SUPERSYMMETRY OF THE STURM–LIOUVILLE AND KORTEVEG–DE VRIES OPERATORS

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Abstract. In 70's A.A. Kirillov interpreted the (stationary) *Sturm–Liouville* operator $L_2 = \frac{d^2}{dx^2} + F$ as an element of the dual space $\hat{\mathfrak{g}}^*$ of the nontrivial central extension $\hat{\mathfrak{g}} = \mathfrak{vir}$, called the Virasoro algebra, of the Witt algebra $\mathfrak{witt} = \mathfrak{der}\ \mathbb{C}[x^{-1}, x]$. He interpreted the (stationary) *KdV operator* $L_3 = \frac{d^3}{dx^3} + \frac{d}{dx}F + F\frac{d}{dx}$ in terms of the stabilizer of L_2. He also found a supersymmetry that reduces solutions of $L_3 f = 0$ to solutions of $L_2 g = 0$ by studying the nontrivial central extension of a simplest super analog of the Virasoro algebra, the Neveu–Schwarz superalgebra. Kirillov also wrote the first superversion of KdV equation.

I extend Kirillov's results and show how to find all supersymmetric extension of the Sturm–Liouville and Korteveg–de Vries operators associated with the 10 *distinguished* stringy superalgebras, i.e., all the simple stringy superalgebras that possess a nontrivial central extension. There are 12 or 14 such extensions, depending on the point of view. I only consider scalar models.

Khesin and Malikov extended Drinfeld–Sokolov's reduction to pseudodifferential operators and related the complex powers of Sturm–Liouville operators with the superized KdV-type hierarchies labelled by complex parameter. Similar approach to our Sturm–Liouville operators is also possible.

Introduction

A. A. Kirillov [Ki1] associated the (stationary) *KdV operator*

$$L_3 = \frac{d^3}{dx^3} + \frac{d}{dx}F + F\frac{d}{dx} \qquad (KdV)$$

and the (stationary) *Sturm–Liouville* operator

$$L_2 = \frac{d^2}{dx^2} + F \qquad (Sch)$$

with the cocycle that determines the nontrivial central extension $\hat{\mathfrak{g}} = \mathfrak{vir}$ — the Virasoro algebra — of the Witt algebra $\mathfrak{g} = \mathfrak{witt} = \mathfrak{der}\mathbb{C}[x^{-1}, x]$. Moreover, he found an explanation of the commonly known useful fact that the product of two solutions f_1, f_2 of the Sturm–Liouville equation $L_2(f) = 0$ satisfies $L_3(f_1 f_2) = 0$. Kirillov's explanation is the existence of a supersymmetry [Ki2].

Kirillov also classified the orbits of the coadjoint representation of $\mathfrak{vir}$ and clarified its equivalence to the following important classification problems: the classification of symplectic leaves of the second Gelfand–Dickey structure on the second order differential operators, of projective structures on the circle, and of Hill equation (i.e., (Sch) with periodic potential F). Kirillov's approach clarifies some earlier results by Poincaré, Kuiper, Lazutkin and

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Pankratova. The recent announcement of the classification of the simple stringy superalgebras and their central extensions [GLS] describes the scope of the problem: there are exactly 12 (or 14, depending on the interpretation) ways to superize the above results of Kirillov.

Kirillov himself partly considered one of these 12 or 14 possibilities, sseveral more were considered by Kuperschmidt, Chaichian–Kulish, P. Mathieu, Khovanova, V. Ovsienko and O. Ovsienko, Khesin, Ivanov–Krivonos–Bellucci–Delduc–Toppan, and many others; from the sea of results we point out [Ku1]-[Ku3], [Kh], [KM], [BIK], [DI], [DIK], [IKT]. So far, the examples of N-extended superKdV equations are only connected with a part of the distinguished stringy superalgebras.

In this paper I do not consider all the cases either: only scalar ones. The four vector-valued cases are much more difficult technically and will be considered elsewhere.

Our (Kirillov's) construction brings the KdV-type equations directly in the Lax form (analog of Euler's equation for a solid body) guaranteeing their complete integrability, cf. [Ku1], [OKh]. One should not forget here that the most profound dynamics, as Shander showed [Sh], is obtained with the help of $1|1$-dimensional time.

0.1. Kirillov's interpretation of the Sturm–Liouville and Korteveg–de Vries operators. Let $\mathfrak{g} = \mathfrak{witt} = \mathfrak{der}\mathbb{C}[x^{-1}, x]$ and let $\hat{\mathfrak{g}} = \mathfrak{vir}$ be the nontrivial central extension of $\mathfrak{g}$ given by the bracket

$$[f\frac{d}{dx} + az, g\frac{d}{dx} + bz] = (fg' - f'g)\frac{d}{dx} + c \cdot \mathrm{Res} fg''' \cdot z \text{ for } c \in \mathbb{C},$$

where z is the generator of the center of $\hat{\mathfrak{g}}$. Let $\mathcal{F} = \mathbb{C}[x^{-1}, x]$ be the algebra of functions; let $\mathcal{F}_\lambda$ for $\lambda \in \mathbb{C}$ be the rank 1 module over $\mathcal{F}$ spanned by dx^λ, where the λth power of dx is determined via analiticity of the formula for the $\mathfrak{g}$-action:

$$(f\frac{d}{dx})(dx^\lambda) = \lambda f' dx^\lambda.$$

In particular, $\mathfrak{g} \cong \mathcal{F}_{-1}$, as $\mathfrak{g}$-modules. Since the module *Vol* of volume forms is $\mathcal{F}_1$, the module dual to $\mathfrak{g}$ is $\mathfrak{g}^* = \mathcal{F}_2$: we use one dx to kill $\frac{d}{dx}$ and another dx to integrate the product of functions. (We confine ourselves to *regular* generalized functions, i.e., we ignore the elements from the space of functionals on $\mathfrak{g}$ with 0-dimensional support, see [Ki1].) Explicitely,

$$F(dx)^2(f\frac{d}{dx}) = \mathrm{Res}\ Ff.$$

0.1.1. The Korteveg–de Vries operator. The Lie algebra of the stationary group of the element $\hat{F} = (F, c) \in \hat{\mathfrak{g}}^* = (\mathfrak{g}^*, \mathbb{C} \cdot z^*)$ is

$$\mathfrak{st}_{\hat{F}} = \{\hat{X} \in \hat{\mathfrak{g}} : \hat{F}([\hat{X}, \hat{Y}]) = 0 \text{ for any } \hat{Y} \in \hat{\mathfrak{g}}\}.$$

Let us describe $\mathfrak{st}_{\hat{F}}$ explicitely. Take $\hat{X} = g\frac{d}{dx} + az$, $\hat{Y} = f\frac{d}{dx} + bz$. Then

$$\hat{F}([\hat{X}, \hat{Y}]) = \widehat{F}[(fg' - f'g)\tfrac{d}{dx} + \mathrm{Res} fg''' \cdot z] =$$
$$\mathrm{Res}[F(fg' - f'g) + cfg'''] \overset{\text{(partial integration)}}{=} \mathrm{Res} f[Fg' + (Fg)' + cg'''].$$

Hence, $\hat{X} \in \mathfrak{st}_{\hat{F}}$ if and only if g is a solution of the equation $L_3 g = 0$, where L_3 is given by formula (KdV) above. If $c \neq 0$ we can always rescale the equation and assume that

$$c = 1. \tag{0.1.1}$$

In what follows this is understood. We call L_3 the *KdV operator*, it is the famous operator of the second Hamiltonian structure for the KdV. Explicitely, the KdV operator is of the form

$$L_3 = (\text{the cocycle operator that determines } \hat{\mathfrak{g}}\ ) + \frac{d}{dx}F + F\frac{d}{dx}. \quad (0.1.2)$$

In what follows we consider the 14 super analogs of this operator.

0.1.2. The Sturm–Liouville operator. The Sturm–Liouville operator $L_2 = \frac{d^2}{dx^2} + F$ is, clearly, selfadjoint. The factorization

$$F(dx)^2 + a \cdot z^* = (dx)^2(F + a\frac{d^2}{dx^2}z^*) \quad (0.1.3)$$

suggests to ignore z^* (this action must be BUT IS IT? justified by comparison of transformation rules of L_2 and $F(dx)^2 + a \cdot z^*$) and represent the elements of $\hat{\mathfrak{g}}^*$ as 2nd order selfadjoint differential operators: $\mathcal{F}_\lambda \longrightarrow \mathcal{F}_{\lambda+2}$. The selfadjointness (i.e., $1-(\lambda+2)=\lambda$) fixes λ, namely, $\lambda = -\frac{1}{2}$. In what follows we consider the 10 super analogs of this operator, 4 more will be considered elsewhere.

0.1.3. KdV hierarchy. Assume that F depends on time, t. The *KdV hierarchy* is the series of evolution equations for $L = L_2$ or, equivalently, for F :

$$\dot{L} = [L, A_k], \text{ where } A_k = (\sqrt{L}^{2k-1})_+ \text{ for } k = 1, 3, 5, \dots. \quad (0.1.4)$$

Here the subscript + singles out the differential part of the pseudodifferential operator. The case $k = 1$ is trivial and $k = 3$ corresponds to the original KdV equation.

• Khesin and Malikov ([KM]) observed that we can also consider evolution equations for psedodifferential operators thus arriving to a continuous KdV hierarchy. Such an approach to evolution equations for L_2 is, as we will see, even more natural in the supersetting, when the Sturm–Liouville operator L_2 itself becomes a pseudodifferential one.

0.2. Kirillov's interpretation of supersymmetry of the Sturm–Liouville and Korteveg–de Vries operators. (To better understand this subsection, the reader has to know the technique of C-points, or *superfields*, see [L1], [L2] or [D]; Berezin called the set of C-points the "Grassmann envelope", [B].) Kirillov suggested to replace in the above scheme $\mathfrak{g} = \mathfrak{witt}$ with the Lie superalgebra $\mathfrak{g} = \mathfrak{k}^L(1|1)$ of contact vector fields on the $1|1$-dimensional supercircle associated with the trivial bundle. The superalgebra $\mathfrak{g}$ has a nontrivial central extension, called the Neveu–Schwarz superalgebra $\mathfrak{ns}$ and the above scheme leads us to the $\mathfrak{ns}(1)$-analog of the KdV operator

$$\mathcal{L}_5 = K_\theta K_1^2 + 2FK_1 + 2K_1F + (-1)^{p(F)}K_\theta F K_\theta \quad (0.2.1)$$

and the $\mathfrak{ns}(1)$-analog of the Sturm–Liouville operator

$$\mathcal{L}_3 = K_\theta K_1 + F. \quad (0.2.2)$$

Here $F \in \Pi(\mathbb{C}[x^{-1}, x, \theta])$ and K_f is the contact vector field generated by $f \in \mathbb{C}[x^{-1}, x, \theta]$, see 1.4, and Π is the change of parity functor: e.g., in $\mathbb{C}[x^{-1}, x, \theta]$ there is a natural parity defined on each monomial as the number of all θ's, well, Π renames all even elements calling them odd and the other way round.

Indeed, let us calculate the stabilizer of an element of $\mathfrak{ns}(1)^*$. In doing so we will use the *C-points* of all objects encountered. Observe that since the integral (or residue) pairs 1 with $\frac{\theta}{x}$, this pairing is odd, and, therefore, $\mathfrak{ns}(1)^* = \Pi(\mathcal{F}_3)$. The straightforward calculations yield: $X = K_f \in \mathfrak{st}_{\hat{F}}$ if and only if f is a solution of the equation

$$\left(cK_\theta\frac{d^2}{dx^2} + 2\frac{d}{dx}F + 2F\frac{d}{dx} + (-1)^{p(F)}K_\theta F K_\theta\right)f = 0. \quad (0.2.3)$$

The operator

$$\mathcal{L}_5 = (\text{the cocycle operator that determines } \hat{\mathfrak{g}}) + 2FK_1 + 2K_1F + (-1)^{p(F)}K_\theta K_1 K_\theta \quad (0.2.4)$$

from the lhs of (0.2.3) will be called the $\mathfrak{ns}(1)$-*KdV operator*.

In components we have: $f = f_0 + f_1\theta$, $F = F_0 + F_1\theta$, where f_0 and F_1 are even functions (of x with values in an auxiliary supercommutative superalgebra C) while f_1 and F_0 are odd ones. Suppose $F_0 = 0$. Since formula (1.4.5) below implies that $\{f(x)\theta, g(x)\theta\}_{K.b.} = fg$, we see that the product of two solutions of the Sturm–Liouville equation $L_2f = 0$ is a solution of the KdV equation $L_3X = 0$. **This is Kirillov's supersymmetry**.

Remark . Kirillov only considered $\mathbb{C}$-points of $\mathfrak{g}$ and $\hat{\mathfrak{g}}$, that is why he missed all odd parameters ($F_0 \neq 0$) of the supersymmetry he found.

0.3. The result. I extend Kirillov's result from $\mathfrak{witt}$ to all simple distinguished stringy Lie superalgebras — an elaboration of Remark from [L1], p. 167, where the importance of odd parameters in this problem was first observed and the problem solved here was raised. To consider *all* superized KdV and Sturm–Liouville operators was impossible before the list of stringy superalgebras and their cocycles ([GLS], [KvdL]) was completed.

0.4. Open problems (for students). Passing to superization of the steps of sec. 0.1, I interpret the elements of $\hat{\mathfrak{g}}^*$ for the distinguished stringy superalgebras $\mathfrak{g}$ as selfadjoint operators, perhaps, pseudodifferential, rather than differential. This, together with ideas applied by Khesin–Malikov to the usual Sturm–Liouville operator, requires generalizations of the Lie superalgebra of matrices of complex size associated with the analogs of superprincipal embeddings of $\mathfrak{osp}(N|2)$ for $N \leq 4$ (considered in [LSS] for $N = 1$). Such superizations were recently described (only for $N = 1$, see [GL]). It still remains to describe the corresponding W-superalgebras and Gelfand–Dickey superalgebras, present the result given below in components in order to compare with the results of physicists ([BIK], [DI], [DIK], [IKT] to name a few), describe superprincipal embeddings for $N > 1$ needed for a detailed superization of Drinfeld–Sokolov's and Khesin–Malikov's constructions.

There also remain four non-scaral cases: $\mathfrak{vect}^L(1|1)$, $\mathfrak{m}^L(1)$, $\mathfrak{vect}^L(1|2)$, and the most interesting $\mathfrak{svect}^L_\lambda(1|2)$.

Closely related to nontrivial central extensions of distinguished stringy superalgebras are superizations of the Schwarz derivative and Bott cocycle. When Radul gave his examples [Ra] several distinguished algebras were unknown; the omission should be mended.

Lastly, the simplest problem: in sec. 1.3 there are described simplest modules more general than $\mathcal{F}_\lambda$. Are there Sturm-Liouville operators acting in them? (For $\mathfrak{vect}$ the simplest modules are volume forms; they do not fit .)

§1. Distinguished stringy superalgebras

We recall all the neccessary data. For the details of classification of simple vectorial Lie superalgebras see [LS] and [GLS]; for a review of the representation theory of simple Lie superalgebras including infinite dimensional ones see [L2], for basics on supermanifolds see [D], [L1, L2] or [M]. The ground field is $\mathbb{C}$.

1.1. Supercircle. A *supercircle* or (for a physicist) a *superstring* of dimension $1|n$ is the real supermanifold $S^{1|n}$ associated with the rank n trivial vector bundle over the circle. Let $x = e^{i\varphi}$, where φ is the angle parameter on the circle, be the even indeterminate of the Fourier transforms; let $\theta = (\theta_1, \dots, \theta_n)$, be the odd coordinates on the supercircle formed by a basis of the fiber of the trivial bundle over the circle. Then (x, θ) are the coordinates on $(\mathbb{C}^*)^{1|n}$, the complexification of $S^{1|n}$.

Denote by $vol = vol(x, \theta)$ the volume element on $(\mathbb{C}^*)^{1|n}$. (Roughly speaking, vol"="$dx \cdot \frac{\partial}{\partial\theta_1} \cdot \ldots \cdot \frac{\partial}{\partial\theta_n}$. Recall that this is not equality: as shown in [BL], the change of variables acts differently on the lhs and rhs and only coinsides for the simplest transformations.)

Let the contact form be

$$\alpha = dx - \sum_{1\leq i\leq n} \theta_i d\theta_i.$$

Usually, if $\left[\frac{n}{2}\right] = k$ we rename the first $2k$ indeterminates and express α as follows for $n = 2k$ and $n = 2k+1$, respectively:

$$\alpha' = dx - \sum_{1\leq i\leq k} (\xi_i d\eta_i + \eta_i d\xi_i) \text{ or } \alpha' = dx - \sum_{1\leq i\leq k} (\xi_i d\eta_i + \eta_i d\xi_i) - \zeta d\zeta.$$

On $(\mathbb{C}^*)^{1|n}$, there are 5 series of simple "stringy" Lie superalgebras of vector fields and 4 exceptional such superalgebras. The 10 of them (or 12 if we distinguish different regradings) are distinguished: they admit nontrivial central extensions.

The "main" 3 series are: $\mathfrak{vect}^L(1|n) = \mathfrak{der}\mathbb{C}[x^{-1}, x, \theta]$, of all vector fields, its subalgebra $\mathfrak{svect}^L_\lambda(1|n)$ of vector fields that preserve the volume form $x^\lambda vol$, and $\mathfrak{k}^L(1|n)$ that preserves the Pfaff equation $\alpha = 0$. The superscript ^{L} indicates that we consider vector fields with Laurent coefficients, not polynomial ones.

The Lie superalgebras of these 3 series are simple with the exception of $\mathfrak{svect}^L_\lambda(1|1)$ for any λ, $\mathfrak{svect}^L_\lambda(1|n)$ for $n > 1$ and $\lambda \in \mathbb{Z}$, and $\mathfrak{k}^L(1|4)$.

The fourth series is a simple ideal $\mathfrak{svect}^{\circ L}(1|n)$ of $\mathfrak{svect}^L_\lambda(1|n)$ for $n > 1$ and $\lambda \in \mathbb{Z}$ contains, the quotient being spanned by $\theta_1 \ldots \theta_n \partial_x$.

The *twisted supercircle* of dimension $1|n$ is the supermanifold that we denote $S^{1|n-1;M}$ is associated with the Whitney sum of the trivial vector bundle of rank $n-1$ and the Möbius bundle. Since the Whitney sum of the two Möbius bundle s is isomorphic to the trivial rank 2 bundle, we will only consider either $S^{1|n}$ or $S^{1|n-1;M}$.

Let $\theta_n^+ = \sqrt{x}\theta_n$ be the corresponding to the Möbius bundle odd coordinate on $\mathbb{C}S^{1|n-1;M}$, the complexification of $S^{1|n-1;M}$. Set

$$\alpha^M = dx - \sum_{1\leq i\leq n-1} \theta_i d\theta_i - x\theta_n d\theta_n;$$

simetimes the following form is more convenient:

$$\alpha'^M = dx - \sum_{1\leq i\leq k} (\xi_i d\eta_i + \eta_i d\xi_i) - x\theta_n d\theta_n; \text{ or } \alpha'^M = dx - \sum_{1\leq i\leq k} (\xi_i d\eta_i + \eta_i d\xi_i) - \zeta d\zeta - x\theta_n d\theta_n.$$

The fifth series is the Lie superalgebra $\mathfrak{k}^M(n)$ that preserves the Pfaff equation $\alpha^M = 0$.

One exceptional superalgebra, $\mathfrak{m}^L(1)$, is the Lie subsuperalgebra in $\mathfrak{vect}^L(1|2)$ that preserves the Pfaff equation given by the *even* contact form

$$\beta = d\tau + \pi dq - q d\pi$$

corresponding to the "odd mechanics" on $1|1$-dimensional supermanifold with $0|1$-dimensional time. Though the following regradings demonstrate the isomorphism of this superalgebra with the nonexceptional ones *considered as abstract* superalgebras, they are distinct as filtered superalgebras and to various realizations of these Lie superalgebras different Sturm–Liouville and KdV operators correspond.

Let t, θ be the indeterminates for $\mathfrak{vect}(1|1)$; let $x\xi, \eta$ be same for $\mathfrak{k}(1|2)$ (in the realization that preserves the Pfaff eq. $\alpha' = 0$); and let τ, q, π be the indeterminates for $\mathfrak{m}(1)$. Denote $\mathfrak{vect}(t, \theta)$ with the grading $\deg t = 2$, $\deg \theta = 1$ by $\mathfrak{vect}(t, \theta; 2, 1)$, etc. Then the following exceptional nonstandard degrees indicated after a semicolon provide us with the isomorphisms:

$$\begin{array}{ll}\mathfrak{vect}(t,\theta;2,1)\cong\mathfrak{k}(1|2); & \mathfrak{k}(t,\xi,\eta;1,2,-1)\cong\mathfrak{m}(1);\\ \mathfrak{vect}(t,\theta;1,-1)\cong\mathfrak{m}(1); & \mathfrak{m}(\tau,q,\pi;1,2,-1)\cong\mathfrak{k}(1|2).\end{array}$$

Another, serious, exception is the Lie superalgebra $\mathfrak{k}^{L\circ}(1|4)$, the simple ideal of codimension 1 in $\mathfrak{k}^L(1|4)$, the quotient being generated by $\frac{\theta_1\theta_2\theta_3\theta_4}{x}$. The remaining exceptions, listed in [GLS], are not distinguished, so we ignore them in this paper.

1.2. The modules of tensor fields. To advance further, we have to recall the definition of the modules of tensor fields over the general vectorial Lie superalgebra $\mathfrak{vect}(m|n)=\mathfrak{der}\,\mathbb{C}[X]$, where $X=(x,\theta)$, and its subalgebras, see [BL]. Let $\mathfrak{g}=\mathfrak{vect}(m|n)$ (for any other $\mathbb{Z}$-graded vectorial Lie superalgebra the construction is identical) and $\mathfrak{g}_\geq=\underset{i\geq 0}{\oplus}\mathfrak{g}_i$, where $\deg X_i=1$ for all i. Clearly, $\mathfrak{g}_0\cong\mathfrak{gl}(m|n)$. Let V be the $\mathfrak{gl}(m|n)$-module with the *lowest* weight $\lambda=\mathrm{lwt}(V)$. Make V into a $\mathfrak{g}_\geq$-module setting $\mathfrak{g}_+\cdot V=0$ for $\mathfrak{g}_+=\underset{i>0}{\oplus}\mathfrak{g}_i$. Let us realize $\mathfrak{g}$ by vector fields on the $m|n$-dimensional linear supermanifold $\mathcal{C}^{m|n}$ with coordinates X. The superspace $T(V)=\mathrm{Hom}_{U(\mathfrak{g}_\geq)}(U(\mathfrak{g}),V)$ is isomorphic, due to the Poincaré–Birkhoff–Witt theorem, to $\mathbb{C}[[X]]\otimes V$. Its elements have a natural interpretation as formal *tensor fields of type* V (or λ). When $\lambda=(a,\dots,a)$ we will simply write $T(\vec{a})$ instead of $T(\lambda)$. We usually consider irreducible $\mathfrak{g}_0$-modules.

Examples: $T(\vec{0})$ is the superspace of functions; $\mathit{Vol}(m|n)=T(1,\dots,1;-1,\dots,-1)$ (the semicolon separates the first m coordinates of the weight with respect to the matrix units E_{ii} of $\mathfrak{gl}(m|n)$) is the superspace of *densities* or *volume forms*. We denote the generator of $\mathit{Vol}(m|n)$ corresponding to the ordered set of coordinates X by $\mathit{vol}(X)$. The space of λ-densities is $\mathit{Vol}^\lambda(m|n)=T(\lambda,\dots,\lambda;-\lambda,\dots,-\lambda)$. In particular, $\mathit{Vol}^\lambda(m|0)=T(\vec{\lambda})$ but $\mathit{Vol}^\lambda(0|n)=T(\overrightarrow{-\lambda})$.

1.3. Modules of tensor fields over stringy superalgebras. Denote by $T^L(V)=\mathbb{C}[t^{-1},t]\otimes V$ the $\mathfrak{vect}(1|n)$-module that differs from $T(V)$ by allowing the Laurent polynomials as coefficients of its elements instead of just polynomials. Clearly, $T^L(V)$ is a $\mathfrak{vect}^L(1|n)$-module. Define the *twisted with weight* μ version of $T^L(V)$ by setting:

$$T^L_\mu(V)=\mathbb{C}[x^{-1},x]x^\mu\otimes V. \tag{1.3.1}$$

• **The "simplest" modules — the analogues of the standard or identity representation of the matrix algebras**. The simplest modules over the Lie superalgebras of series $\mathfrak{vect}$ are, clearly, the modules of λ-densities, Vol^λ. These modules are characterized by the fact that they are of rank 1 over $\mathcal{F}$, the algebra of functions. Over stringy superalgebras, we can also twist these modules and consider Vol^λ_μ. Observe that for $\mu\notin\mathbb{Z}$ this module has only one submodule, the image of the exterior differential d, see [BL], whereas for $\mu\in\mathbb{Z}$ there is, additionally, the kernel of the residue:

$$\mathrm{Res}:\mathit{Vol}^L\longrightarrow\mathbb{C}\,,\quad f\mathit{vol}(x,\theta)\mapsto\text{ the coefficient of }\frac{\theta_1\dots\theta_n}{x}\text{ in the expantion of }f. \tag{1.3.2}$$

• Over $\mathfrak{svect}^L(1|n)$ all the spaces Vol^λ are, clearly, isomorphic, since their generator, $\mathit{vol}(x,\theta)$, is preserved. So all rank 1 modules over the algebra of functions are isomorphic to the module module $\mathcal{F}_{0;\mu}=t^\mu\mathcal{F}$ of twisted functions.

Over $\mathfrak{svect}^L_\lambda(1|n)$, the simplest modules are generated (over functions, perhaps, twisted) by $x^\lambda\mathit{vol}(x,\theta)$. The submodules of the simplest modules over $\mathfrak{svect}^L(1|n)$ and $\mathfrak{svect}^L_\lambda(1|n)$ are the same as those over $\mathfrak{vect}^L(1|n)$; but if the twist $\mu \in \mathbb{Z}$ there is, additionally, the trivial submodule generated by (a power of) $vol(x,\theta)$ or $x^\lambda vol(x,\theta)$, respectively.

• Over contact superalgebras, it is more natural to express the simplest modules not in terms of λ-densities but via powers of the form α which in what follows denotes either α' itself for the $\mathfrak{k}^L$ series, or α^M for the $\mathfrak{k}^M$ series, or β for $\mathfrak{m}^L(1)$. Set:

$$\mathcal{F}_\lambda = \begin{cases} \mathcal{F}\alpha^\lambda & \text{for } n = 0 \\ \mathcal{F}\alpha^{\lambda/2} & \text{otherwise .}\end{cases} \tag{1.3.3}$$

Observe that $Vol^\lambda \cong \mathcal{F}_{\lambda(2-n)}$ as $\mathfrak{k}(1|n)$-modules. In particular, the Lie superalgebra of series $\mathfrak{k}$ does not distinguish between $\frac{\partial}{\partial x}$ and α^{-1}: their transformation rules are identical. Hence,

$$\mathfrak{k}(1|n) \cong \begin{cases} \mathcal{F}_{-1} & \text{if } n = 0 \\ \mathcal{F}_{-2} & \text{otherwise .}\end{cases}$$

We denote the twisted versions by $\mathcal{F}_{\lambda;\mu}$.

For $n = 2$ (and $\alpha = dx - \xi d\eta - \eta d\xi$) there are other rank 1 modules over $\mathcal{F} = \mathcal{F}_0$, the algebra of functions, namely:

$$T(\lambda,\nu)_\mu = \mathcal{F}_{\lambda;\mu} \cdot \left(\frac{d\xi}{d\eta}\right)^{\nu/2}. \tag{1.3.4}$$

• Over $\mathfrak{k}^M$, we should replace α with α^M and the definition of the $\mathfrak{k}^L(1|n)$-modules $\mathcal{F}_{\lambda;\mu}$ should be replaced with

$$\mathcal{F}^M_{\lambda;\mu} = \begin{cases} \mathcal{F}_{\lambda;\mu}(\alpha^M)^\lambda & \text{for } n = 1 \\ \mathcal{F}_{\lambda;\mu}(\alpha^M)^{\lambda/2} & \text{for } n > 1.\end{cases} \tag{1.3.5}$$

For $n = 3$ and $\alpha^M = dx - \xi d\eta - \eta d\xi - t\theta d\theta$ there are other rank 1 modules over the algebra of functions $\mathcal{F}$, namely:

$$T^M(\lambda,\nu)_\mu = \mathcal{F}^M_{\lambda,\mu} \cdot \left(\frac{d\xi}{d\eta}\right)^{\nu/2}. \tag{1.3.6}$$

Examples . 1) The $\mathfrak{k}(2m+1|n)$-module of volume forms is $\mathcal{F}_{2m+2-n}$. In particular, $\mathfrak{k}(1|2) \subset \mathfrak{svect}(1|2)$.

2) As $\mathfrak{k}^L(1|n)$-module, $\mathfrak{k}^L(1|m)$ is isomorphic to $\mathcal{F}_{-1}$ for $m = 0$ and $\mathcal{F}_{-2}$ otherwise. As $\mathfrak{k}^M(1|n)$-module, $\mathfrak{k}^M(1|n)$ is isomorphic to $\mathcal{F}_{-1}$ for $m = 1$ and $\mathcal{F}_{-2}$ otherwise. In particular, $\mathfrak{k}^L(1|4) \simeq Vol$ and $\mathfrak{k}^M(1|5) \simeq \Pi(Vol)$.

1.4. Convenient formulas. The four main series of stringy superalgebras are $\mathfrak{vect}^L(1|n)$, $\mathfrak{svect}^L_\lambda(1|n)$, $\mathfrak{k}^L(1|n)$ and $\mathfrak{k}^M(1|n)$. Obviously,

$$D = f\partial_x + \sum f_i\partial_i \in \mathfrak{svect}^L_\lambda(1|n) \quad \text{if and only if} \quad \lambda f = -x\operatorname{div} D. \tag{1.4.1}$$

A laconic way to describe $\mathfrak{k}$, $\mathfrak{m}$ and their subalgebras is via *generating functions.*

• Odd form $\alpha = \alpha_1$. For $f \in \mathbb{C}[x,\theta]$ set :

$$K_f = (2-E)(f)\frac{\partial}{\partial x} - H_f + \frac{\partial f}{\partial x}E,$$

where $E = \sum\limits_i \theta_i \frac{\partial}{\partial \theta_i}$, and H_f is the hamiltonian field with Hamiltonian f that preserves $d\alpha_1$:

$$H_f = -(-1)^{p(f)}\left(\sum_{j\leq m}\frac{\partial f}{\partial\theta_j}\frac{\partial}{\partial\theta_j}\right).$$

The choice of α' instead of α only affects the form of H_f. We give it for $m = 2k+1$:

$$H_f = -(-1)^{p(f)} \sum_{j\leq k} (\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}).$$

• Even form $\beta = \alpha_0$. For $f \in \mathbb{C}[q, \pi, \tau]$ set:

$$M_f = (2-E)(f)\frac{\partial}{\partial \tau} - Le_f - (-1)^{p(f)}\frac{\partial f}{\partial \tau}E,$$

where $E = q\frac{\partial}{\partial q} + \pi\frac{\partial}{\partial \pi}$, and

$$Le_f = \frac{\partial f}{\partial q}\ \frac{\partial}{\partial \pi} + (-1)^{p(f)}\frac{\partial f}{\partial \pi}\ \frac{\partial}{\partial q}.$$

Since

$$L_{K_f}(\alpha_1) = K_1(f)\alpha_1, \qquad L_{M_f}(\alpha_0) = -(-1)^{p(f)}M_1(f)\alpha_0, \tag{1.4.2}$$

it follows that $K_f \in \mathfrak{k}(1|m)$ and $M_f \in \mathfrak{m}(1)$. Observe that

$$p(Le_f) = p(M_f) = p(f) + \bar{1}.$$

• To the (super)commutators $[K_f, K_g]$ or $[M_f, M_g]$ there correspond *contact brackets* of the generating functions:

$$[K_f, K_g] = K_{\{f,g\}_{k.b.}}; \qquad [M_f, M_g] = M_{\{f,g\}_{m.b.}}$$

The explicit formulas for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on x (resp. τ). The *Poisson bracket* $\{\cdot,\cdot\}_{P.b.}$ is given by the formula

$$\begin{aligned} \{f,g\}_{P.b.} &= -(-1)^{p(f)}\sum_{j\leq m}\frac{\partial f}{\partial \theta_j}\frac{\partial g}{\partial \theta_j} \qquad \text{or} \\ \{f,g\}_{P.b.} &= -(-1)^{p(f)}[\sum_{j\leq m}(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\ \frac{\partial g}{\partial \xi_j}) + \frac{\partial f}{\partial \theta}\ \frac{\partial g}{\partial \theta}]. \end{aligned} \tag{1.4.3}$$

The *Buttin bracket* $\{\cdot,\cdot\}_{B.b.}$ is given by the formula (given here for $n = 1$)

$$\{f,g\}_{B.b.} = \sum_{i\leq n}(\frac{\partial f}{\partial q_i}\ \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)}\ \frac{\partial f}{\partial \xi_i}\ \frac{\partial g}{\partial q_i}). \tag{1.4.4}$$

In terms of the Poisson and Buttin brackets the contact brackets take the form

$$\{f,g\}_{k.b.} = (2-E)(f)\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}(2-E)(g) - \{f,g\}_{P.b.}\ , \tag{1.4.5}$$

respectively,

$$\{f,g\}_{m.b.} = (2-E)(f)\frac{\partial g}{\partial \tau} + (-1)^{p(f)}\frac{\partial f}{\partial \tau}(2-E)(g) - \{f,g\}_{B.b.}. \tag{1.4.6}$$

It is not difficult to prove the following isomorphisms (as superspaces):

$$\mathfrak{k}(1|n) \cong Span(K_f : f \in \mathbb{C}[x,\theta]); \quad \mathfrak{m}(1) \cong Span(M_f : f \in \mathbb{C}[\tau, q, \pi]).$$

• Define the *Möbius contact field* by the formula

$$\tilde{K}_f = (2-\tilde{E})(f)\mathcal{D} + \mathcal{D}(f)\tilde{E} + \tilde{H}_f, \tag{1.4.7}$$

where

$$\tilde{E} = \sum_{i<n}\theta_i\frac{\partial}{\partial \theta_i} + \frac{1}{2}\theta\frac{\partial}{\partial \theta} \quad \text{and} \quad \mathcal{D} = \frac{\partial}{\partial x} - \frac{\theta}{2x}\frac{\partial}{\partial \theta} = \frac{1}{2}\tilde{K}_1 \tag{1.4.8}$$

and where

$$\tilde{H}_f \;=\; (-1)^{p(f)}(\sum \frac{\partial f}{\partial \theta_i}\frac{\partial}{\partial \theta_i} + \frac{1}{x}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta})$$

in the realization with form $\tilde{\alpha}^M$; in the realization with form $\tilde{\alpha}'^M$ for $n = 2k$ and $n = 2k+1$ we have, respectively:

$$\begin{aligned}\tilde{H}_f \;&=\; (-1)^{p(f)}(\sum (\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i}\frac{\partial}{\partial \xi_i}) + \frac{1}{x}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}),\\ \tilde{H}_f \;&=\; (-1)^{p(f)}(\sum (\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i}\frac{\partial}{\partial \xi_i}) + \frac{\partial f}{\partial \zeta}\frac{\partial}{\partial \zeta} + \frac{1}{x}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}).\end{aligned}$$

The corresponding contact bracket of generating functions will be called the *Ramond bracket*; its explicit form is (see (1.4.8))

$$\{f, g\}_{R.b.} = (2 - \tilde{E})(f)\mathcal{D}(g) - \mathcal{D}(f)(2 - \tilde{E})(g) - \{f, g\}_{MP.b.}, \tag{1.4.9}$$

where the *Möbius-Poisson bracket* $\{\cdot, \cdot\}_{MP.b}$ is defined to be

$$\{f, g\}_{MP.b} = (-1)^{p(f)}\left(\sum \frac{\partial f}{\partial \theta_i}\frac{\partial g}{\partial \theta_i} + \frac{1}{x}\frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \theta}\right) \quad \text{in the realization with form } \tilde{\alpha}^M. \tag{1.4.10}$$

Since (cf. (1.4.2))

$$L_{\tilde{K}_f}(\tilde{\alpha}) = \tilde{K}_1(f)\tilde{\alpha}. \tag{1.4.11}$$

It is easy to verify that $\mathfrak{k}(1|n) \cong Span(K_f : f \in \mathbb{C}[x^{-1}, t, \theta])$ whereas $\mathfrak{k}^M(1|n) = Span(\tilde{K}_f : f \in \mathbb{C}[x^{-1}, t, \theta])$. In other words, the spaces are identical but the brackets are nonisomorphic.

1.5. Distinguished stringy superalgebras. In the literature there are several definitions of what we call stringy superalgebras, mostly self-contradicting ones. For example, in almost all physical papers stringy superalgebra are called "superconformal" ones (meaning conformal superalgebras). In reality, only $\mathfrak{witt}$, $\mathfrak{k}^L(1|1)$ and $\mathfrak{k}^M(1|1)$ are conformal in the original sence of the term ([GLS]); but even if other superalgebras were conformal, there central extensions are not, though the term "superconformal" is magnanimously applied to all. In [KvdL] and in several subsequent papers the "superconformal" superalgebras are defined as "simple ... such that considered as $\mathfrak{witt}$-module ...", whereas everybody considers Virasoro, Neveu-Schwarz and Ramond superalgebras as "superconformal" ones, though they are not simple, and condition after "such that" depends on the embedding of $\mathfrak{witt}$. Therefore, we suggest the term *stringy* for the general class of algebras inside of which some are conformal, some are simple, etc. An intrinsic definition of this class is given in [GLS] together with that for the loop superalgebras (which, unlike simple loop algebras, may have no Cartan matrix, a usual key notion in the definition): both types of algebras are $\mathbb{Z}$-graded $\mathfrak{g} = \mathop{\oplus}\limits_{i=-d}^{\infty} \mathfrak{g}_i$ of infinite depth d but in the adjoint representation they act differently:

for the loop algebras every root vector corresponding to any real root

acts locally nilpotently in the adjoint representation,

for the stringy algebras this is not so.

In other words, for all stringy superalgebras there is an analog of the operator $\frac{d}{dx}$ which acts nontrivially on each homogeneous component, whereas for loop algebras there is no such operator. For the list of simple stringy superalgebras see [GLS]. It is instructive to compare [FL, KvdL, GLS]. Under *distinguished* stringy superalgebras we understand both the simple stringy superalgebras admitting the nontrivial central extension and the result of this extension.

Theorem . ([FL, KvdL, GLS]) *The only nontrivial central extensions of the simple stringy Lie superalgebras are those given in the following table.*

The operator ∇ introduced in the second column of the table by the formula $c : D_1, D_2 \mapsto \mathrm{Res}(D_1, \nabla(D_2))$ for an appropriate pairing $(\cdot, \cdot)$ will be referred to as the *cocycle operator*.

Let in this sebsection and in sec. 2.1 K_f be the common notation of both K_f and $\tilde{K}_f$, depending on whether we consider $\mathfrak{k}^L$ or $\mathfrak{k}^M$, respectively. Let further $\mathcal{K} = (2\theta\frac{\partial}{\partial\theta} - 1)\frac{\partial^2}{\partial x^2}$.

algebra	the cocycle $c : K_f, K_g \mapsto \mathrm{Res}(K_f, \nabla(K_g))$	The name of the extended algebra
$\mathfrak{k}^L(1\vert 0)$	$\mathrm{Res} f K_1^3(g)$	Virasoro or $\mathfrak{vir}$
$\left.\begin{matrix}\mathfrak{k}^L(1\vert 1)\\ \mathfrak{k}^M(1\vert 1)\end{matrix}\right\}$	$\mathrm{Res} f K_\theta K_1^2(g)$	Neveu-Schwarz or $\mathfrak{ns}$ Ramond or $\mathfrak{r}$
$\left.\begin{matrix}\mathfrak{k}^L(1\vert 2)\\ \mathfrak{k}^M(1\vert 2)\end{matrix}\right\}$	$(-1)^{p(f)}\mathrm{Res} f K_{\theta_1} K_{\theta_2} K_1(g)$	2-Neveu-Schwarz or $\mathfrak{ns}(2)$ 2-Ramond or $\mathfrak{r}(2)$
$\left.\begin{matrix}\mathfrak{k}^L(1\vert 3)\\ \mathfrak{k}^M(1\vert 3)\end{matrix}\right\}$	$\mathrm{Res} f K_\xi K_\theta K_\eta(g)$	3-Neveu-Schwarz or $\mathfrak{ns}(3)$ 3-Ramond or $\mathfrak{r}(3)$
$\left.\begin{matrix}\mathfrak{k}^{L\circ}(4)\\ \mathfrak{k}^M(1\vert 4)\end{matrix}\right\}$	(1) $(-1)^{p(f)}\mathrm{Res} f K_{\theta_1} K_{\theta_2} K_{\theta_3} K_{\theta_4} K_1^{-1}(g)$ (2) $\mathrm{Res} f(x K_{x^{-1}}(g))$ (3) $\mathrm{Res} f K_1(g)$	(1) $\begin{cases}\text{4-Neveu-Schwarz} = \mathfrak{ns}(4)\\ \text{4-Ramond} = \mathfrak{ns}(4)\end{cases}$ (2) $\begin{cases}\text{4}'\text{-Neveu-Schwarz} = \mathfrak{ns}(4')\\ \text{not defind for } \mathfrak{k}^M(1\vert 4)\end{cases}$ (3) $\begin{cases}\text{4}^0\text{-Neveu-Schwarz} = \mathfrak{ns}(4^0)\\ \text{not defind for } \mathfrak{k}^M(1\vert 4)\end{cases}$
$\mathfrak{vect}^L(1\vert 1)$	$D_1 = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial\theta},\ D_2 = \tilde{f}\frac{\partial}{\partial x} + \tilde{g}\frac{\partial}{\partial\theta} \mapsto$ $\mathrm{Res}(f\mathcal{K}(\tilde{g}) + (-1)^{p(D_1)p(D_2)} g\mathcal{K}(\tilde{f}) +$ $2(-1)^{p(D_1)p(D_2)+p(D_2)} g\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta}(\tilde{g})$	$\widehat{\mathfrak{vect}}^L(1\vert 1)$
$\mathfrak{m}^L(1)$	$M_f, M_g \mapsto \mathrm{Res} f(M_\xi)^3(g)$	$\widehat{\mathfrak{m}^L(1)}$

§2. Superized KdV and Sturm–Liouville operators

2.1. The KdV operators for the distinguished contact superalgebras. Let in this subsection K_f be the common term for both K_f and $\tilde{K}_f$, as Table 1.5, for $\mathfrak{g} = \mathfrak{k}^L$ or $\mathfrak{k}^{L\circ}$ or $\mathfrak{k}^M$. The equation for $K_f \in \mathfrak{st}_{(F,1)}$, where $(F, 1) \in \hat{\mathfrak{g}}^*$, is of the form $KdV(f) = 0$, where the operators KdV are listed in the following table with the cocycle operators being the symmetrizations of the operators ∇ from sec. 1.5; we write $\oplus$ insted of $+$ to grafically separate "the standard part".

n	$KdV=$ the cocycle operator$\oplus$ "the standard part"
0	$K_1^3\oplus FK_1+K_1F$
1	$K_\theta(K_1)^2\oplus 2(FK_1+K_1F)+(-1)^{p(F)}K_\theta FK_\theta$
2	$(K_\xi K_\eta-K_\eta K_\xi)K_1\oplus 2(FK_1+K_1F)+(-1)^{p(F)}(K_\xi FK_\eta-K_\eta FK_\xi)$
3	$(K_\xi K_\eta-K_\eta K_\xi)K_\theta\oplus$ $2(FK_1+K_1F)+(-1)^{p(F)}(K_\xi FK_\eta-K_\eta FK_\xi+K_\theta FK_\theta)$
4_1	$(K_{\xi_1}K_{\eta_1}-K_{\eta_1}K_{\xi_1})(K_{\xi_2}K_{\eta_2}-K_{\eta_2}K_{\xi_2})\int_x\oplus$ $2(FK_1+K_1F)+(-1)^{p(F)}\sum_{i=1,2}(K_{\xi_i}FK_{\eta_i}-K_{\eta_i}FK_{\xi_i})$
4_2	$xK_{x^{-1}}\oplus$ the standard part from 4_1
4_3	$K_1\oplus$ the standard part from 4_1

Clearly, the KdV operators corresponding to the supercircles associated with the cylinder and the Möbius bundle are absolutely different. To establish that similar is the situation with the Schröedinger operators, let us compair $\mathfrak{g}$ with $\mathfrak{g}^*$ for $\mathfrak{k}^L$ and $\mathfrak{k}^M$:

$\mathfrak{g}=\mathfrak{k}^L(1\|n)$	0	1	2	3	4	5	6	n
$\mathfrak{g}$	$\mathcal{F}_{-1}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$
Vol	$\mathcal{F}_1$	$\Pi(\mathcal{F}_1)$	$\mathcal{F}_0$	$\Pi(\mathcal{F}_{-1})$	$\mathcal{F}_{-2}$	$\Pi(\mathcal{F}_{-3})$	$\mathcal{F}_{-4}$	$\Pi^n(\mathcal{F}_{2-n})$
$\mathfrak{g}^*$	$\mathcal{F}_2$	$\Pi(\mathcal{F}_3)$	$\mathcal{F}_2$	$\Pi(\mathcal{F}_1)$	$\mathcal{F}_0$	$\Pi(\mathcal{F}_{-1})$	$\mathcal{F}_{-2}$	$\Pi^n(\mathcal{F}_{4-n})$

$\mathfrak{g}=\mathfrak{k}^M(1\|n)$	1	2	3	4	5	6	7	n
$\mathfrak{g}$	$\mathcal{F}_{-1}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$
Vol	$\Pi(\mathcal{F}_1)$	$\mathcal{F}_1$	$\Pi(\mathcal{F}_0)$	$\mathcal{F}_{-1}$	$\Pi(\mathcal{F}_{-2})$	$\mathcal{F}_{-3}$	$\Pi(\mathcal{F}_{-4})$	$\Pi^n(\mathcal{F}_{3-n})$
$\mathfrak{g}^*$	$\Pi(\mathcal{F}_2)$	$\mathcal{F}_3$	$\Pi(\mathcal{F}_2)$	$\mathcal{F}_1$	$\Pi(\mathcal{F}_0)$	$\mathcal{F}_{-1}$	$\Pi(\mathcal{F}_{-2})$	$\Pi^n(\mathcal{F}_{5-n})$

The comparison of $\mathfrak{g}$ with $\mathfrak{g}^*$ shows that there is a nondegenerate bilinear form on $\mathfrak{g}=\mathfrak{k}^L(1|6)$ and $\mathfrak{g}=\mathfrak{k}^M(1|7)$, even and odd, respectively. These forms are supersymmetric and given by the formula

$$(K_f,K_g)=\text{Res } fg.$$

2.2. The Sturm–Liouville operators as selfadjoint differential operators.

• For the Neveu–Schwarz superalgebras we have the exact sequences

$$0\longrightarrow\mathfrak{z}\longrightarrow\mathfrak{ns}(n)\longrightarrow\mathcal{F}_2\longrightarrow 0. \tag{2.3.1}$$

Here $\mathfrak{z}=\mathbb{C}\cdot z$ if $n\neq 4$ and $\mathfrak{z}=\mathbb{C}\cdot z$, or $\mathbb{C}\cdot z_2$, or $\mathbb{C}\cdot z_3$ if $n=4$, and centers correspond to the three cocycles.

Using the identification $Vol\cong\Pi^n(\mathcal{F}_{2-n})$ we dualize the above exact sequence and get:

$$\begin{aligned}&0\longrightarrow\Pi^n(\mathcal{F}_{4-n})\longrightarrow\mathfrak{ns}^*(n)\longrightarrow\mathfrak{z}^*\longrightarrow 0\quad\text{for } n=1,2,3,\\ &0\longrightarrow\mathcal{F}_0/\mathbb{C}\longrightarrow\mathfrak{ns}^*(\mathbf{4})\longrightarrow\mathfrak{z}^*\longrightarrow 0,\ \text{where } \mathbf{4}=4\text{ or }4'\text{ or }4^0\end{aligned} \tag{2.3.2}$$

• For the Ramond superalgebras we similarly have the exact sequences

$$\begin{aligned} &0 \longrightarrow \mathfrak{z} \longrightarrow \mathfrak{r}(1) \longrightarrow \mathcal{F}_{-1} \longrightarrow 0\\ &0 \longrightarrow \mathfrak{z} \longrightarrow \mathfrak{r}(n) \longrightarrow \mathcal{F}_{-2} \longrightarrow 0 \quad \text{for } n>1. \end{aligned} \tag{2.3.3}$$

Using the identification $Vol \cong \begin{cases}\Pi(\mathcal{F}_1) & \text{for } n=1\\ \Pi^n(\mathcal{F}_{3-n}) & \text{for } n>1\end{cases}$ we dualize the above exact sequence and get:

$$\begin{aligned} &0 \longrightarrow \Pi(\mathcal{F}_2) \longrightarrow \mathfrak{r}^*(1) \longrightarrow \mathfrak{z}^* \longrightarrow 0\\ &0 \longrightarrow \Pi^n(\mathcal{F}_{5-n}) \longrightarrow \mathfrak{r}^*(n) \longrightarrow \mathfrak{z}^* \longrightarrow 0. \end{aligned} \tag{2.3.4}$$

Let us realize the elements of $\mathfrak{ns}^*(n)$ and $\mathfrak{r}^*(n)$ by selfadjoint (pseudo)differential operators $\hat{F}: \mathcal{F}_\lambda \longrightarrow \Pi^n(\mathcal{F}_\mu)$. We have already done this for $\mathfrak{vir}$ in Introduction.

If $\mathfrak{g}^*$ is of the form $\mathcal{F}_A$ of $\Pi(\mathcal{F}_A)$, then the order of the Sturm–Liouville operator determined by $\hat{\mathfrak{g}}^*$ is equal to A. In particular, the order of $\hat{F}$ is equal to $4-n$ for $\mathfrak{ns}^*(n)$ and 0 for $n=4'$ or 4^0 as well; it is equal to $5-n$ for $\mathfrak{r}^*(n)$ if $n>1$ and 2 for $\mathfrak{r}^*(1)$.

Now, let us solve the systems of two equations, of which the first equation counts the order of Sch and the second one is the dualization condition:

$$\begin{array}{lll} \mu = 2+(2-n)+\lambda, & \mu+\lambda = 2-n & \text{for } \mathfrak{ns}(n)\\ \mu = 1+1+\lambda, & \mu+\lambda = 1 & \text{for } \mathfrak{r}(1)\\ \mu = 2+(3-n)+\lambda, & \mu+\lambda = 3-n & \text{for } \mathfrak{r}(n),\ n>1. \end{array}$$

The solutions are:

$$\begin{array}{lll} \mu = 3-n, & \lambda=-1 & \text{for } \mathfrak{ns}(n)\\ \mu = \frac{3}{2}, & \lambda = -\frac{1}{2} & \text{for } \mathfrak{r}(1)\\ \mu = 4-n, & \lambda=-1 & \text{for } \mathfrak{r}(n),\ n>1. \end{array}$$

For the remaining four distinguished stringy superalgebras the Sturm-Liouville operators are matrix ones and the corresponding calculations are more involved.

2.3. The list of Sturm-Liouville operators. For $\mathfrak{k}^L(1|n)$ and $n=0,1$ we can deduce the form of the Sturm-Liouville operators by factorization. For $n>1$ and for $\mathfrak{k}^M(1|n)$ we define the Sturm-Liouville operators as self-adjoint operators equal to the sum of the operator given in the tables with a potential F, where $F \in \Pi^n(\mathcal{F})$ for $\mathfrak{k}^L(1|n)$ and $F \in \Pi^{n+1}(\mathcal{F})$ for $\mathfrak{k}^M(1|n)$: the parity of the potential should be equal to that of the operator. Set $\Delta = \tilde{K}_{\theta_n}\tilde{K}_1^{-1} - \tilde{K}_1^{-1}\tilde{K}_{\theta_n}$. The operators are given with respect to forms α and α^M; in realizations where the contact fields preserve the forms α' and α'^M the expressions are more involved.

n	0	1	2	3
$\mathfrak{k}^L(1\vert n)$	K_1^2	$K_\theta K_1$	$K_{\theta_1}K_{\theta_2}$	$K_{\theta_1}K_{\theta_2}K_{\theta_3}(K_1)^{-1}$
$\mathfrak{k}^M(1\vert n)$	–	$\Delta\tilde{K}_1^2$	$\Delta\tilde{K}_\zeta\tilde{K}_1$	$\Delta(\tilde{K}_\xi\tilde{K}_\eta - \tilde{K}_\eta\tilde{K}_\xi)$

$\mathfrak{k}^L(1\vert 4)$	(1)	$K_{\theta_1}K_{\theta_2}K_{\theta_3}K_{\theta_4}(K_1)^{-2}$
	(2)	$xK_{x^{-1}}(K_1)^{-1} - (K_1)^{-1}xK_{x^{-1}}$
	(3)	any constant $c \neq 0$
$\mathfrak{k}^M(1\vert 4)$		$\Delta\tilde{K}_\zeta K_{\theta_1}K_{\theta_2}(\tilde{K}_1)^{-1}$

For the Lie superalgebra $\mathfrak{vect}^L(1|1)$ the Sturm-Liouville operator is the same operator as for $\mathfrak{k}^L(1|2)$ but rewritten in the form of a matrix and with η replaced with ∂_ξ. We leave as an exercise to the reader the pleasure to write this matrix explicitely as well as to reexpress it in terms of the fields M_f for $\mathfrak{m}^L(1)$. For $\mathfrak{vect}^L(1|2)$ and $\mathfrak{svect}^L_\lambda(1|2)$ the Sturm-Liouville operators can be obtained from the Sturm-Liouville operator for $\mathfrak{k}^L(1|4)$ after restriction. All this will be done explicitely elsewhere.

2.4. The KdV hierarchies associated with the Sturm–Liouville operators. Let L_r be the Sturm–Liouville operator of order r, see sec. 2.3. As Shander taught us [Sh], the time parameter should run in super setting over a 1|1-dimensional supermanifold, cf. [MR], and define the KdV-type equations as the following Lax pairs:

$$D_{\mathcal{T}}(L) = [L, A_k], \text{ where } A_k = (L^{k/r})_+ \text{ for } k \not\equiv r \pmod r \tag{2.5.1}$$

and where

$$D_{\mathcal{T}} = \begin{cases} \frac{d}{dt} & \text{if } p(A_k) = \bar{0} \\ \frac{\partial}{\partial\tau} + \tau\frac{\partial}{\partial t} & \text{if } p(A_k) = \bar{1}. \end{cases}$$

Here the subscript + singles out the differential part of the pseudodifferential operator. For complex k and for $\mathfrak{ns}(\mathbf{4})$ when L is a pseudodifferential operator, the differential part is not well-defined and we shall proceed, *mutatis mutandis*, as Khesin–Malikov. The details is subject of another paper.

§3. D'inechvé

3.1. The Schwarz derivative and Bott cocycle. Affine actions. Let $S = \mathbb{R}/2\pi\mathbb{Z}$; $\mathfrak{witt}$ the Lie algebra of complex-valued polynomial vector fields on S.

As is not difficult to verify, to the Gelfand-Fuchs 2-cocycle

$$a\frac{d}{dx}, b\frac{d}{dx} \mapsto \int_S ab'''dx \tag{3.1}$$

on $\mathfrak{witt}$ with trivial coefficients there corresponds a 1-cocycle with values in $\mathfrak{witt}^*$:

$$a\frac{d}{dx} \mapsto a'''dx^2 \tag{3.2}$$

Two problems arise in connection with this:

1) describe precisely relation between $H^2(\mathfrak{g}; M)$ and $H^1(\mathfrak{g}; M \otimes \mathfrak{g}^*)$. Since the latter are easier to calculate than the former, this is rather important. The question was solved by Dzhumadildaev [Dz] and in the Ph.D. thesis of his and S. Shnider's student P. Zusmanovich (Bar-Ilan U., 1992, regrettably, unpublished). Here we reproduce a part of his letter to Leites.

"First of all, observe that there is no natural map $H^1(\mathfrak{g}; \mathfrak{g}) \longrightarrow H^2(\mathfrak{g})$ (only if $\mathfrak{g}$ has a nondegenerate invariant form, i.e., $fg \simeq \mathfrak{g}^*$, we may consider a map $H^2(\mathfrak{g}) \longrightarrow H^1(\mathfrak{g}; \mathfrak{g})$.).

Contrarywise, there is a natural map $H^2(\mathfrak{g}) \longrightarrow H^1(\mathfrak{g}; \mathfrak{g}^*)$. Indeed, to any $\psi : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ assign $A\psi \in \mathrm{Hom}(\mathfrak{g}; \mathfrak{g}^*)$ setting

$$(A\psi(x))(y) = \psi(x, y).$$

It turns out that $A\psi \in H^1(\mathfrak{g}; \mathfrak{g}^*)$ and the map A is monomorphism.

Generally A is not an epimorphism, and to describe the cokernel, we need *pure Leibniz central extensions*. Namely, the following sequence is exact:

$$0 \longrightarrow H^2(\mathfrak{g}) \longrightarrow H^1(\mathfrak{g}; \mathfrak{g}^*) \longrightarrow HS^0(\mathfrak{g}) \longrightarrow H^3(\mathfrak{g}) \longrightarrow H^2(\mathfrak{g}; \mathfrak{g}^*) \longrightarrow HS^1(\mathfrak{g}) \longrightarrow ...$$

Here all the signs, except for $HS^i(\mathfrak{g})$, are well-known, whereas $HS^*(\mathfrak{g})$ is precisely what I call *symmetric cohomology of Lie algebras*. The corresponding cochain complex and the differential are constructed in my paper [Dz].

The term of interest to us, $HS^0(\mathfrak{g})$, has a nice description: it is precisely the space of invariant symmetric bilinear forms on $\mathfrak{g}$.

As is well-known, to any invariant symmetric bilinear form $(\cdot,\cdot)$ a 3-cocycle corresponds: $C_3 : (x, y, z) \mapsto ([x, y], z)$. Let

$$B : HS^0(\mathfrak{g}) \longrightarrow H^3(\mathfrak{g})$$

be the corresponding map. I claim that

$$\ker B = \operatorname{coker}(H^2(\mathfrak{g}) \longrightarrow H^1(\mathfrak{g}; \mathfrak{g}^*)).$$

For example, for complex simple finite dimensional Lie algebras, the 3-cocycle $([x, y], z)$ is nontrivial, i.e., $\ker B = 0$, hence, $H^2(\mathfrak{g}) \cong H^1(\mathfrak{g}; \mathfrak{g}^*) = 0$.

If $\mathfrak{g}$ is a vectorial Lie algebra, it can happen that $\ker B$ is nontrivial. For example, for $\mathfrak{svect}(3)$ and for Hamiltonian Lie algebras (on tori, when there are invariant forms on $\mathfrak{g}$). Then the “pure Leibniz central extrensions” arise, i.e., a central extension of the Lie algebra, the result of which is not a Lie algebra: the result only satisfies the Jacobi identity, but not the skew-symmetry one.

Thus, a part of $H^1(\mathfrak{g}; \mathfrak{g}^*)$ describes the Lie central extensions and another part describes pure Leibniz central extensions”.

2) Generally, the cohomology $H^1(\mathfrak{g}; M)$ correspond to *affine actions* of $\mathfrak{g}$ in M which are of particular interest for stringy algebras, cf. [FL]. In her M.A. thesis Poletaeva calculated some of these actions, namely she obtained the following statement.

Theorem . 1) *For the contact stringy Lie superalgebras the possible nonzero values of* $H^1(\mathfrak{g}; \mathcal{F}_\lambda)$ *and the corresponding cocycles are as follows*

$\lambda \setminus n$	0	1	2	3	$n > 3$
0	$\frac{f}{t}, \frac{df}{dt}$	$\frac{(2-E)(f)}{t}, \frac{\partial f}{\partial t}$	$\frac{(2-E)(f)}{t}, \frac{\partial f}{\partial t}, K_\xi K_\eta f$	$\frac{(2-E)(f)}{t}, \frac{\partial f}{\partial t}$	$\frac{(2-E)(f)}{t}, \frac{\partial f}{\partial t}$
1	–	$K_\theta \frac{\partial f}{\partial t}$	–	$K_\xi K_\theta K_\eta$	–
2	$\frac{d^2 f}{t^2}$	–	$K_\xi K_\eta f$	–	–
3	–	$K_\theta \frac{\partial^2 f}{\partial t^2}$	–	–	–
4	$\frac{d^3 f}{dt^3}$	–	–	–	–

2) *For* $\mathfrak{g} = \mathfrak{k}^L(2)$ *the nonzero cocycles* $H^(\mathfrak{g}; \mathcal{T}_{\lambda,\mu})$ *and the corresponding cocycles are as follows*

λ, μ	*cocycles*
$0, 0$	$\frac{\partial f}{\partial t}, \quad \frac{(2-E)(f)}{t}, \quad K_\xi K_\eta f$
$1, 1$	$K_\xi(\frac{(2-E)(f)}{t}), \quad K_\xi K_1 f$
$1, -1$	$K_\xi(\frac{(2-E)(f)}{t}), \quad K_\eta K_1 f$
$2, 0$	$K_\xi K_\eta K_1 f$

The cocycles (3.1) and (3.2) can be integrated to the following cocycles on the diffeomorphism group of S. Ten years ago, as far as I know, no realization of this complex group corresponding to $\mathfrak{witt}$ existed. So far let us consider Diff S, we identify the elements of

Diff S with functions, the images of $x = e^{i\varphi}$, where φ is the angle parameter on S. Polnaya putanitsa.

This problem must be discussed with Yura Neretin. Since there are THREE real forms of $\mathfrak{witt}$, it is interesting to understand what are the other two groups, if any, and how to proceed as the odd dimension grows.

Define the *Bott cocycle* to be

$$\mathrm{Bott}(g_1, g_2) = \int_S \ln(g_1 \cdot g_2)' d \ln g_2' \qquad ()$$

Define the *Schwarz derivative* to be

$$S(g) = \left(\frac{g'''}{g'} - \frac{3}{2}\frac{(g'')^2}{(g')^2}\right) dx^2 \qquad ()$$

Consider a 2nd order differential operator $L : \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\mu$:

$$L(f dx^\lambda) = (af'' + bf' + cf)dx^\mu$$

If L is selfadjoint ($b = 0$) and $a = 1$, then the change of variable $x = g(y)$ transforms L as follows:

??

so cdx^2 accrues Schwarz derivative of c.

Passing to $\mathfrak{k}^L(n)$, Radul observed that the conventional expression of the Schwarz derivative is better to rewrite in terms of the *multiplier* of the contact form α under the action of the supergroup of contactomorphisms $\mathrm{Diff}_\alpha S^{1|n}$, the subsupergroup of diffeomorphisms preserving the Pfaff equation $\alpha = 0$. Namely, let

$$G : \begin{cases} x = f(X, \Theta) \\ \theta_i = \varphi_i(X, \Theta) \end{cases}$$

be a contact transformation. (As is easy to verify, IS THIS TRUE? HOW ABOUT $\mathfrak{m}(1)$? $\mathfrak{k}^M(n)$? this means that

$$K_{\theta_i} f + \sum_j \varphi_j K_{\theta_i} \varphi_j = 0 \text{ for all } i$$

and the form α expressed in coordinates X, Θ accrues a factor, the *multiplier* m given by the formula

$$2m = K_1 f - \sum_j \varphi_j K_1 \varphi_j$$

Moreover,

$$K_{\theta_i} = \sum a_{ij} K_{\Theta_i},$$

where

$$(a_{ij}) = (K_{\theta_i} \varphi_j)^{-1}.$$

As is easy to see,

$$\sum_j a_{kj} a_{lj} = m^{-1} \delta_{k,l},$$

i.e., (a_{kj}) is the conformal matrix preserving the symplectic structure on $\mathrm{Ker} d\alpha$.

for $n = 2$ determine the matrix multiplier M from the formula

$$G_*(e_i) = \sum_j M_{ij}(G) e_j,$$

where (e_1, e_2) are the odd vectors that span the fiber of the trivial bundle with which $S^{1|2}$ is associated and selected to be orthogonal with respect to $\mathrm{Ker} d\alpha$. Observe that $M(G)$

does not depend on the choice of the basis. VERNO LI chto $M = (a_{kj})$? (U Radula eto pochemu-to otmecheno tol'ko pri $n = 2$.)

For $n = 2$ set

$$\left\langle \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \right\rangle = ad - bc.$$

Observe that for $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in CO(2; (C^\infty(S^{1|2}) \otimes \Lambda)_{\bar 0})$ we have

$$a^2 + b^2 \in C^\infty(S, \mathbb{R}_+)$$

since $G_{rd} \in \mathrm{Diff}_+(S)$; therefore $\ln(a^2 + b^2)$ is well-defined. So the expression

$$\ln \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \ln(a^2 + b^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varphi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\cos\varphi = \frac{a}{a^2+b^2}$ and $\cos\varphi = \frac{b}{a^2+b^2}$, i.e., φ is determined modulo $2\pi\mathbb{Z}$.

Schwarz derivatives and Bott cocycles $(a, b) \mapsto \int_{S^{1|n}} c(a, b)$ computed so far ([Ra]) are:

n	Schwarz derivative	
0	$\frac{1}{2}\left(\frac{m''}{m} - \frac{3}{2}\left(\frac{m'}{m}\right)^2\right)\alpha^2$	$\int_S \ln(a\cdot b)' d\ln b'$
1	$\frac{1}{2}\left(\frac{K_\theta(m')}{m} - \frac{3}{2}\frac{m' K_\theta(m)}{m^2}\right)\alpha^{3/2}$	$\int_{S^{1\mid 1}} \ln m(a\cdot b) K_\theta \ln m(b) vol(x,\theta)$
2	$\frac{1}{2}\left(\frac{K_{\theta_1}K_{\theta_2}(m)}{m} - \frac{3}{2}\frac{K_{\theta_1}(m)K_{\theta_2}(m)}{m^2}\right)\alpha$	$\exp\left(\int_{S^{1\mid 2}} \langle \ln M(a\cdot b), \ln M(b)\rangle vol(x,\theta)\right)$
3	$-3m^{3/2}\left(v_1(v_2), v_3\right)\alpha^{1/2}$	
4	$\left(??\right)\alpha^{-1/2}$	
4^o	$\left(??\right)\alpha^{-1/2}$	
$4'$	$\left(??\right)\alpha^{-1/2}$	

****

N.Definitions. The equation of the form

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0, \tag{N.1}$$

where $\sigma(x)$ is a polynomial of degree ≤ 2, $\tau(x)$ is a polynomial of degree ≤ 1 and λ is a constant, is called a *hypergeometric type equation* and its solutions are called functions of *hypergeometric type*. By multiplying (N.1) by an appropriate $\rho(x)$ one reduces (N.1) to the selfadjoint form

$$(\sigma(x)\rho(x)y')' + \lambda\rho(x)y = 0, \text{ where } (\sigma(x)\rho(x))' = \tau(x)\rho(x). \tag{N.2}$$

Let y_m and y_n be solutions of (N.2) with distinct eigenvalues λ_m and λ_n, respectively. If for some a and b, not necessarily finite, $\rho(x)$ satisfies the conditions

$$\sigma(x)\rho(x)x^k|_{x=a,b} = 0 \text{ for } k = 0, 1, \dots,$$

then

$$\int_a^b y_m(x)y_n(x)\rho(x)dx = 0. \tag{N.3}$$

(Clearly, if a and b are finite, it suffices to require that $\sigma(x)\rho(x)|_{x=a,b} = 0$.)

In 1929 Bochner [B] classified all orthogonal polynomials which arise as solutions of (N.1). Up to a linear change of variable, there are only four distinct types of them, all of which are called *classical orthogonal polynomials*. They are

1) the *Jacobi polynomials* $P_n^{(\alpha,\beta)}(x)$ defined for $\alpha, \beta > -1$ as polynomial solutions of (0.2) for $\sigma(x) = 1 - x^2$, $\rho = (1-x)^\alpha(1+x)^\beta$, $(a, b) = (-1, 1)$ and $\tau = \beta - \alpha - (\alpha + \beta + 2)x$; $\lambda = n(n+\alpha+\beta+1)$. For $\alpha = \beta = \frac{1}{2}$ the Jacobi polynomials are called *Chebyshev* polynomials.

2) the *Bessel polynomials* $B_n^{(A,B)}(x)$ defined for $A \notin \mathbb{Z}_+$, $B \neq 0$ as polynomial solutions of (0.2) for $\sigma(x) = x^2$, $\rho =??$, $(a, b) =??$ and $\tau = Ax + B$; $\lambda = -n(n + A - 1)$.

3) the *Laguerre polynomials* $L_n^{(\alpha)}(x)$ defined for $\alpha \notin -\mathbb{N}$ as polynomial solutions of (0.2) for $\sigma(x) = x$, $\rho =??$, $(a, b) =??$ and $\tau = \alpha + 1 - x$; $\lambda = n$.

4) the *Hermite polynomials* $H_n^{(\alpha)}(x)$ defined for $\alpha \notin -\mathbb{N}$ as polynomial solutions of (0.2) for $\sigma(x) = 1$, $\rho =??$, $(a, b) =??$ and $\tau = -2x$; $\lambda = 2n$.

Now, consider operator in SEVERAL variables $X = (x_1, \dots, x_n)$:

$$L = A_{ij}(X)D_iD_j + B_i(X)D_i + C$$

where $\deg A_{ij} \leq 2$, $\deg B_i \leq 1$, $C = \text{const}$.

What are its canonical forms? What are the canonical forms of the EQUATION $Ly = 0$?

For $n = 2$ the question was considered (solved?) by ??? Frydriszak [**?**] considered several purely odd indeterminates and very degenerate case $A_{ij} = \text{const}$; he did not study normal forms.

I am not so sure that Frydriszak's setting of the problem is natural, whereas the above examples of Sturm-Liouville operator suggest to investigate the cases of one even x and several odd θ's. Moreover, the least order of self-adjoint operators in these cases is not 2.

References

[B] Berezin F., *Introduction to superanalysis.* Edited and with a foreword by A. A. Kirillov. With an appendix by V. I. Ogievetsky. Translated from the Russian by J. Niederle and R. Kotecký. Translation edited by Dimitri Leites. Mathematical Physics and Applied Mathematics, 9. D. Reidel Publishing Co., Dordrecht, 1987. xii+424 pp.

[BIK] Bellucci S., Ivanov E., Krivonos S. On $N = 3$ super Korteveg – de Vries Equation, J. Math. Phys., 34, 7, 1993, 3087–3097

[BL] Bernstein J., Leites D., Invariant differential operators and irreducible representations of Lie superalgebras of vector fields, Selecta Math. Sov., v. 1, 2, 1981,143–160

[Bo] Bochner S., Über Sturmliouvillesche Polynomsysteme, Math. Z. 29, 1929, 730–736

[CK] Chaichian M., Kulish P. P. Superconformal algebras and their relation to integrable nonlinear systems. Phys. Lett. B 183 (1987), no. 2, 169–174; Kulish, P. P. An analogue of the Korteweg–de Vries equation for the superconformal algebra. (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 155 (1986), Differentsialnaya Geometriya, Gruppy Li i Mekh. VIII,142–149, 194 translation in J. Soviet Math. 41 (1988), no. 2, 970–975

[D] Deligne P. et al (eds.) *Quantum fields and strings: a course for mathematicians.* Vol. 1, 2. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. AMS, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999. Vol. 1: xxii+723 pp.; Vol. 2: pp. i–xxiv and 727–1501

[DI] Delduc F., Ivanov E. $N = 4$ super KdV equation, Phys. Lett. B, v. 309, 1993, 312–319

[DIK] Delduc F., Ivanov E., Krivonos S., $N = 4$ super KdV hierarchy in $N = 4$ and $N = 2$ superspaces, J. Math. Phys., v. 37, 3, 1996, 1356–1381

[Dzh] Dzhumadildaev, A. S. Central extensions of infinite-dimensional Lie algebras. (Russian) Funktsional. Anal. i Prilozhen. 26 (1992), no. 4, 21–29, 96 translation in Funct. Anal. Appl. 26 (1992), no. 4, 247–253 (1993)

[Dz] Dzhumadildaev A., Symmetric (co)homologies of Lie algebras. C. R. Acad. Sci. Paris Sér. I Math. 324 (1997), no. 5, 497–502

[FL] Feigin B., Leites D., New Lie superalgebras of string theories. In: Markov M. et. al. (eds.) *Group–theoretical methods in physics* (Zvenigorod, 1982), v. 1, Nauka, Moscow, 1983, 269–273 (Harwood Academic Publ., Chur, 1985, Vol. 1–3 , 623–629)

[Fr] Frydriszak A., On the Grassmannian special functions I. Lett. Math. Phys. v. 20, 1990, 159–163

[GL] Grozman P., Leites D., Lie superalgebras of supermatrices of complex size. Their generalizations and related integrable systems. In: Vasilevsky N. et. al. (eds.) *Proc. Internatnl. Symp. Complex Analysis and related topics*, Mexico, 1996. Operator Theory: Advances and Applications, 114. Birkhäuser Verlag, Basel, 2000, 73–105

[GLS] Grozman P., Leites D., Shchepochkina I., Lie superalgebras of string theories, hep-th 9702120; ; Acta Mathematica Vietnamica, v. 26, 2001, no. 1, 27–63

[IKT] Ivanov E., Krivonos S., Toppan F., $N = 4$ Sugawara construction on $\widehat{\mathfrak{sl}(2|1)}$, $\widehat{\mathfrak{sl}(3)}$ and mKdV-type superhierarchies. Modern Phys. Lett. A 14 (1999), no. 38, 2673–2686; id., $N = 4$ super NLS–mKdV hierarchies. Phys. Lett. B 405 (1997), no. 1-2, 85–94

[KvdL] Kac V., van de Leur, J. W., On classification of superconformal algebras. In: Gates S.J., Jr. et. al (eds.) *Strings '88. Proceedings of the workshop held at the University of Maryland, College Park, Maryland, May 24–28, 1988*, 77–106, World Sci. Publishing, Teaneck, NJ, 1989; Kac V., Superconformal algebras and transitive group actions on quadrics. Comm. Math. Phys. 186 (1997), no. 1, 233–252

[K2] Kac V., *Infinite dimensional Lie algebras*, Cambridge Univ. Press, Cambridge, 1991

[KM] Khesin B., Malikov F., Universal Drinfeld–Sokolov reduction and matrices of complex size, Commun. Math. Phys., **175** (1996), 113–134.

[Kh] Khovanova, T. G., Lie superalgebra $\mathfrak{osp}(1|2)$, Neveu-Schwarz superalgebra and superization of Korteweg-de Vries equation. In: Markov M. et. al (eds.) *Group theoretical methods in physics*, Vol. I (Yurmala, 1985), VNU Sci. Press, Utrecht, 1986, 307–314; id., The Korteweg-de Vries superequation connected with the Lie superalgebra of Neveu-Schwarz-2 string theory. (Russian) Teoret. Mat. Fiz. 72 (1987), no. 2, 306–312 (English translation: Theoret. and Math. Phys. 72 (1987), no. 2, 899–904)

[Ki1] Kirillov A., *Orbits of the group of diffeomorphisms of the circle and Lie superalgebras*, Sov. J. Funct. Analysis, v. 15, n.2, 1981, 75–76

[Ki2] Kirillov A., *The geometry of moments*, Lect. Notes in Math., Springer, Berlin ea., 1982, 101–123

[Ku1] Kupershmidt B., A super Korteweg–de Vries equation: an integrable system. Phys. Lett. A 102 (1984), no. 5-6, 213–215

[Ku2] Kupershmidt B. (ed.) *Integrable and superintegrable systems*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1990. x+388 pp.; id. *Elements of superintegrable systems. Basic techniques and results.* Mathematics and its Applications, 34. D. Reidel Publishing Co., Dordrecht, 1987. xvi+187 pp.; id., A review of superintegrable systems. Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), Lectures in Appl. Math., 23, Amer. Math. Soc., Providence, RI, 1986, 83–121

[Ku3] Kupershmidt B., ?$\longrightarrow$ Neveu–Schwarz–Ramond superalgebra$\rightarrow$Virasoro algebra. Phys. Lett. A 113 (1985), no. 3, 117–120

[L1] Leites D. *Supermanifold Theory*, Karelia Branch of the USSR Acad. Sci., Petrozavodsk, 1983, 200 pp. (In Russian)

[L2] Leites D. (ed.), *Seminar on supermanifolds*, vv.1–34. Reports of Stockholm University, 1987–1992, 2100 pp.

[LSS] Leites D., Saveliev M. V., Serganova V. V., Embeddings of $\mathfrak{osp}(N|2)$ and completely integrable systems. In: Markov M., Man'ko V. (eds.) *Proc. International Conf. Group-theoretical Methods in Physics*, Yurmala, May, 1985. Nauka, Moscow, 1986, 377–394 (English translation: VNU Sci Press, 1987, 255–297.

[LS] Leites D., Shchepochkina I., Classification of simple Lie superalgebras of vector fields with polynomial coefficients (to appear)

[M] Manin Y., *Gauge field theory and complex geometry.* Second edition. Grundlehren der Mathematischen Wissenschaften, 289. Springer-Verlag, Berlin, 1997. xii+346 pp.

[MR] Manin Yu. I., Radul A. O., A supersymmetric extension of the Kadomtsev–Petviashvili hierarchy. Comm. Math. Phys. 98 (1985), no. 1, 65–77

[OKh] Ovsienko V. Yu., Khesin B. A., The super Korteweg-de Vries equation as an Euler equation.) Funktsional. Anal. i Prilozhen. 21 (1987), no. 4, 81–82. (Russian; English translation: Functional Anal. Appl. 21 (1988), no. 4, 329–331)

[Ra] Radul A. O., Superstring Schwarz derivative and the Bott cocycle. In: Yu. Manin et. al (eds.) *Numerical methods for solutions of boundary and initial value problems of differential equations*, Moscow univ. Press, Moscow, 1986, 53–67 (Russian) and in: B. Kuperschmidt (ed.) *Integrable and superintegrable systems*, World Sci. Publishing, Teaneck, NJ, 1990, 336–351

[Sh] Shander V. N., Vector fields and differential equations on supermanifolds. (Russian) Funktsional. Anal. i Prilozhen. 14 (1980), no. 2, 91–92 (English translation: Functional Anal. Appl. 14 (1980), no. 2, 160–162)

2.3. The list of Sturm-Liouville operators. For $\mathfrak{k}^L(1|n)$ and $n = 0, 1$ we can deduce the form of the Sturm-Liouville operators by factorization. For $n > 1$ and for $\mathfrak{k}^M(1|n)$ we define the Sturm-Liouville operators as self-adjoint operators. In the following table we use an abbreviation: $\Delta = \tilde{K}_\theta(\tilde{K}_1)^{-1} - (\tilde{K}_1)^{-1}\tilde{K}_\theta$. The Sturm-Liouville operator is the sum of the operator given in the tables with a potential F, where $F \in \mathcal{F}$ or $F \in \Pi(\mathcal{F})$: the parity of the potential should be equal to that of the operator.

n	0	1	2	3
$\mathfrak{k}^L(1\|n)$	K_1^2	$K_\theta K_1$	$K_\xi K_\eta - K_\eta K_\xi$	$(K_\xi K_\theta K_\eta - K_\eta K_\theta K_\xi)(K_1)^{-1}$
$\mathfrak{k}^M(1\|n)$	–	$\Delta\tilde{K}_1^2$	$\Delta\tilde{K}_{\theta_1}\tilde{K}_1$	$\Delta(\tilde{K}_\xi\tilde{K}_\eta - \tilde{K}_\eta\tilde{K}_\xi)$

$\mathfrak{k}^L(1\|4)$	(1) $(K_{\xi_1}K_{\eta_1} - K_{\eta_1}K_{\xi_1})(K_{\xi_2}K_{\eta_2} - K_{\eta_2}K_{\xi_2})(K_1)^{-2}$ (2) $xK_{x^{-1}}(K_1)^{-1} - (K_1)^{-1}xK_{x^{-1}}$ (3) $c \neq 0$ any constant
$\mathfrak{k}^M(1\|4)$	(1) $\Delta\tilde{K}_{\theta_1}(\tilde{K}_\xi\tilde{K}_\eta - \tilde{K}_\eta\tilde{K}_\xi)(\tilde{K}_1)^{-1}$

So far I did not write an explicit expression for the Sturm-Liouville operator corresponding to cocycle (3) or for $\mathfrak{vect}^L(1|2)$ and $\mathfrak{vect}^L_\lambda(1|2)$.

For the Lie superalgebra $\mathfrak{vect}^L(1|1)$ the Sturm-Liouville operator is the same operator as for $\mathfrak{k}^L(1|2)$ but rewritten in the form of a matrix and with η replaced with ∂_ξ. We leave as an exercise to the reader the pleasure to write this matrix explicitely as well as to reexpress it in terms of the fields M_f for $\mathfrak{m}^L(1)$.

For $\mathfrak{vect}^L(1|2)$ and $\mathfrak{svect}^L_\lambda(1|2)$ the Sturm-Liouville operators can be obtained from the Sturm-Liouville operator for $\mathfrak{k}^L(1|4)$ after restriction.

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