HOW TO RUN A ROACH

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ABSTRACT. In this paper we study the topology of the configuration space of a device with d legs (alias 'roach') such that at most k of them are allowed to be off the ground. We suggest control strategies providing that this roach moves stably forward starting from a 'maximal' subset of admissible initial positions of legs. The meaning of the term 'maximal' will be explained below. Advantages of the suggested control strategies are their simplicity in realization and stability.

1. Introduction

Consider a device which we call a *roach* whose primary goal is to move forward using its legs. Each leg is a single limb rotating around its horizontal axis. Mechanics of a such motion suggests that there exists a 'forbidden' subset of the configuration space, e.g. if more than a certain number of legs points up (i.e. are sufficiently close to the vertical direction) then our device fails. Our objective is to find a controlling strategy for rotation of roach's legs to provide its stable forward motion starting from a largest possible set in the configuration space of the initial position of legs.

In order to describe our set-up mathematically we assume that the total number of roach's legs is d and we will concentrate here on the situation when at least $k \leq d$ legs up leads to a failure. This means that on the d-dimensional torus T^d which is the configuration space of legs positions we have a chosen *set of failure positions* $\mathcal{F}_k \subset T^d$ which is a certain open domain with the smooth boundary which retracts to the k-skeleton $Sk_k \subset T^d$ (w.r.t. the standard cell stratification of T^d). In other words, $\mathcal{F}_k$ is the set of all legs positions where at least k angles are close to $\frac{\pi}{2}$. Its complement $\Omega_k = T^d \setminus \mathcal{F}_k$ is called the *set of admissible initial positions.*

By a *autonomous stable control* we mean a piecewise smooth vector field $\mathbb{V}$ defined at least in $\Omega_k \subset T^d$ (including its boundary $\partial\Omega_k$) and such that:
a) $\mathbb{V}$ points inside Ω_k on $\partial\Omega_k$;

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b) as large subset $\mathcal{A}_k \subset \Omega_k$ as possible is the basin of a single attractor of $\mathbb{V}$'s trajectories, this attractor being is a simple closed curve in Ω_k with the winding number $(1, ..., 1)$.

If these requirements are met then $\mathbb{V}$ provides a control strategy such that the roach stably runs forward when the initial position of its legs belongs to $\mathcal{A}_k$. We are deliberately vague here about the meaning of the expression 'as large as possible' which will be clarified below.

The set $\mathcal{A}_k$ will be further referred to as the *set of controllable initial positions* and its complement $\mathcal{B}_k = \Omega_k \setminus \mathcal{A}_k$ is called the *set of uncontrollable initial positions.*

By a *time-dependent stable control* we mean a vector field $\mathbb{V}_t$ in $\Omega_k \times [0, \infty)$ which has similar properties to the above one but, in general, is time-dependent.

Below we describe what we think is a natural choice of $\mathcal{A}_k$, a time-dependent and an autonomous controls, which work for a wide class of choices of $\mathcal{F}_k$. Finally, we describe an intriguing discrete dynamical system associated with our choice of $\mathcal{A}_k$.

The structure of the paper is as follows. In § 2 we describe important topological preliminaries. In § 3-4 we describe a non-autonomous and an autonomous controls for the suggested device respectively for the case when $\mathcal{F}_k$ is exactly the k-skeleton of T^d. Finally, in § 6 we describe our discrete dynamical system.

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2. Topology of $\mathcal{A}_k$ and its complement in Ω_k

2.1. **Basic facts.** To start with notice that the set $\mathcal{F}_k$ of failure positions is retractible to the k-skeleton $Sk_k \subset T^d$ and thus the embedding of its complement Ω_k to $T^d \setminus Sk_k$ is a homotopy equivalence. The later space $T^d \setminus Sk_k$ is in its turn is retractible to Sk_{d-k-1}, see e.g. [1]. In particular, the integer (co)homology groups of Ω_k coincide with these of T^d up to the dimension $d-k-1$ and are trivial otherwise. The latter isomorphism of (co)homology groups is simply induces by the inclusion $\Omega_k \subset T^d$. Also the fundamental group $\pi_1(\Omega_k)$ of Ω_k is isomorphic to that of T^d and thus it s coincides with $\mathbb{Z}^d$ for $k \leq d-2$ and is the free group with d generators for $k = d-1$.

Recall that, by our assumptions the flow defined by $\mathbb{V}$ should retract $\mathcal{A}_k$ to a single closed trajectory with the winding number $(1, ..., 1)$ and thus $\mathcal{A}_k$ has the homotopy type of S^1.

Choose some closed simple oriented curve Γ in Ω_k (referred further as 'gait') with the winding number $(1, ..., 1)$. If $\mathcal{F}_k$ is a sufficiently small neighborhood of Sk_k then we can choose Γ among the geodesics of the flat metrics on T^d, i.e. among curves of the form $\Gamma_\gamma = \gamma + t(1, ..., 1)$ on T^d where $t \in \mathbb{R}$ and γ is a sufficiently generic point in T^d.

Definition 1. We say that Γ is *unknotted* if it is homotopy equivalent as a knot to the flat curve above (i.e. there exists a diffeomorphism of the ambient space Ω_k sending one curve to the other disregarding there parametrizations). We say that the set $\mathcal{A}_k$ is *unknotted* if it is homotopy equivalent to some δ-neighborhood of an unknotted Γ in Ω_k.

Definition 2. A set $\mathcal{A}_k$ of controllable initial positions is called *set-theoretically maximal* in Ω_k if no proper superset of $\mathcal{A}_k$ in Ω_k has the same the homotopy type as $\mathcal{A}_k$.

In what follows we will restrict our attention only to the class of unknotted $\mathcal{A}_k$'s and to the set-theoretical maximality of $\mathcal{A}_k$ although other more geometrical versions of this notion make perfect sense.

Let us present our construction of $\mathcal{A}_k$ in the basic case $\mathcal{F}_k = Sk_k$ and, therefore, $\Omega_k = T^d \setminus Sk_k$ for some $0 \leq k \leq d-2$. (WHAT ABOUT OUR ASSUMPTION THAT $\mathcal{F}_k$IS AN OPEN DOMAIN?

Main construction. Represent T^d as the d-dimensional cube $K_d = [-1, 1]^d$ with its parallel sides pairwise identified. We use the agreement that the origin $O = (0, 0, ..., 0) \in K_d$ corresponds to the position 'all legs down'. Take the cone $\mathcal{C}_d$ in T^d over the skeleton Sk_{d-2} with the vertex at O. This cone $\mathcal{C}_d$ is a singular hypersurface in T^d stratified by the cones over different coordinate subtori contained in Sk_{d-2}. Notice that $\mathcal{C}_d$ contains $\binom{d}{2}$ strata of maximal dimension which are cones over $\binom{d}{2}$ individual coordinate subtori of codimension 2. The complement $T^d \setminus \mathcal{C}_d$ consists of d 'polytopes' each being the union of two pyramids over $(d-1)$-dimensional base cubes, see Fig.1. The closure of each such cube coincides with one of the coordinate tori of codimension 1. Let us denote these polytopes as $Pyr_i,\ i = 1, ..., d$ where i is the coordinate missing in the $(d-1)$-dimensional cube which is coned. The gait Γ hits the boundary of each Pyr_i in two points belonging to some faces on $\mathbb{C}_d$. Each such face is the cone over some $(d-2)$-dimensional open cube. The face where Γ enters (resp. leaves) Pyr_i is called the ith entrance face F_i (resp. the ith exit face G_i) and the corresponding points are called the entrance/exit points. Another important point within Pyr_i is the point where Γ hits the base of Pyr_i, i.e. the corresponding $(d-1)$-dimensional cube, see Fig. 2. Notice also that Γ defines a cyclic order on the set of all Pyr_i according to the cyclic order in which it hits them, see Fig.1. Note that the exit face for any pyramid is at the same time the entrance face of the next one in this cyclic order. Wlog let us assume that after we fix Γ we can enumerate the coordinates $(x_1, ..., x_d)$ on T^d in such a way that this cyclic order is $x_1 < x_2 < x_3 < ... < x_d < x_1$.

Finally, let $\widetilde{\mathcal{C}}_d = \mathcal{C}_d \setminus \cup_{i=1}^d F_i = \mathcal{C}_d \setminus \cup_{j=1}^d E_j$ be the cone with all open entrance (or, equivalently, exit) faces removed and define the *universal set of controllable positions* $\widetilde{\mathcal{A}}_d$ as

$$\widetilde{\mathcal{A}}_d = T^d \setminus \widetilde{\mathcal{C}}_d.$$

Notice that $\widetilde{\mathcal{A}}_d$ belongs to Ω_k for all $k \geq 2$. We set $\mathcal{A}_k = \widetilde{\mathcal{A}}_d$ for any such k.

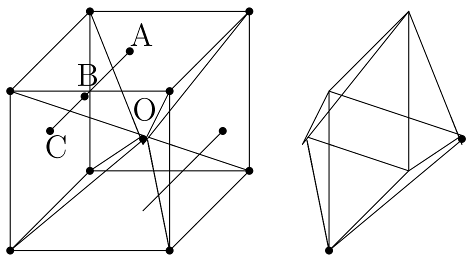

Figure 1. T^3 represented as the 3-cube and the cone $\mathcal{C}_3$.

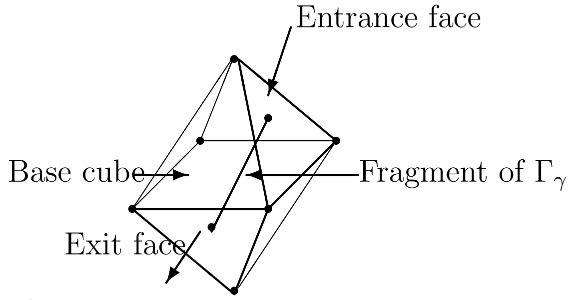

Figure 2. A single pyramid Pyr_i in three dimensions.

Description of $\widetilde{\mathcal{A}}_d$ in terms of leg positions. For the sake of clarity let us present a simple description of $\widetilde{\mathcal{A}}_d$ in terms of configurations of legs. Let $(x_1, ..., x_d)$, $-3\pi/2 \leq x_i \leq \pi/2$ be the usual angular coordinates on the torus T^d. Notice that we assume we have a cyclic order on $x_1 < x_2 < ... < x_d < x_1$ defined by the gait Γ, see above. The unusual interval $-3\pi/2 \leq x_i \leq \pi/2$ is motivated by the fact that the origin of the above cube K_d is chosen at the position 'all legs down', i.e. $x_i = -\pi/2$. The coordinates $(t_1, ..., t_d)$ on K_d are thus given by $t_i = (x_i + \pi/2)/\pi$, $i = 1, ..., d$. The i-th open pyramid Pyr_i consists then of exactly those leg positions, (i.e. d-tuples of labelled points on S^1) for which the i-th leg has the maximal height, i.e. $\sin x_i > \sin x_j, \forall j \neq i$. Its entrance face is the set of all leg positions when exactly the $(i-1)$-st leg and the i-th leg are at the maximal height among all legs. Additionally, they are not allowed to coincide and they are in the correct cyclic position. Analogously. its exit face is the set of all leg positions when exactly the i-th and the $(i+1)$-st leg are at the maximal height among all legs. As before, they are not allowed to coincide and they are in the correct cyclic position, see below.

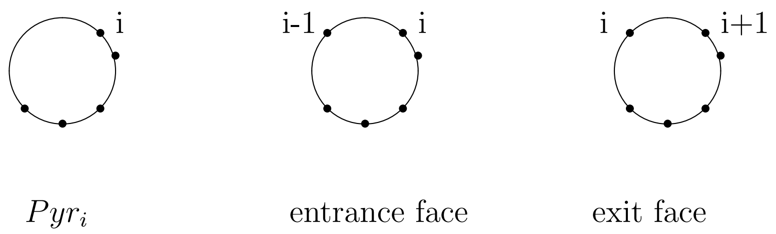

Figure 3. Intrepretation Pyr_i in terms of leg configurations.

Proposition 1. *The set $\widetilde{\mathcal{A}}_d$ retracts in Ω_k to Γ (and is, therefore, homeomorphic to $S^1 \times B^{d-1}$) and is set-theoretically maximal in T^d and, therefore, in each $\Omega_k \subset T^d$.*

Proof. The argument proving the retraction is the same for all k and is as follows. Recall that $\widetilde{\mathcal{A}}_d$ is glued of a chain of d open convex and cyclically ordered polytopes $Pyr_i,\ i = 1, ..., d$ where each Pyr_i is glued to the next one along their common face. The curve Γ can be thought as a piecewise linear closed curve within $\widetilde{\mathcal{A}}_d$. We define a certain piecewise linear and continuous on the whole $\widetilde{\mathcal{A}}_d$ 'projection' retracting it to Γ. Namely, identify the pyramid Pyr_i with a convex polytope in $\mathbb{R}^d$ as on Fig.2. Take two affine hyperplanes which are the linear spans of the entrance and exit faces respectively and intersect them. This intersection is an affine subspace H of codimension 2 in $\mathbb{R}^d$ which lies outside the open pyramid and intersects it in the cube of codimension 3. Finally, consider the family of all hyperplanes containing H. This family foliates Pyr_i is such a way that the entrance and exit faces are two of the fibers. Each fiber is a convex polytope intersecting Γ at a single point. Project now each fiber on this unique point of Γ. This map is the required retraction of Pyr_i. Notice that since it sends the entrance face to the entrance point and the exit face to the exit point one each pyramid these maps together define a continuous map from $\widetilde{\mathcal{A}}_d$ to Γ.

Let us show now that any proper superset of $\widetilde{\mathcal{A}}_d$ in T^d has a different homotopy type than $\widetilde{\mathcal{A}}_d$. In fact, we show that adding any point to $\widetilde{\mathcal{A}}_d$ in T^d we create a new element in $H_1(T^d)$ in addition to the element $(1, 1, 1, ..., 1)$ realized by $H_1(\widetilde{\mathcal{A}}_d)$. Since this new element is nontrivial already in T^d it can not disappear if we add to $\widetilde{\mathcal{A}}_d$ more points. The problem is, therefore, reduced to adding a single point to $\widetilde{\mathcal{A}}_d$. Assume that we glue to $\widetilde{\mathcal{A}}_d$ some point $p \in T^d$ which is not in $\widetilde{\mathcal{A}}_d$. Notice that by our construction any such point p belongs to the common boundary of at least two pyramids Pyr_i and Pyr_j which are not neighbors in the cyclic order. Using the above realization of each pyramid as a usual polytope in R^d connect p within Pyr_i and Pyr_j respectively to the intersection points of Γ with the i-th and j-th base cubes B_i and B_j respectively along the straight segments. Obviously, these segments belong to $\widetilde{\mathcal{A}}_d$ (except for the point p itself). The union of Γ and these segments is topologically a circle with a chord connecting its two distinct points and the cycle $(1, 1, ..., 1)$ in $H_1(T^d)$ represented by Γ is now the sum of two cycles where we use the new chord as the shortcut. Notice, finally, that neither of these two cycles is homologically trivial and, therefore, both of them represent elements in $H_1(T^d)$ different from $(1, 1, ..., 1)$. Indeed, each base cube B_i is the $(d-1)$-cycle in T^d representing the homology class dual to the corresponding 1-cocycle in $H^1(T^d)$ and the pairing of $H_1(T^d)$ and $H^1(T^d)$ can be realized using the intersection indices of a curve in T^d with the base cubes. Finally, we notice that each of the shortcuts cross some nonempty and not complete (which follows from the assumption that Pyr_i and Pyr_j are not neighbors) subset of the base cubes once each. $\diamondsuit$

3. Time-dependent continuous control

In the case of $\mathcal{F}_k = Sk_k \subset T^d$ and the above choice of $\widetilde{\mathcal{A}}_d$ let us present a simple time-dependent control which converges starting with any initial leg configuration belonging to $\widetilde{\mathcal{A}}_d$ to the rotation of the equally spaced and cyclically ordered according to a given in advance cyclic order leg configuration, this rotation being performed with a constant angle speed. (As we just mentioned our control depends on an arbitrary but fixed in advance choice of a cyclic order of legs.)

We prefer to present our control in the form of an algorithm which uses essentially three legs at a time: the leading leg (i.e. the one with the maximal sine of its angle), its consecutive leg in the chosen cyclic order, and the leg which is geometrically next after the leading leg in the clockwise direction. (We are assuming that the forward motion of a roach corresponds to the counterclockwise rotation of legs.)

Initial step. At the initial time moment we check the initial position of legs. If it belongs to the set $\mathcal{F}_k$ of failure positions then our roach can not start moving and the control terminates. Otherwise, the initial position is admissible and can be either controllable or uncontrollable. If it is uncontrollable go to the *accidental displacement step* below. Otherwise, since it is controllable we find in which Pyr_i it belongs. (Here we for each $i = 1, ..., d$ include its entrance face into Pyr_i.) The we declare the i-th leg *leading* and proceed to the *forward motion step* below.

Forward motion step. A) All legs except the $(i+1)$-st leg (in the cyclic order) start moving forward with the unit speed.

B) Determine the length of the period of the leadership of the i-th leg and the speed V_{i+1} of the $(i+1)$-st leg during this period as follows.

(i) To calculate T_i let the j-th leg be the geometrically next leg after the i-th leg in the clockwise direction and which is different from the $(i+1)$-st leg. (In other words, if if the $(i+1)$-st leg is geometrically next after the i-th leg clockwise then the j-th leg is the next one after the $(i+1)$-st. Notice that if there are several legs at the same position as the j-th leg we can choose any of them as the j-th.) Calculate the leadership time T_i for the i-th leg by the formula:

$$T_i = \frac{\pi - x_i - x_j}{2}.$$

(Recall that by our assumptions on i and j one has that $-\pi/2 < x_j < x_i < \pi/2$.) One can easily see that T_i is exactly the time required for the legs i and j moving forward with the unit speed to reach the same (and maximal among all the legs except possibly for the $(i+1)$-st) height. At this moment the i-th leg will be located in the left half-circle while the j-th leg will still be in the right half-circle.

(ii) Set the constant speed V_{i+1} of the $(i+1)$-st leg during the period T_i to be equal to $\frac{\pi/2 - x_{i+1}}{T_i}$.

(Notice that this choice of constant speed provides that after time T_i, i.e. when the i-th and the j-th legs reach equal height the $(i+1)$-st leg will reach the 12 o'clock position.)

C) Reenter the substep A of the forward motion step with the $(i+1)$-st leg chosen as *leading*.

Accidental displacement step. Make a random displacement of the initial position of legs within the admissible set Ω_k and return to the *initial step* above.

Proposition 2. *For any initial leg position in $\tilde{\mathcal{A}}_d$ the motion will converge to the motion with the constant unit speed and equally spaced leg for which their internal cyclic order coincides with their cyclic order on the circle.*

Proof. Our algorithm keeps us within $\tilde{\mathcal{A}}_d$. Namely, at each moment one either has a single leg at the maximal height or the i-th and the $(i+1)$-st legs at the maximal height with the correct clockwise order. When some leg makes a complete turn then the clockwise order of all legs will coincide with their cyclic order which we have chosen in advance. But their positions can be very closely grouped together on the circle. What remains to observe that after the first turn the relative position of all legs will quickly converge to the equally spaced situation. This follows immediately from the circumstance that starting from the second turn during the period of leadership of the i-th leg one places the $(+1)$-st leg in the position of the bisector of the angle between the i-th and the $(i+2)$-nd legs. This averaging process quickly converges to the equally spaced and rotating at the unit speed configuration of legs. $\diamondsuit$

Remark 1. The above dynamics consists of two phases: the 1st turn of the legs and the remaining motion. During the first turn all the legs are placed in the clockwise order coinciding with their cyclic order. This is done within a rather small time interval and might be difficult to technically realize in practice since it requires quick motions of legs and quick stops. One observes that small measurement mistakes can result in the instability of the motion since the order of leading legs can experience big changes. The second phase, on the other hand, presents no difficulties, and the motion quickly converges to the rotation of the equally spaced legs with the unit speed.

4. Autonomous continuous control

The next stable vector field $\mathbb{V}$ on $\tilde{A}_d$ was recently suggested by the first author. It is piece-wise smooth and analytic in each of the open pyramids where the single leg is the highest one. (In principle, the idea behind this dynamics is very similar to that of the time-dependent dynamics described in the next section.) Take a pyramid Pyr_i where the i-th leg has the strictly largest height among all legs, i.e. $h_i = \sin x_i$ is greater than all the other h_j's. (Here $x_j,\ j = 1, ..., d$ are the angle coordinates on our torus T^d.

Define $\mathbb{V}$ as

$$\begin{cases} \dot{x_j} = 1 \text{ for } j \neq i+1, \\ \dot{x}_{i+1} = \frac{C}{\sqrt{h_i - \max_{j\neq i,i+1} h_j}}. \end{cases}$$

In other words, the heir-apparent of the current leader is hurried forward, with a singular velocity if there is a competitor (the exponent 1/2 is to avoid some non-complete flows; the constant C should be chosen large enough to ensure overtaking of all the competitors).

It seems that this choice of a vector field works! To be continued ...

5. Final remarks and open problems

1. Different types of failure sets different from Sk_k and their tubular neighborhoods might be natural to impose on the roach motion. For example, since a roach has left and right legs it seems natural to forbid that too many right or left legs should be off the ground. However the same universal set $\widetilde{\mathcal{A}}_d$ of controllable leg positions does the job for

any set of failure positions within Sk_{d-2}. Repeating word by word the proof of Proposition 1 we get the following.

Lemma 1. *For any failure set $\mathcal{F} \subseteq Sk_{d-2}$ the set $\widetilde{\mathcal{A}}_d$ is set-theoretically maximal.*

2. In the present note we introduced and discussed the notion of a set-theoretical maximality of the set $\mathcal{A}_k$ of controllable positions. As an example of another notion of maximality that makes sense one can suggest the *volume maximality* of $\mathcal{A}_k \subset T^d$, i.e. one requires that the volume of its complement in the flat metric on T^d should be minimal. In case, when the complement has positive codimension one should consider the volume form of the appropriate dimension. Notice that the set $\mathcal{A}_k$ which was constructed and used in § 2 is not volume minimal in the above sense since the $(d-1)$-dimensional volume of the set $\widetilde{\mathcal{C}}_d$ can be made smaller by local smoothening. The problem of finding of the set $\mathcal{A}_k$ of the maximal volume is very tempting even in the standard case $\mathcal{F}_k = Sk_k$ considered above.

3. WHAT ABOUT THE FACT THAT WE ASSUMED IN THE BEGINNING THAT $\mathcal{F}_k$ IS AN OPEN DOMAIN? WHAT ABOUT LARGE NEIGHBORHOODS?

6. Appendix. Discrete autonomous control

Below we describe an interesting discrete dynamical system associated with our construction in § 2. It does not immediately solve our problem since it does not have a single attractor but it has many remarkable mathematical features and can be easily modified to serve our purposes. Notice that we want to construct a flow through the union of the pyramids Pyr_i such that on each individual open pyramid this flow enters only through its entrance face, say F and leaves through the exit face, say G, see Fig.2.

Both faces are cones over different and intersecting $(d-2)$-dimensional cubes and the flow should move from the entrance face F through the $(d-1)$-cube which is the base B of the whole pyramid and then further to the exit face F. Let us define two natural maps from the (open) entrance face F to the (open) base cube B and then from B to the (open) exit face G. Each such map can be transformed into a (continuous) flow by connecting the preimage and its image by a straight line within the pyramid. Thus each trajectory of such a flow within Pyr_i will be the union of two straight segments.

The most natural way to do it is by using the so-called blow-up/blow down rational transformations. We present these transformations explicitly below for the cases $d = 3$ and $d \geq 4$. (The essential distinction of these two cases is explained by the fact that for $d = 3$ the entrance/exit faces are the usual triangles and, therefore, they allow additional symmetry transformations unavailable for $d \geq 4$.)

Case $d = 3$. The entrance/exit faces F and G are usual triangles and the base cube B is a usual square. Let us identify the entrance triangle F with the triangle with the vertices $(0,0),(1,0),(1,1)$ in $\mathbb{R}^2$; the base square B with the square whose vertices are $(0,0),(1,0),(0,1)$, $(1,1)$ and, finally, the exit triangle G with the triangle with the vertices $(0,0),(0,1),(1,1)$.

The *blow-up* map $\Phi : (x,y) \to (x, \frac{y}{x})$ sends F to B. (It sends the pencil of lines through the origin to the pencil of horizontal lines.) Its inverse *blow-down* map $\Psi : (s,t) \to (st, t)$ maps B to G. It sends the pencil of vertical lines to the pencil of lines through the origin. Their composition $\chi = \Psi \circ \Phi : (x,y) \to (y, \frac{y}{x})$ sends F to G, see Fig. 4.

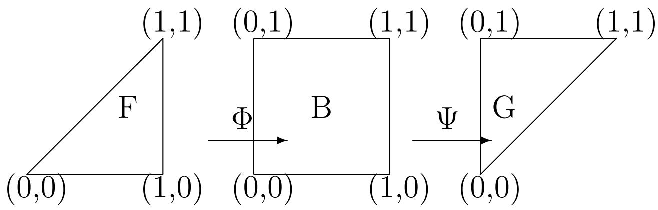

Figure 4. Blow-up and blow-down transformations.

To get the complete discrete dynamical system assume that the three (since $d = 3$) pyramids Pyr_1, Pyr_2, Pyr_3 are cyclically ordered as $1 < 2 < 3 < 1$ by the choice of Γ. Denote their entrance faces as F_1, F_2, F_3 and their exit faces as G_1, G_2, G_3. (One has, $F_1 = G_2, F_2 = G_3, F_3 = G_1$.) Assume now that we apply our transformation χ three times consecutively, i.e first from F_1 to $G_1 = F_2$, then from F_2 to $G_2 = F_3$, and, finally back to $G_3 = F_1$. The resulting self-map $\Theta : F_1 \to F_1$ is classically referred to as the *Poincare return map* of the dynamical system. To calculate it explicitly we need to find a suitable affine transformation A sending G back to F in the above example. Then we get the self-map Θ by composing χ with A and taking the 3-rd power of the resulting composition. As such a map A one can choose $A : (u,v) \to (1-u, 1-v)$ which implies that the required Poincare return map is the third power of $\Theta = A \circ \chi$ where:

$$\Theta : (x,y) \to \left(1-y, 1-\frac{y}{x}\right).$$

Lemma 2. *The above map Θ has and unique fixed point within the triangle F_1 and its fifth power is identity.*

Proof. The system of equations defining fixed points reads as

$$\begin{cases} x = 1-y, \\ y = 1-\frac{y}{x} \end{cases}$$

and its two solutions are $x_1 = \frac{2\sqrt{2}-1}{2}$, $y_1 = \frac{3-2\sqrt{2}}{2}$ and $x_2 = -\frac{1+2\sqrt{2}}{2}$, $y_2 = \frac{3+2\sqrt{2}}{2}$. One can easily check that only the first solution belongs to F_1. Direct calculations show that

$$\Theta^2 : (x,y) \to \left(\frac{y}{x}, \frac{y(1-x)}{x(1-y)}\right), \quad \Theta^3 : (x,y) \to \left(\frac{x-y}{x(1-y)}, \frac{x-y}{(1-x)}\right)$$

$$\Theta^4 : (x,y) \to \left(\frac{1-x}{1-y}, 1-x\right), \quad \Theta^5 : (x,y) \to (x,y).$$

♢

The Poincare return map is thus equals to $\Theta^3 : (x,y) \to \left(\frac{x-y}{x(1-y)}, \frac{x-y}{(1-x)}\right)$.

Case $d \geq 4$. Analogously, we have d pyramids each being a cone over a $(d-1)$-cube. Their entrance and exit faces are cones over $(d-2)$-cubes respectively. We need a map Φ sending the open entrance face F to the open base $(d-1)$-cube B and a map Ψ sending the open base cube B to the open exit face G. They can be given explicitly as follows. Let us identify F with the domain $\{0 < x_2 < x_1 < 1; 0 < x_3 < x_1 < 1; ... \; 0 < x_{d-1} < x_1 < 1\}$, i.e. with the cone over the square $\{0 < x_2 < 1, \; 0 < x_{d-1} < 1\}$ with the vertex at the origin. The base B will be identified with the cube $\{0 < x_1 < 1, \; 0 < x_2 < 1, \, 0 < x_{d-1} < 1\}$, and, finally, the exit face G with $\{0 < x_1 < x_2 < 1; \; 0 < x_3 < x_2 < 1, ..., 0 < x_{d-1} < x_2 < 1\}$. Then the *blow-up map* Φ and the *blow-down map* Ψ can be chosen as follows:

$$\Phi : (x_1, x_2, ..., x_{d-1}) \to \left(x_1, \frac{x_2}{x_1}, \frac{x_3}{x_1}, ..., \frac{x_{d-1}}{x_1}\right)$$

$$\Psi : (y_1, y_2, ..., y_{d-1}) \to (y_1 y_2, y_2, ..., y_{d-1} y_2).$$

Their composition $\chi : F \to G$ coincides with

$$\chi : (x_1, x_2, ..., x_{d-1}) \to \left(x_2, \frac{x_2}{x_1}, \frac{x_2 x_3}{x_1^2}, ..., \frac{x_2 x_{d-1}}{x_1^2}\right).$$

An appropriate linear map A sending G back to F is just a cyclic permutation of coordinates:

$$A : (z_1, z_2, ..., z_{d-1}) \to (z_2, z_3, ..., z_1).$$

Thus we get the composition $\Theta = A \circ \chi : F \to F$ (whose d-th power is the Poincare return map) given by:

$$\Theta : (x_1, x_2, ..., x_{d-1}) \to \left(\frac{x_2}{x_1}, \frac{x_2 x_3}{x_1^2}, ..., \frac{x_2 x_{d-1}}{x_1^2}, x_2\right).$$

Proposition 3. *The above map Θ has a curve of fixed points parameterized by $(t, t^2, t^2, ..., t^2)$, $t \in \mathbb{R}$. Moreover, for any $d \geq 3$ one has that $\Theta^{d-1} = id$.*

Proof. Indeed, the system of equations defining fixed points reads as

$$x_1 = \frac{x_2}{x_1},\ x_2 = \frac{x_2x_3}{x_1^2},\ x_3 = \frac{x_2x_4}{x_1^2},\ \cdots x_{d-2} = \frac{x_2x_{d-1}}{x_1^2},\ x_{d-1} = x_2.$$

which immediately implies $x_1^2 = x_2 = x_3 = ... = x_{d-1}$. To show that $\Theta^{d-1} = id$ notice that since Θ is a monomial map it suffices to show that $M_d^{d-1} = id_{d-1}$ where M_d is the matrix of the exponents of the map Θ and id_{d-1} is the identity matrix of size $d-1$. (Indeed, the matrix of exponents for Θ^i coincides with M_d^i.) This is done in the following lemma.

Lemma 3. *The characteristic polynomial of the* $(d-1)\times(d-1)$*-matrix* M_d *equals* $(-1)^d(1-t^{d-1})$*. Therefore, by the Hamilton-Cayley theorem* $M_d^{d-1} = id_{d-1}$*.*

Proof. Looking at the exponents of Θ we see that the matrix M_d has the form

$$M_d = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ -2 & 1 & 1 & 0 & \dots & 0 \\ -2 & 1 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -2 & 1 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

To make our calculations easy we introduce two families of $(k\times k)$-matrices D_k and E_k given by:

$$D_k = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & -t & 1 & 0 & \dots & 0 \\ 1 & 0 & -t & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \cdots & -t & 1 \\ 1 & 0 & 0 & \cdots & \cdots & -t \end{pmatrix}, \quad E_k = \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ 2 & -t & 1 & 0 & \dots & 0 \\ 2 & 0 & -t & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 0 & 0 & \cdots & -t & 1 \\ 0 & 0 & 0 & \cdots & 0 & -t \end{pmatrix}.$$

Expanding by the first row one obtains the following recurrences

$$Det(D_k) = (-t)^{k-1} - Det(D_{k-1}) \quad Det(E_k) = 2(-t)^{k-1} - Det(E_{k-1})$$

resulting in the formulas

$$Det(D_k) = (-1)^{k-1}(t^{k-1}+t^{k-2}+...+1), \quad Det(E_k) = (-1)^{k-1}2(t^{k-1}+t^{k-2}+...+t).$$

Expanding now the characteristic polynomial $Ch_d(t)$ of M_d by the first row (after the sign change in the first row) we get the relation

$$-Ch_d(t) = (t+1)[(1-t)(-t)^{d-3} - Det(D_{d-3})] - Det(E_{d-2}).$$

Substituting of the expressions for $Det(D_{d-3})$ and $Det(E_{d-2})$ in the latter formula one gets $Ch_d(t) = (-1)^d(1-t^{d-1})$. ♢

Proposition 3 is now settled. ♢

Corollary 1. *For* $d \geq 4$ *the Poincare return map of our dynamical system equals* $\Theta^d = \Theta$*.*

References

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