INTUITIONISTIC RAMIFIED TYPE THEORY (EXTENDED ABSTRACT)

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In this talk we examine the natural interpretation of a ramified type hierarchy into Martin-Löf type theory with an infinite sequence of universes. It is shown that under this interpretation some useful special cases of Russell's reducibility axiom are valid. This is enough to make the type hierarchy usable for development of constructive mathematics. We present a ramified type theory IRTT suitable for this purpose. IRTT allows for all the basic constructions of set theory: products, exponents, quotient sets, disjoint unions, equalisers. Their category-theoretic universal properties can be established.

1. Ramified Type Theory

Russell introduced in his ramified type theory a distinction between different levels of propositions in order to solve logical paradoxes, notably the Liar Paradox and the paradox he found in Frege's system (Russell 1908). A history and a modern reconstruction of Russell's type theory can be found in the article by Kamareddine, Laan and Nederpelt (2002). To be able to carry out certain mathematical constructions he then introduced the reducibility axiom, which had the effect of making the system impredicative. In this talk we introduce an alternative axiom of reducibility only for functional relations, which in the context of intuitionistic logic does not lead to impredicativity.

We turn to the formal presentation of our theory. The set of *ramified type symbols* $\mathcal{R}$ is inductively defined by the constructions $\mathbf{P}_n(\cdot)$ (nth level power set), $\times$ (products) from basic symbols $\mathbf{1}$ (the one element type) and $\mathbf{N}$ (natural numbers). The *level* of a type symbol A, $\mathrm{lv}(A)$, is defined recursively $\mathrm{lv}(\mathbf{1}) = \mathrm{lv}(\mathbf{N}) = 0$,

$$\mathrm{lv}(A \times B) = \max(\mathrm{lv}(A), \mathrm{lv}(B)), \qquad \mathrm{lv}(\mathbf{P}_n(A)) = \max(n+1, \mathrm{lv}(A)).$$

Our system of *intuitionistic ramified type theory* (IRTT) is based on many-sorted intuitionistic logic. The sorts are the symbols in $\mathcal{R}$. We define simultaneously the set of terms $\mathrm{Term}(A)$ of type $A \in \mathcal{R}$ and the set of formulas of level $k \in \mathbb{N}$, denoted $\mathrm{Form}(k)$. The crucial constructions are those related to subsets

- For each $A \in \mathcal{R}$ there is a infinite sequence of variables of sort A: $\mathsf{v}_0^A, \mathsf{v}_1^A, \mathsf{v}_2^A, \ldots$ in $\mathrm{Term}(A)$;

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- If $\varphi \in \mathrm{Form}(k)$ and x is a variable in $\mathrm{Term}(A)$, then the set abstraction term $\{x : A \mid \varphi\} \in \mathrm{Term}(\mathbf{P}_k(A))$.
- If $\mathrm{lv}(A) \leq k$ and $a, b \in \mathrm{Term}(A)$, then $(a =_A b) \in \mathrm{Form}(k)$;
- If $a \in \mathrm{Term}(A)$ and $b \in \mathrm{Term}(\mathbf{P}_n(A))$, then $(a \,\epsilon\, b) \in \mathrm{Form}(k)$ for any $k \geq n$;
- $\mathrm{Form}(k)$ is closed under propositional connectives. If $\varphi \in \mathrm{Form}(k)$ and x is a variable in $\mathrm{Term}(A)$ where $\mathrm{lv}(A) \leq k$, then $(\forall x : A)\varphi, (\exists x : A)\varphi \in \mathrm{Form}(k)$.

The axioms of ramified type theory are the following. First there is a group of standard axioms stating that each $=_A$ is an equivalence relation and that operations and predicates respect these equivalence relations. The arithmetical axioms are standard and there is a full induction scheme.

For subsets we have the following axioms

- Axiom of extensionality:
$$(\forall X, Y : \mathbf{P}_k(A))((\forall z : A)(z \,\epsilon\, X \Leftrightarrow z \,\epsilon\, Y) \Rightarrow X =_{\mathbf{P}_k(A)} Y)$$
- Defining axiom for restricted comprehension:
$$(\forall z : A)(z \,\epsilon\, \{x : A \mid \varphi\} \Leftrightarrow \varphi[z/x]).$$

To state the special reducibility axiom, which is the final axiom, we introduce some terminology. Mimicking the terminology in topos logic (Bell 1988) we let a *local set* be a type A together with an element X of some restricted power set $\mathbf{P}_n(A)$ of A. It is thus specified by a triple (A, X, n), where A is the underlying type, X is the propositional function defining the subset of A and n the level of the propositional function. A *map* from (A, X, m) to (B, Y, n) is some $R : \mathbf{P}_k(A \times B)$ which is a functional relation between X and Y.

- **Special reducibility axiom:** For $A, B \in \mathcal{R}$, $m, n \in \mathbb{N}$, we have for $k = \max(\mathrm{lv}(B), m, n)$ that for any $r \in \mathbb{N}$
$$\begin{aligned}&(\forall X : \mathbf{P}_m(A))(\forall Y : \mathbf{P}_n(B))(\forall F : \mathbf{P}_r(A \times B))\\ &\quad\Big[F \text{ map from } (A, X, m) \text{ to } (B, Y, n) \Rightarrow\\ &\qquad(\exists G : \mathbf{P}_k(A \times B))(\forall z : A \times B)(z \,\epsilon\, F \Leftrightarrow z \,\epsilon\, G)\Big]\end{aligned}$$

We may also extend the basic theory IRTT with the principle of Relativized Dependent Choice (RDC).

2. Setoids

As interpreting theory we consider Martin-Löf type theory (Martin-Löf 1984) with an infinite sequence of universes $\mathrm{U}_0 \subseteq \mathrm{U}_1 \subseteq \mathrm{U}_2 \subseteq \cdots$, where also $U_n : U_{n+1}$. This theory $\mathrm{ML}_{<\omega}$ is predicative in the strict sense of Feferman and Schütte and its proof-theoretic ordinal is Γ_0.

Theorem 2.1. *IRTT + RDC can be interpreted in Martin-Löf type theory with an infinite sequence of universes.*

We indicate some important ingredients in the proof.

On the propositions-as-types interpretation U_n can be regarded as the type of propositions of level n. A setoid $A = (|A|, =_A)$ *is of index* (m, n), or is an (m, n)-*setoid,* if $|A| : \mathrm{U}_m$ and $=_A$: $|A| \to |A| \to \mathrm{U}_n$. Let $\Omega_n = (\mathrm{U}_n, \leftrightarrow)$, where equality is logical equivalence. This is an $(n+1, n)$-setoid of index.

Lemma 2.2. *If A is an (m, n)-setoid and B is a (k, ℓ)-setoid then function space setoid $B^A = [A \to B]$ has index* $(\max(m, n, k, \ell), \max(m, \ell))$.

The type symbols of $\mathcal{R}$ interpret naturally as an extensional hierarchy of setoids in the theory $\mathrm{ML}_{<\omega}$. Define setoids S^* by recursion on the structure of $S \in \mathcal{R}$: $\mathbf{1}^* = (\mathrm{N}_1, \mathrm{Id}(\mathrm{N}_1, \cdot, \cdot))$, $\mathbf{N}^* = (\mathrm{N}, \mathrm{Id}(\mathrm{N}, \cdot, \cdot))$, $(S \times T)^* = S^* \times T^*$ and

$$\mathbf{P}_k(S)^* = [S^* \to \Omega_k].$$

Lemma 2.3. *If $S \in \mathcal{R}$ and* $\mathrm{lv}(S) \leq n$*, then S^* is an (n, n)-setoid.*

The interpretation $(-)^*$ is now extended according to the standard practice for propositions-as-types interpretations of many-sorted intuitionistic logic. Each formula φ is interpreted as a type φ^*. Each term a of sort A is interpreted as an element a^* of type $|A^*|$.

Lemma 2.4. *For* $\varphi \in \mathrm{Form}(n)$*, the interpretation satisfies* $\varphi^* : \mathrm{U}_n$.

Next we consider the semantic version of a local set. A pair $M = (S_M, \chi_M)$ consisting of S_M, an (m, n)-setoid, and a propositional function $\chi_M \in [S_M \to \Omega_k]$ is called a *local set.* It gives rise to a setoid

$$\widehat{M} = ((\Sigma x : S_M)\chi_M(x), =')$$

where $(x, p) =' (y, q) \Longleftrightarrow_{\mathrm{def}} x =_{S_M} y$. This setoid has index $(\max(m, k), n)$. The validity of the special reducibility axiom under the interpretation, is verified by considering the setoids $\widehat{(A^*, X^*)}$ and $\widehat{(B^*, Y^*)}$ and using the principle of unique choice to show that all maps are respresented as graphs of elements of the setoid $[\widehat{(A^*, X^*)} \to \widehat{(B^*, Y^*)}]$.

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