Intuitionistic Ramified Type Theory

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(The main results in here were first presented at workshop talks in Kanazawa, May 26 – 30, MF Oberwolfach, April 6 - 12 and Russell '08 in Swansea, 15 - 16 March in 2008.)

2/ Russell's Theory of Types

Bertrand Russell (1908): Mathematical Logic as Based on the Theory of Types.

Though at the time considered a marvel of mathematical rigor, Russell's presentation was partly informal, e.g. regarding substitution.

A modern reconstruction using lambda calculus notation is in

F. Kamareddine, T. Laan, R. Nederpelt. *Types in Logic and Mathematics Before 1940.* Bulletin of Symbolic Logic 2002.

3/ Simple type theory

The simple types are defined inductively

- Basic types of individuals are $\mathbf{N}$ $(= \{0, s(0), s(s(0)), \ldots\})$ and $\mathbf{1}$ $(= \{\star\})$
- For types A and B the product $A \times B$ is a type.
- For a type A the propositional functions on A constitute a type $\mathbf{P}(A)$

If $\varphi(x)$ is a proposition then $(\lambda x : A)\varphi(x) : \mathbf{P}(A)$ is a propositional function.

The propositional functions can equivalently be regarded as subsets: Write

- $\{x : A \mid \varphi(x)\} =_{\text{def}} (\lambda x : A)\varphi(x)$
- $a \in F =_{\text{def}} F(a)$ where $F : \mathbf{P}(A)$ and $a : A$

There is no restriction on the proposition $\varphi(x)$. It may very well contain a quantifier $(\forall F : \mathbf{P}(A))$. Thus

$$\{x : A \mid \varphi(x)\} : \mathbf{P}(A)$$

is constructed using quantification over the totality to which it self belongs.

This is however an impredicative construction and instance of the vicious circle principle.

5/ Ramified types

To avoid possible paradoxes and vicious circles Russell introduced a stratification of the propositions and hence also the propositional functions. Thus for every type S we have a stratified sequence of power sets of S

$$\mathrm{P}_0(S) \subseteq \mathrm{P}_1(S) \subseteq \mathrm{P}_2(S) \subseteq \cdots$$

To form

$$\{x : A \mid \varphi(x)\} : \mathrm{P}_k(A)$$

it is required that $\varphi(x)$ contains only quantifiers over individuals or over $\mathrm{P}_n(S)$ where $n < k$. It quantifies only over objects already constructed or given. Predicativity is retained.

6/ Ramified type symbols

Define the ramified type symbols $\mathcal{R} = \cup_{n \geq 0} \mathcal{R}_n$. inductively:

- $\mathcal{R}_0$ contains **1**, **N** and is closed under $\times$
- $\mathcal{R}_{n+1}$ contains $\mathcal{R}_n$ and is closed under $\times$ and $\mathbf{P}_k(\cdot)$ for $k \leq n$.

We have e.g.

$$A = \mathbf{P}_3(\mathbf{N} \times \mathbf{P}_2(\mathbf{P}_1(\mathbf{1}))) \in \mathcal{R}_4$$

$$B = \mathbf{P}_1(\mathbf{P}_2(\mathbf{1})) \in \mathcal{R}_3$$

(According to Russell's typing (K-L-N 2002) only A would be ramified. B is however rarely useful in his system.)

7/ Ramified formulas

The set of formulas of level n, $\mathcal{F}_n$, include

- $\perp$, $s =_A t$ for $A \in \mathcal{R}_n$,
- $X(t)$ for $X : \mathbf{P}_k(A)$ and $k \leq n$

and is closed under $\wedge$, $\vee$, $\longrightarrow$ and quantifiers $(\forall x : A)$ and $(\exists x : A)$ where $A \in \mathcal{R}_n$.

Restricted (predicative) comprehension principle: for $\varphi(x) \in \mathcal{F}_n$

$$\{x : A \mid \varphi(x)\} : \mathbf{P}_n(A).$$

8/ Reducibility axiom

Russell's **axiom of reducibility** can be phrased

$$(\forall X : \mathbf{P}_n(A))(\exists Y : \mathbf{P}_0(A)(\forall z : A)(z \in X \Leftrightarrow z \in Y).$$

This has the effect of collapsing the levels of propositions, as noted by Ramsey.

Then one may as well consider simple type theory, as the theory becomes impredicative.

In an intuitionistic setting, some special instance the axiom of reducibility are indeed predicatively valid, as we shall show later.

9/ Martin-Löf type theory: Type universes

Standard formulations of Martin-Löf type theory (1984) include a cumulative hierarchy of type universes $U_0, U_1, U_2, \ldots$.

$$\frac{A : U_n}{A \text{ type}} \qquad \frac{A : U_n}{A : U_{n+1}} \qquad U_n : U_{n+1}$$

U_0 contains basic types: $N_0, N_1, N : U_0$

Each U_n is closed under type operations $\Sigma, \Pi, +, \mathrm{Id}(\cdot,\cdot,\cdot)$. E.g.

$$\frac{A : U_n \qquad B(x) : U_n \; (x : A)}{(\Sigma x : A)B(x) : U_n}.$$

Note: Most of the constructive mathematical analysis in, say Bishop and Bridges (1985), can be carried out using one universe in this theory, or even without a universe.

10/ Martin-Löf type theory: Propositions-as-types

In MLTT everything is a type or an element of a type. A type A can also be interpreted as a proposition A (the type of proofs of the proposition). Translation table for type constructions:

$(\Sigma x : A)B(x)$	$(\exists x : A)B(x)$	$\bigvee_{x:A} B(x)$
$(\Pi x : A)B(x)$	$(\forall x : A)B(x)$	$\bigwedge_{x:A} B(x)$
$A \times B$	$A \wedge B$	
$A \rightarrow B$	$A \supset B$	
$A + B$	$A \vee B$	
$\emptyset$	$\perp$	

The term *Formulae-as-types* was originally used by Howard (1969).

11/ Martin-Löf type theory: Universes as truth-predicates

In the strictly typed version of universes they are understood as families of types

$$U \text{ type} \qquad \frac{a : U}{T(a) \text{ type}}$$

- U is considered as a set of codes for types, and T is the decoding function.
- Alternatively, under the propositions-as-types interpretation, U can be considered as a set of (infinitary) formulas, and T as a truth predicate

12/ Martin-Löf type theory: Universes as truth-predicates

For the natural numbers type introduce a code:

$$\frac{}{\ulcorner N \urcorner : U} \qquad \frac{}{T(\ulcorner N \urcorner) = N}$$

The closure rules of the universe are now, e.g. for Σ

$$\frac{A : U \qquad B(x) : U_n \; (x : T(A))}{(\ulcorner \Sigma \urcorner x : A)B(x) : U}$$

$$\frac{A : U \qquad B(x) : U \; (x : T(A))}{T(\ulcorner \Sigma \urcorner x : A)B(x)) = (\Sigma x : T(A))T(B(x)).}$$

Thus e.g.

$$T(\ulcorner \Sigma \urcorner x : \ulcorner N \urcorner)B(x)) = (\exists x : N)T(B(x)).$$

13/ Setoids $\sim$ Errett Bishop's notion of set

- A **setoid** $A = (|A|, =_A)$ is a type $|A|$ together with an equivalence relation $=_A$.
- An **(extensional) function** $f : A \longrightarrow B$ between setoids is a function (operation) $|A| \longrightarrow |B|$ together with a proof that the operation respects the equalities $=_A$ and $=_B$.

When based on Martin-Löf type theory this forms a good category of sets for constructive mathematics, supporting several choice principles: *Axiom of Unique Choice, Dependent Choice* and *Aczel's Presentation Axiom*.

14/ Stratified setoids

A setoid A is an (m, n)-setoid if

$$|A| : U_m \qquad\qquad =_A : |A| \longrightarrow |A| \longrightarrow U_n.$$

- m-setoid $=_{\text{def}}$ (m, m)-setoid
- m-classoid $=_{\text{def}}$ $(m+1, m)$-setoid
- ("Replacement") $f : A \longrightarrow B$, A m-setoid, B m-classoid $\Longrightarrow$ $\mathrm{Im}(f)$ m-setoid. — reason for the name classoid.

15/ Examples of stratified setoids

- $\mathbb{N} = (N, \mathrm{Id}(N, \cdot, \cdot))$ is a 0-setoid.
- Aczel's model of CZF $(V, =_V)$ forms a 0-classoid in ML type theory (if built from the universe U_0).
- $\Omega_n = (U_n, \leftrightarrow)$ propositions of level n with logical equivalence constitute an n-classoid.
- For an n-setoid A, the setoid of extensional propositional functions of level n

$$P_n(A) = [A \longrightarrow \Omega_n]$$

  is an n-classoid.

16/ Exponent setoid

For setoids A and B the **exponent setoid** $[A \longrightarrow B] = B^A$ is given by

$$|B^A| =_{\mathrm{def}} (\Sigma f : |A| \longrightarrow |B|)(\forall x, y : |A|)(x =_A y \Rightarrow f(x) =_B f(y))$$

and

$$(f, p) =_{B^A} (g, q) \Longleftrightarrow_{\mathrm{def}} (\forall x : |A|)(f(x) =_B g(x))$$

If A is an (m, n)-setoid and B is an (m', n')-setoid, then $[A \longrightarrow B]$ is an $(\max(m, m', n, n'), \max(m, n'))$-setoid.

In particular, the category of n-setoids are closed under exponentiation, but the n-classoids are not.

17/ A natural model of ramified types in MLTT

For each type symbol $S \in \mathcal{R}$ define a setoid S^* in Martin-Löf type theory, by recursion:

- $\mathbf{N}^* = (N, \mathrm{Id}(N, \cdot, \cdot))$
- $\mathbf{1}^* = (N_1, \mathrm{Id}(N_1, \cdot, \cdot))$
- $(A \times B)^* = A^* \times B^*$ (cartesian product)
- $\mathbf{P}_n(A)^* = [A^* \longrightarrow \Omega_n]$ (power construction)

This gives a hierarchy of types which satisfies the extensionality axioms. We have

$$A \in \mathcal{R}_n \Longrightarrow A^* \ n\text{-setoid}$$

18/ Local sets

Following practice in topos theory (Bell 1988) define a **local set** to be a type together with a subset.

A **local set of grade** (m, n) is a pair $A = (\tau_A, \rho_A)$ where

$$\tau_A : \mathcal{R}_m \qquad\qquad \rho_A : \mathbf{P}_n(\tau_A)^*.$$

Let A and B be local sets. A **map** $F : A \longrightarrow B$ is a relation $F : \mathbf{P}_\ell(\tau_A \times \tau_B)^*$, for some ℓ, such that

$$(\forall x : {\tau_A}^*)(\rho_A(x) \Rightarrow (\exists! y : {\tau_B}^*)(\rho_B(y) \wedge F(x, y)))$$

and

$$(\forall x : {\tau_A}^*)(\forall y : {\tau_B}^*)(F(x, y) \Rightarrow \rho_A(x) \wedge \rho_B(y))$$

We remark that the types used in local sets are almost as simple as those used in impredicative simple type theory or topos logic (higher order logic).

In particular, dependent types are not used.

For a each local (m, n)-set $A = (\tau_A, \rho_A)$ we associate a corresponding setoid

$$\check{A} = ((\Sigma x : {\tau_A}^*)\rho_A(x), =')$$

where

$$(x, p) =' (y, q) \Longleftrightarrow_{\mathrm{def}} x =_{{\tau_A}^*} y.$$

This setoid has index $(\max(m, n), m)$.

For local sets A and B, we have a setoid $[\check{A} \longrightarrow \check{B}]$ which can represent all maps $A \longrightarrow B$. For $f : [\check{A} \longrightarrow \check{B}]$ define

$$G_f(x, y) = (\exists p : \rho_A(x))(\exists q : \rho_B(y))(f(x, p) =_{\check{B}} (y, q)).$$

Then for any map $F : A \longrightarrow B$ there is by Axiom of Unique Choice some unique $f : [\check{A} \longrightarrow \check{B}]$ with

$$F = G_f.$$

22/ A reducibility principle for functions

Reducibility for functions:

Thus for local sets A and B there is some level ℓ so that for a map $F : A \longrightarrow B$ (of arbitrary level) there is a map $G : A \longrightarrow B$ with $G : \mathbf{P}_\ell(\tau_A \times \tau_B)^*$ such that

$$F = G.$$

Note that this would not work predicatively in a classical setting, as it implies a full comprehension principle.

23/ Quotient sets

Quotient sets are constructed as sets of equivalence classes.
Let $A = (\tau_A, \rho_A)$ be an (m, n)-subset and suppose
$E : \mathbf{P}_k(\tau_A \times \tau_A)^*$ is an equivalence relation on A, i.e. it satisfies

- $\rho_A(x) \Leftrightarrow E(x, x)$
- $E(x, y) \Rightarrow E(y, x)$
- $E(x, y) \wedge E(y, z) \Rightarrow E(x, z)$

Define the quotient subset $A/E = (\tau_B, \rho_B)$ by $\tau_B = \mathbf{P}_\ell(\tau_A)$ where $\ell = \max(m, n, k)$, and

$$\rho_B(S) = (\exists x : {\tau_A}^*)(\rho_A(x) \wedge (\forall y : {\tau_A}^*)(S(y) \Leftrightarrow E(x, y))).$$

24/ Category of local sets

The category of local sets admits constructions of function sets, quotients, pullbacks etc.

The local sets *should* form a locally cartesian closed pretopos ("predicative topos") when modelled in Martin-Löf type theory with an infinite sequence of universes (a theory of strength Γ_0). Remains to verify: the messy existence of Π_f.

25/ An intuitionistic ramified type theory

Intuitionistic ramified type theory is based on intuitionistic logic and has axioms

1. Defining axioms for ramified comprehension terms $\{x : A : \phi(x)\} : \mathbf{P}_k(A)$
2. Extensionality axiom
3. Arithmetical axioms with full induction scheme
4. Reducibility for function spaces.

No vicious impredicativity. Axiom scheme 4 is benevolent thanks to the BHK-intrepretation.

(Can we reach Γ_0 with "mathematically natural axioms" valid in the model?)

26/ Additional axioms true in the standard model

- Fullness.
  - Verify using the fact that every setoid A has projective cover $\underline{A} = (|A|, \mathrm{Id}(|A|, \cdot, \cdot))$ and consider $[\underline{A} \longrightarrow B]$ as representing relations instead of $[A \longrightarrow B]$.
- Dependent choice (see below).
- Generalised inductive definitions

27/ Dependent choice

RDC: Let A be any sort and $m, n \geq 0$. Then we have the axiom: for any $D : \mathbf{P}_m(A)$, any $R : \mathbf{P}_n(A \times A)$, and any $x : A$ satisfying

$$x \in D \wedge (\forall x : A)(x \in D \Rightarrow (\exists y : A)(y \in D \wedge \langle x, y \rangle \in R))$$

there is $F : \mathbf{P}_k(\mathbf{N} \times A)$ a map from $\mathbb{N}$ to (A, D), satisfying

(a) $\langle 0, x \rangle \in F$,

(b) $(\forall i : \mathbf{N})(\forall y, z : A)(\langle i, y \rangle \in F \wedge \langle i + 1, z \rangle \in F \Rightarrow \langle y, z \rangle \in R)$.

Here $k = \mathrm{lv}(A)$.

28/ Inductive definitions à la Aczel (1977) in IRTT

Typical rule for inductive generation:

$$[b] \quad \frac{a_1 \quad a_2 \quad \cdots}{a'}$$

The data for an inductive definition $\mathcal{D}$ is given by four local sets

(i) $X : \mathbf{P}_m(A)$ — the underlying set of abstract propositions

(ii) $R : \mathbf{P}_n(B)$ — the set of rule instances

(iii) $C : \mathbf{P}_r(A \times B)$ a relation between X and R. The intention is that $\langle a, b \rangle \,\epsilon\, C$ says that a is a conclusion of the rule instance b.

(iv) $P : \mathbf{P}_s(A \times B)$ a relation between X and R. Here the intention is that $\langle a, b \rangle \,\epsilon\, C$ expresses that a is a premiss of the rule instance b.

29/ Inductive definitions à la Aczel (1977) – cont'd

A subset $S \subseteq X$ with $S : \mathbf{P}_k(A)$ is $\mathcal{D}$-*closed* if for all $x : A$ and $r : B$

$$\langle r, x \rangle \,\epsilon\, C \wedge (\forall y : A)(\langle r, y \rangle \,\epsilon\, P \Rightarrow y \,\epsilon\, S) \Longrightarrow x \,\epsilon\, S$$

The *Principle of General Inductive Definition* (PGID) says that there is a smallest $\mathcal{D}$-closed subset of X. More precisely, for $\ell = \max(\mathrm{lv}(A), \mathrm{lv}(B), r, s)$ there is a $M : \mathbf{P}_\ell(X)$ such that

(a) M is $\mathcal{D}$-closed,

(b) For any $t \geq 0$: if $T : \mathbf{P}_t(A)$ is $\mathcal{D}$-closed, then $M \subseteq T$.

To verify this principle in type theory we first construct an operator $\Gamma : [\mathbf{P}_\ell(A) \longrightarrow \mathbf{P}_\ell(A)]$: for $S : \mathbf{P}_\ell(A)$ let $\Gamma(S)$ be

$$\{x : A \mid (\exists r : B)(\langle r, x\rangle \mathrel{\epsilon} C \wedge (\forall y : A)(\langle r, y\rangle \mathrel{\epsilon} P \Rightarrow y \mathrel{\epsilon} S))\}$$

Define types

$$I = (\Sigma x : A)(\Sigma r : B)(\langle r, x\rangle \mathrel{\epsilon} C)$$

and for $i : I$,

$$D(i) = (\Sigma y : A)(\langle \pi_1(\pi_2(i)), y\rangle \mathrel{\epsilon} P).$$

Construct the W-type $V = (Wi : I)D(i)$. By the assumption on universes being closed under W-types, we have $V : U_\ell$.

Define subsets $S_\alpha : \mathbf{P}_\ell(A)$ $(\alpha : V)$ by V-recursion

$$S_{\sup(i,h)} = \bigcup_{z:D(i)} \Gamma(S_{h(z)}). \tag{1}$$

Let

$$M = \bigcup_{\alpha:V} \Gamma(S_\alpha).$$

We verify that it is the smallest $\mathcal{D}$-closed subset of X.

Suppose $x \epsilon \Gamma(M)$. Thus there is $r : B$ with $\langle r, x\rangle \epsilon C$ and

$$(\forall y : A)(\langle r, y\rangle \epsilon P \Rightarrow (\exists \alpha : V) y \epsilon \Gamma(S_\alpha)).$$

Rewriting this using explicit proof objects we get $t : (\langle r, x\rangle \epsilon C)$ and

$$(\forall y : A)(\forall p : (\langle r, y\rangle \epsilon P))(\exists \alpha : V)(y \epsilon \Gamma(S_\alpha)).$$

In terms of D this gives

$$(\forall q : D(\langle x, \langle r, t\rangle\rangle))(\exists \alpha : V)(\pi_1(q) \epsilon \Gamma(S_\alpha)).$$

Let $i = \langle x, \langle r, t\rangle\rangle$. By the type-theoretic axiom of choice there is $h : D(i) \longrightarrow V$ so that

$$(\forall q : D(i))(\pi_1(q) \epsilon \Gamma(S_{h(q)})).$$

Let $\beta = \sup(i, h)$. So $\Gamma(S_{h(q)}) \subseteq S_\beta$ by (1). Thus

$$(\forall y : A)(\langle r, y\rangle \epsilon P \Rightarrow y \epsilon S_\beta).$$

Thus by definition of Γ, we get $x \epsilon \Gamma(S_\beta)$. But then $x \epsilon M$ as required.

34/ Pros and Cons of IRTT as a Foundation

- +/- Similar to topos logic (see e.g. Bell 1988) but predicative and ramified.
- \+ Model of IRTT is straightforward and does not require infinite tree constructions (W-types) as standard CZF.
- \+ Simpler type structure than Martin-Löf type theory.
- \- Ramfication levels are annoying but is dealt with automatically if one focusses on using the universal properties the constructions (Exponents, quotients etc.).
- \- No treatment of big universes of sets yet.
- \+ Non-committal regarding choice principles. Probably amenable to a purely category-theoretic treatment.

35/ IRTT in modern proof assistants

Two modern proof assistants that are based on Martin-Löf type theory: Coq (Paris) and Agda (Göteborg).

They both use an infinite hierarchy of type universes, and each universe is closed under strong principles for inductive definitions.

In Agda, the universe levels are explicitly given as numerical variables.

In Coq, the type universe levels are (mostly) hidden and a system of constraints on universe levels is maintained, to make sure that explicit numeric levels can be assigned if needed. (This is formally reminiscent of the stratifications in New Foundations, NF.)

36/ IRTT in modern proof assistants (cont.)

In Coq a seemingly inconsistent typing judgement

```
(Type -> Type) : Type
```

will be understood as

```
(Type@{i} -> Type@{j}) : Type@{k}
```

with the constraint $i < k \,\&\, j < k$ on levels.

Using explicit universe levels we can formally check the *reducibility principle for functions* in Coq (or Agda) and we expect the other axioms to be readily checked similarly.

37/ IRTT in Coq

If proved wisely (and using the **type-theoretic axiom of choice**) universe level i will be independent of level j:

```
Theorem FlatFunReducibility
    (A:Type@{i})(EA: A -> A -> Type@{i})
    (qA : Is_Equiv_Relation EA)
    (B:Type@{i})(EB: B -> B -> Type@{i})
    (qB : Is_Equiv_Relation EB)
    (X:A -> Type@{i})(xX : Is_Ext_Pred A EA X)
    (Y:B -> Type@{i})(xY : Is_Ext_Pred B EB Y)
    (F: (prod A B) -> Type@{j})
    (xF : Is_Ext_Rel A EA B EB F) :
      is_map (mk_ls _ (mk_Pw A EA qA X xX))
             (mk_ls _ (mk_Pw B EB qB Y xY))
             (mk_Pw2 A EA qA B EB qB F xF) ->
 ∃ G: (prod A B) -> Type@{i},
  Is_Ext_Rel A EA B EB G ∧ ∀ a b, (F (a,b) <-> G (a, b)).
```

38/ References

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