of V_y. Since the affine part of V_y may be dropped, this is equivalent to the study of the stalk of the IH sheaf at the vertex of an even-dimensional cone. It turns out that such a cone cone is a suspension of the spherization of the tangent bundle to an odd-dimensional sphere.

Indeed, one can introduce new variables and rewrite the equation of the cone in (2) in the form

$$z_1^2 + \cdots + z_{2(i-j)+1}^2 = 0.$$

Real and imaginary parts considered separately yield the equations

$$\begin{gathered} (\operatorname{Re} z_1)^2 + \cdots + (\operatorname{Re} z_{2(i-j)+1})^2 - \big((\operatorname{Im} z_1)^2 + \cdots + (\operatorname{Im} z_{2(i-j)+1})^2\big) = 0, \\ \operatorname{Re} z_1 \cdot \operatorname{Im} z_1 + \cdots + \operatorname{Re} z_{2(i-j)+1} \cdot \operatorname{Im} z_{2(i-j)+1} = 0. \end{gathered}$$

These equations describe two orthogonal vectors $(\operatorname{Re} z_1, \dots, \operatorname{Re} z_{2(i-j)+1})$ and $(\operatorname{Im} z_1, \dots, \operatorname{Im} z_{2(i-j)+1})$ in $\mathbb{R}^{2(i-j)+1}$ that have equal lengths; in other words, it is a cone over the spherization of the tangent bundle to the sphere $S^{2(i-j)+1}$.

The intersection homology of this object can be computed easily using formula (3.3) from [KL2], which says that if Y is smooth and X is a cone over Y, than

$$IH^l(X) = \begin{cases} H^l(Y), & \text{if } l < \dim X, \\ 0, & \text{if } l \geqslant \dim X. \end{cases}$$

On the other hand,

$$H^l(STS^{2(i-j)+1}) = \begin{cases} \mathbb{Z}, & \text{if } l = 0,\ 2(i-j),\ 2(i-j)+1,\ 4(i-j)-1, \\ 0, & \text{otherwise,} \end{cases}$$

and the proof is completed. □

Proof of Theorem 2. Claim (i) follows immediately from the Lakshmibai–Sandhya criterion mentioned in the introduction.

Let $M = \cup_\gamma M_\gamma$ be a stratified manifold with an open dense nonsingular locus M_0 and nonsingular strata M_γ, and Z be a smooth manifold. Following [Ze, Mc], we say that $Z \to M$ is a *small resolution of singularities* if 1) fibers $Z \to M_\gamma$ are locally trivial, and 2) the preimage Z_x of any point $x \in M_\gamma$ satisfies the inequality

$$2 \dim Z_x < \operatorname{codim} M_\gamma.$$

If a stratified manifold admits a small resolution of singularities, then the stalk of its IH sheaf at any point is isomorphic to the ordinary homology of the preimage of this point, see [Ze].

Let $y = (i, 1, \dots, n, j)^{-1} \in \underline{\mathcal{M}}_{1,n-1}$, $i > j+2$, and V_y be the corresponding Schubert variety (with respect to a fixed complete flag $f = \{f^1 \subset f^2 \subset \dots \subset f^{n-1}\}$). Evidently, V_y is given by the following conditions:

$$\begin{aligned} V_y = \{\varphi = \{\varphi^1 \subset \varphi^2 \subset \dots \subset \varphi^{n-1}\} \in F_n :& \\ \varphi^1 \subset f^i,\ \varphi^2 \supset f^1,\ \varphi^3 \supset f^2,\ \dots, \varphi^j \supset f^{j-1},& \\ \dim(\varphi^{j+1} \cap f^{j+1}) \geqslant j, \dots, \dim(\varphi^{i-1} \cap f^{i-1}) \geqslant i-2,& \\ \varphi^{i-1} \subset f^i, \dots, \varphi^{n-2} \subset f^{n-1}\}.& \end{aligned}$$

Let $\Pi\colon V_y \to F_{n-j+1}$ be the natural quotient mapping,

$$\Pi\colon \{\varphi^1 \subset \varphi^2 \subset \dots \subset \varphi^{n-1}\} \mapsto \{\varphi^j/f^{j-1}, \varphi^{j+1}/f^{j-1}, \dots, \varphi^{n-1}/f^{j-1}\},$$

and $\Theta\colon V_y \to F_i$ be the natural restriction mapping,

$$\Theta\colon \{\varphi^1 \subset \varphi^2 \subset \dots \subset \varphi^{n-1}\} \mapsto \{\varphi^1, \varphi^2, \dots, \varphi^{i-1}\}.$$

To describe the images of Π and Θ we need the following two operators on permutations. The first of them, $\pi\colon S_n \to S_{n-j+1}$, acts as follows: given $\sigma \in S_n$, it removes all nonpositive entries from the sequence $\sigma(1)-j+1, \sigma(2)-j+1, \dots, \sigma(n)-j+1$ and takes the inverse of the obtained permutation in S_{n-j+1}. The second one, $\theta\colon S_n \to S_i$, removes all the entries exceeding i from the sequence $\sigma(1), \sigma(2), \dots, \sigma(n)$ and takes the inverse of the obtained permutation in S_i.

Lemma 1. (i) $\Pi(V_y) = V_{\pi(y^{-1})}$;
(ii) $\Theta(V_y) = V_{\theta(y^{-1})}$;
(iii) *both* $\Pi|_{V_y}$ *and* $\Theta|_{V_y}$ *are locally trivial bundles with smooth fibers.*

Proof. Almost evident. $\square$

Lemma 1 implies

$$IH^*(V_y) = IH^*(V_{\pi(y^{-1})}) = IH^*(V_{\theta(y^{-1})}) = IH^*(V_{\theta(\pi(y^{-1}))}).$$

However, it is easy to see that $\theta(\pi(i,1,\dots,n,j)) = (i-j+1,2,\dots,i-j,1)$; thus, in order to find $IH^*(V_y)$ it suffices to consider only y's of the form $(n,2,\dots,n-1,1)$.

Let $\mu = \{\mu_1,\dots,\mu_k\}$ be a set of integers satisfying $2 \leqslant \mu_1 < \mu_2 < \dots < \mu_k \leqslant n-2$. We define $\Sigma_\mu = \{\varphi \in V_y \colon \varphi^j = f^j,\ j \in \mu;\ \varphi^j \neq f^j,\ j \notin \mu\}$. It is easy to see that each Σ_μ (for a nonempty μ) is a nonsingular subset of V_y.

Lemma 2. *The set* $\Sigma_\varnothing = V_y \setminus \cup_{\mu\neq\varnothing}\Sigma_\mu$ *is an open dense nonsingular subset of* V_y.

Proof. Evidently, $\Sigma_\varnothing$ is an open dense subset of V_y. To prove nonsingularity, we shall introduce smooth coordinates in a neighborhood U of an arbitrary point $\varphi_0 \in \Sigma_\varnothing$. Indeed, denote by $\chi^k(\varphi)$ the intersection $\varphi^{k+1} \cap f^{k+1}$ for $k = 1,\dots,n-3$. For $\varphi \in \Sigma_\varnothing$ one readily gets $\dim \xi^k(\varphi) = k$; thus, χ maps $\Sigma_\varnothing$ to F_{n-3}. Moreover, it is easy to see that the image of U under χ is an open $(n-3)$-dimensional disk. So, all we need is to introduce smoothly varying coordinates on the fibers of the bundle defined by χ.

By definition, one has $\varphi_0^{n-2} \neq f^{n-2}$. Let l^2 denote an affine two-dimensional plane in $\mathbb{C}^n$ transversally intersecting φ_0^{n-2} at some point far enough from f^{n-2}, and let $q(\varphi)$ denote the intersection point of l^2 and φ^{n-2}. Evidently, the pair $(\chi^{n-3}(\varphi), q(\varphi))$ defines φ^{n-2} in a unique way.

Next, let w_l^1, $l = 2,\dots,n-3$, denote an affine one-dimensional line in φ_0^{l+1} intersecting φ_0^l transversally at some point far enough from f^l. We consider the projection $\Psi_l \colon \mathbb{C}^n \to \varphi_0^{l+1}$ along an arbitrary subspace ψ^{n-l-1} transversal to φ_0^{l+1}. Let $r_l(\varphi)$ be the intersection point of w_l^1 and $\Psi_l(\varphi^l)$; then the triple $(\varphi^{l+1}, \chi^{l-1}(\varphi), r_l(\varphi))$ defines φ^l in a unique way.

Finally, for φ^2 and φ^{n-2} fixed, one can choose in an obvious way coordinates $(\kappa_1(\varphi), \kappa_2(\varphi)) \in \mathbb{C}^2$ defining φ^1 and φ^{n-1} for $\varphi \in U$ such that $\varphi^1 \subset \varphi^2$, $\varphi^{n-1} \supset \varphi^{n-2}$.

Therefore, the set $(\chi(\varphi), q(\varphi), r_2(\varphi), \dots, r_{n-3}(\varphi), \kappa_1(\varphi), \kappa_2(\varphi))$ defines coordinates in U, thus proving the smoothness of $\Sigma_\varnothing$ in a neighborhood of φ_0. $\square$

Let now $\varphi \in \Sigma_\mu$, $\mu \neq \varnothing$; we say that a flag $\chi \in F_{n-3}$ is compatible with φ if the following condition holds:

$$\begin{aligned} &\chi^l = \varphi^{l+1} \cap f^{l+1}, && \text{if } l+1 \notin \mu, \\ &\chi^l \subset f^{l+1}, && \text{if } l+1 \in \mu. \end{aligned}$$

Let $\mathcal{Z}$ denote the set of all pairs (φ, χ) such that $\varphi \in V_y$ and χ is compatible with φ.

Lemma 3. (i) *$\mathcal{Z}$ is nonsingular.*

(ii) *The projection $(\varphi, \chi) \mapsto \varphi$ is a small resolution of singularities for V_y.*

Proof. The proof of the first claim is similar to the proof of Lemma 2, and is thus omitted.

To prove the second claim, observe that $\dim V_y = 2n - 3$, while

$$\dim \Sigma_\mu = \sum_{l=1}^{k+1} \max\{2(\mu_l - \mu_{l-1}) - 3, 0\}$$

(provided we stipulate $\mu_0 = 0$ and $\mu_{k+1} = n$). The latter formula can be rewritten as follows. We say that integers p, q, $1 \leqslant p \leqslant q \leqslant n-1$, belong to the same connected component with respect to μ if $p, q \notin \mu$ implies $r \notin \mu$ for any $p \leqslant r \leqslant q$. Let $\#_\mu$ denote the number of connected components with respect to μ. Then one easily gets $\dim \Sigma_\mu = 2(n-k-1) - \#_\mu$. Now, the dimension of the preimage of any element of Σ_μ is equal to k. Therefore, the inequality in the definition of small resolutions is equivalent to $\#_\mu > 1$. However, by the definition of μ, one has $1 \notin \mu$ and $n-1 \notin \mu$, which means that each μ defines at least two connected components. $\square$

To accomplish the proof of Theorem 2 it is enough to calculate the ordinary homology of the preimage of any element $\varphi \in \Sigma_\mu$. However, from the homological point of view this preimage is equivalent to the direct product of k copies of $\mathbb{C}P^1 \approx S^2$. Hence, $P(x, y) = (1+q)^k$. Here k is the number of subspaces of φ coinciding with the corresponding subspaces of f, and thus k is equal to the number of solutions of the following equation and two inequalities in z:

$$z(z+1) = 2\sum_{p=1}^{z} x_p, \qquad j+1 \leqslant z \leqslant i-2. \qquad \square$$

Proof of Theorem 3. The proof follows from the general combinatorial procedure of finding R-polynomials described in [SSV2]. For the sake of self–completeness, we borrow from [SSV2] several notions related to permutations.

A *decreasing subsequence* in an arbitrary permutation $\pi = (i_1, \ldots, i_n)$ is a subsequence $s = (i_{j_1}, i_{j_2}, \ldots, i_{j_k})$ such that $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant k$ and $i_{j_1} > i_{j_2} > \cdots > i_{j_k}$.

The *reduced length* of a decreasing subsequence is equal to the number of its elements minus one. The *domination* of a decreasing subsequence is equal to the number of elements $i_j \in \pi$ for which there exists an element $i_l \in s$ such that $j < l$ and $i_j < i_l$.

The *cyclic shift* of π with respect to a decreasing subsequence $s = (i_{j_1}, i_{j_2}, \ldots, i_{j_k})$ is the transformation sending i_{j_1} onto i_{j_k}, i_{j_2} onto $i_{j_1}, \ldots$, i_{j_k} onto $i_{j_{k-1}}$ and preserving the rest of the elements. (If s is trivial, that is, consists of just one element, then the transformation is identical.)

According to the procedure, to find $R_{x,y}(q)$ we start from the following three permutations: $\alpha = y^{-1} = (n-1, n-2, \ldots, n, \ldots, 1, \ldots, 3, 2)$ (with 1 at position k and n at position l), $\beta = x^{-1}w_0 = (2, 3, \ldots, n, \ldots, 1, \ldots, n-$

$2, n-1)$ (with 1 at position $n+1-i$ and n at position $n+1-j$), and $\sigma = w_0 = (n, \dots, 1)$.

The procedure builds a tree, whose vertices are permutations. The root of the tree is σ. The tree is built level by level. At the ith step of the procedure we find decreasing subsequences starting at element $\alpha(i)$ and ending at position $\beta(i)$ in each permutation on level $i-1$. We next perform cyclic shifts with respect to each of these decreasing subsequences and thus obtain the set of the children for each permutation of level $i-1$. In each of the obtained permutations we block the largest element of the corresponding decreasing subsequence. (Blocking just means that this element cannot be included in a decreasing subsequence on all subsequent steps of the algorithm, and that it is not counted in the domination of such a subsequence.) Each edge of the tree (joining a parent with its child) thus corresponds to a decreasing subsequence in the parent permutation. Such an edge gets a weight (w_1, w_2), where w_1 is the reduced length of the corresponding decreasing subsequence, and w_2 is the domination of the decreasing subsequence. The weight (W_1, W_2) of a vertex of the tree is the sum of the weights of edges on the unique path from this vertex to the root. According to [SSV2], $R_{x,y}(q)$ equals the sum of the products $(q-1)^{W_1} q^{W_2}$ over all vertices of the nth level (and, thus, equals 0 if the tree has less than n levels).

Now we can prove the theorem. First, let $k + j < n + 1$. Then all the permutations on level $k-1$ have 1 at position n. Thus, on step k there are no decreasing subsequences starting at 1 and ending at any position different from n, and hence the tree has only $k-1$ levels.

Let now $l+i > n+1$. Then all the permutations on level $n-i$ have n at the first position. Thus, on step $n+1-i$ there are no decreasing subsequences ending at the first position and starting at any element different from n; hence, the tree has only $n-i$ levels.

Let now $\Omega_{j,k} \cap \Omega_{l,i} = \varnothing$. There are two possibilities: $k < l$ and $k > l$. Assume that $k < l$ (the proof for the other case is similar). Then on the first $n-j$ steps of the procedure we always have only one decreasing subsequence, namely, the trivial (one–element) one. So, the tree after $n-j$ steps is just a path, and the weight of each of its edges is $(0, 0)$. Step $n+1-j$ suggests a variety of decreasing subsequences starting at $j-1$ and ending at position n. However, for each such subsequence not including element $j-2$ the resulting permutation (on level $n+1-j$) does not have children, since the position of $j-2$ in any such permutation is $n+3-j$, while the end of the subsequence starting at $j-2$ at step $n+2-j$ should be at position $n+2-j$. For similar reasons, for each subsequence starting at $j-1$ and avoiding $j-3, \dots, n+1-k$ (at step $n+1-j$), the resulting permutation does not have descendants at level $n+3-j, \dots, k-1$, respectively. On the other hand, if a decreasing subsequence on step $n+1-j$ starts at $j-1$ and includes any element distinct from $j-2, \dots, n+1-k$ and 1, then the resulting permutation does not have descendants at level k. Therefore, the tree has only one vertex at level k, and it corresponds to the decreasing subsequence $(j-1, j-2, \dots, n-k+1, 1)$ at level $n+1-j$. Accordingly, the weight of this vertex is $(k-n-1+j, 0)$.

On the steps $k+1, \dots, l-1$ we again have each time only the trivial decreasing subsequence, and thus the weight of this part of the tree is $(0, 0)$. Each of the steps $l, \dots, n-i$ gives rise to exactly one decreasing subsequence (of length 2); these are $(n, n+1-l)$, $(n+1-l, n-l), \ldots, (i+2, i+1)$. Since each edge of this part of the tree has weight $(1, 0)$, the total weight of the part is $(n+1-l-i, 0)$. Finally, at all the steps $n+1-i, \ldots, n$ there is again only the trivial decreasing subsequence. Thus, the tree has only one vertex at level n, and its weight is $(k+j-l-i, 0)$. This proves claim 2(i) of the theorem.

Let now $\Omega_{j,k} \cap \Omega_{l,i} \neq \varnothing$, and one of these segments be degenerate. Assume without loss of generality that $\Omega_{j,k}$ is nondegenerate. This means that $n+1-j < l = n+1-i < k$. The proof in this case goes along the same lines that in the previous case. The only difference is that the decreasing subsequence that survives at step $n+1-j$ is $(j-1, j-2, \ldots, n+2-k, 1)$, and thus the weight of the unique vertex at level n is $(k-n-2+j, 0)$. Since in this case $(k+j)-(l+i)-1 = k+j-n-2$, we get claim 2(ii).

Finally, let us consider the case of nondegenerate intersecting segments. There are four possibilities described by the inequalities $n+1-j \leqslant l < n+1-i \leqslant k$, $n+1-j \leqslant l < k \leqslant n+1-i$, $l \leqslant n+1-j < n+1-i \leqslant k$, and $l \leqslant n+1-j < k \leqslant n+1-i$, respectively. Since the proof in all theses cases goes along the same lines, we restrict ourselves to the first case. The reasoning is similar to that for the case 2(i). Namely, we get that there is no branching at levels $1, \ldots, n-j$, and that the elements $j-1, j-2, \ldots, n+1-l$ should be included in a decreasing subsequence at step $n+1-j$ in order to survive up to $l-1$. The only decreasing subsequence at step l has length 2; it starts at n (at position 1) and ends at position l. Now, if the element $n-l$ was included in a surviving decreasing subsequence at step $n+1-j$, then it will appear at position 1 after step l, and thus, the only decreasing subsequence at step $l+1$ is the two–element subsequence starting at position 1 and ending at position $l+1$. However, if $n-l$ was not included in a surviving decreasing subsequence at step $n+1-j$, then the element at position 1 after step l is smaller than $n-l$; thus, on step $l+1$ this element will be dominated by $n-l$ (which will be the only element of the only decreasing subsequence at this step). Therefore, $n-l$ may be or may be not included in a surviving decreasing subsequence at step $n+1-j$. In the first case its contribution to the weight of any of its descendants at levels below $l+1$ is $(2, 0)$: $(1, 0)$ for participation in a “long” decreasing subsequence at step $n+1-j$ and $(1, 0)$ more for participation in the “short” decreasing subsequence at step $l+1$. In the second case, the contribution equals to $(0, 1)$, for participation in the trivial decreasing subsequence with domination number 1 at step $l+1$. The same statement holds also for the elements $n-1-l, \ldots, i+1$. All the elements $i, \ldots, n+2-k$ should be necessarily included in any surviving decreasing subsequence at step $n+1-j$ (for the same reasons as the elements $j-1, \ldots, n+1-l$). Finally, there is no branching at steps $k+1, \ldots, n$, and the elements $n-k+1, \ldots, 2$ should not be included in a surviving decreasing subsequence at step $n+1-j$.

Therefore, the weight of a vertex of the tree at level n equals to $(k+l+j+i-2n-3+2m, n-l-i-m)$, where m is the number of the elements among $n-l, \ldots, i+1$ included in the corresponding decreasing subsequence at step $n+1-j$. Thus, for any fixed m, $0 \leqslant m \leqslant n-l-i$, there are exactly $\binom{n-l-i}{m}$ vertices at level n having the same weight. So, the R-polynomial in this case equals

$$(q-1)^{k+l+j+i-2n-3}\sum_{m=0}^{n-l-i}\binom{n-l-i}{m}(q-1)^{2m}q^{n-l-i-m}$$
$$= (q-1)^{k+l+j+i-2n-3}(q^2-q+1)^{n-l-i}.$$

Finally, for $n+1-j \leqslant l \leqslant n+1-i \leqslant k$ one has

$$|k+i-n-1| + |l+j-n-1| - 1 = k+l+j+i-2n-3,$$
$$\frac{1}{2}((k+j)-(l+i)-|k+i-n-1|+|l+j-n-1|)-1 = n-l-i,$$

and thus claim 2(iii) is proved. □

Problem 1. *Calculate explicitly $P_{x,y}(q)$, where $x \prec y$ comes from the flag variety F_{i_1,i_2} for all $i_1 < i_2$.*

Problem 2. *Calculate explicitly $P_{x,y}(q)$, where y avoids forbidden sequences of type* 3412.

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